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Coprime extensions of indecomposable solutions to the Yang–Baxter equation

Carsten Dietzel

We introduce a method to extend involutive nondegenerate set-theoretic solutions to the Yang–Baxter equation by means of equivariant mappings to graded modules, thus leading to the notion of a *twisted extension*. Furthermore, we define coprime extensions of solutions and prove that each coprime extension of indecomposable solutions can be obtained as a suitable twisted extension. We then apply our results to obtain a full description of indecomposable solutions of size pqr, where p, q and r are different primes, from a structure theorem of Cedó and Okniński. We close with some remarks on a cohomology theory for solutions developed by Lebed and Vendramin.

We express our results in the language of cycle sets.

Introduction

Let V be a finite-dimensional vector space over a commutative field. For an automorphism $R : V^{\otimes 2} \to V^{\otimes 2}$, we say that R satisfies the (*linear*) *Yang–Baxter equation* if the identity

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{YBE1}$$

is satisfied in $\mathrm{End}(V^{\otimes 3})$, where $R_{12}(u \otimes v \otimes w) = R(u \otimes v) \otimes w$ and $R_{23}(u \otimes v \otimes w) = u \otimes R(v \otimes w)$.

Equation (YBE1) was first studied by the physicists Baxter [1] and Yang [25] in the context of statistical physics and integrable systems. Later, Turaev discovered that solutions to the (linear) Yang–Baxter equation give rise to link invariants [22], such as the Jones polynomial, therefore providing a framework for both the unification of known link invariants and the construction of novel ones. Therefore an understanding of the Yang–Baxter equation and its solutions sheds new light on the theory of knots.

However, solutions to (YBE1) are, in general, hard to obtain, as their construction requires sophisticated tools, such as the representation theory of quantum groups. By 1992, most solutions Drinfeld [8] was aware of were deformations of the *flip* solution $R(v \otimes w) = w \otimes v$. This motivated him to ask for a systematic study of *set-theoretic solutions* to the Yang–Baxter equation which correspond to linear solutions that permute a basis of $V^{\otimes 2}$, with the goal of obtaining novel families of solutions by deformation.

Let X be a nonempty finite set and $r : X \times X \to X \times X$ a bijection. The pair (X, r) is then said to be a *set-theoretic solution to the Yang–Baxter equation* if the functional equation for maps $X^3 \to X^3$

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23} \tag{YBE2}$$

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holds, where $r_{ij} : X^3 \to X^3$ is the mapping that acts as r on the coordinates i and j and as identity on the others.

It was quickly discovered by Etingof, Schedler and Soloviev [9] and Gateva-Ivanova and Van den Bergh [11] that set-theoretic solutions can be analyzed extraordinarily well by means of group-theoretic tools. This observation attracted the attention of a growing and diverse community of researchers. Over the last years, new tools, such as cycle sets [18] and braces [19] were introduced into this field and connections have been drawn to other fields of algebra, such as Garside theory [4].

The algebraic approach to set-theoretic solutions is most efficient after imposing some restrictions on r. Writing $r(x, y) = (\lambda_x(y), \rho_y(x))$, we call a set-theoretic solution to the Yang–Baxter equation *nondegenerate* if the maps λ_x and ρ_x are bijective for all $x \in X$. In this work, solutions will always be nondegenerate, and we will furthermore assume that X is *involutive*, meaning that $r^2 = \mathrm{id}_{X\times X}$. These assumptions are necessary for an approach by means of nondegenerate cycle sets and left braces (of abelian type). However, it is notable that an algebraic theory can also be set up for solutions where r is not necessarily involutive [17; 12] and for solutions where r is not necessarily bijective [5].

From now on, we will always assume that X is involutive and nondegenerate.

By a result of Rump [18], set-theoretic solutions to the Yang–Baxter equation are in bijective correspondence with simple algebraic structures, the *nondegenerate cycle sets* (*cycle sets*, for short):

Definition. A *cycle set* is a set X with a binary operation $X \times X \to X$; $(x, y) \mapsto x * y$ such that the following axioms are satisfied:

$$(x * y) * (x * z) = (y * x) * (y * z) \quad \text{for all } x, y, z \in X, \tag{C1}$$

$$\sigma_x : X \to X; \quad y \mapsto x * y \quad \text{is bijective for all } x \in X. \tag{C2}$$

If furthermore the *square map*

$$\mathrm{Sq} : X \to X; \quad x \mapsto x * x \tag{C3}$$

is bijective, a cycle set X is *nondegenerate*.

A *homomorphism* from a cycle set X to a cycle set Y is defined as a map $f : X \to Y$ such that $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Convention. By a cycle set we mean a nonempty finite nondegenerate cycle set. We will refer to a cycle set $(X, *)$ by its underlying set X.

With each set-theoretic solution $r(x, y) = (\lambda_x(y), \rho_y(x))$, one can associate a cycle set structure on X by putting $x * y = \lambda_x^{-1}(y)$. On the other hand, for each cycle set X, there is a unique set-theoretic solution on X that satisfies $r(x * y, x) = (y * x, y)$. By these assignments, cycle sets and solutions are in bijective correspondence [18, Proposition 1].

A central problem in the theory of set-theoretic solutions is the classification of solutions. One way to approach this problem is via the classification of cycle sets by size. However, a full classification by size has only been obtained for cycle sets of size $\leqslant 11$ by Van Caudenberg, Bogaerts and Vendramin [23], by

means of a novel computational approach. The results of their computations suggest that the number of cycle sets explodes as the size increases, so that a full classification of cycle sets of larger sizes is out of reach. Therefore we restrict the issue of classification to a smaller class of cycle sets, the *indecomposable* cycle sets, that is, cycle sets X with *no* nontrivial partition $X = X_1 \sqcup X_2$ such that $X_i * X_i = X_i$ for $i \in \{1, 2\}$. Under the correspondence between cycle sets and set-theoretic solutions, this means that there is no nontrivial partition $X = X_1 \sqcup X_2$ such that $r(X_i \times X_i) = X_i \times X_i$ for $i \in \{1, 2\}$.

The restriction to indecomposable cycle sets is reasonable insofar as they can be considered as building blocks for general cycle sets, in that they can be recursively assembled via a matching process to finally obtain all cycle sets. In fact, in the indecomposable case, there are several classification results by size:

(1) Etingof, Schedler and Soloviev classified indecomposable cycle sets of size p, where p is a prime [9].

(2) The results of Van Caudenberg, Bogaerts and Vendramin encompass the classification of indecomposable cycle sets of size $\leqslant 11$ [23].

(3) Using a theorem of Cedó and Okniński on cycle sets of squarefree size [2], the classification of indecomposable cycle sets of size pq, where p and q are different primes, follows from a result of Jedlička and Pilitowska [14].

(4) In collaboration with Properzi and Trappeniers, the author of this article has recently obtained a classification of indecomposable cycle sets of size p^2, where p is a prime [7].

Defining the *retraction* of a cycle set X as the quotient $X_{\mathrm{ret}} = X/\!\sim$, where $\sim$ is the equivalence relation given by $x \sim y \iff \sigma_x = \sigma_y$, one can define a cycle set structure on X_{ret} by putting $[x] * [y] = [x * y]$. Under this structure, the canonical quotient map $\mathrm{ret} : X \to X_{\mathrm{ret}}$ becomes a homomorphism of cycle sets. In [2], Cedó and Okniński prove that if X is a cycle set of squarefree size > 1, then $|X_{\mathrm{ret}}| < |X|$ and X_{ret} is again a cycle set of squarefree size. Therefore X is an *extension* of X_{ret}, which opens up the possibility to parametrize X in terms of dynamical cocycles over its retraction X_{ret}, as has been discovered by Vendramin [24]. Lebed and Vendramin [16] refined this approach and showed that certain extensions can be parametrized by 2-cocycles with values in an abelian group.

The Cedó–Okniński results imply that indecomposable cycle sets of squarefree size can be obtained by a recursive extension process from smaller ones of their kind. However, direct computation of dynamical cocycles in order to extend solutions is difficult. In order to circumvent this problem, we develop a method to parametrize a certain class of extensions that we call *coprime*. It will be shown that all coprime extensions of an indecomposable cycle set can be obtained as *twisted extensions* by maps into graded modules. This method enables us to recover an indecomposable cycle set X from X_{ret}, whenever the size of the *permutation group* $\mathcal{G}(X_{\mathrm{ret}})$ is coprime to the size of the fibers of $X \twoheadrightarrow X_{\mathrm{ret}}$ (Section 2). We then develop a method to check if a twisted extension is indecomposable and describe the permutation group whenever the resulting extension is coprime (Section 3). We then apply this method to give a full description of indecomposable cycle sets of size pqr in terms of multiplicative characters whenever p, q and r are different primes (Section 4). We close with a brief discussion of Lebed–Vendramin cohomology.

Due to the highly nontrivial descriptions obtained for the cycle sets of size pqr and multipermutation level 3 (see Theorems 4.8 and 4.10), we refrain from an explicit description of the corresponding set-theoretic solutions.

1. Preliminaries

In this section, we collect basic results on cycle sets and braces.

Having already defined cycle sets in the introduction, we recall the exact definition of indecomposability.

Definition 1.1 [9, Definition 2.5; 18]. A cycle set X is called *indecomposable* if for each partition $X = X_1 \sqcup X_2$ into subcycle sets, we have either $X_1 = \varnothing$ or $X_2 = \varnothing$.

We will rely a lot on the following fundamental fact:

Proposition 1.2. *Let $f : Y \twoheadrightarrow X$ be a surjective homomorphism of finite indecomposable cycle sets. Then all fibers of f have the same size, that is, $|f^{-1}(x)| = |Y|/|X|$ for all $x \in X$. In particular, $|X|$ divides $|Y|$.*

Proof. See [2, Lemma 3.3]. □

A remarkable property of finite cycle sets is that a significant proportion of them have repeated permutations. This gives rise to the notion of the *retraction* of cycle sets.

Definition 1.3 [9, §3.2; 18]. Let X be a cycle set. Then the *retraction* of X is the quotient $X_{\mathrm{ret}} = X/\!\sim$, where $\sim$ is the equivalence relation given by $x \sim y \iff \sigma_x = \sigma_y$, together with the cycle set operation $[x] * [y] = [x * y]$.

That X_{ret} indeed is a cycle set under the given operation is proven in [18, Lemma 2].

A cycle set can often be retracted more than once, which motivates the notion of *multipermutation level* [9, Definition 3.1]: define iteratively $X^{(0)} = X$, and $X^{(n+1)} = X^{(n)}_{\mathrm{ret}}$ for $n \geqslant 0$. Then the *multipermutation level* of X is defined as

$$\mathrm{mpl}(X) = \begin{cases} \min\{n \geqslant 0 : |X^{(n)}| = 1\} & \text{if there exists } n \geqslant 0 \text{ such that } |X^{(n)}| = 1, \\ \infty & \text{else.} \end{cases}$$

We will most of the time denote X_{ret} by $X^{(1)}$.

An important invariant of a cycle set is its *permutation group*, which is the subgroup

$$\mathcal{G}(X) = \langle \sigma_x \mid x \in X \rangle \leqslant \mathrm{Sym}_X,$$

where, as in the axioms, $\sigma_x(y) = x * y$. Note that X is indecomposable if and only if $\mathcal{G}(X)$ acts transitively on X.

It turns out that $\mathcal{G}(X)$ can be enriched by another group structure that turns $\mathcal{G}(X)$ into what is called a left brace of abelian type. As we will nowhere delve into the very rich theory of skew braces, we restrict the definition of a *brace* to this special case. In this article, we use the definition of Cedó, Jespers and Okniński, but remark that braces of abelian type go back to the work of Rump [19]. In the following, we collect some basic results on braces, most of which can be found in [3]. If not, we explicitly state the reference.

Definition 1.4. A *brace* is a triple $B = (B, +, \circ)$ such that $B^+ = (B, +)$ is an abelian group — the *additive group* of B — $B^\circ = (B, \circ)$ is a group — the *multiplicative group* of B — and the identity

$$a \circ (b + c) = a \circ b - a + a \circ c$$

is satisfied for all $a, b, c \in B$.

A *homomorphism* from a brace A to a brace B is defined as a map $f : A \to B$ that is a homomorphism for both the additive groups and the multiplicative groups. Everything that comes with the concept of homomorphism (isomorphism, ...) is defined in the obvious way.

In general, we will suppress the $\circ$ and abbreviate $a \circ b = ab$. Note that it can be shown that B^+ and B° share the same identity, which we denote by 0.

The λ-*action* of a brace is defined by $\lambda_a(b) = ab - a$ $(a, b \in B)$. As it satisfies the equations

$$\lambda_a(b + c) = \lambda_a(b) + \lambda_a(c); \quad \lambda_{ab} = \lambda_a \circ \lambda_b,$$

the map

$$\lambda : (B, \circ) \to \mathrm{Aut}(B^+); \quad a \mapsto \lambda_a$$

is a well-defined homomorphism of groups. In particular, B^+ becomes a B°-module.

A brace can be seen as a *ring-like* structure; accordingly there is a notion of (left) ideal.

Definition 1.5. Let B be a brace. A *left ideal* of B is a subset $I \subseteq B$ such that I is a subgroup of B^+ that is invariant under the λ-action, in that $\lambda_a(I) = I$ for all $a \in I$.

If furthermore I is a normal subgroup of B°, then I is called an *ideal* of B.

Note that each left ideal $I \subseteq B$ is also a multiplicative subgroup of B. As in ring theory, the quotient

$$B/I = \{a + I : a \in B\} = \{a \circ I : a \in B\}$$

can be formed, and if I is an ideal, B/I is again a brace under the operations inherited by B.

A distinguished ideal of a brace B that will play an important role throughout this work is the *socle*

$$\mathrm{Soc}(B) = \{a \in B : ab = a + b \text{ for all } b \in B\} = \ker(\lambda).$$

If B is a brace and $a, b \in B$, we define ${}^a b = aba^{-1}$. We quickly reprove a well-known result about the socle that will be useful later.

Proposition 1.6. *Let B be a brace. Then for $a \in B$ and $b \in \mathrm{Soc}(B)$, we have*

$$\lambda_a(b) = {}^a b.$$

Proof. Let a and b be as in the statement. The normality of $\mathrm{Soc}(B)$ implies that ${}^a b \in \mathrm{Soc}(B)$. We can therefore rewrite the identity $ab = {}^a ba$ as $a + \lambda_a(b) = {}^a b + \lambda_{{}^a b}(a) = {}^a b + a$. Canceling a, we obtain $\lambda_a(b) = {}^a b$. □

Convention. For an integer n, we denote by $\pi(n)$ the set of primes dividing n. In order to prevent potential confusion, we clarify here that a term of the form $\pi(-)$ will always refer to the set of prime divisors of the argument while π without an argument will be used to refer to some fixed set of primes.

We repeat the definition of an important kind of left ideal that we often consider:

Definition 1.7. Let B be a finite brace and π a set of primes. Then the *π-primary component* of B is defined as

$$B_\pi = \{a \in B : \pi(o_+(a)) \subseteq \pi\},$$

the (additive) Hall-π-subgroup of B_π, where $o_+(a)$ denotes the order of a in B^+.

Note that B_π is a characteristic subgroup in B^+, and thus a left ideal in B.

We will furthermore rely on the notion of a *semidirect product* of braces. As it is sufficient for our needs, we will restrict to *external* semidirect products.

Definition 1.8. Let B be a brace and let $B_1, B_2 \leqslant B$ be left ideals where B_1 is an ideal. If the additive group of B^+ decomposes as a direct sum $B^+ = B_1 \oplus B_2$, we call B an (*external*) *semidirect product* of B_1 and B_2. In such a case, we write $B = B_1 \rtimes B_2$ (or $B_2 \ltimes B_1$, depending on the factor that we want to put the emphasis on).

The relevance of braces for the theory of cycle sets lies in the fact that the permutation group $\mathcal{G}(X)$ of a cycle set X carries a unique additive group structure $\mathcal{G}(X)$ such that $(\mathcal{G}(X), +, \circ)$ is a brace. The multiplication $\circ$ is the usual composition of permutations and, defining the element $\lambda_x = \sigma_x^{-1} \in \mathcal{G}(X)$ for $x \in X$, there is a unique brace structure on $\mathcal{G}(X)$ with

$$\lambda_x + \lambda_y = \lambda_x \circ \lambda_{\lambda_x^{-1}(y)}$$

for all $x, y \in X$ or, equivalently, $\sigma_x^{-1}(y) = \lambda_x(y)$, the λ-operation of the brace.

Note that, under this brace structure, $\lambda_{x*y} = \lambda_{\lambda_x}^{-1}(\lambda_y)$ for $x, y \in X$.

Convention. Sym_X and all elements therein act from the left. For an element $g \in \mathcal{G}(X) \leqslant \mathrm{Sym}_X$, the notation λ_g has a threefold meaning: it refers to the action of g on X, to the λ-action of g on $\mathcal{G}(X)$ and, as $\mathcal{G}(X)$ is a permutation group, λ_g ultimately refers to the element g itself. There is no danger of confusion, though, as the λ-action on $\mathcal{G}(X)$ is compatible with the λ-action on X and it will always be clear which set is acted upon by $\mathcal{G}(X)$. However, we will avoid equating $g = \lambda_g$ when no actions on other elements are involved.

Furthermore, for an element $g \in \mathcal{G}(X)$, we will write $\sigma_g = \lambda_g^{-1}$ whenever it is more convenient to do so.

It is important to realize that the construction of the permutation group of a cycle set is *not* functorial without restriction. However, it becomes so after restricting to surjective homomorphisms.

Theorem 1.9. *Let $\varphi : Y \twoheadrightarrow X$ be a surjective homomorphism of cycle sets. Then there is a unique brace homomorphism $\mathcal{G}(\varphi) : \mathcal{G}(Y) \to \mathcal{G}(X)$ such that $\mathcal{G}(\varphi)(\lambda_y) = \lambda_{\varphi(y)}$ for all $y \in \mathcal{G}(Y)$.*

Proof. See [15, p. 5]. □

The following result will be fundamental for our investigations, and due to its ubiquity, we will not refer to it all the time.

Theorem 1.10. *Let* $\mathrm{ret} : X \twoheadrightarrow X^{(1)}$ *be the retraction morphism. Then* $\ker(\mathcal{G}(\mathrm{ret})) = \mathrm{Soc}(\mathcal{G}(X))$. *In particular, if* $f : Y \twoheadrightarrow X$ *is an epimorphism such that* $\mathrm{ret} : Y \twoheadrightarrow Y_{\mathrm{ret}}$ *factors through* f, *then* $\ker(\mathcal{G}(f)) \subseteq \mathrm{Soc}(\mathcal{G}(Y))$.

Proof. See [3, Lemma 6.1]. □

2. Construction of coprime extensions of cycle sets

Notation. Given a group G, acting from the left on a set X via the operation $(g, x) \mapsto g \cdot x$, we denote the orbit of an element $x \in X$ under the action of G by

$$\mathcal{O}_G(x) = \{g \cdot x : g \in G\}.$$

Definition 2.1. Let X and Y be cycle sets where X is indecomposable. We call a surjective homomorphism $f : Y \twoheadrightarrow X$ a *constant extension* (*extension*, for short) if for all $x, y \in Y$, we have the implication $f(x) = f(y) \implies \sigma_x = \sigma_y$. If Y is also indecomposable, we call f an *extension of indecomposable cycle sets*.

An extension of indecomposable cycle sets $f : Y \twoheadrightarrow X$ is called *coprime* if $\gcd(|\mathcal{G}(X)|, |Y|/|X|) = 1$.

Note that by Proposition 1.2, $|Y|/|X| = |f^{-1}(x)|$ for all $x \in X$.

A cycle set can occur as an extension of a given cycle set in very different ways. In order to take this into account, we introduce the notion of *equivalence* of extensions which refines the notion of isomorphism.

Definition 2.2. Let $f_i : Y_i \twoheadrightarrow X$ for $i = 1, 2$ be two extensions of a cycle set X. We call f_1 and f_2 *equivalent* if there is an isomorphism $\iota : Y_1 \to Y_2$ such that $f_1 = f_2 \circ \iota$.

The next proposition follows directly from Theorem 1.10.

Proposition 2.3. *Let* $f : Y \twoheadrightarrow X$ *be an extension of cycle sets. Then* $\ker(\mathcal{G}(f)) \subseteq \mathrm{Soc}(\mathcal{G}(Y))$.

Before giving the construction of general coprime extensions of indecomposable cycle sets, we need to introduce and recall some concepts:

Let X be a set and A an abelian group. It is easy to see that A^X, the group of tuples $a = (a_x)_{x \in X}$ under coordinatewise addition, acts on $X \times A$ by

$$a + (x, b) = (x, b + a_x).$$

We want to give a description of this action that is equivariant with respect to an action of a group G on X. To this end, let X be a set that is acted upon by G by means of an operation denoted by $(g, a) \mapsto g \cdot x$ for $g \in G$ and $x \in X$. Define an *X-graded G-module* as an abelian group A with a direct decomposition into subgroups

$$A = \bigoplus_{x \in X} A_x,$$

and an action by automorphisms $\rho : G \to \mathrm{Aut}(A)$, also denoted by $g \cdot a = (\rho(g))(a)$, that respects the grading in the sense that for all $g \in G$ and $x \in X$ we have that

$$g \cdot A_x \subseteq A_{g \cdot x}.$$

We write the elements of A as tuples $(a_x)_{x \in X}$ with $a_x \in A_x$ and refer to the subgroups A_x as the *graded components* of A. Furthermore, given two X-graded G-modules A and B, we call an additive homomorphism $f : A \to B$ a *homomorphism of X-graded G-modules* if f is G-equivariant, i.e., $f(g \cdot a) = g \cdot f(a)$ for all $g \in G$ and $a \in A$, and if f respects the grading by X, meaning that $f(A_x) \subseteq B_x$ for all $x \in X$. All notions that come along with homomorphisms (isomorphism, ...) are defined in the obvious way.

Given an X-graded G-module A, we define the *scattering* of A as the set

$$A^{\mathrm{sc}} = \bigsqcup_{x \in X} \{x\} \times A_x,$$

which has a canonical G-action given by

$$g \cdot (x, a) = (g \cdot x, g \cdot a).$$

It can easily be shown that A acts on A^{sc} in a G-equivariant way.

Proposition 2.4. *Let A be an X-graded G-module. Then the binary operation*

$$A \times A^{\mathrm{sc}} \to A^{\mathrm{sc}}; \quad (a, (x, b)) \mapsto a + (x, b) = (x, b + a_x),$$

is an action of A on A^{sc} that is G-equivariant in the sense that

$$g \cdot (a + (x, b)) = g \cdot a + g \cdot (x, b) \tag{2-1}$$

for $g \in A$, $a \in A$ and $(x, b) \in A^{\mathrm{sc}}$.

Proof. As A acts componentwise on A^{sc} by regular actions of the respective factors A_x, it is easy to check this is indeed a group action. For (2-1), we calculate

$$g \cdot (a + (x, b)) = g \cdot (x, b + a_x) = (g \cdot x, g \cdot b + g \cdot a_x) = g \cdot a + g \cdot (x, b),$$

where in the last equality, we use that $(g \cdot a)_{g \cdot x} = g \cdot a_x$. □

Definition 2.5. Let X be a cycle set and let A be an X-graded group with respect to the action of $\mathcal{G}(X)$ on X. Let $\Phi : X \times X \to A$ be a map such that

$$\Phi(x, y) \in A_y, \tag{2-2}$$

$$g \cdot \Phi(x, y) = \Phi(\lambda_g(x), \lambda_g(y)), \tag{2-3}$$

for $x, y \in X$ and $g \in \mathcal{G}(X)$. Then define the *twisted extension* $X \otimes_\Phi A$ as the set A^{sc} with the binary operation

$$(x, a) * (y, b) = \lambda_x^{-1} \cdot (y, b + \Phi(x, y)) = (x * y, \lambda_x^{-1} \cdot (b + \Phi(x, y))), \tag{2-4}$$

together with the projection $\mathrm{pr} : X \otimes_\Phi A \twoheadrightarrow X; (x, a) \mapsto x$.

Proposition 2.6. *The projection* $\mathrm{pr} : X \otimes_\Phi A \twoheadrightarrow X$ *defines an extension of cycle sets.*

Proof. First note that $\lambda_x\lambda_{x*y} = \lambda_y\lambda_{y*x}$, which is a mere reformulation of (C1) in terms of the λ-action. Using that, we calculate

$$\begin{aligned}((x,a)*(y,b))*((x,a)*(z,c)) &= (x*y, \lambda_x^{-1}\cdot(b+\Phi(x,y)))*(x*z, \lambda_x^{-1}\cdot(c+\Phi(x,z)))\\ &= \Big((x*y)*(x*z), \lambda_{x*y}^{-1}\cdot\big(\lambda_x^{-1}\cdot(c+\Phi(x,z)) + \underbrace{\Phi(x*y, x*z)}_{=\lambda_x^{-1}\cdot\Phi(y,z)}\big)\Big)\\ &= \big((x*y)*(x*z), (\lambda_x\lambda_{x*y})^{-1}\cdot(c+\Phi(x,z)+\Phi(y,z))\big)\\ &= \big((y*x)*(y*z), (\lambda_y\lambda_{y*x})^{-1}\cdot(c+\Phi(x,z)+\Phi(y,z))\big)\\ &= ((y,b)*(a,x))*((y,b)*(z,c)).\end{aligned}$$

Furthermore, it is quickly seen from the nondegeneracy of X that the square map

$$\mathrm{Sq}(x,a) = \big(\mathrm{Sq}(x), \lambda_x^{-1}\cdot(a+\Phi(x,x))\big)$$

is bijective as well, so $X\otimes_\Phi A$ is indeed a nondegenerate cycle set. Furthermore, it is clear that pr is a homomorphism of cycle sets. As, moreover, $(x,a)*(y,b) = \lambda_x^{-1}\cdot(y, b+\Phi(x,y))$ is independent of a, the map pr is indeed an extension of cycle sets. □

Note that different maps Φ can define isomorphic twisted extensions. This will be needed later.

Proposition 2.7. *Let* $\mathrm{pr} : X\otimes_\Phi A \twoheadrightarrow X$ *be a twisted extension and let* $f : A \to A$ *be an isomorphism of* X*-graded* G*-modules. Then, with* $\Phi' = f\circ\Phi$*, the map* $\mathrm{pr}' : X\otimes_{\Phi'} A \twoheadrightarrow X$ *is also a twisted extension, and the map*

$$g : X\otimes_\Phi A \to X\otimes_{\Phi'} A; \quad (x,a)\mapsto (x, f(a))$$

is an equivalence between pr *and* pr′.

Proof. It is quickly checked that Φ' still satisfies (2-2) and (2-3), so Φ' again defines a twisted extension. The equivalence property follows from a short calculation. □

Corollary 2.8. *Let* $\mathrm{pr} : X\otimes_\Phi A \twoheadrightarrow X$ *be a twisted extension and let* k *be an integer such that* $\gcd(k, |A|) = 1$*. Then the extension* $\mathrm{pr}' : X\otimes_{k\cdot\Phi} A \twoheadrightarrow X$ *is equivalent to* pr.

Proof. Under the given conditions, $A\to A$; $a\mapsto k\cdot a$ is an isomorphism of X-graded G-modules. □

In the special case when the X-graded $\mathcal{G}(X)$-module A is a permutation module, i.e., there is an isomorphism $A\cong B^X$ for some abelian group B, where $(g\cdot f)(x) = f(\lambda_g^{-1}(x))$, the action on $A^{\mathrm{sc}} = X\times B$ simplifies to $g\cdot(x,a) = (\lambda_g(x), a)$, so we can obtain twisted extensions in a simpler way: in this case, they can be described by a map $\Gamma : X\times X\to B$ that satisfies the invariance property

$$\Gamma(\lambda_g(x), \lambda_g(y)) = \Gamma(x,y) \quad \text{for } x, y\in X \text{ and } g\in\mathcal{G}(X). \tag{2-5}$$

In this special case, (2-4) turns into the following operation on $X \times B$:

$$(x, a) * (y, b) = (x * y, b + \Gamma(x, y)). \tag{2-6}$$

We denote the resulting cycle set by $X \times_\Gamma B$ and call it a *parallel extension* of X by B (via Γ).

Despite trying to make our approach as independent of a basepoint as possible, it will occasionally be useful to make use of the *fundamental group* of a cycle set [20].

Definition 2.9. Let X be an indecomposable cycle set and $x_0 \in X$. Then the *fundamental group* of X with *basepoint* x_0 is the subgroup

$$\Pi_1(X, x_0) = \{g \in \mathcal{G}(X) : \lambda_g(x_0) = x_0\} \leqslant \mathcal{G}(X).$$

Remark 2.10. The fundamental group $\Pi_1(X, x_0)$ also plays a role in Rump's covering theory [21], as its subgroups are in correspondence with *strong coverings*, i.e., surjective homomorphisms $f : (Y, y_0) \twoheadrightarrow (X, x_0)$ where Y is an indecomposable cycle set with basepoint y_0 and $\mathcal{G}(f) : \mathcal{G}(Y) \to \mathcal{G}(X)$ is an isomorphism. It can be shown that $|Y|/|X|$ divides $|\mathcal{G}(X)|$ in these cases, so no nontrivial strong covering is a coprime extension. Surprisingly, the fundamental group $\Pi_1(X, x_0)$ will nevertheless play a significant role in the description of coprime extensions.

As we are mostly concerned with indecomposable cycle sets, it will sometimes be favorable to work with the following version of $\mathcal{G}(X)$-equivariance:

Let X be indecomposable. Choosing a point $x_0 \in X$, we consider the map $\Phi^0(x) = \Phi(x_0, x)$. Then (2-2) tells us that

$$\Phi^0(y) \in A_y \quad \text{for all } \ y \in X. \tag{2-7}$$

Putting a $g \in \Pi = \Pi_1(X, x_0)$ in (2-3), equivariance furthermore specializes to

$$\Phi^0(\lambda_g(y)) = g \cdot \Phi^0(y). \tag{2-8}$$

On the other hand, given a mapping $\Phi^0 : X \to A$ such that (2-7) is satisfied and (2-8) holds for all $g \in \Pi$, a mapping $\Phi : X \times X \to A$ satisfying (2-2) and (2-3) is easily constructed by

$$\Phi(\lambda_g(x_0), y) = g \cdot \Phi^0(\lambda_g^{-1}(y)). \tag{2-9}$$

Therefore we can write the operation on $X \otimes_\Phi A$ as

$$(x, a) * (y, b) = \big(x * y, (\lambda_x^{-1} g) \cdot (b + \Phi^0(\lambda_g^{-1}(y)))\big), \tag{2-10}$$

where $g \in \mathcal{G}(X)$ satisfies $\lambda_g(x_0) = x$.

We see from the above discussion that, given an X-graded $\mathcal{G}(X)$-module A, any map $\Phi^0 : X \to A$ satisfying (2-7) and (2-8) defines an equivariant map $\Phi : X \times X \to A$ and therefore can be used to construct the cycle set $X \otimes_\Phi A$ by means of (2-10).

In the special case of a parallel extension $X \times_\Gamma B$, we similarly define $\Gamma^0(x) = \Gamma(x_0, x)$ which satisfies $\Gamma^0(\lambda_g(x)) = \Gamma^0(x)$ whenever $g \in \Pi_1(X, x_0)$ and Γ can be recovered from such a map by $\Gamma(x, y) = \Gamma^0(\lambda_g^{-1}(y))$ for $g \in \mathcal{G}(X)$ with $\lambda_g(x_0) = y$.

The expression in (2-10) now becomes

$$(x, a) * (y, b) = \big(x * y, b + \Gamma^0(\lambda_g^{-1}(y))\big), \tag{2-11}$$

where $g \in \mathcal{G}(X)$ satisfies $\lambda_g(x_0) = x$.

We will later see that an understanding of the orbits of $\Pi_1(X, x_0)$ on X will be of high relevance for our classification.

We now state one main theorem of this section:

Theorem 2.11. *Let $f : Y \twoheadrightarrow X$ be a coprime extension of indecomposable cycle sets. Then f is equivalent to a twisted extension* $\mathrm{pr} : X \otimes_\Phi A \to X$.

The following lemma is a special case of a more general, unpublished, observation that was made in former collaboration with Silvia Properzi and Senne Trappeniers. In this special case, we give a simplified proof that makes use of a fixed-point argument similar to the one discovered by Trappeniers.

Lemma 2.12. *Let $f : Y \twoheadrightarrow X$ be an extension of indecomposable cycle sets and $K = \ker \mathcal{G}(f)$. Then $\pi(|K|) \subseteq \pi(|Y|/|X|)$.*

Proof. Take any prime $p \notin \pi(|Y|/|X|)$ and let K_p the unique additive p-Sylow subgroup of K which is also the unique multiplicative p-Sylow subgroup of K as K is a trivial ideal. Therefore K_p is characteristic in K°. K leaves the fibers $f^{-1}(x)$ $(x \in X)$ setwise fixed. As $p \nmid |f^{-1}(x)| = |Y|/|X|$, the action of K_p has a fixed point on Y, say y. Therefore $K_p \leqslant \mathcal{G}(Y)_y$, the stabilizer of y. But K_p is characteristic in K°, which is normal in $\mathcal{G}(Y)^\circ$. Therefore K_p is normal in $\mathcal{G}(Y)^\circ$, and therefore acts trivially on Y. This implies that $K_p = 1$. □

Recall furthermore that the *holomorph* of a group G is defined as the semidirect product (of groups)

$$\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G),$$

where $\mathrm{Aut}(G)$ acts by the respective automorphisms on the first factor. There is a canonical action of $\mathrm{Hol}(G)$ by affine maps on G via $(g, \alpha) \cdot h = g\alpha(h)$.

Lemma 2.13. *Let A be a finite group and let $G \leqslant \mathrm{Hol}(A)$ be a subgroup with* $\gcd(|G|, |A|) = 1$. *Then G has a fixed point on A.*

Proof. Consider the group $H = AG \leqslant \mathrm{Hol}(A)$ and the projection $\pi : \mathrm{Hol}(A) \twoheadrightarrow \mathrm{Aut}(A)$; $(a, \alpha) \mapsto \alpha$. Since $|G|$ is coprime to $|A|$, the map π sends G isomorphically onto its image $\pi(G)$. The section $s : \mathrm{Aut}(A) \to \mathrm{Hol}(A)$; $\alpha \mapsto (0, \alpha)$ now induces the section $G' = (s\pi)(G)$ of $\pi(G)$ that obviously fixes the point $0 \in A$. By the Schur–Zassenhaus theorem [13, §18.3], $G = {}^gG'$ for some $g \in \mathrm{Hol}(A)$, so G fixes the point $g \cdot 0$. □

We can finally prove the first main result of this section:

Proof of Theorem 2.11. Let $\pi = \pi(|\mathcal{G}(X)|)$, and consider the subgroups $\mathcal{G}(Y)_\pi \leqslant \mathcal{G}(Y)$ and $K = \ker(f) \trianglelefteq \mathcal{G}(Y)$. By Lemma 2.12 and the assumption of coprimality, $\gcd(|K|, |\mathcal{G}(Y)_\pi|) = 1$. Hence $\mathcal{G}(f)$ maps $\mathcal{G}(Y)_\pi$ isomorphically onto $\mathcal{G}(X)$, and we have a semidirect product of braces $\mathcal{G}(Y) = K \rtimes \mathcal{G}(Y)_\pi$.

By definition, K fixes the fibers $\mathcal{F}_x = f^{-1}(x)$, for $x \in X$, setwise. Consider the orbit set $K \setminus Y$. As K is normal in $\mathcal{G}(Y)$, the orbits $\mathcal{O}_K(y)$ for $y \in Y$ form blocks for the action of $\mathcal{G}(Y)$ on Y. As they have the same size and refine the partition of Y by the family $(\mathcal{F}_x)_{x\in X}$, we see that $|K \setminus Y| = |X| \cdot m$ for some $m \mid |\mathcal{F}_x|$.

On the other hand, the decomposition $\mathcal{G}(Y) = K \rtimes \mathcal{G}(Y)_\pi$ shows that $\mathcal{G}(Y)_\pi$ acts transitively on $|K \setminus Y|$, so $\pi(|K \setminus Y|) \subseteq \pi$. In particular, $\gcd(|K \setminus Y|, |\mathcal{F}_x|) = 1$, so $|K \setminus Y| = |X|$.

This shows that K acts transitively on each fiber. As $K \leqslant \mathrm{Soc}(K)$ by Proposition 2.3, we see that K° is abelian, so K induces regular transitive actions of abelian groups on the fiber $\mathcal{F}_x$ that are pairwise conjugate. Writing the factors as $A_x = \mathrm{im}(K \to \mathrm{Sym}_{\mathcal{F}_x})$, we can consider K as a subgroup of $A = \prod_{x\in X} A_x$ in a way that K is a subdirect product of the $(A_x)_{x\in X}$.

Denote by $s : \mathcal{G}(X) \hookrightarrow \mathcal{G}(Y)$ the canonical section that identifies $\mathcal{G}(X)$ with $\mathcal{G}(Y)_\pi$. Now pick a point $x_0 \in X$ and consider the fundamental group $\Pi_{x_0} = \Pi_1(X, x_0)$ and the fundamental subgroup $\Pi'_{x_0} = s(\Pi_{x_0})$. Then Π'_{x_0} fixes the fiber $\mathcal{F}_{x_0}$ as a set and, as K is normalized by Π'_{x_0}, we deduce that the image of Π'_{x_0} in $\mathrm{Sym}(\mathcal{F}_{x_0})$ normalizes the regular transitive action of A_{x_0} on $\mathcal{F}_{x_0}$. As $\gcd(|\Pi'_{x_0}|, |A_{x_0}|) = 1$, Lemma 2.13 implies that Π'_{x_0} has a fixed point $y_0 \in \mathcal{F}_{x_0}$.

By standard arguments concerning group actions, $\Pi_{\lambda_g(x_0)} = {}^g\Pi_{x_0}$. We furthermore see that $\lambda_{s(g)}(y_0)$ is a fixed point of $s(\Pi_{\lambda_g(x_0)})$ acting on the fiber $\mathcal{F}_{\lambda_g(x_0)}$.

Now consider the orbit $X' = \mathcal{O}_{\mathcal{G}(Y)_\pi}(y_0)$. As $\mathcal{G}(X)$ acts transitively on X, we have $X' \cap \mathcal{F}_x \neq \varnothing$ for all $x \in X$. Furthermore, if $\{y_1, y_2\} \in \mathcal{F}_x \cap X'$ for some $x \in X$, then $y_2 = \lambda_{s(g)}(y_1)$ for some $g \in \mathcal{G}(X)$. As $\lambda_{s(g)}$ then fixes the fiber $\mathcal{F}_x$ setwise, we conclude that $g \in \Pi_x$. But we have seen that the elements in $X' \cap \mathcal{F}_x$ are all fixed by $s(\Pi_x)$, so $y_1 = y_2$. Therefore $|\mathcal{F}_x \cap X'| = 1$ for all $x \in X$. For $x \in X$, define the point y_x by $\{y_x\} = X' \cap \mathcal{F}_x$.

Recalling that $A_x = \mathrm{im}(K \to \mathrm{Sym}_{\mathcal{F}_x})$ and that K is normalized by $\mathcal{G}(Y)$, we get an action of $\mathcal{G}(X)$ by automorphisms of $A = \bigoplus_{x\in X} A_x$ by

$$\mathcal{G}(X) \times A_x \to A_{\lambda_g(x)}; \quad (g, \lambda_k|_{\mathcal{F}_x}) \mapsto g \cdot (\lambda_k|_{\mathcal{F}_x}) = (\lambda_{s(g)k})|_{\mathcal{F}_{\lambda_g(x)}}.$$

With this rule A is an X-graded G-module.

We can now use the fixed points y_x for $x \in X$ to coordinatize Y by means of the bijection

$$\gamma : A^{\mathrm{sc}} \to Y; \quad (x, a) \mapsto a + y_x.$$

Here we suggestively use additive notation for the action of A_x on $\mathcal{F}_x$. More precisely, for $a = \lambda_k|_{\mathcal{F}_x}$, we mean $a + y_x = \lambda_k(y_x)$. Note that $\gamma(x, 0) = y_x$.

Now γ is, in fact, an isomorphism of $\mathcal{G}(X)$-actions: writing $a = \lambda_k|_{\mathcal{F}_x}$, we calculate for $g \in \mathcal{G}(X)$

$$\begin{aligned}\lambda_{s(g)}(\gamma(x, a)) &= \lambda_{s(g)}(\lambda_k(y_x)) = \lambda_{s(g)k}(\lambda_{s(g)}(y_x)) = (g \cdot (\lambda_k|_{\mathcal{F}_x}))(y_{\lambda_g(x)}) \\ &= \gamma(\lambda_g(x), g \cdot a) = \gamma(g \cdot (x, a)).\end{aligned}$$

Under the coordinatization provided by γ, we can therefore identify $Y = A^{\mathrm{sc}}$ for some X-graded $\mathcal{G}(X)$-module A.

Due to the fact that $f : Y \to X$ is an extension, $\lambda_{(x,a)}$ is only dependent on x. Moreover, the decomposition $\mathcal{G}(Y) = K \rtimes \mathcal{G}(X)_\pi$ tells us that for each $(x, a) \in Y$, there is a unique decomposition

$$\lambda_{(x,a)} = k_x + s(\lambda_x),$$

where $k_x \in K$. As $K \leqslant A$, there is a map $\Phi : X \times X \to A$ with $\Phi(x, y) \in A_y$ such that for all $x, y \in X$, we have

$$\lambda_{k_x}(y, a) = (y, a - \Phi(x, y)).$$

As $K \subseteq \mathrm{Soc}(\mathcal{G}(Y))$, we can rewrite

$$\lambda_{(x,a)} = k_x + s(\lambda_x) = k_x s(\lambda_x);$$

therefore

$$(x, a) * (y, b) = \lambda^{-1}_{(x,a)}(y, b) = \lambda^{-1}_{s(\lambda_x)}\lambda^{-1}_{k_x}(y, b) = \lambda^{-1}_{s(\lambda_x)}(y, b + \Phi(x, y)) = \lambda_x^{-1} \cdot (y, b + \Phi(x, y)).$$

We are left with proving equivariance. Pick any $g \in \mathcal{G}(X)$ and make use of Proposition 1.6 to obtain that

$$\lambda_{(\lambda_g(x),a)} = \lambda_{s(g)}(k_x + s(\lambda_x)) = \lambda_{s(g)}(k_x) + \lambda_{s(g)}(s(\lambda_x)) = {}^{s(g)}k_x + s(\lambda_{\lambda_g(x)}).$$

which implies $k_{\lambda_g(x)} = {}^{s(g)}k_x$ and $s(g)k_x = k_{\lambda_g(x)}s(g)$. Therefore, for $(y, b) \in Y$,

$$\begin{aligned}
\lambda_{s(g)k_x}(y, b) &= \lambda_{k_{\lambda_g(x)}s(g)}(y, b),\\
g \cdot (y, b + \Phi(x, y)) &= \Phi(\lambda_g(x), \lambda_g(y)) + g \cdot (y, b), \quad \text{(because } g \cdot (y, b) \in \lambda_g(y))\\
g \cdot (y, b) + g \cdot \Phi(x, y) &= g \cdot (y, b) + \Phi(\lambda_g(x), \lambda_g(y)).
\end{aligned}$$

Canceling $g \cdot (y, b)$, we have proven that $g \cdot \Phi(x, y) = \Phi(\lambda_g(x), \lambda_g(y))$. □

Given a cycle set of the form $Z = X \otimes_\Phi A$, we write $Z_0 = \{(x, 0) : x \in X\} \subseteq Z$. A careful investigation of the construction shows that we can choose any $\mathcal{G}(Y)_\pi$-invariant section of $f : Y \twoheadrightarrow X$ to be the set Z_0 under our coordinatization procedure.

Corollary 2.14. *Let $f : Y \twoheadrightarrow X$ be a coprime extension of indecomposable cycle sets. Let $\pi = \pi(|\mathcal{G}(X)|)$ and let $\mathfrak{O} \subseteq Y$ be a $\mathcal{G}(Y)_\pi$-orbit such that the restriction $f|_{\mathfrak{O}} : \mathfrak{O} \to X$ is bijective. Then there is a twisted extension $\mathrm{pr} : Y' = X \otimes_\Phi A \twoheadrightarrow X$, together with an isomorphism $g : Y' \xrightarrow{\sim} Y$ such that $f \circ g = \mathrm{pr}$ and $g(Y'_0) = \mathfrak{O}$.*

Recall that a cycle set X is called *uniconnected* if X is indecomposable and $\Pi_1(X, x_0) = 1$, which is the same as saying that $\mathcal{G}(X)$ acts regularly on X. In this case, each X-graded $\mathcal{G}(X)$-module is isomorphic to a permutation module, and Theorem 2.11 specializes to the following corollary:

Corollary 2.15. *Let $f : Y \to X$ be a coprime extension of indecomposable cycle sets where X is uniconnected. Then f is equivalent to a parallel extension $\mathrm{pr} : X \times_\Gamma A \twoheadrightarrow X$.*

3. Permutation groups of coprime extensions of cycle sets

In this section, we give a simple description of $\mathcal{G}(Y)$ for $Y = X \otimes_\Phi A$ if X and A are chosen in a suitable way.

Given a map $\Phi : X \times X \to A$ satisfying (2-2) and (2-3), we define the elements $\Phi_x = \sum_{y \in X} \Phi(x, y) \in A$ for $x \in A$. Using these elements, we can also express (2-3) as

$$g \cdot \Phi_x = \Phi_{\lambda_g(x)}. \tag{3-1}$$

The following result describes the permutation brace of a twisted extension that results in a coprime extension of indecomposable cycle sets. Note that for an X-graded G-module A, we identify A with the subgroup of $\mathrm{Sym}_{A^{\mathrm{sc}}}$ that acts by $f \cdot (x, b) = (x, b + f_x)$ for $f \in A$ and $(x, a) \in A^{\mathrm{sc}}$.

Theorem 3.1. *Let X be indecomposable, $\pi = \pi(|\mathcal{G}(X)|)$, and $\mathrm{pr} : Y = X \otimes_\Phi A \twoheadrightarrow X$ be a twisted extension with $\gcd(|\mathcal{G}(X)|, |A_x|) = 1$ for $x \in X$, where Y is indecomposable. Then there is a semidirect product decomposition*

$$\mathcal{G}(Y) = \mathcal{G}(Y)_\pi \ltimes K$$

where $\mathcal{G}(Y)_\pi = \{(x, a) \mapsto (\lambda_g(x), g \cdot a) : g \in \mathcal{G}(X)\} \cong \mathcal{G}(X)$ and $K = \langle \Phi_x : x \in X \rangle \leqslant A$ is a trivial brace such that the λ-action of $\mathcal{G}(Y)_\pi$ on K coincides with $\mathcal{G}(X)$-module structure on K in the following sense: for $f \in K$ acting by $\lambda_f(x, a) = (x, a + f_x)$ and $g \in \mathcal{G}(X)$, we have $\lambda_{s(g)}(f) = g \cdot f$, where $s : \mathcal{G}(X) \to \mathcal{G}(Y)$ denotes the unique section of $\mathcal{G}(\mathrm{pr}) : \mathcal{G}(Y) \twoheadrightarrow \mathcal{G}(X)$ with $s(\mathcal{G}(X)) = \mathcal{G}(Y)_\pi$.

Before proving the result, note that this statement is not completely trivial. It is tempting to reverse the construction from the proof of Theorem 2.11 and decompose the cycle set permutation $(y, b) \mapsto \lambda_x^{-1} \cdot (y, b + \Phi(x, y))$ into components $(y, b) \mapsto (y, b + \Phi(x, y))$ and $(y, b) \mapsto \lambda_x^{-1} \cdot (y, b)$, the latter coming from the action of the π-subgroup in the proof. However, it is not clear at all that for an arbitrary extension $\mathrm{pr} : X \otimes_\Phi A \twoheadrightarrow X$ the construction in the proof will reproduce the parameter Φ as there might be many parameters giving the same solution. We need some lemmata before proving the result.

Lemma 3.2. *Let X be an indecomposable cycle set and let A be an X-graded $\mathcal{G}(X)$-module with $\gcd(|\mathcal{G}(X)|, |A_x|) = 1$ for all x. Suppose that $\mathrm{pr} : Y = X \otimes_\Phi A \twoheadrightarrow X$ is a twisted extension that is not necessarily indecomposable. Let $\pi = \pi(|\mathcal{G}(X)|)$. Then there is a (set-theoretic) section $s : X \hookrightarrow Y$ with $\mathrm{pr} \circ s = \mathrm{id}_X$ such that $\mathcal{G}(Y)_\pi$ fixes $s(X)$.*

Proof. We construct an affine action of $\mathcal{G}(Y)$ on A as follows: for $g \in \mathcal{G}(Y)$, write $\bar{g} = \mathcal{G}(\mathrm{pr})(g)$. It is easy to check that $g \cdot (a_x)_{x \in X} = \lambda_g(a_{\bar{g}^{-1}(x)})_{x \in X}$ defines an action of $\mathcal{G}(Y)$ on A. Writing the action of a $g \in \mathcal{G}(Y)$ on Y as $\lambda_g(x, a_x) = (\lambda_{\bar{g}}(x), \bar{g} \cdot (a_x + b_{g,x}))$, this action can be described by the formula

$$g \cdot (a_x)_{x \in X} = (\bar{g} \cdot (a_{\lambda_{\bar{g}}^{-1}(x)} + b_{g, \lambda_{\bar{g}}^{-1}(x)}))_{x \in X}.$$

It is now immediate that the action represents $\mathcal{G}(Y)$ as a subgroup of $\mathrm{Hol}(A)$. As $\gcd(|\mathcal{G}(Y)_\pi|, |A|) = 1$, we can again use Lemma 2.13 to conclude that $\mathcal{G}(Y)_\pi$ has a fixed point on A. Let $(a_x)_{x \in X}$ be such a fixed point. Then $\lambda_g(a_x) = a_{\lambda_{\bar{g}}(x)}$ for all $x \in X$ and $g \in \mathcal{G}(Y)_\pi$, so $\mathcal{G}(Y)_\pi$ fixes the set $\mathcal{O} = \{(x, a_x) : x \in X\}$, which is the image of the set-theoretic section $s : X \hookrightarrow Y$; $x \mapsto (x, a_x)$. □

Recall that for a twisted extension $Y = X \otimes_\Phi A$, we define $Y_0 = \{(x, 0) : x \in X\} \subseteq Y$.

Lemma 3.3. *Let X be an indecomposable cycle set, take a point $x_0 \in X$, and let A be an X-graded $\mathcal{G}(X)$-module with $\gcd(|\mathcal{G}(X)|, |A_x|) = 1$. Let $Y = X \otimes_\Phi A$ be a twisted extension where Y is not necessarily indecomposable. Furthermore, let $B \leqslant A$ be an X-graded submodule of A and let $Z = X \otimes_\Gamma B$ be another extension where Z is not necessarily indecomposable. Let $f : Z \hookrightarrow Y$ be an embedding of cycle sets such that $f(x, a) = (x, a + \delta_x)$ for a fixed family of $\delta_x \in A_x$ for $x \in X$, and suppose that $\mathcal{O}_{\mathcal{G}(Y)_\pi}(x_0, 0) = Y_0$. Then $f(Z_0)$ is a $\mathcal{G}(Y)_\pi$-orbit in Y and $\Gamma = \Phi$.*

Proof. The homomorphism property reads

$$\begin{aligned} f((x, a) * (y, b)) &= f(x, a) * f(y, b), \\ \big(x * y, \lambda_x^{-1} \cdot (b + \Gamma(x, y)) + \delta_{x*y}\big) &= \big(x * y, \lambda_x^{-1} \cdot (b + \delta_y + \Phi(x, y))\big) \\ \implies \lambda_x^{-1} \cdot (\Gamma(x, y) + \delta_{x*y}) &= \lambda_x^{-1} \cdot (\delta_y + \Phi(x, y)), \\ \lambda_x \cdot \delta_{x*y} - \delta_y &= \Phi(x, y) - \Gamma(x, y). \end{aligned} \tag{$\dagger$}$$

As the right side is $\mathcal{G}(X)$-equivariant, the left side is, as well, meaning that

$$\lambda_x \cdot \delta_{x*y} - \delta_y = \lambda_g^{-1} \cdot (\lambda_{\lambda_g(x)} \cdot \delta_{\lambda_g(x)*\lambda_g(y)} - \delta_{\lambda_g(y)}). \tag{3-2}$$

Taking $\lambda_g = \lambda_x^{-1}$ in (3-2), we obtain

$$\begin{aligned} \lambda_x \cdot \delta_{x*y} - \delta_y &= \lambda_x \cdot (\lambda_{\lambda_x^{-1}(x)} \cdot \delta_{\lambda_x^{-1}(x)*\lambda_x^{-1}(y)} - \delta_{\lambda_x^{-1}(y)}) = \lambda_x \cdot (\lambda_{\mathrm{Sq}(x)} \cdot \delta_{(x*x)*(x*y)} - \delta_{x*y}) \\ &= \lambda_x \lambda_{\mathrm{Sq}(x)} \cdot \delta_{(x*x)*(x*y)} - \lambda_x \cdot \delta_{x*y} = \lambda_{2x} \cdot \delta_{2x*y} - \lambda_x \cdot \delta_{x*y}, \end{aligned}$$

where we write $kx * y = \lambda_{kx}^{-1}$ with $\lambda_{kx} = k \cdot \lambda_x$. By induction over k, using $g + h = g\lambda_g^{-1}(h)$ and the fact that $\mathrm{Sq}(x) = \lambda_x^{-1}(x)$, one shows that

$$\lambda_{kx} = \lambda_x \lambda_{\mathrm{Sq}(x)} \ldots \lambda_{\mathrm{Sq}^{k-1}(x)}.$$

An iterated application of the argument above proves for $k \geqslant 0$ that

$$\lambda_x \cdot \delta_{x*y} - \delta_y = \lambda_{(k+1)x} \cdot \delta_{(k+1)x*y} - \lambda_{kx} \cdot \delta_{kx*y}. \tag{3-3}$$

Let d be the exponent of $\mathcal{G}(X)^+$, which is also the smallest positive integer such that $k \cdot \lambda_x = 0$ for all $x \in X$. With this d, we get, using (3-3):

$$d \cdot (\lambda_x \cdot \delta_{x*y} - \delta_y) = \sum_{k=0}^{d-1} \lambda_{(k+1)x} \cdot \delta_{(k+1)x*y} - \lambda_{kx} \cdot \delta_{kx*y} = \lambda_{dx} \cdot \delta_{dx*y} - \lambda_{0x} \cdot \delta_{0x*y} = \delta_y - \delta_y = 0.$$

Now d divides $|\mathcal{G}(X)|$ — see also [10, Proposition 2.7] — therefore $\gcd(d, |A|) = 1$, from which we infer that $\lambda_x \cdot \delta_{x*y} - \delta_y = 0$ for all $x, y \in X$, and $\delta_{\lambda_x^{-1}(y)} = \lambda_x^{-1} \cdot \delta_y$. As the λ_x ($x \in X$) generate $\mathcal{G}(X)$, it follows that $\delta_{\lambda_g(y)} = g \cdot \delta_y$ holds for all $g \in \mathcal{G}(X)$, $y \in X$; therefore the set

$$\{(x, \delta_x) : x \in X\} \subseteq X \otimes_\Phi A$$

is a $\mathcal{G}(Y)_\pi$-orbit in Y. The observation that $f(x,0)=(x,\delta_x)$ for $(x,0)\in Z_0$ concludes the proof that $f(Z_0)$ is a $\mathcal{G}(Y)_\pi$-orbit in Y.

Furthermore, $\lambda_x\cdot\delta_{x*y}=\delta_{\lambda_x(x*y)}=\delta_y$, so (†) implies that $\Phi(x,y)-\Gamma(x,y)=0$, i.e., $\Phi=\Gamma$. □

Proof of Theorem 3.1. By Theorem 2.11, pr is isomorphic to $\mathrm{pr}':Y'=X\otimes_{\Phi'}A'\twoheadrightarrow X$ for some $\mathcal{G}(X)$-equivariant map $\Phi':X\times X\to A'$. Note that the structure of the underlying X-graded $\mathcal{G}(X)$-module is determined, up to isomorphism, by the λ-action of $\mathcal{G}(Y)_\pi$ on the fibers $\mathcal{F}_x=A_x$ and A'_x (see the proof of Theorem 2.11), so we may assume that $A'=A$.

It is sufficient to show that $\mathcal{G}(Y)_\pi$ consists of the permutations $(x,a)\mapsto(\lambda_g(x),g\cdot a)$ with $g\in\mathcal{G}(X)$ as then it follows that K is generated by the maps $(y,b)\mapsto(y,b+\Phi(x,y))$ for $x\in X$.

This is done by showing that $Y_0\subseteq Y$ is a $\mathcal{G}(Y)_\pi$-orbit, as then the $\mathcal{G}(X)$-module structure on A forces the elements of $\mathcal{G}(Y)_\pi$ to act as $(x,a)\mapsto(\lambda_g(x),g\cdot a)$.

Note first that Lemma 3.2 tells us there is a $\mathcal{G}(Y)_\pi$-orbit $\mathfrak{O}\subseteq Y$ of the form $\mathfrak{O}=\{(x,a_x):x\in X\}$, so by Corollary 2.14 to Theorem 2.11, there is a twisted extension $\mathrm{pr}':Y'=X\otimes_\Gamma A\twoheadrightarrow X$ and an equivalence $g:Y\to Y'$ between the extensions pr and pr′ such that $g(x,a_x)=(x,0)$ for all $x\in X$. As g respects the regular action of A on A^{sc}, g must be of the form $g(x,a)=(x,a-a_x)$. Now Lemma 3.3 tells us that $\Gamma=\Phi$. On the other hand, the same lemma says that $f(Y_0)$ is a $\mathcal{G}(Y')_\pi$-orbit in Y'. By the isomorphism $\mathcal{G}(g):\mathcal{G}(Y)\xrightarrow{\sim}\mathcal{G}(Y')$, it follows that Y_0 is a $\mathcal{G}(Y)_\pi$-orbit, which is what we wanted to show.

As $K\leqslant\mathrm{Soc}(\mathcal{G}(Y))$, it is clear that K is trivial. From Proposition 1.6 it follows that $\lambda_{s(g)}(f)={}^{s(g)}f$ for $g\in\mathcal{G}(X)$ and $f\in K$. By our proof, $\lambda_{s(g)}(x,a)=(\lambda_g(x),g\cdot a)$ holds for $(x,a)\in Y$. Therefore we calculate

$$\begin{aligned}\lambda_{\lambda_{s(g)}(f)}(x,a)&=\lambda_{s(g)}(\lambda_f(\lambda_g^{-1}(x),g^{-1}\cdot a))=\lambda_{s(g)}(\lambda_g^{-1}(x),g^{-1}\cdot a+f(\lambda_g^{-1}(x)))\\&=(x,a+g\cdot f(\lambda_g^{-1}(x)))=\lambda_{g\cdot f}(x,a).\end{aligned}$$ □

We now reformulate this result in terms of parallel extensions. For a map $\Gamma:X\times X\to B$, where B is an (ungraded) group, we define the element $\Gamma_x\in B^X$ by $\Gamma_x(y)=\Gamma(x,y)$.

Corollary 3.4. *Let X be indecomposable, $\pi=\pi(|\mathcal{G}(X)|)$, and $\mathrm{pr}:Y=X\times_\Gamma B\twoheadrightarrow X$ be a parallel extension where $\gcd(|\mathcal{G}(X)|,|B|)=1$, and where Y is indecomposable. Then there is a semidirect product decomposition*

$$\mathcal{G}(Y)=\mathcal{G}(Y)_\pi\ltimes K,$$

where $\mathcal{G}(Y)_\pi=\{(x,a)\mapsto(\lambda_g(x),a):g\in\mathcal{G}(X)\}\cong\mathcal{G}(X)$ and $K=\langle\Gamma_x:x\in X\rangle\leqslant B^X$ is a trivial brace. Furthermore, for $f\in K$ acting by $\lambda_f(x,a)=(x,a+f(x))$ and $g\in\mathcal{G}(X)$, we have $\lambda_{s(g)}(f)={}^gf$, where $s:\mathcal{G}(X)\to\mathcal{G}(Y)$ is the unique section of $\mathcal{G}(\mathrm{pr}):\mathcal{G}(Y)\twoheadrightarrow\mathcal{G}(X)$ with $s(\mathcal{G}(X))=\mathcal{G}(Y)_\pi$ and where ${}^gf(x)=f(\lambda_g^{-1}(x))$.

We can now characterize the cases in which $X\otimes_\Phi A$ is indecomposable.

Theorem 3.5. *Suppose X is an indecomposable cycle set and A is an X-graded $\mathcal{G}(X)$-module with $\gcd(|\mathcal{G}(X)|,|A_x|)=1$. Then a twisted extension $X\otimes_\Phi A$ is indecomposable if and only if for any $x_0\in X$,*

$$A_{x_0}=\langle\Phi(x,x_0):x\in X\rangle.\tag{3-4}$$

Proof. Assume first that $Y = X \otimes_\Phi A$ is indecomposable. Then Theorem 3.1 implies that $\mathcal{G}(Y) = \mathcal{G}(X)_\pi \ltimes K$, where $K = \langle \Phi_x : x \in X \rangle$. From the proof of Theorem 3.1 it follows that

$$\mathcal{O}_{\mathcal{G}(X)_\pi}(x_0, 0) \cap (\{x_0\} \times A_{x_0}) = \{(x_0, 0)\}.$$

Therefore K acts transitively on $\{x_0\} \times A_{x_0}$, which is the same as saying that $\langle \Phi_x(x_0) : x \in X \rangle = A_{x_0}$.

Suppose now that (3-4) is satisfied. Then by Lemma 3.2, Y contains a $\mathcal{G}(Y)_\pi$-orbit of the form $\mathfrak{O} = \{(x, a_x) : x \in X\} \subseteq Y$. Fixing this orbit and the elements a_x, we let $Y' = \mathcal{O}_{\mathcal{G}(Y)}(x_0, a_{x_0})$ for some $x_0 \in X$ and consider the map $\mathrm{pr}' = \mathrm{pr}\,|_{Y'} : Y' \to X$. As X is indecomposable and $\mathcal{G}(\mathrm{pr})$ is surjective, it follows that pr' is indeed surjective. As the permutations $\sigma_{(x,a)}$ for $(x, a) \in X \otimes_\Phi A$ only depend on the X-coordinate, it follows that $\mathcal{G}(Y') = \mathcal{G}(Y)$. So $Y' = \mathcal{O}_{\mathcal{G}(Y')}(x_0, a_{x_0})$, which implies that Y' is indecomposable.

Furthermore, $\mathrm{pr}' : Y' \twoheadrightarrow X$ is a coprime extension: Let $\pi = \pi(|\mathcal{G}(X)|)$ and let π' be the set of primes not in π. Then

$$\pi\left(\frac{|Y'|}{|X|}\right) = \pi\left(\frac{|Y'|}{|\mathfrak{O}|}\right) = \pi\left(\frac{|\mathcal{G}(Y)|}{|\mathcal{G}(Y)_\pi|} \cdot \frac{|\Pi_1(Y, y_0) \cap \mathcal{G}(Y)_\pi|}{|\Pi_1(Y, y_0)|}\right) \subseteq \pi\left(\frac{|\mathcal{G}(Y)|}{|\mathcal{G}(Y)_\pi|}\right) = \pi'.$$

It follows that there is an isomorphism $X \otimes_\Gamma B \cong Y'$ for some submodule $B \leqslant A$ as $Y' \subseteq X \otimes_\Phi A$; the graded components B_x of B are the regular subgroups induced by $\ker(\mathcal{G}(\mathrm{pr}'))) \leqslant A$ on the fibers $\mathrm{pr}'^{-1}(x)$. Furthermore, $\mathfrak{O} = \{(x, a_x) : x \in X\}$ is a $\mathcal{G}(Y')_\pi$-orbit, so by Corollary 2.14, we can identify the embedding $Y' \hookrightarrow Y$ with an embedding $X \otimes_\Gamma B \hookrightarrow X \otimes_\Phi A$; $(x, a) \mapsto (x, a + a_x)$. Lemma 3.3 now implies $\Gamma = \Phi$. As Y' is indecomposable, we conclude from the first part of the theorem that

$$B_{x_0} = \langle \Phi(x, x_0) : x \in X \rangle = A_{x_0}$$

for each $x_0 \in X$. So $B = A$, which implies that $Y = Y'$, from which we infer that Y is indecomposable. □

The corresponding corollary for parallel extensions is stated as follows.

Corollary 3.6. *Let X be an indecomposable cycle set and B a group that satisfies* $\gcd(|\mathcal{G}(X)|, |B|) = 1$. *Then a parallel extension $X \times_\Gamma B$ is indecomposable if and only if*

$$B = \langle \Gamma(x, y) : x \in X \rangle. \tag{3-5}$$

4. Indecomposable cycle sets of size pqr

The goal of this section is to give a full description of indecomposable cycle sets of size pqr. We will rely on the following result:

Theorem 4.1 (Cedó and Okniński [2, Theorems 4.1 and 4.5]). *Let X be an indecomposable cycle set of squarefree size. Then* $\mathrm{mpl}(X) < \infty$. *Furthermore,* $\pi(|\mathcal{G}(X)|) = \pi(|X|)$.

Corollary 4.2. *Let X be an indecomposable cycle set of squarefree size. Then the homomorphism* $\mathrm{ret} : X \twoheadrightarrow X^{(1)}$ *is a coprime extension.*

Proof. Let $|X|$ be squarefree. Then $\pi(|\mathcal{G}(X^{(1)})|) = \pi(|X^{(1)}|)$, by Theorem 4.1. Furthermore, the squarefreeness of $|X|$ implies

$$\varnothing = \pi\left(\frac{|X|}{|X^{(1)}|}\right) \cap \pi(|X^{(1)}|) = \pi\left(\frac{|X|}{|X^{(1)}|}\right) \cap \pi(|\mathcal{G}(X^{(1)})|),$$

so $\gcd(|X|/|X^{(1)}|, |\mathcal{G}(X)|) = 1$, that is, $\mathrm{ret} : X \twoheadrightarrow X^{(1)}$ is a coprime extension. □

Together with the apparatus developed in Section 2, Corollary 4.2 allows us to implement the following strategy to construct all indecomposable cycle sets of squarefree size n:

(1) Suppose that all indecomposable cycle sets of sizes $m \mid n$ with $m \neq n$ have already been constructed.

(2) For given m with $n = m \cdot k$, consider the cycle sets X' with $|X'| = m$.

(3) Construct all X'-graded $\mathcal{G}(X')$-modules A with $|A_{x_0}| = k$.

(4) Find all $\mathcal{G}(X')$-equivariant mappings $\Phi : X' \times X' \to A$ that satisfy the indecomposability criterion in Theorem 3.5.

We start by repeating a well-known case:

Proposition 4.3. *If X is an indecomposable cycle set with $\mathrm{mpl}(X) = 1$ and $|X| = n$, then $X \cong \mathbb{Z}_n$ with $x * y = y + 1$. In particular, X is uniconnected.*

Proof. As $\mathrm{mpl}(X) = 1$, we can write $x * y = \sigma(y)$ for some permutation $\sigma \in \mathrm{Sym}_X$. Clearly $\mathcal{G}(X) = \langle \sigma \rangle$, which is transitive if and only if σ is an n-cycle. Therefore we can choose an $x_0 \in X$ and construct the desired isomorphism as

$$f : \mathbb{Z}_n \to X; \quad k \mapsto \sigma^k(x_0).$$

Uniconnectivity is obvious. □

The following result concerns cycle sets of squarefree size and multipermutation level 2 which have been constructed in greater generality by Jedlička and Pilitowska [14]. We nevertheless show how to obtain these cycle sets by the method developed in this article; furthermore, it gives us the opportunity to express those cycle sets in a way that is handy for further computations concerning cycle sets of multipermutation level 3.

Proposition 4.4. *Let X be an indecomposable cycle set of squarefree size $n = m \cdot k$. If $\mathrm{mpl}(X) = 2$ and $|X^{(1)}| = m$, then there is an isomorphism with a parallel extension $X \cong \mathbb{Z}_m \times_\Gamma \mathbb{Z}_k$, where $\mathbb{Z}_m$ carries the cycle set operation $x * y = y + 1$.*

Putting $\Gamma^0(x) = \Gamma(0, x)$, the cycle set operation of $\mathbb{Z}_m \times_\Gamma \mathbb{Z}_k$ can be expressed by

$$(x, a) * (y, b) = (y + 1, b + \Gamma^0(y - x)). \tag{4-1}$$

Vice versa, each map $\Gamma^0 : \mathbb{Z}_m \to \mathbb{Z}_k$ provides an indecomposable cycle set with $\mathrm{mpl}(X) = 2$ and $|X^{(1)}| = m$ if $\mathbb{Z}_k = \langle \Gamma^0(x) : x \in \mathbb{Z}_m \rangle$ and if for all $0 \neq d \in \mathbb{Z}_m$, there is an $x \in \mathbb{Z}_m$ with $\Gamma^0(x + d) \neq \Gamma^0(x)$.

Proof. If X is indecomposable and $\mathrm{mpl}(X) = 2$, then $X^{(1)}$ is indecomposable with $\mathrm{mpl}(X^{(1)}) = 1$. By Proposition 4.3, $X^{(1)} \cong \mathbb{Z}_m$, with the cycle set operation $x * y = y + 1$. Identifying $X^{(1)} = \mathbb{Z}_m$, the retraction map $X \twoheadrightarrow \mathbb{Z}_m$ is a coprime extension, by Corollary 4.2. As $\mathbb{Z}_m$ is uniconnected, Corollary 2.15 implies that X is a parallel extension of $\mathbb{Z}_m$ by an abelian group of order k which must be isomorphic to $\mathbb{Z}_k$. It follows that $X \cong \mathbb{Z}_m \times_\Gamma \mathbb{Z}_k$ for a $\mathcal{G}(X^{(1)})$-invariant map $\Gamma : \mathbb{Z}_m \times \mathbb{Z}_m \to \mathbb{Z}_k$.

Putting $\Gamma^0(x) = \Gamma(0, x)$, we can use (2-6) to express the operation on $\mathbb{Z}_m \times_\Gamma \mathbb{Z}_k$ as

$$(x, a) * (y, b) = (y + 1, b + \Gamma^0(\lambda_g^{-1}(y)))$$

with a fixed map $\Gamma^0 : \mathbb{Z}_m \to \mathbb{Z}_k$ where $g \in \mathcal{G}(X^{(1)})$ is chosen so that $\lambda_g(0) = x$. Obviously, g must satisfy $\lambda_g(z) = z + x$; therefore $\lambda_g^{-1}(y) = y - x$, which leads to the description by (4-1).

By Corollary 3.6, given $\Gamma^0 : \mathbb{Z}_m \to \mathbb{Z}_k$, the cycle set on $\mathbb{Z}_m \times \mathbb{Z}_k$ defined by (4-1) is indecomposable if and only if

$$\mathbb{Z}_k = \langle \Gamma(x, y) : x, y \in \mathbb{Z}_m \rangle = \langle \Gamma^0(y - x) : x, y \in \mathbb{Z}_m \rangle = \langle \Gamma^0(x) : x \in \mathbb{Z}_m \rangle.$$

If the projection $\mathbb{Z}_m \times \mathbb{Z}_k \to \mathbb{Z}_m$; $(x, a) \mapsto x$ is a retraction, then for any $0 \neq d \in \mathbb{Z}_m$ we have $\sigma_{(0,0)} \neq \sigma_{(-d,0)}$, which implies that there is an $x \in \mathbb{Z}_m$ with $\Gamma^0(x-0) \neq \Gamma^0(x-(-d))$, i.e., $\Gamma^0(x) \neq \Gamma^0(x+d)$. If, on the other hand, $\sigma_{(x,0)} = \sigma_{(y,0)}$ for distinct $x, y \in \mathbb{Z}_m$, then for all $z \in \mathbb{Z}_m$, we have $\Gamma^0(z-x) = \Gamma^0(z-y)$. Substituting $z' = z-x$ and putting $d = x-y$, we see that for all $z' \in \mathbb{Z}_m$, we have $\Gamma^0(z') = \Gamma^0(z'+d)$, while $0 \neq d \in \mathbb{Z}_m$. □

We now want to tackle the classification of indecomposable cycle sets X of size pqr, where p, q and r are distinct primes and where $\mathrm{mpl}(X) = 3$. Without restriction, we can assume that $|X^{(1)}| = pq$ and $|X^{(2)}| = p$. Using Proposition 4.4, we can write $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$. As q is prime, Proposition 4.4 implies that $\Gamma : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_q$ indeed defines an indecomposable cycle set if and only if $\Gamma^0 = \Gamma(0, -)$ is nonzero and, by the primality of p, the multipermutation level of this cycle set is 2 if and only if Γ^0 is nonconstant, which is equivalent to Γ being nonconstant.

We first want to understand the stabilizers in $X^{(1)}$ better. To this end, we prove the following result:

Proposition 4.5. *Let $X = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$ where Γ is nonconstant. Let $x_0 = (0, 0)$. Then the orbits of $\Pi_1(X, x_0)$ on X are given by*

(1) *the singletons $\{(x, a)\}$ if Γ^0 is **proportional to a one-dimensional character**, that is, $\Gamma^0 = \alpha \cdot \chi$ for some $\alpha \in \mathbb{Z}_q^*$ and some $\chi : \mathbb{Z}_p \to \mathbb{Z}_q^\times$ with $\chi(x + y) = \chi(x)\chi(y)$, and otherwise*

(2) *the singletons $\{(0, a)\}$ for $a \in \mathbb{Z}_q$ and the blocks $\{x\} \times \mathbb{Z}_q$ $(x \in \mathbb{Z}_p^*)$.*

Our approach is to exploit the functional description of $K = \ker(\mathcal{G}(X^{(1)}) \twoheadrightarrow \mathcal{G}(X^{(2)}))$ from Theorem 3.1 to connect stabilization to the vanishing of functions.

First of all, we present a very broad framework for this idea: let M be a set and $N \subseteq M$. Let A be an abelian group and consider a subgroup $H \leqslant A^M$. We want to measure to what extent the vanishing of a function $f \in H$ on N forces f to vanish on other points too. To this end, we define the set

$$\mathcal{Z}_{H,N} = \{x \in M : \forall f \in H \text{ such that } f|_N \equiv 0 \implies f(x) = 0\}.$$

Obviously $N \subseteq \mathcal{Z}_{H,N}$.

Our motivation to introduce this concept comes from the fact that the orbits of the stabilizer can be described in terms of $\mathcal{Z}_{H,N}$ if we consider extensions of uniconnected cycle sets. Recall that for $\Gamma : X \times X \to B$ where B is an (ungraded) group, we define $\Gamma_x \in B^X$ by $\Gamma_x(y) = \Gamma(x, y)$.

Lemma 4.6. *Let X be a uniconnected cycle set, q a prime, and suppose that $\mathrm{pr} : Y = X \times_\Gamma \mathbb{Z}_q \twoheadrightarrow X$ is a coprime parallel extension of indecomposable cycle sets.*

Let $y_0 = (x_0, 0) \in Y$ and put

$$H = \langle \Gamma_x \mid x \in X \rangle \leqslant \mathbb{Z}_q^X.$$

Then the orbit of $(x, a) \in Y$ under $\Pi_1(Y, y_0)$ is

$$\mathcal{O}_{\Pi_1(Y,y_0)}(x, a) = \begin{cases} \{(x, a)\} & \text{if } x \in \mathcal{Z}_{H,\{x_0\}}, \\ \{x\} \times \mathbb{Z}_q & \text{if } x \notin \mathcal{Z}_{H,\{x_0\}}. \end{cases}$$

Proof. The induced action of $\Pi_1(Y, y_0)$ on the quotient X fixes x_0, and thus it fixes the entirety of X, as X is uniconnected. It follows that

$$\Pi_1(Y, y_0) \subseteq \ker(\mathcal{G}(Y) \to \mathcal{G}(X)) \leqslant \mathrm{Soc}(G(Y)) \overset{\text{Corollary 3.4}}{=} \langle \Gamma_x \mid x \in X \rangle = H.$$

As an element $f \in H$ acts as $f \cdot (x, a) = (x, a + f(x))$, so it is now obvious that $\Pi_1(Y, y_0)$ can be identified with $H_0 = \{f \in H : f(0) = 0\}$.

Furthermore, (x, a) is fixed under the action of H_0 if and only if $f(x) = 0$ for all $f \in H_0$, i.e., $x \in \mathcal{Z}_{H,\{x_0\}}$. On the other hand, if $x \notin \mathcal{Z}_{H,\{x_0\}}$, we see similarly that $\mathcal{O}_{H_0}(x, a) \supseteq \{x\} \times \mathbb{Z}_p$. As H_0 fixes all sets $\{x\} \times \mathbb{Z}_q$ setwise, we conclude that $\mathcal{O}_{H_0}(x, a) = \{x\} \times \mathbb{Z}_q$. □

We can now state our lemma. For this, we denote by $\mathbb{F}_q$ the field with q elements.

Lemma 4.7. *Let p and q be different primes and let $U \leqslant \mathbb{F}_q^{\mathbb{Z}_p}$ be a $\mathbb{Z}_p$-invariant subspace with respect to the regular representation. Then*

$$\mathcal{Z}_{U,\{0\}} = \begin{cases} \mathbb{Z}_p & \text{if } \dim_{\mathbb{F}_q} U \leqslant 1, \\ \{0\} & \text{if } \dim_{\mathbb{F}_q} U \geqslant 2. \end{cases}$$

Proof. It is immediate that $0 \in \mathcal{Z}_{U,\{0\}}$. Suppose that there is a $d \in \mathbb{Z}_p$ such that $0 \neq d \in \mathcal{Z}_{U,\{0\}}$. Then for $f \in U$, the implication $f(0) = 0 \implies f(d) = 0$ is valid by definition. As U is $\mathbb{Z}_p$-invariant, the implication $f(x) = 0 \implies f(x + d) = 0$ holds for $f \in U$ as well. This shows that $\mathcal{Z}_{U,\{0\}}$ is invariant under the mapping $x \mapsto x + d$. Therefore $\mathcal{Z}_{U,\{0\}} = \mathbb{Z}_p$. In this case, we argue that $\dim_{\mathbb{F}_q} U \leqslant 1$: if $\dim_{\mathbb{F}_q} U \geqslant 1$, there is an $x_0 \in \mathbb{Z}_p$ such that $f(x_0) \neq 0$ for at least one $f \in U$. By shift-invariance, we can choose x_0; therefore the $\mathbb{F}_q$-linear map $\varphi : U \to \mathbb{F}_q$; $f \mapsto f(0)$ is surjective with $\dim_{\mathbb{F}_q} \ker(\varphi) = \dim_{\mathbb{F}_q} U - 1$. But for all $f \in \ker(\varphi)$, we have $f(0) = 0$, so as $\mathcal{Z}_{U,\{0\}} = \mathbb{Z}_p$ it follows that $f(x) = 0$ for all $x \in \mathbb{Z}_p$. Therefore $\ker \varphi = \{0\}$, which implies $\dim_{\mathbb{F}_q} U = 1$. Therefore we have shown that $\mathcal{Z}_{U,\{0\}} = \{0\}$ if $\dim_{\mathbb{F}_q} U \geqslant 2$.

If $\dim_{\mathbb{F}_q} U = 0$, it is obvious that $\mathcal{Z}_{U,\{0\}} = \mathbb{Z}_p$. On the other hand, if $\dim_{\mathbb{F}_q} U = 1$, then we can pick an $f \in U$ with $f(0) = 1$. By $\mathbb{Z}_p$-invariance, $\dim_{\mathbb{F}_q} U = 1$ implies there is a $\xi \in \mathbb{F}_q$ such that $f(x+1) = \xi \cdot f(x)$ for all x. Therefore $f(x) = \xi^x$ and $U = \{x \mapsto \alpha \cdot \xi^x : \alpha \in \mathbb{F}_q\}$. As $\xi^x \neq 0$ for all x, it now follows immediately that indeed $\mathcal{Z}_{U,\{0\}} = \mathbb{Z}_p$ in this case. □

Now the proof of Proposition 4.5 is almost immediate:

Proof of Proposition 4.5. Consider the cycle set $X = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$ where Γ is nonconstant. Then $H = \langle \Gamma_x \rangle \leqslant \mathbb{Z}_q^{\mathbb{Z}_p} \cong \mathbb{F}_q^{\mathbb{Z}_p}$, where $\Gamma_x(y) = \Gamma^0(y - x)$, can be considered a $\mathbb{Z}_p$-invariant subspace of $\mathbb{F}_q^{\mathbb{Z}_p}$. If $\Gamma_0 = \alpha \cdot \chi$ for some character $\chi : \mathbb{Z}_p \to \mathbb{Z}_q^*$ and $\alpha \neq 0$, then $\Gamma_x = \chi(-x) \cdot \Gamma^0$, which implies $\dim_{\mathbb{F}_q} H = 1$. By Lemma 4.7, $\mathcal{Z}_{H,\{0\}} = \mathbb{Z}_p$, which implies by Lemma 4.6 that the orbits of $\Pi_1(X, x_0)$ on X are given by all singletons $\{(x, a)\}$.

If Γ^0 is *not* proportional to a character, then $\dim_{\mathbb{F}_q} H \geqslant 2$ and Lemma 4.7 implies that $\mathcal{Z}_{H,\{0\}} = \{0\}$. By Lemma 4.6, the orbits of $\Pi_1(X, x_0)$ on X are the singletons $\{(0, a)\}$ for $a \in \mathbb{Z}_q$ and the sets $\{x\} \times \mathbb{Z}_q$ for $x \in \mathbb{Z}_p^*$. □

We are now in position to provide a description of the indecomposable cycle sets X with $|X| = pqr$ and $\mathrm{mpl}(X) = 3$. Recall that we assume $|X^{(1)}| = pq$ and $|X^{(2)}| = p$. By Proposition 4.4, we can write $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$ for some nonconstant $\Gamma : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_q$.

We begin with the case where $\Gamma^0 = \Gamma(0, -)$ is proportional to a character. Corollary 2.8 shows that scalings of Γ^0 do not affect the isomorphism type $X^{(1)}$, so we may assume that Γ^0 is itself a nontrivial character, i.e., $\Gamma^0(x) = \xi^x$ for some $\xi \in \mathbb{Z}_q^*$ with $\xi^p = 1$. The extension of Γ^0 to a $\mathcal{G}(X^{(1)})$-invariant map is clearly given by $\Gamma(x, y) = \xi^{y-x}$.

Theorem 4.8. *Suppose that X is an indecomposable cycle set with $|X| = pqr$ and let $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$ with $\Gamma(x, y) = \xi^{y-x}$ for some $\xi \in \mathbb{Z}_q^*$ such that $\xi^p = 1$. Then $X \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ with the binary operation*

$$(x, a, s) * (y, b, t) = (y + 1, b + \xi^{y-x}, t + \Phi^0(y - x, b - \xi^{y-x}a)) \tag{4-2}$$

for some nonconstant $\Phi^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_r$. On the other hand, each nonconstant map $\Phi^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_r$ defines an indecomposable cycle set by means of (4-2).

Note that we do not discuss the issue when the cycle set described by (4-2) is exactly of multipermutation level 3. This comment applies to all subsequent classifications, as well.

Proof. By Proposition 4.5, the cycle set $X^{(1)}$ is uniconnected. It is clear that each element of $\mathcal{G}(X^{(1)})$ is of the form

$$\tau_{(d,k)} : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_p \times \mathbb{Z}_q; \quad (x, a) \mapsto (x + d, a + k \cdot \xi^x)$$

for $d \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_q$. As $|\mathcal{G}(X^{(1)})| = pq$, we have $\tau_{(d,k)} \in \mathcal{G}(X^{(1)})$ for all $d \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_q$.

By Corollary 2.15, there is an isomorphism $X \cong X^{(1)} \times_\Phi \mathbb{Z}_r$. Using (2-11), we use the map

$$\Phi^0 = \Phi((0, 0), -) : X^{(1)} \to \mathbb{Z}_r$$

to express the multiplication on $X^{(1)} \times_\Phi \mathbb{Z}_r$ by

$$((x, a), s) * ((y, b), t) = ((x, a) * (y, b), t + \Phi^0(\lambda_g^{-1}(y, b))), \quad \text{where } \lambda_g(0, 0) = (x, a). \tag{$*$}$$

Given $(x, a), (y, b) \in \mathbb{Z}_p \times \mathbb{Z}_q$, we see that $\tau_{(x,a)}(0, 0) = (x, a)$. There is a $g \in \mathcal{G}(X^{(1)})$ such that $\lambda_g = \tau_{(x,a)}$. This g satisfies $\lambda_g^{-1}(y, b) = (y - x, b - \xi^{y-x}a)$. Plugging this into $(*)$ results in (4-2).

It is obvious that the construction results in a cycle set as (4-2) expresses the operation in a parallel extension in terms of Φ^0. This cycle set is seen to be indecomposable by an application of Corollary 3.6. □

We now consider the case when $X^{(1)}$ is *not* uniconnected, i.e., $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$ where the map $\Gamma^0 : \mathbb{Z}_p \to \mathbb{Z}_q$ is *not* proportional to a character.

To this end, we begin by constructing the $X^{(1)}$-graded $\mathcal{G}(X^{(1)})$-modules A with $|A_x| = r$, that is $A_x \cong \mathbb{Z}_r$ for all $x \in X^{(1)}$. The experienced reader may have noticed that if a set Y is acted upon transitively by a group G, then each Y-graded G-module $A = \bigoplus_{x\in Y} A_x$ is isomorphic to the induced module $\mathrm{Ind}_{G_0}^G A_{x_0}$ where G_0 is the stabilizer of some element $x_0 \in Y$, as the A_x form a *system of imprimitivity* in the sense of [6, §50.1]. But even without this knowledge, it is not hard to see that A is already determined up to isomorphism by the G_0-module A_{x_0}.

Considering $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$, we want to find the $X^{(1)}$-graded $\mathcal{G}(X^{(1)})$-modules A with $|A_x| = \mathbb{Z}_r$ for all $x \in X^{(1)}$. We fix $(0, 0) \in X^{(1)}$ as our basepoint. Let $K = \ker(\mathcal{G}(X^{(1)}) \to \mathcal{G}(X^{(2)}))$. It turns out that

$$\mathcal{G}(X^{(1)})_0 = \Pi_1(X^{(1)}, (0, 0)) \leqslant K.$$

This can be seen as follows: if $\bar{g}$ denotes the image of $g \in \mathcal{G}(X^{(1)})_0$ under $\mathcal{G}(\mathrm{ret}) : \mathcal{G}(X^{(1)}) \to \mathcal{G}(X^{(2)})$, we observe that $\lambda_{\bar{g}}(0) = 0$, which implies $\bar{g} = 0$ as $X^{(2)}$ is uniconnected.

We use Corollary 3.4 to describe K as the trivial brace over the abelian group

$$K = \langle \Gamma_x \mid x \in \mathbb{Z}_p \rangle \leqslant \mathbb{Z}_q^{\mathbb{Z}_p},$$

where $\Gamma_x : \mathbb{Z}_p \to \mathbb{Z}_q$ is given by $\Gamma_x(y) = \Gamma(x, y) = \Gamma^0(y - x)$. We therefore identify K and $\mathcal{G}(X^{(1)})_0$, in particular, with a subgroup of $\mathbb{Z}_q^{\mathbb{Z}_p}$ and write its elements by functions $f : \mathbb{Z}_q \to \mathbb{Z}_p$ that act on $X^{(1)}$ as $\lambda_f(x, a) = (x, a + f(x))$. It is now easy to determine the $\mathcal{G}(X^{(1)})_0$-module structures on $\mathbb{Z}_r$ — each such structure is given by a character $\bar{\chi} : \mathcal{G}(X^{(1)})_0 \to \mathbb{Z}_r^*$, and as the group $\mathbb{Z}_q^{\mathbb{Z}_p}$ and its subgroup K are elementary abelian, $\mathcal{G}(X^{(1)})_0$ is a summand thereof. Therefore each character on $\mathcal{G}(X^{(1)})_0$ is the restriction of a character on K and $\mathbb{Z}_q^{\mathbb{Z}_p}$. Identifying $\mathcal{G}(X^{(1)})_0$ with an (additive) subgroup of $\mathbb{Z}_q^{\mathbb{Z}_p}$, it follows that we can write

$$\bar{\chi} = \chi|_{\mathcal{G}(X^{(1)})_0}, \quad \text{where } \chi : K \to \mathbb{Z}_r;\ f \mapsto \prod_{x\in\mathbb{Z}_p^*} \xi_x^{f(x)}.$$

Here the elements $\xi_x \in \mathbb{Z}_r$ $(x \in \mathbb{Z}_p)$ satisfy $\xi_x^q = 1$. Note that we exclude $x = 0$ because $f(0) = 0$, due to f stabilizing $(0, 0)$.

Therefore $\mathcal{G}(X^{(1)})_0$ acts on $\mathbb{Z}_r$ via $f \cdot z = z \cdot \prod_{x\in\mathbb{Z}_p^*} \xi_x^{f(x)}$. The unique extension to an $X^{(1)}$-graded $\mathcal{G}(X^{(1)})$-group A with $A_{(0,0)} = \mathbb{Z}_r$ is given by $A_{(x,a)} = \mathbb{Z}_r$ for all $(x, a) \in X^{(1)}$, where $\mathcal{G}(X^{(1)})$ acts by

$$\rho_{(d,f)} : A_{(x,a)} \to A_{(x+d,a+f(x))}, \tag{4-3}$$

$$z \mapsto z \cdot \prod_{y\in\mathbb{Z}_p^*} \xi_y^{f(y+x)}. \tag{4-4}$$

Here (d, f) denotes the element in $\mathcal{G}(X^{(1)}) \leqslant \mathbb{Z}_p \ltimes \mathbb{Z}_q^{\mathbb{Z}_p}$ that maps (x, a) to $(x + d, a + f(x))$.

Note that given the character $\bar{\chi}$ on $\mathcal{G}(X^{(1)})_0$, the extension to an X-graded $\mathcal{G}(X)$-module is unique up to isomorphism as $\mathcal{G}(X^{(1)})$ acts transitively on $X^{(1)}$, so we only have to check that this is indeed an $X^{(1)}$-graded $\mathcal{G}(X^{(1)})$-module. We consider the action of $\rho_{(e,g)}\rho_{(d,f)}$ on an element $z \in A_{(x,a)}$:

$$(\rho_{(e,g)}\rho_{(d,f)})(z) = \rho_{(e,g)}\left(z \cdot \prod_{y\in\mathbb{Z}_p^*} \xi_y^{f(y+x)}\right) = z \cdot \prod_{y\in\mathbb{Z}_p^*} \xi_y^{g(y+x+d)} \cdot \prod_{y\in\mathbb{Z}_p^*} \xi_y^{f(y+x)} = z \cdot \prod_{y\in\mathbb{Z}_p^*} \xi_y^{g(y+x+d)+f(y+x)}$$
$$= \rho_{(e,g)(d,f)}(z).$$

It will occasionally be practical to take shifts into account. Hence we define for $x \in \mathbb{Z}_p$ the characters

$$\chi_x : K \to \mathbb{Z}_r; \quad f \mapsto \prod_{y\in\mathbb{Z}_p^*} \xi_y^{f(y+x)}. \tag{4-5}$$

Note that $\chi = \chi_0$. With this definition, (4-4) can be rewritten for $z \in A_{(x,a)}$ as

$$\rho_{(d,f)}(z) = \chi_x(f) \cdot z \in A_{(x+d,a+f(x))}. \tag{4-6}$$

We fix the $X^{(1)}$-graded $\mathcal{G}(X^{(1)})$-module $A = \mathbb{Z}_r^{X^{(1)}}$ where $\mathcal{G}(X^{(1)})$ acts as in (4-4). We have to find conditions for a function $\Phi^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to A$ to satisfy the $\mathcal{G}(X^{(1)})_0$-equivariance given by (2-8). Rewritten in terms of the module structure given by (4-6), this translates to

$$\Phi^0(x, a + f(x)) = \Phi^0(x, a) \cdot \chi_x(f) \tag{4-7}$$

for $f \in \mathcal{G}(X^{(1)})_0$. Note that $f \in \mathcal{G}(X^{(1)})_0$ implies $f(0) = 0$. Furthermore, we observe that the existence of an $f \in \mathcal{G}(X^{(1)})_0$ with $\chi_x(f) \neq 1$ and $f(x) = 0$ forces $\Phi^0(x, a) = 0$. On the other hand, suppose that $x \in \mathbb{Z}_p^*$. If $\chi_x(f) = 1$ for all $f \in \mathcal{G}(X^{(1)})_0$ with $f(x) = 0$, then given the value $\Phi(x, 0)$, the values $\Phi^0(x, a)$ can easily be reconstructed as

$$\Phi^0(x, a) = \Phi^0(x, 0) \cdot \chi_x(f), \tag{4-8}$$

where $f \in \mathcal{G}(X^{(1)})_0$ is chosen so that $f(x) = a$. Observe that such an f exists because of Proposition 4.5 and the assumption that $x \neq 0$, and independence from the choice of f follows from the assumption that $\chi_x(f) = 1$ for all $f \in \mathcal{G}(X^{(1)})_0$ with $f(x) = 0$. We define the set

$$\mathcal{N}_\chi = \{x \in \mathbb{Z}_p : \forall f \in \mathcal{G}(X^{(1)})_0 : f(x) = 0 \implies \chi_x(f) = 1\}. \tag{4-9}$$

We now investigate two cases:

Case 1: $0 \in \mathcal{N}_\chi$. This implies that $\bar{\chi} = \chi|_{\mathcal{G}(X^{(1)})_0}$ is the trivial character. Therefore the module A is isomorphic to the permutation module $\mathbb{Z}_r^{X^{(1)}}$. In this case, we have a parallel extension $X^{(1)} \times_\Phi \mathbb{Z}_r \cong X$.

We reconstruct Φ from the restriction $\Phi^0 = \Phi((0, 0), -) : X^{(1)} \to \mathbb{Z}_r$, which is $\Pi(X^{(1)}, (0, 0))$-invariant in the sense of (2-5). By Proposition 4.5, this is the same as saying that Φ^0 is constant on $\{x\} \times \mathbb{Z}_q$ for $x \in \mathbb{Z}_p^*$. Therefore there are functions $\Phi_1^0 : \mathbb{Z}_q \to \mathbb{Z}_r$ and $\Phi_2^0 : \mathbb{Z}_p^* \to \mathbb{Z}_r$ such that

$$\Phi^0(x, a) = \begin{cases} \Phi_1^0(a) & \text{if } x = 0, \\ \Phi_2^0(x) & \text{if } x \neq 0. \end{cases}$$

It is quickly checked that the extension of Φ^0 to a $\mathcal{G}(X^{(1)})$-invariant map $\Phi : X^{(1)} \times X^{(1)} \to \mathbb{Z}_r$ is given by

$$\Phi((x,a),(y,b)) = \begin{cases} \Phi_1^0(b-a) & \text{if } x = y, \\ \Phi_2^0(y-x) & \text{if } x \neq y. \end{cases}$$

Plugging everything into (2-6), the resulting cycle set is then described as $X = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ with the operation

$$(x,a,s) * (y,b,t) = \begin{cases} (y+1, b+\Gamma^0(y-x), t+\Phi_1^0(b-a)) & \text{if } x = y, \\ (y+1, b+\Gamma^0(y-x), t+\Phi_2^0(x-y)) & \text{if } x \neq y. \end{cases}$$

By Corollary 3.6, X is indecomposable if and only if Φ_1^0 or Φ_2^0 is nonconstant.

Remark 4.9. Assume that we are given another prime $s \notin \{p,q,r\}$. Then there is an obvious way to extend this construction: Let $\Omega_1^0 : \mathbb{Z}_r \to \mathbb{Z}_s$, $\Omega_2^0 : \mathbb{Z}_q^* \to \mathbb{Z}_s$ and $\Omega_3^0 : \mathbb{Z}_p^* \to \mathbb{Z}_s$. Then the cycle set structure constructed on $X = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ can be extended to give one on $X \times \mathbb{Z}_s$ by means of

$$(x,a,s,v) * (y,b,t,w) = \begin{cases} (y+1, b+\Gamma^0(y-x), t+\Phi_1^0(b-a), w+\Omega_1^0(t-s)) & \text{if } x = y,\ a = b, \\ (y+1, b+\Gamma^0(y-x), t+\Phi_1^0(b-a), w+\Omega_2^0(b-a)) & \text{if } x = y,\ a \neq b, \\ (y+1, b+\Gamma^0(y-x), t+\Phi_2^0(y-x), w+\Omega_3^0(y-x)) & \text{if } x \neq y. \end{cases}$$

It is obvious how to iterate this construction.

Case 2: $0 \notin \mathcal{N}_\chi$. For $x \in \mathcal{N}_\chi$, we now define a map $\phi_x^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to A$ by

$$\phi_x^0(y,a) \in A_{y,a}, \qquad \phi_x^0(y, f(x)) = \begin{cases} 0 & \text{if } x \neq y, \\ \chi_x(f) & \text{if } x = y, \end{cases} \quad \text{for } f \in \mathcal{G}(X^{(1)})_0.$$

It is easy to see that ϕ_x^0 is well defined and that $\phi_x^0(y, a + f(y)) = f \cdot \phi_x^0(y,a)$ for all $f \in \mathcal{G}(X^{(1)})_0$ and $(y,a) \in \mathbb{Z}_p \times \mathbb{Z}_q$, and any $\mathcal{G}(X^{(1)})_0$-equivariant $\Phi^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to A$ can be assembled as

$$\Phi^0 = \sum_{x \in \mathcal{N}_\chi} c_x \phi_x^0 \tag{4-10}$$

with $c_x \in \mathbb{Z}_r$ for $x \in \mathcal{N}_\chi$. Note that it is necessary and sufficient for $X \otimes_\Phi A$ to be indecomposable that $c_x \neq 0$ for some $x \in \mathcal{N}_\chi$.

We are left with expressing the operation on X in terms of Φ^0. We make the assumption that there is an $x_0 \in \mathbb{Z}_p$ with $\Gamma_{x_0}(0) = 1$. By Corollary 2.8, this assumption does not pose any restriction. Recall that $\Gamma_x(y) = \Gamma(x,y) = \Gamma^0(y-x)$.

We now define for $d \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_q$ the permutations

$$\tau_{(d,k)} : X^{(1)} \to X^{(1)}; \quad (x,a) \mapsto (x+d, a + k \cdot \Gamma_{x_0}(x)).$$

By Corollary 3.4, $\tau_{(d,k)} \in \mathcal{G}(X^{(1)})$. We now identify A^{sc} with $X^{(1)} \times \mathbb{Z}_r$ and, using (2-10), we express the operation on $X = X^{(1)} \times \mathbb{Z}_r$ as

$$((x,a),s) * ((y,b),t) = ((x,a)*(y,b), \underbrace{\lambda_{(x,a)}^{-1} \cdot (t + g \cdot \Phi^0(\lambda_g^{-1}(y,b)))}_{=:F(x,a,y,b,t)}), \quad \text{where } \lambda_g(0,0) = (x,a). \quad (*)$$

It turns out that $\tau_{(x,a)}(0,0) = (x,a)$, so let $g \in \mathcal{G}(X^{(1)})$ be such that $\lambda_g = \tau_{(x,a)}$. With this g, we furthermore get $\lambda_g^{-1}(y,b) = (x-y, b - a\cdot\Gamma_{x_0}(y-x))$. Note that $\lambda_{(x,a)}^{-1}(y,b) = (y+1, b+\Gamma_x(y))$. Using the description (4-4) for the module structure, we can therefore calculate

$$\begin{aligned} F(x,a,y,b,t) &= \lambda_{(x,a)}^{-1}\cdot \underbrace{(t + g\cdot \overbrace{\Phi^0(\lambda_g^{-1}(y,b))}^{\in A_{(x-y,b-a\cdot\Gamma^0(y-x-x_0))}})}_{\in A_{(y,b)}} \\ &= \chi_y(\Gamma_x)\cdot\big(t + \chi_{x-y}(a\cdot\Gamma_{x_0})\cdot\Phi^0(x-y, b-a\cdot\Gamma_{x_0}(y-x))\big). \end{aligned}$$

We can now summarize our findings. In the following we write, as usual, $K = \langle \Gamma_x \mid x \in \mathbb{Z}_p\rangle$. Furthermore, we identify $\mathcal{G}(X^{(1)})_0$ with $K_0 = \{f \in K : f(0) = 0\}$.

Theorem 4.10. *Suppose X is an indecomposable cycle set with* $\mathrm{mpl}(X) = 3$, $|X| = pqr$ *and* $X^{(1)} = \mathbb{Z}_p \times_\Gamma \mathbb{Z}_q$, *where Γ^0 is nonzero but* ***not*** *proportional to a character. Then one of the following holds*:

(1) $X \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, *with the operation*

$$(x,a,s)*(y,b,t) = \begin{cases} (y+1, b+\Gamma^0(y-x), t+\Phi_1^0(b-a)) & \text{if } x = y, \\ (y+1, b+\Gamma^0(y-x), t+\Phi_2^0(x-y)) & \text{if } x \neq y, \end{cases}$$

where $\Phi_1^0 : \mathbb{Z}_q \to \mathbb{Z}_r$ and $\Phi_2^0 : \mathbb{Z}_p^* \to \mathbb{Z}_r$ are maps that are not both identically zero.

(2) $X \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, *with the operation*

$$(x,a,s)*(y,b,t) = \big(y+1, b+\Gamma^0(y-x),\ \chi_y(\Gamma_x)\cdot\big(t+\chi_{x-y}(a\cdot\Gamma_{x_0})\cdot\Phi^0(x-y, b-a\cdot\Gamma_{x_0}(y-x))\big)\big).$$

Here $\Gamma^0(-x_0) = 1$ and $\chi : K \to \mathbb{Z}_r^*$ is a character whose restriction to K_0 is not identically 1, *and with*

$$\mathcal{N}_\chi = \{x \in \mathbb{Z}_p : \forall f \in K_0 \text{ such that } f(x) = 0 \implies \chi_x(f) = 1\},$$

we have $\Phi^0 = \sum_{x\in\mathcal{N}_\chi} c_x\phi_x^0$, with $c_x \in \mathbb{Z}_r$, not all 0, *where*

$$\phi_x^0 : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_r, \quad \phi_x^0(y,a) = \begin{cases} 0 & \text{if } x \neq y, \\ \chi_x(f) & \text{if } x = y, \text{ where } f \in K_0 \text{ satisfies } f(x) = a. \end{cases}$$

In both cases, the conditions for Φ_1^0, Φ_2^0 and Φ^0 are sufficient to define an indecomposable cycle set (*that might be of multipermutation level* < 3).

Note that the second case of the theorem can only occur when $q \mid r-1$.

The theorem suggests that it is very unlikely to achieve a general explicit description of cycle sets of squarefree order whose size has more than three prime factors.

5. A comment on Lebed–Vendramin cohomology

We recapitulate the concept of an *abelian extension* of cycle sets that has been investigated by Lebed and Vendramin [16].

Definition 5.1 [16, Definition 9.9]. Let $p : Y \twoheadrightarrow X$ be a (constant) extension of cycle sets and let B be an abelian group with an action $B \times Y \to Y$; $(a, y) \mapsto y + a$ that restricts to regular actions on the fibers $p^{-1}(x)$ for $x \in X$. Then the pair $(p : Y \to X, B)$ is called an *abelian extension of X by B* if

$$y * (z + a) = (y * z) + a$$

holds for all $y, z \in Y$ and $a \in B$ (note that the condition $(y + a) * z = y * z$ in the original definition is contained in the assumption that p is a constant extension).

They prove that each extension of a cycle set X by B can be parametrized by a 2-*cocycle* $\Phi : X \times X \to B$ that satisfies

$$\Phi(x, z) + \Phi(x * y, x * z) = \Phi(y, z) + \Phi(y * x, y * z)$$

for all $x, y, z \in X$. Note that parallel extensions are an instance of this construction.

Given a 2-cocycle Φ, one can recover, up to equivalence of extensions, Y as $X \times B$ with the operation

$$(x, a) * (y, b) = (x * y, b + \Phi(x, y)),$$

together with the projection $Y \twoheadrightarrow X$; $(x, a) \mapsto x$. Denote this cycle set by $X \times_\Phi B$. Furthermore, 2-cocycles $\Phi, \Gamma : X \times X \to B$ define equivalent extensions[1] if and only if there is a map $\gamma : X \to B$ such that

$$\Phi(x, y) - \Gamma(x, y) = \gamma(x * y) - \gamma(y). \tag{5-1}$$

In view of the proof of Theorem 2.11, it is natural to generalize the concept as follows:

Definition 5.2. Let X be a cycle set and A an X-graded $\mathcal{G}(X)$-module. Then a (constant) extension $p : Y \twoheadrightarrow X$ is called an *abelian extension of X by A* if there is an action of A on Y given by $(a, y) \mapsto y + a$ such that

(1) for $x \in X$, the graded component $A_x \leqslant A$ acts regularly on the fiber $p^{-1}(x)$,

(2) for $x, y \in X$ with $y \neq x$, the graded component $A_y \leqslant A$ has trivial action on $p^{-1}(x)$,

(3) for $x \in X$, $y \in Y$ and $a \in A$, the action satisfies

$$x * (y + a) = x * y + \lambda_x^{-1} \cdot a.$$

Paralleling the proofs of Lebed and Vendramin, one establishes that an extension of X by A can be represented as $X \otimes_\Phi A = A^{\mathrm{sc}}$, together with a binary operation

$$(x, a) * (y, b) = (x * y, \lambda_x^{-1} \cdot (b + \Phi(x, y))) \tag{5-2}$$

for some map $\Phi : X \times X \to A$ with $\Phi(x, y) \in A_y$ for $x, y \in X$. Equation (5-2) defines a cycle set if and only if Φ is a *twisted* 2-cocycle, meaning that

$$\Phi(x, z) + \lambda_x \cdot \Phi(x * y, x * z) = \Phi(y, z) + \lambda_y \cdot \Phi(y * x, y * z). \tag{5-3}$$

[1] Here an equivalence of extensions furthermore respects the actions of B.

Note that if equivariance in the form of (2-3) is satisfied, then $\Phi(x * y, x * z) = \lambda_x^{-1} \cdot \Phi(y, z)$, and $\Phi(y * x, y * z) = \lambda_y^{-1} \cdot \Phi(x, z)$, so (5-3) is indeed satisfied. Furthermore, Φ and Γ define equivalent extensions by A, if there is a $\gamma : X \to A$ with $\gamma(x) \in A_x$ for $x \in X$ such that

$$\Phi(x, y) - \Gamma(x, y) = c_y - \lambda_x \cdot c_{x*y}.$$

Considering the proof of Theorem 2.11, it becomes apparent that an extension $p : Y \to X$ of indecomposable cycle sets can be parametrized as an extension by an X-graded $\mathcal{G}(X)$-module if $K = \ker(\mathcal{G}(Y) \twoheadrightarrow \mathcal{G}(X))$ induces regular actions on the fibers $p^{-1}(x)$ for $x \in X$. This condition is satisfied in the case of a coprime extension.

What we have essentially shown is that for an X-graded $\mathcal{G}(X)$-module A with $\gcd(|\mathcal{G}(X)|, |A|) = 1$, each twisted 2-cocycle $\Phi : X \times X \to A$ is cohomologous to a unique equivariant 2-cocycle. This observation implies the need for a development of *twisted* Lebed–Vendramin cohomology in order to get more conceptual proofs of our results.

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References

[1] R. J. Baxter, "Partition function of the eight-vertex lattice model", *Ann. Physics* **70** (1972), 193–228. MR

[2] F. Cedó and J. Okniński, "Indecomposable solutions of the Yang–Baxter equation of square-free cardinality", *Adv. Math.* **430** (2023), art. id. 109221. MR

[3] F. Cedó, E. Jespers, and J. Okniński, "Braces and the Yang–Baxter equation", *Comm. Math. Phys.* **327**:1 (2014), 101–116. MR

[4] F. Chouraqui, "Garside groups and Yang–Baxter equation", *Comm. Algebra* **38**:12 (2010), 4441–4460. MR

[5] I. Colazzo, E. Jespers, A. Van Antwerpen, and C. Verwimp, "Left non-degenerate set-theoretic solutions of the Yang–Baxter equation and semitrusses", *J. Algebra* **610** (2022), 409–462. MR

[6] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **XI**, Interscience, New York, 1962. MR

[7] C. Dietzel, S. Properzi, and S. Trappeniers, "Indecomposable involutive set-theoretical solutions to the Yang–Baxter equation of size p^2", *Comm. Algebra* **53**:3 (2025), 1238–1256. MR

[8] V. G. Drinfeld, "On some unsolved problems in quantum group theory", pp. 1–8 in *Quantum groups* (Leningrad, 1990), edited by P. P. Kulish, Lecture Notes in Math. **1510**, Springer, 1992. MR

[9] P. Etingof, T. Schedler, and A. Soloviev, "Set-theoretical solutions to the quantum Yang–Baxter equation", *Duke Math. J.* **100**:2 (1999), 169–209. MR

[10] E. Feingesicht, "Dehornoy's class and Sylows for set-theoretical solutions of the Yang–Baxter equation", *Internat. J. Algebra Comput.* **34**:1 (2024), 147–173. MR

[11] T. Gateva-Ivanova and M. Van den Bergh, "Semigroups of I-type", *J. Algebra* **206**:1 (1998), 97–112. MR

[12] L. Guarnieri and L. Vendramin, "Skew braces and the Yang–Baxter equation", *Math. Comp.* **86**:307 (2017), 2519–2534. MR

[13] B. Huppert, *Endliche Gruppen, I*, Grundl. Math. Wissen. **134**, Springer, 1967. MR

[14] P. Jedlička and A. Pilitowska, "Indecomposable involutive solutions of the Yang–Baxter equation of multipermutation level 2 with non-abelian permutation group", *J. Combin. Theory Ser. A* **197** (2023), art. id. 105753. MR

[15] A. Kanrar and W. Rump, "Involutive Yang–Baxter groups never act as Frobenius groups", preprint, 2023. arXiv 2312.06691

[16] V. Lebed and L. Vendramin, "Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation", *Adv. Math.* **304** (2017), 1219–1261. MR

[17] J.-H. Lu, M. Yan, and Y.-C. Zhu, "On the set-theoretical Yang–Baxter equation", *Duke Math. J.* **104**:1 (2000), 1–18. MR

[18] W. Rump, "A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation", *Adv. Math.* **193**:1 (2005), 40–55. MR

[19] W. Rump, "Braces, radical rings, and the quantum Yang–Baxter equation", *J. Algebra* **307**:1 (2007), 153–170. MR

[20] W. Rump, "Classification of indecomposable involutive set-theoretic solutions to the Yang–Baxter equation", *Forum Math.* **32**:4 (2020), 891–903. MR

[21] W. Rump, "Classification of indecomposable involutive set-theoretic solutions to the Yang–Baxter equation", *Forum Math.* **32**:4 (2020), 891–903. MR

[22] V. G. Turaev, "The Yang–Baxter equation and invariants of links", *Invent. Math.* **92**:3 (1988), 527–553. MR

[23] D. Van Caudenberg, B. Bogaerts, and L. Vendramin, "Incremental SAT-based enumeration of solutions to the Yang–Baxter equation", pp. 3–22 in *Tools and algorithms for the construction and analysis of systems*, edited by A. Gurfinkel and M. Heule, Springer, 2025.

[24] L. Vendramin, "Extensions of set-theoretic solutions of the Yang–Baxter equation and a conjecture of Gateva-Ivanova", *J. Pure Appl. Algebra* **220**:5 (2016), 2064–2076. MR

[25] C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction", *Phys. Rev. Lett.* **19** (1967), 1312–1315. MR

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Shadowing maps

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This article is about the shadowing property of homeomorphisms on compact metric spaces with the focus on the map associating a point of the space to each pseudoorbit — this is what we call a "shadowing map". Based on some particular dynamical properties, such as expansivity, we develop a brief theory and a hierarchy of such maps. We consider examples as odometers, shifts on infinite spaces, topologically hyperbolic homeomorphisms and north–south dynamics. We revisit a well-known technique for proving shadowing of expansive homeomorphisms with canonical coordinates due to R. Bowen to obtain a shadowing map with the property we call "self-tuning" from a hyperbolic bracket. This notion is introduced as part of the hierarchy of shadowing maps studied herein.

1. Introduction

Let $f : M \to M$ be a homeomorphism of the compact metric space (M, d). For $\delta > 0$ given, we say that a sequence $x_i \in M$ for $i \in \mathbb{Z}$ is a δ-*pseudoorbit* (or simply a δ-*orbit*) if $d(x_{i+1}, f(x_i)) \leq \delta$ for all $i \in \mathbb{Z}$. The homeomorphism is said to have the *shadowing property* (also known as *pseudoorbit tracing property*) if for all $\varepsilon > 0$ there is $\delta > 0$ such that for each δ-orbit x there is $p \in M$ such that $d(f^i(p), x_i) \leq \varepsilon$ for all $i \in \mathbb{Z}$. We say that such p is an ε-*shadow* of the δ-orbit x.

The definition of the shadowing property is about a function associating to each δ-orbit a point of the space. We study some properties of such function, their relations and the dynamical consequences. We assume two basic axioms:

- continuity, and
- orbits are shadowed by themselves.

For the continuity, we consider the δ-orbits as elements of $M^{\mathbb{Z}}$ endowed with the product topology. For the second axiom notice that for $\delta = 0$ we have a *real orbit*, i.e., $x_i = f^i(x_0)$ for all $i \in \mathbb{Z}$. For such an orbit x we demand the *shadowing point* to be x_0. These are quite natural conditions but, alone, they do not imply the shadowing property, as we will see in Example 2.6. A map satisfying these two axioms will be called *pseudoorbit map*. A pseudoorbit map inducing the shadowing property will be called *shadowing map*. In the next subsections we explain the additional properties we consider for shadowing maps and the results we obtained.

1.1. ***A hierarchy of shadowing maps.*** We develop a hierarchy of shadowing maps (see Table 1 below) by adding to these elementary axioms some properties that will be explained in what follows.

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topological hyperbolicity $\overset{\text{Theorem 7.6}}{\Longleftrightarrow}$ dynamically and translationally equivariant shadowing map
$\overset{\text{obvious}}{\Longrightarrow}$ translationally equivariant shadowing map $\overset{\text{Remark 6.2}}{\Longrightarrow}$ self-tuning shadowing map
$\overset{\text{Proposition 5.2}}{\Longrightarrow}$ L-shadowing map $\overset{\text{Proposition 4.6}}{\Longrightarrow}$ shadowing map $\overset{\text{Proposition 3.2}}{\Longrightarrow}$ shadowing

Table 1. Hierarchy of shadowing maps.

1.1.1. *Topological hyperbolicity.* The expansive homeomorphisms with shadowing will be called *topologically hyperbolic* and have a canonical shadowing map (explained by R. Bowen in [9] and P. Walters in [26]). These systems have strong properties and guided our research. We say that f is *expansive* if there is $\xi > 0$ such that if $p \neq q$ for $p, q \in M$, then there is $i \in \mathbb{Z}$ such that $d(f^i(p), f^i(q)) > \xi$. From the expansivity we see an important property: if $\varepsilon \leq \frac{1}{2}\xi$ and δ is taken from the definition of shadowing, then for a δ-orbit there is a unique shadowing point. This uniqueness implies two properties:

- dynamical equivariance, and
- translational equivariance.

Let us explain these properties. To fix notation, denote by $\mathrm{Sh}_f(x)$ the point that shadows the pseudoorbit x. The pseudoorbits can be modified in two natural ways. On the one hand, we can transform x by applying f to obtain the pseudoorbit $y = \tilde{f}(x)$ with $y_i = f(x_i)$ for all $i \in \mathbb{Z}$, which is shadowed by $\mathrm{Sh}_f(y)$. But $f(\mathrm{Sh}_f(x))$ also shadows y. The uniqueness of the shadowing point implies the dynamical equivariance or f-equivariance:

$$\mathrm{Sh}_f(\tilde{f}(x)) = f(\mathrm{Sh}_f(x)).$$

On the other hand, x can be transformed by a translation of indices τ to obtain $z = \tau(x)$ where $z_i = x_{i+1}$ for all $i \in \mathbb{Z}$. Again $f(\mathrm{Sh}_f(x))$ is a shadow of $\tau(x)$ and the uniqueness implies the translational equivariance or τ-equivariance:

$$\mathrm{Sh}_f(\tau(x)) = f(\mathrm{Sh}_f(x)).$$

In Theorem 7.6 we will show that if f admits a shadowing map which is dynamically and translationally equivariant then f is topologically hyperbolic. This is the strongest notion we consider.

1.1.2. *Translationally equivariant shadowing maps.* To obtain a weaker concept, we can simply forget expansivity and dynamical equivariance and just have a translationally equivariant shadowing map. In Section 6.1 we show that the shift homeomorphism of a compact space with infinitely many points has a translationally equivariant shadowing map which is neither dynamically equivariant nor expansive.[1] The τ-equivariance implies a semiconjugacy with τ; thus more dynamical consequences are expected. Indeed, in Sections 6.2–6.4 we show that if f admits a τ-equivariant shadowing map then

- f is topologically stable in the sense of Walters,
- $R(f) = \mathrm{clos}(\mathrm{Per}(f))$,

[1] If M is not totally disconnected the nonexpansivity of the shift can be deduced from a result of R. Mañé [20] stating that only finite-dimensional spaces can admit expansive homeomorphisms. The other case, as well as the general case, is quite direct; see Proposition 6.5.

- $f|_{\Omega(f)}$ has a τ-equivariant shadowing map,
- the induced bracket has uniform contraction.

1.1.3. *L-shadowing.* There is a stronger form of shadowing called *L-shadowing* (related to the *two-sided s-limit shadowing* of [13]) which was introduced in [10] by B. Carvalho and W. Cordeiro. Following this paper, we say that f has the *L-shadowing property* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if x is a δ-orbit and $d(f(x_i), x_{i+1}) \to 0$ as $|i| \to \infty$ then there is $p \in M$ that ε-shadows and $d(f^i(p), x_i) \to 0$ as $|i| \to \infty$. Among other things, they show that if a positively N-expansive homeomorphism is defined in a compact metric space and has L-shadowing, then the space has finitely many points. By [10, Proposition 2] L-shadowing implies shadowing, and [10, Proposition 3] proves that topologically hyperbolic homeomorphisms have L-shadowing. We prove that topologically hyperbolic homeomorphisms have a stronger property, called *self-tuning*. This property is between τ-equivariance and L-shadowing.

1.1.4. *Hyperbolic brackets and self-tuning shadowing maps.* In [9] Bowen proves the shadowing property for expansive homeomorphisms with canonical coordinates. The canonical coordinates induce a bracket as $[p, q] = r$, where r is the point in the local stable set of q and in the local unstable set of p, provided that p and q are close enough. The shadowing property is proved by considering the hyperbolic contraction of local stable and unstable sets. These ideas are captured in [23], where D. Ruelle defines a *Smale space* as a homeomorphism with a hyperbolic bracket; see Section 7.3 for the details. In Section 5 we revisit Bowen's technique for hyperbolic brackets. We forget the expansivity condition and look for minimal conditions on the bracket in order to induce a shadowing map. In Section 5 we prove that Bowen's technique gives more than a shadowing map; see Theorem 5.14. The new technical condition obtained will be called *self-tuning*. It has the following dynamical meaning: if $p \in M$ shadows the pseudoorbit x and for some $j \in \mathbb{Z}$ and $n \geq 1$ large the distances $d(f(x_i), x_{i+1})$ are small for all $|i - j| \leq n$, then the shadowing is more precise, i.e., $d(f^j(p), x_j)$ is small. In other words, the shadowing is better where the pseudoorbit has smaller jumps. In Section 5.2 we show that Bowen's technique for a north–south homeomorphism gives rise to a self-tuning shadowing map which is not translationally equivariant.

During our research, the notion of self-tuning arose as a link, a natural intermediate property, between τ-invariance and L-shadowing. It was a surprise to the author that Bowen's techniques in fact implied the self-tuning. The idea of the proof of Theorem 5.14 is simple but technical: we rewrite Bowen's proof but taking care of the jumps, that is, instead of assuming just a δ-orbit and only knowing that $d(f(x_i), x_{i+1}) < \delta$, we control the shadowing accuracy depending on the jumps $d(f(x_i), x_{i+1})$ performed by the pseudoorbit at each integer i.

1.2. ***Brackets.*** Throughout the paper we develop some brackets and explore their relation with shadowing maps. As we mentioned previously, a hyperbolic bracket induces a shadowing map with self-tuning, but it is easier to induce a bracket from a shadowing map as follows. Given close points $p, q \in M$ define a pseudoorbit x such that $x_i = f^i(p)$ for $i < 0$ and $x_i = f^i(q)$ for $i \geq 0$. Then the induced bracket is $[p, q] = \mathrm{Sh}_f(x)$; see Section 2.6.

The bracket inherits some properties of the shadowing map. For instance, the bracket induced by a shadowing map (Section 3.4) satisfies: for all $\varepsilon > 0$ there is $\gamma > 0$ such that if $d(p, q) \leq \gamma$ and $r = [p, q]$ then $d(f^n(r), f^n(q)) \leq \varepsilon$ for all $n \geq 0$ and $d(f^{-n}(r), f^{-n}(p)) \leq \varepsilon$ for all $n \geq 0$. In turn, an L-shadowing map induces a bracket (Section 4.3) which in addition satisfies $d(f^n(r), f^n(q)) \to 0$ and $d(f^{-n}(r), f^{-n}(p)) \to 0$ as $n \to +\infty$. With respect to these ideas, the case of the shift homeomorphism is interesting because its natural shadowing map induces a bracket which, in turn, induces the shadowing map in a natural way. This natural way is Bowen's technique, which we mentioned before; this is explained in Section 6.1. The properties of the shift homeomorphism are applied in Section 6.4 to conclude that the bracket induced by a τ-equivariant shadowing map has uniform contractions.

In Section 7.3 we analyze Ruelle's set of axioms for the hyperbolic bracket of a Smale space.

1.3. ***Weak forms of expansivity.*** Besides developing this brief theory of shadowing maps and brackets we obtained results concerning weak forms of expansivity:

- In Theorem 3.3 we show that if M is locally connected, f is cw-expansive (H. Kato [15]) and has a shadowing map then f is expansive. We say that f is *cw-expansive* if there is $\xi > 0$ such that if $C \subset M$ is connected and $\operatorname{diam}(f^i(C)) \leq \xi$ for all $i \in \mathbb{Z}$ then C is a singleton
- In Theorem 3.11 we show that if f is cw-expansive and has a shadowing bracket then f is cw_1-expansive. Shadowing brackets are defined in Section 3.4 as continuous maps $[\,\cdot\,,\cdot\,] : \Delta_\delta \to M$ satisfying:

(1) $[p, p] = p$ for all $p \in M$, and

(2) for all $\varepsilon > 0$ there is $\gamma > 0$ such that if $d(p, q) \leq \gamma$ then $[p, q] \in W^s_\varepsilon(q) \cap W^u_\varepsilon(p)$.

We say that f is *cw_1-expansive* if there is $\xi > 0$ such that if $C_s, C_u \subset M$ are ξ-stable and ξ-unstable connected sets then $C_s \cap C_u$ is a singleton.

- In Theorem 8.6 we show that if f is N-expansive, has the $\mathcal{T}_0$-continuous shadowing property (see [18] and the definition in Section 8) and M is a closed manifold then f is topologically hyperbolic. For $N \geq 1$ f is *N-expansive* (C. Morales [21]) if there is $\xi > 0$ such that if $A \subset M$ and $\operatorname{diam}(f^i(A)) \leq \xi$ for all $i \in \mathbb{Z}$ then A has at most N points.

1.4. ***Examples considered.*** We briefly explain the counterexamples from Table 2:

$$\begin{aligned}
&\text{topological hyperbolicity} \xleftarrow{\text{Remark 6.11 (shift maps)}} \text{translationally equivariant shadowing map} \\
&\xleftarrow{\text{Proposition 5.18 (north–south)}} \text{self-tuning shadowing map} \xleftarrow{?} \text{L-shadowing map} \\
&\xleftarrow{\text{Proposition 4.11 (h-shadowing)}} \text{shadowing map} \\
&\xleftarrow{\text{Example 3.4 (pseudo-Anosov of the two-sphere)}} \text{shadowing}
\end{aligned}$$

Table 2. Summary of counterexamples. We could not solve whether or not every L-shadowing map has self-tuning.

With shadowing but without shadowing map (Example 3.4). The pseudo-Anosov homeomorphism of the two-dimensional sphere appears first in the literature known by the author in Walters' book [27, p. 140]. It was previously used by the author [1; 4] and its interesting properties seem endless. By [1, Theorem 5.5] it is cw-expansive and has shadowing. It is not expansive; see [27, p. 140] or [4, Proposition 2.2.2]. Therefore it has no shadowing map; this follows directly from Theorem 3.3, as the sphere is locally connected.

With shadowing map but without L-shadowing (Proposition 4.11). There are some systems for which $\mathrm{Sh}_f(x) = x_0$ is a shadowing map. This property is called *h-shadowing* in the literature; see Section 3.3. A simple example with this property is the identity of the Cantor set. This contradicts the L-shadowing property, as systems with h-shadowing have no asymptotic points.

With self-tuning but without τ-equivariant shadowing map (Proposition 5.18). We consider a north–south diffeomorphism of a sphere, and in Section 5.2 we construct a hyperbolic bracket for it. From Bowen's techniques it induces a shadowing map with self-tuning. Then we prove that this particular shadowing map is not τ-equivariant. This counterexample is weak as there could still be another τ-equivariant shadowing map (we could not solve this problem).

With τ-equivariance but not topologically hyperbolic (Remark 6.11). As we said, the shift homeomorphism of a compact metric space X has a natural τ-equivariant shadowing map. As the shift is not expansive if X has infinitely many points we conclude that it is not topologically hyperbolic.

1.5. ***Outline.*** The paper goes from the bottom to the top of Table 1. In Section 2 we present pseudoorbit maps and the elementary notions we need. In Section 3 we introduce shadowing maps. In Section 4 we consider shadowing maps related to the L-shadowing property. Section 5 is devoted to the study of the self-tuning property. In Section 6 we develop the translationally equivariant shadowing maps. In Section 7 we consider dynamically equivariant shadowing maps and expansivity. In Section 8 we prove some results concerning weak forms of expansivity and the Lee's continuous shadowing property.

2. Pseudoorbit maps

In Section 2.1 we start by presenting the most elementary object of our study, which are the pseudoorbits. In Section 2.2 we define pseudoorbit maps, giving examples and proving basic properties. In Section 2.3 we study discrepancy functions to measure how far a pseudoorbit is from being a real orbit. In Section 2.4 we prove an equivalence for pseudoorbit maps. In Section 2.5 we introduce the connecting map, which is the basis for the brackets considered in Section 2.6.

2.1. ***Preliminaries on pseudoorbits.*** Let (M, d) be a compact metric space. Let $M^{\mathbb{Z}}$ be the set of sequences $x : \mathbb{Z} \to M$. We denote $x(i)$ by x_i for all $i \in \mathbb{Z}$. The set $M^{\mathbb{Z}}$ is endowed with the product topology and, consequently, it is compact. We fix an arbitrary value $\mu \in (0, 1)$ to define the distances in $M^{\mathbb{Z}}$

$$\tilde{d}_{\mathrm{sum}}(x, y) = \sum_{i\in\mathbb{Z}} \mu^{|i|} d(x_i, y_i), \quad \tilde{d}_{\max}(x, y) = \max_{i\in\mathbb{Z}} \mu^{|i|} d(x_i, y_i)$$

for all $x, y \in M^{\mathbb{Z}}$. Notice that as M is compact, the set $\{\mu^{|i|}d(x_i, y_i) : i \in \mathbb{Z}\}$ has indeed a maximum value. Both distances induce the product topology on $M^{\mathbb{Z}}$. On $M^{\mathbb{Z}}$ we consider the *translation homeomorphism*[2] $\tau : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ defined as

$$(\tau(x))_i = x_{i+1} \quad \text{for all } i \in \mathbb{Z}. \tag{1}$$

Define $\mathcal{H}(M)$ as the set of homeomorphisms of M. For $f \in \mathcal{H}(M)$ and $\delta \geq 0$ define

$$M^{\mathbb{Z}}(f, \delta) = \{x \in M^{\mathbb{Z}} : d(x_{i+1}, f(x_i)) \leq \delta \text{ for all } i \in \mathbb{Z}\}.$$

Any element of $M^{\mathbb{Z}}(f, \delta)$ is called a δ-*orbit* of f.

Remark 2.1. The set $M^{\mathbb{Z}}(f, \delta)$ is closed and invariant under τ, i.e., $\tau(M^{\mathbb{Z}}(f, \delta)) = M^{\mathbb{Z}}(f, \delta)$.

Define the *orbit map* $\text{orb} : \mathcal{H}(M) \times M \to M^{\mathbb{Z}}$ as $x = \text{orb}(f, p) = \text{orb}_f(p) \in M^{\mathbb{Z}}$, where $x_i = f^i(p)$ for all $i \in \mathbb{Z}$. From the definitions, the diagram

$$\begin{array}{ccc} M^{\mathbb{Z}} & \xrightarrow{\tau} & M^{\mathbb{Z}} \\ {\scriptstyle \text{orb}_f}\uparrow & & \uparrow{\scriptstyle \text{orb}_f} \\ M & \xrightarrow{f} & M \end{array}$$

commutes and orb_f is injective.

Remark 2.2. In particular, this means that there is a homeomorphic copy of M in $M^{\mathbb{Z}}$ which is the image of orb_f, i.e., the subset of orbits of f. Also, orb_f is a dynamical conjugacy between f and $\tau : \text{orb}_f(M) \to \text{orb}_f(M)$.

2.2. ***Pseudoorbit maps.*** For $\delta > 0$ given, we consider a function $\text{Po}_f : M^{\mathbb{Z}}(f, \delta) \to M$.

Definition 2.3. We say that Po_f is a *pseudoorbit map* if

(1) Po_f is continuous,

(2) $\text{Po}_f(\text{orb}_f(p)) = p$ for all $p \in M$.

The problem now is to find a connection between this elementary definition and the shadowing property. The next definition is natural in this direction.

Definition 2.4. We say that Po_f *induces shadowing* if for all $\varepsilon > 0$ there is $\gamma \in (0, \delta]$ such that if $x \in M^{\mathbb{Z}}(f, \gamma)$ then $\text{Po}_f(x)$ is an ε-shadow of x.

In Section 3 we study when a pseudoorbit map induces shadowing. We finish this subsection with some properties and examples.

Proposition 2.5. *If a function* $\text{Po}_f : M^{\mathbb{Z}}(f, \delta) \to M$ (*not necessarily a pseudoorbit map*) *induces shadowing, then* $\text{Po}_f(\text{orb}_f(p)) = p$ *for all* $p \in M$.

[2]Formally, it is the usual shift homeomorphism but we save the term *shift* for Section 6.1, where it is regarded as a dynamical system.

Proof. Given $p \in M$ we have that $x = \mathrm{orb}_f(p)$ is a δ-orbit for all $\delta > 0$. Then $\mathrm{Po}_f(x)$ ε-shadows x for all $\varepsilon > 0$. This means $d(f^i(\mathrm{Po}_f(x)), x_i) < \varepsilon$ for all $\varepsilon > 0$ and $f^i(\mathrm{Po}_f(x)) = x_i$, i.e., $f^i(\mathrm{Po}_f(x)) = f^i(p)$, and for $i = 0$ we have $\mathrm{Po}_f(\mathrm{orb}_f(p)) = p$. □

Example 2.6. Perhaps the simplest example of a pseudoorbit map is the projection $\mathrm{Po}_f(x) = x_0$ defined for all $x \in M^{\mathbb{Z}}$. It is continuous because the product topology in $M^{\mathbb{Z}}$ makes its projections continuous. It is easy to see that $\mathrm{Po}_f(\mathrm{orb}_f(p)) = p$; thus, the projection on the 0-coordinate is a pseudoorbit map. So every homeomorphism has a pseudoorbit map and its existence has no dynamical consequence. However it is well understood when the projection induces shadowing; see Section 3.3.

In Example 3.4 we will show that a pseudo-Anosov homeomorphism of the two-dimensional sphere has the shadowing property, but has no pseudoorbit map inducing shadowing.

2.3. ***Discrepancy functions.*** We introduce some techniques that will be used in Section 2.4 and in the proof of Proposition 5.2. Following [16], for $x \in M^{\mathbb{Z}}$ define the *f-discrepancy functions*[3]

$$\mathcal{D}_f^1(x) = \sup d(x_{i+1}, f(x_i)), \quad \mathcal{D}_f^2(x) = \sup_{i\in\mathbb{Z}} \tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)).$$

Remark 2.7. It is clear that $x \in M^{\mathbb{Z}}$ is a δ-orbit if and only if $\mathcal{D}_f^1(x) \le \delta$.

The next result proves that both functions are equivalent ways to measure how far $x \in M^{\mathbb{Z}}$ is from being an orbit of f.

Proposition 2.8. *The discrepancy functions satisfy*:

(1) *For all $x \in M^{\mathbb{Z}}$, $\mathcal{D}_f^*(x) \ge 0$ with equality if and only if x is an orbit of f, for $* = 1, 2$.*

(2) *$\mathcal{D}_f^*(x) = \mathcal{D}_f^*(\tau(x))$ for all $x \in M^{\mathbb{Z}}$, for $* = 1, 2$.*

(3) *$\mu\mathcal{D}_f^1(x) \le \mathcal{D}_f^2(x)$ for all $x \in M^{\mathbb{Z}}$ (recall that μ is the parameter used to define $\tilde{d}_{\mathrm{sum}}$).*

(4) *For all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathcal{D}_f^1(x) \le \delta$ then $\mathcal{D}_f^2(x) \le \varepsilon$.*

Proof. It is clear that they are nonnegative and that $\mathcal{D}_f^1$ vanishes only at orbits. The same conclusion for $\mathcal{D}_f^2$ follows from (3) and (4) that we prove below. The proof of (2) for $* = 1$ is direct from the definition. For $* = 2$ we define $y = \tau(x)$ and

$$\mathcal{D}_f^2(y) = \sup_{i\in\mathbb{Z}} \tilde{d}_{\mathrm{sum}}(\tau^i(y), \mathrm{orb}_f(y_i)) = \sup_{i\in\mathbb{Z}} \tilde{d}_{\mathrm{sum}}(\tau^{i+1}(x), \mathrm{orb}_f(x_{i+1})) = \mathcal{D}_f^2(x).$$

To prove (3) notice that for all $i \in \mathbb{Z}$ we have

$$\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) = \sum_{j\in\mathbb{Z}} \mu^{|j|} d([\tau^i(x)]_j, [\mathrm{orb}_f(x_i)]_j) \ge \mu d([\tau^i(x)]_1, [\mathrm{orb}_f(x_i)]_1) = \mu d(x_{i+1}, f(x_i)),$$

where the inequality follows by putting $j = 1$ in the sum. Taking $\sup_{i\in\mathbb{Z}}$ on both sides we obtain (3).

[3] In [16], besides defining the discrepancy functions, the authors consider a property called *continuously shadowing* for a different kind of system called *semihyperbolic maps*.

To prove (4), consider $\varepsilon > 0$ given. Take $n \geq 1$ such that $\sum_{|i|>n} \mu^{|i|} \operatorname{diam}(M) < \frac{1}{2}\varepsilon$. Let

$$\gamma = \tfrac{1}{2}\varepsilon \Big(\sum_{|i|\leq n} \mu^{|i|} \Big)^{-1}.$$

Take $\delta > 0$ such that if x is a δ-orbit then $d(f^i(x_0), x_i) < \gamma$ for $|i| \leq n$. Then

$$\begin{aligned} \tilde{d}_{\text{sum}}(x, \operatorname{orb}_f(x_0)) &= \sum_{i\in\mathbb{Z}} \mu^{|i|} d(x_i, f^i(x_0)) = \sum_{|i|\leq n} \mu^{|i|} d(x_i, f^i(x_0)) + \sum_{|i|>n} \mu^{|i|} d(x_i, f^i(x_0)) \\ &< \sum_{|i|\leq n} \mu^{|i|}\gamma + \sum_{|i|>n} \mu^{|i|} \operatorname{diam}(M) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Since $\tau^i(x)$ is also a δ-orbit, we have

$$\tilde{d}_{\text{sum}}(\tau^i(x), \operatorname{orb}_f((\tau^i(x))_0)) = \tilde{d}_{\text{sum}}(\tau^i(x), \operatorname{orb}_f(x_i)) < \varepsilon$$

for all $i \in \mathbb{Z}$. Taking the supremum again the proof ends. □

2.4. ***A characterization.*** We will prove a characterization of pseudoorbit maps that will be used in the proof of Proposition 3.2 to show that shadowing maps induce shadowing.

Proposition 2.9. *For a continuous map* $\operatorname{Po}_f : M^{\mathbb{Z}}(f, \delta) \to M$ *the following are equivalent*:

(1) Po_f *is a pseudoorbit map.*

(2) *For all* $\varepsilon > 0$ *there is* $\rho > 0$ *such that if* $x \in M^{\mathbb{Z}}(f, \rho)$ *then* $d(x_i, \operatorname{Po}_f(\tau^i(x))) < \varepsilon$ *for all* $i \in \mathbb{Z}$.

(3) *For all* $\varepsilon > 0$ *there is* $\rho > 0$ *such that if* $x \in M^{\mathbb{Z}}(f, \rho)$ *then* $d(x_0, \operatorname{Po}_f(x)) < \varepsilon$.

Proof. It is obvious that (2) implies (3). We have that (3) implies (2) because $(\tau^i(x))_0 = x_i$ for all $i \in \mathbb{Z}$; also $x \in M^{\mathbb{Z}}(f, \rho)$ if and only if $\tau^i(x) \in M^{\mathbb{Z}}(f, \rho)$ for all $i \in \mathbb{Z}$ (Remark 2.1).

To prove that (1) implies (3) consider $\varepsilon > 0$ given and take $\nu > 0$ such that if $\tilde{d}_{\text{sum}}(x, y) < \nu$ then $d(\operatorname{Po}_f(x), \operatorname{Po}_f(y)) < \varepsilon$ (uniform continuity of Po_f). By Proposition 2.8 there is $\rho > 0$ such that if x is a ρ-orbit then $\tilde{d}_{\text{sum}}(x, \operatorname{orb}_f(x_0)) < \nu$. Therefore, given any $x \in M^{\mathbb{Z}}(f, \rho)$, we have $d(x_0, \operatorname{Po}_f(x)) = d(\operatorname{Po}_f(\operatorname{orb}_f(x_0)), \operatorname{Po}_f(x)) < \varepsilon$.

Let us show that (3) implies (1). Given $p \in M$ let $x = \operatorname{orb}_f(p)$. Then, $x_0 = p$ and $x \in M^{\mathbb{Z}}(f, \rho)$ for all $\rho > 0$. Thus $d(x_0, \operatorname{Po}_f(x)) < \varepsilon$ for $\varepsilon > 0$. This implies $x_0 = \operatorname{Po}_f(x)$ and

$$p = x_0 = \operatorname{Po}_f(x) = \operatorname{Po}_f(\operatorname{orb}_f(p)).$$

Therefore Po_f is a pseudoorbit map. □

2.5. ***Connecting map.*** Define the *connecting map*

$$\operatorname{con} : \mathcal{H}(M) \times M^2 \to M^{\mathbb{Z}}$$

as $\mathrm{con}(f, p, q) = \mathrm{con}_f(p, q) = x$, where

$$x_i = \begin{cases} f^i(p) & \text{for } i < 0, \\ f^i(q) & \text{for } i \geq 0. \end{cases}$$

This map is the key to inducing a bracket from a shadowing map in Section 2.6.

On $\mathcal{H}(M)$ we consider the C^0 distance

$$d_{C^0}(f, g) = \sup\{d(f(p), g(p)), d(f^{-1}(p), g^{-1}(p)) : p \in M\}.$$

Proposition 2.10. *The connecting map is continuous and uniformly continuous with respect to the variables in M^2: for all $f \in \mathcal{H}(M)$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $d_{C^0}(f, g) < \delta$, $d(p_1, p_2) < \delta$ and $d(q_1, q_2) < \delta$ then*

$$d(\mathrm{con}_f(p_1, q_1), \mathrm{con}_g(p_2, q_2)) < \varepsilon.$$

Proof. Given $\varepsilon > 0$ take $j \geq 0$ and $\gamma > 0$ such that

$$\sum_{|i|>j} \mu^{|i|} \operatorname{diam}(M) < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \sum_{|i|\leq j} \mu^{|i|}\gamma < \tfrac{1}{2}\varepsilon.$$

Given $f \in \mathcal{H}(M)$ consider $\delta > 0$ such that if $d(r_1, r_2) < \delta$ and $d_{C^0}(g, f) < \delta$ then $d(f^i(r_1), g^i(r_2)) < \gamma$ for all $|i| \leq j$. In this way, if $d_{C^0}(f, g) < \delta$, $d(p_1, p_2) < \delta$, $d(q_1, q_2) < \delta$, $x = \mathrm{con}_f(p_1, q_1)$ and $y = \mathrm{con}_g(p_2, q_2)$ then

$$\tilde{d}_{\mathrm{sum}}(x, y) = \sum_{i\in\mathbb{Z}} \mu^{|i|} d(x_i, y_i) \leq \sum_{|i|>j} \mu^{|i|} \operatorname{diam}(M) + \sum_{|i|\leq j} \mu^{|i|}\gamma < \varepsilon. \qquad \square$$

Corollary 2.11. *The orbit map is continuous and uniformly continuous with respect to the second variable: for all $f \in \mathcal{H}(M)$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $d_{C^0}(f, g) < \delta$ and $d(p, q) < \delta$ then $d(\mathrm{orb}_f(p), \mathrm{orb}_g(q)) < \varepsilon$.*

Proof. It follows from Proposition 2.10 because $\mathrm{orb}_f(p) = \mathrm{con}_f(p, p)$ for all $f \in \mathcal{H}(M)$ and $p \in M$. $\square$

2.6. ***Induced brackets.*** Consider the δ-neighborhood of the diagonal

$$\Delta_\delta = \{(p, q) \in M^2 : d(p, q) \leq \delta\}.$$

Suppose that $\mathrm{Po}_f : M^{\mathbb{Z}}(f, \delta) \to M$ is a pseudoorbit map. Since $\mathrm{con}_f(\Delta_\delta) \subset M^{\mathbb{Z}}(f, \delta)$ we can define a bracket $[\,\cdot\,,\cdot\,] : \Delta_\delta \to M$ as

$$[p, q] = \mathrm{Po}_f(\mathrm{con}_f(p, q)). \tag{2}$$

We say that this bracket is *induced* by Po_f.

Proposition 2.12. *The induced bracket satisfies*:

(1) *It is continuous.*

(2) $[p, p] = p$ *for all* $p \in M$.

Proof. Continuity follows by the continuity of Po_f (by definition) and con_f (Proposition 2.10). For $p = q$ we have $[p, p] = \mathrm{Po}_f(\mathrm{con}_f(p, p)) = \mathrm{Po}_f(\mathrm{orb}_f(p)) = p$. □

At this moment the induced bracket has essentially no dynamical property. In the next sections this will change as the pseudoorbit map becomes more interesting.

3. Shadowing maps

In this section we introduce and study the shadowing maps. In Section 3.1 we prove a useful equivalence. In Section 3.2 we show that cw-expansivity implies expansivity, assuming the existence of a shadowing map and the local connection of the space. In Section 3.3 we present known results characterizing when the projection is a shadowing map. In Section 3.4 we define shadowing brackets. We show that cw-expansivity with a shadowing bracket implies cw_1-expansivity, which in turn implies expansivity on compact surfaces.

3.1. ***Shadowing maps.*** The concept of shadowing map arises by considering two things. On the one hand, at least it should be a pseudoorbit map inducing the shadowing property. On the other hand, we have the strong property of τ-equivariance, which means

$$d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

The formal definition of shadowing map is given as a weak form of this property, and in Proposition 3.2 we prove that both viewpoints coincide.

Definition 3.1. A pseudoorbit map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f, \delta) \to M$ is a *shadowing map* if for all $\varepsilon > 0$ there is $\gamma \in (0, \delta]$ such that if $x \in M^{\mathbb{Z}}(f, \gamma)$ then

$$d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) \leq \varepsilon \quad \text{for all } i \in \mathbb{Z}. \tag{3}$$

This condition is exactly what a pseudoorbit map needs to induce the shadowing property.

Proposition 3.2. *A pseudoorbit map induces shadowing if and only if it is a shadowing map. In particular, every homeomorphism with a shadowing map has the shadowing property.*

Proof. Direct: Given $\varepsilon > 0$, take $\gamma > 0$ such that every γ-orbit x is $\frac{1}{2}\varepsilon$-shadowed by $\mathrm{Sh}_f(x)$. Notice that $\tau^j(x)$ also is a γ-orbit for all $j \in \mathbb{Z}$. Thus

$$d\big(f^i(\mathrm{Sh}_f(\tau^j(x))), (\tau^j(x))_i\big) \leq \tfrac{1}{2}\varepsilon \quad \text{for all } i, j \in \mathbb{Z}.$$

For $j = 0$ we have

$$d(f^i(\mathrm{Sh}_f(x)), x_i) \leq \tfrac{1}{2}\varepsilon \quad \text{for all } i \in \mathbb{Z}$$

and for $i = 0$ we conclude

$$d\big(\mathrm{Sh}_f(\tau^j(x)), (\tau^j(x))_0\big) = d\big(\mathrm{Sh}_f(\tau^j(x)), x_j\big) \leq \tfrac{1}{2}\varepsilon \quad \text{for all } j \in \mathbb{Z}.$$

The triangle inequality gives

$$d\big(f^k(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^k(x))\big) \le d\big(f^k(\mathrm{Sh}_f(x)), x_k\big) + d\big(x_k, \mathrm{Sh}_f(\tau^k(x))\big) \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

for all $k \in \mathbb{Z}$. This proves (3).

Converse: Given $\varepsilon > 0$ take $\gamma > 0$ from Proposition 2.9 such that if $x \in M^{\mathbb{Z}}(f, \gamma)$ then

$$d(x_i, \mathrm{Sh}_f(\tau^i(x))) \le \tfrac{1}{2}\varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

By hypothesis, considering a smaller γ if needed, we can also assume

$$d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) \le \tfrac{1}{2}\varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

The triangle inequality gives us

$$d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) \le \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

This proves that Sh_f induces shadowing. □

3.2. ***Cw-expansivity and shadowing maps.*** We will show that there are homeomorphisms with shadowing that do not admit a shadowing map. For this purpose we recall the notion of continuumwise expansivity.

A subset $C \subset M$ is a *continuum* if it is closed and connected. Following Kato we say that f is *cw-expansive* if there is $\xi > 0$ such that if $C \subset M$ is a continuum and $\mathrm{diam}(f^i(C)) \le \xi$ for all $i \in \mathbb{Z}$ then C is a singleton. If we define the *dynamical ball*

$$\Gamma_\xi(p) = \{q \in M : d(f^i(p), f^i(q)) \le \xi \text{ for all } i \in \mathbb{Z}\},$$

then f is cw-expansive if and only if there is $\xi > 0$ such that $\Gamma_\xi(p)$ is totally disconnected for all $p \in M$.

Theorem 3.3. *If M is locally connected, f is cw-expansive and has a shadowing map then f is expansive.*

Proof. Let 2ξ be a cw-expansivity constant and suppose that $\mathrm{Sh}_f(x)$ is a $\frac{1}{2}\xi$-shadow of each δ-orbit x. Take $\gamma > 0$ such that $\gamma < \min\{\frac{1}{2}\delta, \frac{1}{2}\xi\}$ and

$$d(a, b) < \gamma \implies d(f(a), f(b)) < \tfrac{1}{2}\delta. \tag{4}$$

As M is locally connected there is $c > 0$ such that if $d(a, b) \le c$ then there is a continuum of diameter smaller than γ containing a and b. We will show that c is an expansivity constant.

Suppose that $d(f^i(p), f^i(q)) \le c$ for all $i \in \mathbb{Z}$. For each $i \in \mathbb{Z}$ take a continuum $C_i \subset M$ containing $f^i(p)$ and $f^i(q)$ with $\mathrm{diam}(C_i) < \gamma$. Define the set

$$A = \{x \in M^{\mathbb{Z}} : x_i \in C_i \text{ for all } i \in \mathbb{Z}\}.$$

Note that $\mathrm{orb}_f(p), \mathrm{orb}_f(q) \in A$.

In this paragraph we will show that $A \subset M^{\mathbb{Z}}(f, \delta)$. If $x \in A$ and $i \in \mathbb{Z}$ then $d(f^i(p), x_i) < \gamma$ (because both points belong to C_i) and $d(f^{i+1}(p), f(x_{i+1})) < \frac{1}{2}\delta$ (by (4)). Since $x_{i+1}, f^{i+1}(p) \in C_{i+1}$ we have

$d(x_{i+1}, f^{i+1}(p)) < \gamma < \frac{1}{2}\delta$ and the triangle inequality implies $d(f(x_i), x_{i+1}) \leq \frac{1}{2}\delta$. Thus each $x \in A$ is a δ-orbit and $A \subset M^{\mathbb{Z}}(f, \delta)$.

Since Sh_f is a pseudoorbit map we conclude $\mathrm{Sh}_f(A)$ is a continuum (A is a continuum and Sh_f is continuous) and $p, q \in \mathrm{Sh}_f(A)$.

For $r \in \mathrm{Sh}_f(A)$ there is $x \in A$ such that $r = \mathrm{Sh}_f(x)$. Since $\mathrm{Sh}_f(x)$ is a $\frac{1}{2}\xi$-shadow of x we have $d(f^i(r), x_i) \leq \frac{1}{2}\xi$ for all $i \in \mathbb{Z}$. Since $d(x_i, f^i(p)) \leq \gamma \leq \frac{1}{2}\xi$, by the triangle inequality we have $d(f^i(r), f^i(p)) \leq \xi$ for all $i \in \mathbb{Z}$. This means $\mathrm{Sh}_f(A) \subset \Gamma_\xi(p)$. Since 2ξ is a cw-expansivity constant the set $\Gamma_\xi(p)$ is totally disconnected. Thus $\mathrm{Sh}_f(A)$ is connected (proved above), totally disconnected and contains p and q. This implies that $\mathrm{Sh}_f(A)$ is a singleton and $p = q$. This proves that f is expansive with expansivity constant c. □

Example 3.4. In [1, Theorem 5.5] it is shown that the pseudo-Anosov homeomorphism of the two-dimensional sphere described in [1; 4] is cw-expansive and has shadowing. It is well known that it is not expansive; see for instance [4, Proposition 2.2.2]. Therefore, it is an example with the shadowing property but without a shadowing map (it follows from Theorem 3.3 as it is not expansive).

3.3. ***The h-shadowing property.*** In this section we show examples on totally disconnected spaces having a shadowing map. Sometimes the shadowing point is in front of us.

Definition 3.5 (exact hit shadowing [8, Definition 3.3]). We say that f has *shadowing with exact hit*, or *h-shadowing*, if for all $\varepsilon > 0$ there is $\delta > 0$ such that if x is a δ-orbit then $d(f^i(x_0), x_i) \leq \varepsilon$ for all $i \in \mathbb{Z}$.

Remark 3.6. By definition, f has h-shadowing if and only if $\mathrm{Sh}_f(x) = x_0$ is a shadowing map.

Theorem 3.7 [7; 17]. *For a homeomorphism f of a compact metric space M the following are equivalent*:

(1) *f has h-shadowing.*

(2) *f has shadowing and is equicontinuous.*

(3) *f is equicontinuous and M is totally disconnected.*

Example 3.8. Odometers are the examples of minimal and equicontinuous homeomorphisms of the Cantor set; see [17, Proposition 4.4]. Therefore, odometers have a shadowing map.

3.4. ***Shadowing bracket.*** In this section we continue Section 2.6 about the bracket induced by a shadowing map.

Definition 3.9. A *shadowing bracket* is a continuous map $[\cdot, \cdot] : \Delta_\delta \to M$ such that

(1) $[p, p] = p$ for all $p \in M$,

(2) for all $\varepsilon > 0$ there is $\gamma > 0$ satisfying $d(p, q) \leq \gamma \implies [p, q] \in W^s_\varepsilon(q) \cap W^u_\varepsilon(p)$.

Recall that the induced bracket is $[p, q] = \mathrm{Sh}_f(\mathrm{con}_f(p, q))$.

Proposition 3.10. *The bracket induced by a shadowing map is a shadowing bracket.*

Proof. Given $\varepsilon > 0$ take $0 < \varepsilon' \le \varepsilon$ such that $d(a,b) \le \varepsilon'$ for $a, b \in M$, implies $d(f(a), f(b)) \le \varepsilon$. Since f has shadowing there is $\gamma > 0$ such that every γ-orbit x is ε'-shadowed by $\mathrm{Sh}_f(x)$. For $p, q \in M$ with $d(p,q) \le \gamma$ we have that $x = \mathrm{con}_f(p,q)$ is a γ-orbit. Thus it is ε'-shadowed by $r = [p,q]$. That is, $d(f^n(r), f^n(q)) \le \varepsilon' \le \varepsilon$ for all $n \ge 0$. Also, $d(f^{-n}(r), f^{-n}(p)) \le \varepsilon' \le \varepsilon$ for all $n \ge 1$. For $n = 0$ we have $d(r,p) \le \varepsilon$ from the way we choose ε'. This implies $r = [p,q] \in W^s_\varepsilon(q) \cap W^u_\varepsilon(p)$. □

3.5. ***Cw-expansivity and shadowing brackets.*** In Section 3.2 we proved that the pseudo-Anosov homeomorphism of the two-sphere does not admit a shadowing map. Now we show that it does not even admit a shadowing bracket. This is done through Theorem 3.11, where we show that cw-expansivity implies cw_1-expansivity provided the existence of a shadowing bracket. Unlike Theorem 3.3, here we do not assume the local connection of M. Both proofs have the topological connection as a key property. We learnt this kind of argument from J. Lewowicz; see for instance [19, Lemma 2.3].

We recall the required definitions. A continuum is *ε-stable* if $\mathrm{diam}(f^n(C)) \le \varepsilon$ for all $n \ge 0$. It is *ε-unstable* if $\mathrm{diam}(f^{-n}(C)) \le \varepsilon$ for all $n \ge 0$. For $N \ge 1$ we say that f is *cw_N-expansive* [4] if there is $\xi > 0$ such that if $C_s \subset M$ is an ξ-stable continuum and $C_u \subset M$ is ξ-unstable then $C_s \cap C_u$ contains at most N points. It is stronger than cw-expansivity.

Theorem 3.11. *If f is cw-expansive and has a shadowing bracket then f is cw_1-expansive.*

Proof. Suppose that $2\xi > 0$ is a cw-expansivity constant. Suppose that the bracket is defined in Δ_δ. Taking a smaller δ if needed we can assume that $[p,q] \in W^s_{\xi/3}(q) \cap W^u_{\xi/3}(p)$ whenever $d(p,q) \le \delta$.

Suppose that C_s and C_u are $\frac{1}{3}\xi$-stable/unstable continua, respectively. Assume that $\mathrm{diam}(C_s \cup C_u) \le \delta$. In this way we can define

$$A = [C_u, C_s] = \{[p,q] : p \in C_u, q \in C_s\}.$$

Note that A is connected as it is the image of a connected set by a continuous function. If $r_1, r_2 \in A$ we can write $r_j = [p_j, q_j]$ for $p_j \in C_u$ and $q_j \in C_s$ for $j = 1, 2$. In this way we have, for $i \ge 0$,

$$d(f^i(r_1), f^i(r_2)) \le d(f^i(r_1), f^i(q_1)) + d(f^i(q_1), f^i(q_2)) + d(f^i(q_2), f^i(r_2)) \le \tfrac{1}{3}\xi + \tfrac{1}{3}\xi + \tfrac{1}{3}\xi = \xi.$$

Analogously, for $i < 0$ we have $d(f^i(r_1), f^i(r_2)) \le \xi$. Thus if we fix some $r \in A$ we conclude $A \subset \Gamma_{2\xi}(r)$. Since 2ξ is a cw-expansivity constant $\Gamma_{2\xi}(r)$ and A are totally disconnected. As A is connected, it is a singleton.

Finally notice that for each $p \in C_u \cap C_s$ we have $p = [p,p] \in A$, i.e., $C_u \cap C_s$ is contained in the singleton A and it is a singleton or empty. Let us show that $\gamma = \min\{\frac{1}{2}\delta, \frac{1}{3}\xi\}$ is a cw_1-expansivity constant. Suppose that C_s and C_u are γ-stable/unstable continua, respectively. If C_s intersects C_u then the diameter of their union is at most δ, so we can apply our previous results to conclude that their intersection is a singleton. □

Corollary 3.12. *If M is a compact surface, f is cw-expansive and has a shadowing bracket then f is conjugate to a linear Anosov diffeomorphism of the two-torus.*

Proof. In [4, Theorem 6.8.5] it is shown that every cw_1-expansive homeomorphism of a compact surface is expansive. From Theorem 3.11 we conclude that f is expansive. The shadowing bracket implies the local product structure and the shadowing property. Applying the classification theorem of expansive homeomorphisms of surfaces [19; 14] the proof ends. □

Corollary 3.13. *The pseudo-Anosov homeomorphisms of the two-sphere of Example 3.4 has neither a shadowing bracket nor a shadowing map.*

Proof. It is easy to see that this example is not cw_1-expansive. By Theorem 3.11 it admits no shadowing bracket. By Proposition 3.10 it also admits no shadowing map. □

4. L-shadowing maps

In this brief section we present the L-shadowing property, define L-shadowing maps, discuss some examples and prove elementary properties.

4.1. ***Preliminaries on the L-shadowing property.*** The L-shadowing property captures some features of expansive homeomorphisms with shadowing.

Definition 4.1. We say that f has the *L-shadowing property* if for all $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-orbit x that is also a two-sided limit pseudoorbit (i.e., $d(f(x_i), x_{i+1}) \to 0$ as $|i| \to \infty$), there is $p \in M$ that ε-shadows and two-sided limit shadows x (i.e., $d(f^i(p), x_i) \to 0$ as $|i| \to \infty$).

Examples of homeomorphisms with L-shadowing are structurally stable diffeomorphisms and some pseudo-Anosov maps of the two-sphere [6, Theorem A]. In [6, Theorem D] it is given a homeomorphism with L-shadowing, topologically mixing and without periodic points. We cite some known dynamical properties.

Theorem 4.2 [6, Proposition 4.1]. *If f has shadowing and its restriction to $\Omega(f)$ is expansive then f has L-shadowing.*

Theorem 4.3. *If f has L-shadowing then*:

(1) [10, Proposition 2] *f has shadowing and $R(f)$ has finitely many chain recurrent classes.*

(2) [6, Theorem B] *Each chain recurrent class is either expansive or contains arbitrarily small topological semihorseshoes.*

(3) [6, Theorem F] *If C is a chain recurrent class of f then there are $C_1, \dots, C_n \subset C$, for $n \geq 1$, closed and disjoint sets such that $C = \bigcup_{j=1}^{n} C_j$, $f(C_j) = C_{j+1 \bmod n}$ and f^n restricted to each C_i is topologically mixing.*

For $\varepsilon > 0$ given, let $V_\varepsilon^*(p) = W_\varepsilon^*(p) \cap W^*(p)$ for $* = s, u$.

Theorem 4.4 [6, Theorem E]. *A homeomorphism f has L-shadowing if and only if it has shadowing and for all $\varepsilon > 0$ there is $\delta > 0$ such that $d(p, q) < \delta$ implies $V_\varepsilon^s(p) \cap V_\varepsilon^u(q) \neq \varnothing$.*

4.2. ***L-shadowing map.*** The next definition is natural.

Definition 4.5. A pseudoorbit map Sh_f is an *L-shadowing map* if for all $\varepsilon > 0$ there is $\gamma > 0$ such that for every γ-orbit x that is also a two-sided limit pseudoorbit it holds that $\mathrm{Sh}_f(x)$ ε-shadows and two-sided limit shadows x.

Proposition 4.6. *Every L-shadowing map is a shadowing map.*

Proof. Given $\varepsilon > 0$, take γ from the definition of an L-shadowing map. For x a γ-orbit define the sequence of γ-orbits y_n, for $n \geq 1$, as

$$(y_n)_i = \begin{cases} x_i & \text{if } |i| \leq n, \\ f^{i-n}(x_n) & \text{if } i \geq n, \\ f^{i+n}(x_{-n}) & \text{if } i \leq n. \end{cases}$$

We see that $y_n \to x$ and each y_n is a two-sided limit γ-orbit. By definition, $\mathrm{Sh}_f(y_n)$ is an ε-shadow of y_n. Taking the limit we conclude that $\mathrm{Sh}_f(x)$ is an ε-shadow of x. Thus Sh_f induces shadowing and is a shadowing map (Proposition 3.2). □

Remark 4.7. By definition, every L-shadowing map induces the L-shadowing property.

4.3. ***L-brackets.*** In light of L-shadowing maps we define L-brackets.

Definition 4.8. An *L-bracket* is a continuous map $[\,\cdot\,,\cdot\,] : \Delta_\delta \to M$ such that

(1) $[p, p] = p$ for all $p \in M$,

(2) for all $\varepsilon > 0$ there is $\gamma > 0$ satisfying

$$d(p, q) \leq \gamma \implies [p, q] \in V^s_\varepsilon(q) \cap V^u_\varepsilon(p).$$

Remark 4.9. From Theorem 4.4 we have that, for a homeomorphism with an L-bracket, the shadowing property and the L-shadowing property are equivalent.

Proposition 4.10. *The bracket induced by an L-shadowing map is an L-bracket.*

Proof. By Proposition 4.6 every L-shadowing map is a shadowing map. Thus we can apply Proposition 3.10 to conclude that $[p, q] = \mathrm{Sh}_f(\mathrm{con}_f(p, q))$ is a shadowing bracket. Then, given $\varepsilon > 0$, take $\gamma > 0$ such that if $d(p, q) \leq \gamma$ then $[p, q] \in W^s_\varepsilon(q) \cap W^u_\varepsilon(p)$. Finally, if $x = \mathrm{con}_f(p, q)$ we see that x is a two-sided limit pseudoorbit. Thus, from the definition of L-shadowing map, $\mathrm{Sh}_f(x) \in V^u(p) \cap V^s(q)$ and the proof ends. □

Proposition 4.11. *The homeomorphisms with h-shadowing do not have the L-shadowing property if M has infinitely many points. Consequently, these systems have a shadowing map (Remark 3.6) but not an L-shadowing map (Remark 4.7).*

Proof. By Theorem 4.4 the L-shadowing property implies the existence of asymptotic points (i.e., $p, q \in M$ such that $p \neq q$ and $d(f^n(p), f^n(q)) \to 0$ as $n \to +\infty$) if M has infinitely many points. By definition f^{-1} also has h-shadowing and Theorem 3.7 implies the equicontinuity of f^{-1}. This contradicts the existence of asymptotic points and the proof ends. □

5. Self-tuning

In this section we introduce and study the property we call *self-tuning*. In Section 5.1 we consider hyperbolic brackets and show how they induce self-tuning pseudoorbit maps. It is a technical proof based on Bowen's arguments of [9]. In Section 5.2 this result is applied to north–south dynamics. We start with the definition and some fundamental properties.

Definition 5.1. The pseudoorbit map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f,\delta) \to M$ has *self-tuning* if for all $\varepsilon > 0$ there is $\gamma > 0$ such that if $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma$ for some $x \in M^{\mathbb{Z}}(f,\delta)$ and some $i \in \mathbb{Z}$ then

$$d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) \leq \varepsilon. \tag{5}$$

Proposition 5.2. *Every self-tuning pseudoorbit map is an L-shadowing map.*

Proof. Given $\varepsilon > 0$ take $\gamma > 0$ as in Definition 5.1. In what follows we use the discrepancy function $\mathcal{D}_f^2$ of Section 2.3. By Proposition 2.8 there is $\rho > 0$ such that if x is a ρ-orbit then $\mathcal{D}_f^2(x) < \gamma$ and consequently $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma$ for all $i \in \mathbb{Z}$. Thus every ρ-orbit satisfies (5) for all $i \in \mathbb{Z}$, Sh_f is a shadowing map and it induces shadowing.

Suppose that x is a ρ-orbit that is also a two-sided limit pseudoorbit. That is, $d(f(x_i), x_{i+1}) \to 0$ as $|i| \to \infty$. This implies that $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) \to 0$ as $|i| \to \infty$. Thus $d(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))) \to 0$ as $|i| \to \infty$. Also, from the uniform continuity of Sh_f, we have

$$d\big(\mathrm{Sh}_f(\tau^i(x)), x_i\big) = d\big(\mathrm{Sh}_f(\tau^i(x)), \mathrm{Sh}_f(\mathrm{orb}_f(x_i))\big) \to 0$$

as $|i| \to \infty$. Finally,

$$d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) \leq d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) + d\big(\mathrm{Sh}_f(\tau^i(x)), x_i\big) \to 0.$$

This proves that Sh_f is an L-shadowing map. □

We have neither a counterexample nor a proof for the converse of Proposition 5.2. The next characterization is used to prove that hyperbolic brackets (Definition 5.4) induce the self-tuning property.

Proposition 5.3. *A pseudoorbit map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f,\delta) \to M$ has self-tuning if and only if for all $\varepsilon' > 0$ there is $\gamma' > 0$ such that if $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma'$ for some $x \in M^{\mathbb{Z}}(f,\delta)$ and some $i \in \mathbb{Z}$ then*

$$d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) \leq \varepsilon'. \tag{6}$$

Proof. We start with an argument for both directions. Given $\varepsilon, \varepsilon' > 0$ take $\gamma'' > 0$ such that if $\tilde{d}_{\mathrm{sum}}(y,z) < \gamma''$ then $d(\mathrm{Sh}_f(y), \mathrm{Sh}_f(z)) < \frac{1}{2}\min\{\varepsilon, \varepsilon'\}$.

To prove the forward direction take $\gamma \in (0, \gamma'')$ such that if $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma$ then

$$d\big(f^i(\mathrm{Sh}_f(x)), \mathrm{Sh}_f(\tau^i(x))\big) \leq \tfrac{1}{2}\varepsilon'.$$

Since $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma''$ we have

$$d\big(\mathrm{Sh}_f(\tau^i(x)), x_i\big) = d\big(\mathrm{Sh}_f(\tau^i(x)), \mathrm{Sh}_f(\mathrm{orb}_f(x_i))\big) < \tfrac{1}{2}\varepsilon'.$$

Thus $d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) \leq \varepsilon'$.

The converse is quite similar. Take $\gamma' \in (0, \gamma'')$ such that if $\tilde{d}_{\text{sum}}(\tau^i(x), \text{orb}_f(x_i)) < \gamma'$ then

$$d\big(f^i(\text{Sh}_f(x)), x_i\big) \le \tfrac{1}{2}\varepsilon.$$

Again we conclude $d\big(\text{Sh}_f(\tau^i(x)), x_i\big) < \frac{1}{2}\varepsilon$ and $d\big(f^i(\text{Sh}_f(x)), \text{Sh}_f(\tau^i(x))\big) \le \varepsilon$. □

5.1. ***Hyperbolic brackets.*** In this section we show that hyperbolic brackets induce shadowing maps with self-tuning. For this purpose we will assume that f and f^{-1} are Lipschitz.

Definition 5.4. Suppose that $c \ge 1$ and $\mu \in (0, 1)$. A bracket $[\,\cdot\,,\cdot\,] : \Delta_\gamma \to M$ has (c, μ)-*hyperbolic contraction* if $d(p, q) \le \gamma$ implies

$$d(f^n([p, q]), f^n(q)) \le c\mu^n d(p, q), \quad d(f^{-n}([p, q]), f^{-n}(p)) \le c\mu^n d(p, q)$$

for all $n \ge 0$.

5.1.1. *Lipschitz homeomorphisms.* In this section we give some elementary estimates for Lipschitz maps. For $L > 0$ and $n \ge 1$ define $L_n = 1 + L + L^2 + \cdots + L^{n-1}$. Note that $L_1 = 1$ and $LL_n + 1 = L_{n+1}$. Define

$$\delta_i(x) = d(f(x_{i-1}), x_i)$$

for all $i \in \mathbb{Z}$ and $x \in M^{\mathbb{Z}}$. The consideration of this quantity makes the difference between Bowen's proof and our argument because we need to improve the shadowing where the values of $\delta_i(x)$ are small.

Lemma 5.5. *If f is Lipschitz with Lipschitz constant $L > 0$ then*

$$d(f^n(x_0), x_n) \le \sum_{j=1}^{n} \delta_j(x) L^{n-j} \le L^n \sum_{j=1}^{n} \delta_j(x) \quad \textit{for all } n \ge 1 \tag{7}$$

for all $x \in M^{\mathbb{Z}}$. If x is a δ-orbit then

$$d(f^n(x_0), x_n) \le \delta L_n \quad \textit{for all } n \ge 1. \tag{8}$$

Proof. We prove the first inequality by induction. For $n = 1$ we have $d(f(x_0), x_1) \le \delta_1(x)$ (this is in fact an equality by definition). If the inequality is true for $n \ge 1$ then

$$d(f^{n+1}(x_0), x_{n+1}) \le d(f^{n+1}(x_0), f(x_n)) + d(f(x_n), x_{n+1}) \le L \sum_{j=1}^{n} \delta_j(x) L^{n-j} + \delta_{n+1} = \sum_{j=1}^{n+1} \delta_j(x) L^{n+1-j}.$$

Since M is compact we have $L \ge 1$. Thus $L^n \ge L^{n-j}$ for all $1 \le j \le n$ and

$$\sum_{j=1}^{n} \delta_j(x) L^{n-j} \le L^n \sum_{j=1}^{n} \delta_j(x).$$

If x is a δ-orbit then $\delta_j(x) \le \delta$ for all $j \in \mathbb{Z}$ and

$$\sum_{j=1}^{n} \delta_j(x) L^{n-j} \le \delta \sum_{j=1}^{n} L^{n-j} = \delta L_n.$$

□

5.1.2. *A pseudoorbit map.* In this section we define a pseudoorbit map from a hyperbolic bracket. In Section 5.1.3 we show that in fact it has self-tuning. Suppose that f is Lipschitz and has a hyperbolic bracket $[\,\cdot\,,\cdot\,]:\Delta_\gamma \to M$ with parameters c and μ as explained above. Fix m large enough that

$$c\mu^m \leq \tfrac{1}{2}c. \tag{9}$$

For a δ-orbit x we wish to define the sequences $q_n,\ p_n \in M$ for $n \geq 0$, as

$$q_0 = x_0, \qquad q_n = [f^m(q_{n-1}), x_{nm}] \quad \text{for all } n \geq 1, \qquad p_n = f^{-nm}(q_n) \quad \text{for all } n \geq 1. \tag{10}$$

This makes sense if the bracket can be applied, that is, if $d(f^m(q_{n-1}), x_{nm}) \leq \gamma$ for all $n \geq 1$.

Lemma 5.6. *If $0 < 2\delta L_m \leq \gamma$ and x is a δ-orbit then $d(f^m(q_{n-1}), x_{nm}) \leq 2\delta L_m$ for all $n \geq 1$. In particular, the sequence q_n is well-defined for δ-orbits with $0 < 2\delta L_m \leq \gamma$.*

Proof. Arguing by induction, notice that for $n = 1$ we can apply (8) and

$$d(f^m(q_0), x_m) = d(f^m(x_0), x_m) \leq \delta L_m.$$

Now suppose that $d(f^m(q_{n-1}), x_{nm}) \leq 2\delta L_m$ for some $n \geq 1$. Since $q_n = [f^m(q_{n-1}), x_{nm}]$ we can apply the hyperbolic contraction condition, the fact $c \geq 1$ and (9) to obtain

$$d(f^m(q_n),\ f^m(x_{nm})) = d(f^m([f^m(q_{n-1}), x_{nm}]),\ f^m(x_{nm})) \leq c\mu^m d(f^m(q_{n-1}), x_{nm}) \leq \frac{1}{2c} 2\delta L_m \leq \delta L_m.$$

Also, by (8) we have

$$d(f^m(x_{nm}), x_{(n+1)m}) = d(f^m(x_{nm}), x_{nm+m}) \leq \delta L_m.$$

The triangle inequality gives

$$d(f^m(q_n), x_{(n+1)m}) \leq d(f^m(q_n),\ f^m(x_{nm})) + d(f^m(x_{nm}), x_{(n+1)m}) \leq \delta L_m + \delta L_m = 2\delta L_m. \qquad \square$$

In what follows we assume that x is a δ-orbit with $0 < 2\delta L_m \leq \gamma$.

Lemma 5.7. *For all $k \geq 0$ and $t \geq 1$*

$$d\big(f^{-k}(q_t),\ f^{-k}(f^m(q_{t-1}))\big) \leq c\mu^k d(f^m(q_{t-1}), x_{tm}) \leq c\mu^k (2\delta L_m). \tag{11}$$

Proof. Since $q_t = [f^m(q_{t-1}), x_{tm}]$ we can apply the hyperbolic contraction to obtain the first inequality. The last inequality follows from Lemma 5.6. $\square$

Lemma 5.8. *For all $n \geq 0$ we have $d(p_{n+1},\ p_n) \leq c\mu^{(n+1)m}(2\delta L_m)$.*

Proof. By (11) with $k = (n+1)m$ and $t = n+1$

$$\begin{aligned} d(p_{n+1},\ p_n) &= d(f^{-(n+1)m}(q_{n+1}),\ f^{-nm}(q_n)) \\ &= d\big(f^{-(n+1)m}(q_{n+1}),\ f^{-(n+1)m}(f^m(q_n))\big) \leq c\mu^{(n+1)m}(2\delta L_m). \end{aligned}$$

$\square$

For $n \geq 1$ we can define $\mathrm{Sh}^+_{f,n} : M^{\mathbb{Z}}(f,\delta) \to M$ as $\mathrm{Sh}^+_{f,n}(x) = p_n$. From the definitions it is clear that $\mathrm{Sh}^+_{f,n}$ is continuous. Lemma 5.8 gives the uniform convergence of this sequence of maps. Therefore we can define the continuous map $\mathrm{Sh}^+_f = \lim_{n\to+\infty} \mathrm{Sh}_{f,n}$. Analogously, we can define $\mathrm{Sh}^-_f : M^{\mathbb{Z}}(f,\delta) \to M$ as $\mathrm{Sh}^-_f = \mathrm{Sh}^+_{f^{-1}}$, where $\mathrm{Sh}^+_{f^{-1}}$ is the map obtained from the procedure just explained but with f^{-1} in place of f. Finally consider

$$\mathrm{Sh}_f = [\mathrm{Sh}^-_f, \mathrm{Sh}^+_f]. \tag{12}$$

Proposition 5.9. *The function* Sh_f *is a pseudoorbit map.*

Proof. It is clear that Sh_f is continuous. In order to conclude that it is a pseudoorbit map we need to check $\mathrm{Sh}_f(\mathrm{orb}_f(p)) = p$ for any $p \in M$. Let $x = \mathrm{orb}_f(p)$. Then $x_i = f^i(p)$ for all $i \in \mathbb{Z}$. Thus $p_0 = q_0 = x_0 = p$. For $n \geq 0$ suppose that $p_n = p$ and $q_n = f^{nm}(p)$. Then $q_{n+1} = [f^m(q_n), x_{(n+1)m}] = [f^m(f^{nm}(p)), f^{(n+1)m}(p)] = f^{(n+1)m}(p)$. Then $p_{n+1} = p$. By induction we proved that $p_n = p$ for all $n \geq 0$. Thus $p_n \to p$ and $\mathrm{Sh}^+_f(x) = p$. Analogously, $\mathrm{Sh}^-_f(x) = p$ and $\mathrm{Sh}_f(\mathrm{orb}_f(p)) = \mathrm{Sh}_f(\mathrm{con}_f(p,p)) = [\mathrm{Sh}^-_f(\mathrm{con}_f(p,p)), \mathrm{Sh}^+_f(\mathrm{con}_f(p,p))] = [p,p] = p$. This proves that Sh_f is a pseudoorbit map. □

5.1.3. *Proving the self-tuning property.* In this section we will show that the pseudoorbit map of (12) has self-tuning. First we prove a series of lemmas. The difference between what we do and Bowen's estimates is that for us it is not enough to have a fixed δ for x; instead we need to know how much x is jumping at each $i \in \mathbb{Z}$ as we need to *tune* the shadowing where the jumps from $f(x_i)$ to x_{i+1} are small.

As in Lemma 5.6, suppose that $0 < 2\delta L_m \leq \gamma$ and x is a δ-orbit. Define

$$\delta^m_l(x) = L^m \sum_{j=1}^{m} \delta_{(l-1)m+j}(x).$$

Lemma 5.10. *For all* $u \geq 1$

$$d(f^m(q_{u-1}), x_{um}) \leq \sum_{l=1}^{u} \frac{\delta^m_l(x)}{2^{u-l}}. \tag{13}$$

Proof. Arguing by induction, notice that $u = 1$ implies $l = 1$ and we have to prove

$$d(f^m(q_0), x_m) \leq \delta^m_1(x).$$

This follows from (7) with $u = m$. Now assume that (13) holds for some $u \geq 1$. For $u+1$ we have

$$d(f^m(q_u), x_{(u+1)m}) \leq d(f^m(q_u), f^m(x_{um})) + d(f^m(x_{um}), x_{(u+1)m}) \leq \tfrac{1}{2} d(f^m(q_{u-1}), x_{um}) + \delta^m_{u+1}(x)$$
$$\leq \frac{1}{2} \sum_{l=1}^{u} \frac{\delta^m_l(x)}{2^{u-l}} + \delta^m_{u+1}(x) = \sum_{l=1}^{u} \frac{\delta^m_l(x)}{2^{u+1-l}} + \frac{\delta^m_{u+1}(x)}{2^0} = \sum_{l=1}^{u+1} \frac{\delta^m_l(x)}{2^{u+1-l}}.$$ □

Lemma 5.11. *If* $n \geq 1$, $i \in [0, nm]$ *and* $sm \leq i < (s+1)m$ *then*

$$d(f^i(p_n), f^{i-sm}(q_s)) \leq \frac{4c}{3} \sum_{l=1}^{n} \frac{\delta^m_l(x)}{2^{|l-s-1|}}, \quad d(f^{i-sm}(q_s), f^{i-sm}(x_{sm})) \leq 2c \sum_{l=1}^{s} \frac{\delta^m_l(x)}{2^{s+1-l}},$$
$$d(f^{i-sm}(x_{sm}), x_i) \leq \delta^m_{s+1}(x)$$

and

$$d(f^i(p_n), x_i) \leq \kappa \sum_{l=1}^{\infty} \frac{\delta_l^m(x)}{2^{|l-s-1|}}, \tag{14}$$

where $\kappa = \frac{10}{3}c + 1$.

Proof. First inequality: The hyperbolic contraction and Lemma 5.10 with $u = t$ imply

$$d\big(f^{-k}(q_t),\, f^{-k}(f^m(q_{t-1}))\big) \leq c\mu^k d(f^m(q_{t-1}), x_{tm}) \leq c\mu^k \sum_{l=1}^{t} \frac{\delta_l^m(x)}{2^{t-l}} \tag{15}$$

whenever $1 \leq t \leq k$. The triangle inequality gives

$$\begin{aligned} d(f^i(p_n),\, f^{i-sm}(q_s)) = d(f^{i-nm}(q_n),\, f^{i-sm}(q_s)) &\leq \sum_{t=s+1}^{n} d(f^{i-tm}(q_t),\, f^{i-(t-1)m}(q_{t-1})) \\ &= \sum_{t=s+1}^{n} d\big(f^{i-tm}(q_t),\, f^{i-tm}(f^m(q_{t-1}))\big). \end{aligned}$$

From (15) for $k = tm - i \geq 0$ we have

$$d\big(f^{i-tm}(q_t),\, f^{i-tm}(f^m(q_{t-1}))\big) \leq c\mu^{tm-i} \sum_{l=1}^{t} \frac{\delta_l^m(x)}{2^{t-l}} \leq c\mu^{(t-s-1)m} \sum_{l=1}^{t} \frac{\delta_l^m(x)}{2^{t-l}}$$

because $tm - i > (t-s-1)m$. From (9) $\mu^m \leq \frac{1}{2}$ and $\mu^{(t-s-1)m} \leq \frac{1}{2^{t-s-1}}$. Then

$$d\big(f^{i-tm}(q_t),\, f^{i-tm}(f^m(q_{t-1}))\big) \leq c\frac{1}{2^{t-s-1}} \sum_{l=1}^{t} \frac{\delta_l^m(x)}{2^{t-l}}.$$

Consequently

$$d(f^i(p_n),\, f^{i-sm}(q_s)) \leq \sum_{t=s+1}^{n} \sum_{l=1}^{t} c\frac{1}{2^{t-s-1}} \frac{\delta_l^m(x)}{2^{t-l}}.$$

We transform de double sum as follows:

$$\sum_{t=s+1}^{n} \sum_{l=1}^{t} = \sum_{t=s+1}^{n} \sum_{l=1}^{s} + \sum_{t=s+1}^{n} \sum_{l=s+1}^{t} = \sum_{t=s+1}^{n} \sum_{l=1}^{s} + \sum_{l=s+1}^{n} \sum_{t=l}^{n}.$$

For the first sum

$$\begin{aligned} \sum_{t=s+1}^{n} \sum_{l=1}^{s} \frac{1}{2^{t-s-1}} \frac{\delta_l^m(x)}{2^{t-l}} &= \frac{1}{2^{-s-1}} \sum_{t=s+1}^{n} \frac{1}{2^{2t}} \sum_{l=1}^{s} \frac{\delta_l^m(x)}{2^{-l}} \\ &= \frac{1}{2^{-s-1}} \frac{4}{3} \left(\frac{1}{4^{s+1}} - \frac{1}{4^{n+1}} \right) \sum_{l=1}^{s} \frac{\delta_l^m(x)}{2^{-l}} < \frac{4}{3} \sum_{l=1}^{s} \frac{\delta_l^m(x)}{2^{s+1-l}}. \end{aligned}$$

For the second sum

$$\sum_{t=s+1}^{n}\sum_{l=s+1}^{t}\frac{1}{2^{t-s-1}}\frac{\delta_l^m(x)}{2^{t-l}}=\sum_{l=s+1}^{n}\sum_{t=l}^{n}\frac{1}{2^{t-s-1}}\frac{\delta_l^m(x)}{2^{t-l}}=\sum_{l=s+1}^{n}\frac{\delta_l^m(x)}{2^{-l-s-1}}\sum_{t=l}^{n}\frac{1}{2^{2t}}$$
$$=\sum_{l=s+1}^{n}\frac{\delta_l^m(x)}{2^{-l-s-1}}\frac{4}{3}\left(\frac{1}{4^l}-\frac{1}{4^{n+1}}\right)<\frac{4}{3}\sum_{l=s+1}^{n}\frac{\delta_l^m(x)}{2^{-l-s-1}}\left(\frac{1}{4^l}\right)=\frac{4}{3}\sum_{l=s+1}^{n}\frac{\delta_l^m(x)}{2^{l-s-1}}.$$

Then

$$d(f^i(p_n),\,f^{i-sm}(q_s))\le\frac{4c}{3}\sum_{l=1}^{n}\frac{\delta_l^m(x)}{2^{|l-s-1|}}.$$

Second inequality: Notice that $i-sm\ge 0$ and again the hyperbolic contraction and Lemma 5.10 give

$$d(f^{i-sm}(q_s),\,f^{i-sm}(x_{sm}))\le c\mu^{i-sm}d(f^m(q_{s-1}),x_{sm})\le c\sum_{l=1}^{s}\frac{\delta_l^m(x)}{2^{s-l}}.$$

Third inequality: We will apply (7). From our hypothesis we know $0\le i-sm<m$. Then

$$d(f^{i-sm}(x_{sm}),x_i)=d(f^{i-sm}(x_{sm}),x_{i-sm+sm})\le\sum_{j=1}^{i-sm}\delta_{sm+j}(x)L^{i-sm-j}\le L^m\sum_{j=1}^{i-sm}\delta_{sm+j}(x)\le\delta_{s+1}^m(x).$$

Last inequality: The inequality (14) follows from the triangle inequality on the first three. □

Remark 5.12. Notice that (14) implies the Lipschitz shadowing, i.e., there is $K>0$ such that every $K\varepsilon$-orbit x is ε-shadowed by $\mathrm{Sh}_f(x)$.

Lemma 5.13. *If $1\le j\le n$ then*

$$\delta_j(x)\le Ld(f^{j-1}(x_0),x_{j-1})+d(f^j(x_0),x_j)\le(L+1)\max_{1\le j\le n}d(f^j(x_0),x_j).$$

Proof. For any $j=1,\dots,n$ we have

$$\begin{aligned}\delta_j(x)=d(f(x_{j-1}),x_j)&\le d(f(x_{j-1}),\,f^j(x_0))+d(f^j(x_0),x_j)\\&\le Ld(x_{j-1},\,f^{j-1}(x_0))+d(f^j(x_0),x_j)\\&\le(L+1)\max\{d(x_{j-1},\,f^{j-1}(x_0)),d(f^j(x_0),x_j)\}\\&\le(L+1)\max_{1\le j\le n}d(f^j(x_0),x_j).\end{aligned}$$

□

Theorem 5.14. *The pseudoorbit map induced by a hyperbolic bracket has self-tuning.*

Proof. We will apply Proposition 5.3 for $i\ge 0$ (the case $i<0$ is analogous). Taking the limit as $n\to+\infty$ in (14) we have

$$d(f^i(\mathrm{Sh}_f^+(x)),x_i)\le\kappa\sum_{l=1}^{\infty}\frac{\delta_l^m(x)}{2^{|l-s-1|}}\tag{16}$$

for all $i \geq 0$. For Sh_f^- we have

$$d\big(f^{-i}(\mathrm{Sh}_f^-(x)), x_{-i}\big) \leq \kappa \sum_{l=1}^{\infty} \frac{\delta_l^m(\hat{x})}{2^{|l-s-1|}} \tag{17}$$

for all $i \geq 0$, where $\hat{x}$ is the pseudoorbit defined as $\hat{x}_i = x_{-i}$ for all $i \in \mathbb{Z}$. By the hyperbolic contraction of the bracket we have

$$d\big(f^i(\mathrm{Sh}_f(x)), f^i(\mathrm{Sh}_f^+(x))\big) \leq c\mu^i d(\mathrm{Sh}_f^-(x), \mathrm{Sh}_f^+(x)),$$
$$d\big(f^{-i}(\mathrm{Sh}_f(x)), f^{-i}(\mathrm{Sh}_f^-(x))\big) \leq c\mu^i d(\mathrm{Sh}_f^-(x), \mathrm{Sh}_f^+(x))$$

for all $i \geq 0$. Putting $i = 0$ (and $s = 0$) in (16) and (17) we have

$$d(\mathrm{Sh}_f^+(x), x_0) \leq \kappa \sum_{l=1}^{\infty} \frac{\delta_l^m(x)}{2^{|l-1|}}, \quad d(\mathrm{Sh}_f^-(x), x_0) \leq \kappa \sum_{l=1}^{\infty} \frac{\delta_l^m(\hat{x})}{2^{|l-1|}}.$$

Therefore, for $i \geq 0$,

$$\begin{aligned} d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) &\leq d\big(f^i(\mathrm{Sh}_f(x)), f^i(\mathrm{Sh}_f^+(x))\big) + d\big(f^i(\mathrm{Sh}_f^+(x)), x_i\big) \\ &\leq c\mu^i\kappa\left(\sum_{l=1}^{\infty} \frac{\delta_l^m(x)}{2^{|l-1|}} + \sum_{l=1}^{\infty} \frac{\delta_l^m(\hat{x})}{2^{|l-1|}}\right) + \kappa \sum_{l=1}^{\infty} \frac{\delta_l^m(x)}{2^{|l-s-1|}} \end{aligned}$$

with analogous bound for $d\big(f^{-i}(\mathrm{Sh}_f(x)), x_{-i}\big)$.

In order to apply Proposition 5.3 consider $\varepsilon' > 0$ given. Take $\hat{l} \geq 1$ such that

$$c\mu^i\kappa\left(\sum_{|l-1|\geq\hat{l}} \frac{\mathrm{diam}(M)}{2^{|l-1|}} + \sum_{|l-1|\geq\hat{l}} \frac{\mathrm{diam}(M)}{2^{|l-1|}}\right) + \kappa \sum_{|l-s-1|\geq\hat{l}} \frac{\mathrm{diam}(M)}{2^{|l-s-1|}} \leq 3c\kappa\,\mathrm{diam}(M) \sum_{|t|\geq\hat{l}} \frac{1}{2^{|t|}} \leq \tfrac{1}{2}\varepsilon'.$$

Take $\hat{s}$ such that if $i \geq \hat{s}m$ then

$$c\mu^i\kappa\left(\sum_{|l-1|<\hat{l}} \frac{\mathrm{diam}(M)}{2^{|l-1|}} + \sum_{|l-1|<\hat{l}} \frac{\mathrm{diam}(M)}{2^{|l-1|}}\right) \leq \tfrac{1}{4}\varepsilon'.$$

For $0 \leq sm \leq i < \hat{s}m$, we can apply Lemma 5.13 to obtain γ' such that if $\tilde{d}_{\mathrm{sum}}(\tau^i(x), \mathrm{orb}_f(x_i)) < \gamma'$ then

$$c\kappa\left(\sum_{|l-1|<\hat{l}} \frac{\delta_l^m(x)}{2^{|l-1|}} + \sum_{|l-1|<\hat{l}} \frac{\delta_l^m(\hat{x})}{2^{|l-1|}}\right) \leq \tfrac{1}{4}\varepsilon'.$$

Also by Lemma 5.13 we can assume that γ' is so small that

$$\kappa \sum_{|l-s-1|<\hat{l}} \frac{\delta_l^m(x)}{2^{|l-s-1|}} \leq \tfrac{1}{4}\varepsilon'.$$

Then $d\big(f^i(\mathrm{Sh}_f(x)), x_i\big) \leq \varepsilon'$ for all $i \geq 0$. The case $i < 0$ is analogous and the proof ends. □

5.1.4. *Mutual inductions.* In this section we show that, after solving a technical problem, the bracket induced by the induced shadowing map is the original bracket. That is, if $x = \mathrm{con}_f(p, q)$ for some $p, q \in M$, we consider the question: does $\mathrm{Sh}_f(x) = [p, q]$?

Proposition 5.15. *If* $x = \mathrm{con}_f(p, q)$ *then* $\mathrm{Sh}_f^+(x) = q$.

Proof. We have $x_n = f^n(q)$ for all $n \geq 0$. Thus $p_0 = q_0 = x_0 = q$. For $n \geq 0$ suppose that $p_n = q$ and $q_n = f^{nm}(q)$. Then $q_{n+1} = [f^m(q_n), x_{(n+1)m}] = [f^m(f^{nm}(q)), f^{(n+1)m}(q)] = f^{(n+1)m}(q)$. Then $p_{n+1} = q$. By induction we proved that $p_n = q$ for all $n \geq 0$. Thus $p_n \to q$ and $\mathrm{Sh}_f^+(x) = q$. □

For Sh_f^- there is a problem of symmetry as the 0-th coordinate of $\mathrm{con}_f(p, q)$ is q and not p. This can be changed by considering $\tau^{-1}\big(\mathrm{con}_f(f(p), f(q))\big)$, whose 0-th coordinate is p. Thus we can define $\widehat{\mathrm{Sh}}_f^-(x) = \mathrm{Sh}_{f^{-1}}^+\big(\tau^{-1}(f(x))\big)$ and now $\widehat{\mathrm{Sh}}_f^-(\mathrm{con}_f(p, q)) = p$. Consequently

$$\widehat{\mathrm{Sh}}_f(x) = [\widehat{\mathrm{Sh}}_f^-(x), \mathrm{Sh}_f^+(x)]$$

is a shadowing map satisfying $\widehat{\mathrm{Sh}}_f(\mathrm{con}_f(p, q)) = [p, q]$.

5.2. ***North–south dynamics.*** In this section we construct a hyperbolic bracket for a north–south dynamics to conclude that it admits a shadowing map with self-tuning.

Let M be a sphere of dimension $d \geq 1$ and take $f : M \to M$ a north–south homeomorphism, i.e., there are different fixed points $N, S \in M$ and for any $p \in M \setminus \{N, S\}$, $f^n(p) \to S$ and $f^{-n}(p) \to N$ as $n \to +\infty$. To simplify the exposition we can assume that f is a diffeomorphism, N and S are hyperbolic fixed points, and M has a Riemannian metric for which there are $r > 0$ and $\mu \in (0, 1)$ such that

$$\left.\begin{array}{rl} p, q \in B_r(S) & \Longrightarrow\ d(f^n(p), f^n(q)) \leq \mu^n d(p, q), \\ p, q \in B_r(N) & \Longrightarrow\ d(f^{-n}(p), f^{-n}(q)) \leq \mu^n d(p, q), \end{array}\right\} \quad \text{for all } n \geq 0. \tag{18}$$

We assume r is small enough that $B_r(N)$ and $B_r(S)$ have disjoint closures. Let $L > 0$ be a Lipschitz constant for f and f^{-1}.

To define the bracket consider a map[4] $\varphi : M \to I = \{t \in \mathbb{R} : 0 \leq t \leq 1\}$ such that

$$\varphi(p) = 1 \quad \text{if } p \in B_r(N), \qquad \varphi(p) = 0 \quad \text{if } p \in B_r(S).$$

Let $\rho > 0$ be such that the exponential map $\exp_p : T_pM \to M$ is injective on $\{v \in T_pM : \|v\| \leq \rho\}$ for all $p \in M$. If $d(p, q) \leq \rho$ define

$$[p, q] = \exp_p(\varphi(p) \exp_p^{-1}(q)).$$

The bracket chooses a point in the shortest geodesic segment from p to q. Notice that

$$[p, q] = q \quad \text{if } p \in B_r(N), \qquad [p, q] = p \quad \text{if } p \in B_r(S),$$

[4]We do not use the standard notation $I = [0, 1]$ to avoid confusion with the brackets we are considering.

For $p, q \in B_r(N)$ we have

$$d(f^n(q),\, f^n([p,q])) = d(f^n(q),\, f^n(q)) = 0 \le \mu^n d(p,q),$$
$$d(f^{-n}(p),\, f^{-n}([p,q])) = d(f^{-n}(p),\, f^{-n}(q)) \le \mu^n d(p,q),$$

for all $n \ge 0$. Analogously, for $p, q \in B_r(S)$ we have

$$d(f^n(q),\, f^n([p,q])) = d(f^n(q),\, f^n(p)) \le \mu^n d(p,q),$$
$$d(f^{-n}(p),\, f^{-n}([p,q])) = d(f^{-n}(p),\, f^{-n}(p)) = 0 \le \mu^n d(p,q),$$

for all $n \ge 0$. Take $\delta > 0$ smaller than r and ρ such that if

$$U = \{p \in M : d(p,N) > r - \delta \text{ and } d(p,S) > r - \delta\}$$

then $p, q \in U$ and $d(p,q) < \delta$ implies that the geodesic from p to q (*via* the injective restriction of the exponential map) is disjoint from the $\frac{1}{2}r$-balls centered at N and S. In this way for p and q not in $B_r(N)$ nor in $B_r(S)$ but with $d(p,q) < \delta$ we have that $[p,q]$ is not in $B_{r/2}(N) \cup B_{r/2}(S)$. Thus we can take $u \ge 1$ such that $f^i(p),\, f^i(q),\, f^i([p,q]) \in B_{r/2}(N) \cup B_{r/2}(S)$ whenever $|i| \ge u$. Also we can assume that

$$f^u(B_r(N)) \cup f^{-u}(B_r(S)) = M. \tag{19}$$

This condition is used in the proof of Proposition 5.18. Consequently

$$\left.\begin{aligned} d(f^n(q),\, f^n([p,q])) &\le (L^u/\mu^u)\mu^n d(p,q), \\ d(f^{-n}(p),\, f^{-n}([p,q])) &\le (L^u/\mu^u)\mu^n d(p,q), \end{aligned}\right\} \quad \text{for all } n \ge 0. \tag{20}$$

Taking

$$c = L^u/\mu^u \tag{21}$$

we have that $[\,\cdot\,,\cdot\,]$ is a hyperbolic bracket.

Remark 5.16. This bracket cannot be f-equivariant because by Proposition 7.4 f should have to be expansive.

Question 5.17. *Does every structurally stable C^1 diffeomorphism admit a hyperbolic bracket*?

Proposition 5.18. *The shadowing map induced by the hyperbolic bracket for the north–south dynamics has self-tuning but is not translationally equivariant.*

Proof. The shadowing map has self-tuning by Theorem 5.14. We know $f^u(B_r(N)) \cup f^{-u}(B_r(S)) = M$ by (19), thus for each $p \in M$ there is at most one integer i such that $\varphi(f^{iu}(p)) \in (0,1)$. Take $p, q \in M \setminus (B_r(N) \cup B_r(S))$ with $d(p,q) \le \delta$, $p \ne q$ and $\varphi(p) \in (0,1)$. Let $x = \mathrm{con}_f(p,q)$. In this way, $\mathrm{Sh}_f(x) \ne q$. For $\tau^u(x)$ we have $(\tau^u(x))_i = x_{u+i} = f^{u+i}(q)$. Thus, as in the proof of Proposition 5.15, we have $\mathrm{Sh}_f^+(x) = f^u(q)$. Since $f^u(q) \in B_r(S)$ if δ is small enough we have $\mathrm{Sh}_f^-(\tau^u(x)) \in B_r(S)$ and $\varphi(\mathrm{Sh}_f^-(\tau^u(x))) = 1$. This implies $[\mathrm{Sh}_f^-(\tau^u(x)), \mathrm{Sh}_f^+(\tau^u(x))] = f^u(q)$ and $\mathrm{Sh}_f(\tau^u(x)) = f^u(q)$. We see from this that $\mathrm{Sh}_f(\tau^u(x)) \ne f^u(\mathrm{Sh}_f(x))$. □

Question 5.19. *We do not know whether or not a north–south dynamics admits (another) shadowing map being translationally equivariant.*

For an arbitrary shadowing map we can prove the next proposition.

Proposition 5.20. *Every shadowing map for a north–south dynamics is an L-shadowing map.*

Proof. Let x be a two-sided limit δ-orbit. We will show that x_n converges to N or S as $n \to +\infty$. From (18) we have that if $d(x_{n_0}, S) \leq \delta/(1-\mu)$ then $d(x_n, S) \leq \delta/(1-\mu)$ for all $n \geq n_0$. If x_n does not converge to N, then there are $\varepsilon > 0$ and $n_k \to \infty$, $k \geq 1$, such that $d(x_{n_k}, N) > \varepsilon$ for all $k \geq 1$. As x is a two sided-limit we see that after x_{n_k}, for k large, $x_{n_k+j} \in B_{\delta/(1-\mu)}(S)$ and $x_n \in B_{\delta/(1-\mu)}(S)$ for all $n \geq n_k + j$. This proves that $x_n \to S$. Since the orbits of f also converges to N or S, we see that any shadowing map is L-shadowing. □

6. Translational-equivariance

In this section we study τ-equivariant shadowing maps. In Section 6.1 we develop the shift dynamical system and show that it has a τ-equivariant shadowing map. In Section 6.2 we show that τ-equivariance implies Walters' topological stability. In Section 6.3 we prove some dynamical consequences concerning chain recurrence, periodic points and the nonwandering set. In Section 6.4 we show that τ-equivariant shadowing maps induce brackets with uniform contraction.

We start with the definition and remarks.

Definition 6.1. A pseudoorbit map is τ-*equivariant* if for some $\delta > 0$ the next diagram commutes:

$$\begin{array}{ccc} M^{\mathbb{Z}}(f,\delta) & \xrightarrow{\tau} & M^{\mathbb{Z}}(f,\delta) \\ {\scriptstyle \mathrm{Sh}_f}\downarrow & & \downarrow{\scriptstyle \mathrm{Sh}_f} \\ M & \xrightarrow{f} & M \end{array} \tag{22}$$

From this definition we see that any τ-equivariant pseudoorbit map is a semiconjugacy of the map τ and f.

Remark 6.2. From the definitions we also see that every τ-equivariant pseudoorbit map has self-tuning.

6.1. ***The shift homeomorphism.*** The shift homeomorphism is a well-known dynamical system, and in this section we will exploit its dynamical properties.

6.1.1. *Bracket.* Given a compact metric space (X, d_X) consider $M = X^{\mathbb{Z}}$. Define $[\,\cdot\,,\cdot\,] : M \times M \to M$ as

$$[x, y]_i = \begin{cases} x_i & \text{if } i \leq -1, \\ y_i & \text{if } i \geq 0. \end{cases} \tag{23}$$

Proposition 6.3. *This bracket satisfies*:

(1) *It is continuous.*

(2) $[x, x] = x$ *for all* $x \in M$.

(3) $[x, [y, z]] = [x, z] = [[x, y], z]$ *for all* $x, y, z \in M$.

(4) $\tilde{d}_{\text{sum}}(x,[x,y]) + \tilde{d}_{\text{sum}}([x,y],y) = \tilde{d}_{\text{sum}}(x,y)$ *for all* $x, y \in M$.

(5) $\max\{\tilde{d}_{\max}(x,[x,y]), \tilde{d}_{\max}([x,y],y)\} = \tilde{d}_{\max}(x,y)$ *for all* $x, y \in M$.

The proof of this proposition is direct from definitions. As explained in [23, §7.1] the first three conditions imply a *global product structure* as follows. For $x \in M$ define

$$V^-(x) = \{[x,y] \in M : y \in M\}, \quad V^+(x) = \{[y,x] \in M : y \in M\}.$$

Then

$$[\,\cdot\,,\cdot\,] : V^+(x) \times V^-(x) \to M$$

is a homeomorphism, for any $x \in M$.

6.1.2. *The shift homeomorphism.* Consider the shift homeomorphism $\sigma : M \to M$ given by $(\sigma(x))_i = x_{i+1}$ for all $i \in \mathbb{Z}$ and every $x \in M = X^{\mathbb{Z}}$.

Remark 6.4. This is the same definition of (1), but now σ is regarded as a dynamical system while τ is an *index translation* applied to pseudoorbits.

Proposition 6.5. *The shift homeomorphism* σ *on* $M = X^{\mathbb{Z}}$ *has the following dynamical properties*:

(1) *It is topologically mixing.*

(2) *It is expansive if and only if* X *is finite.*

(3) *It has the shadowing property.*

(4) *The set of periodic points is dense in* M.

(5) *For* $y, z \in V^+(x)$ *we have* $\tilde{d}_{\text{sum}}(\sigma^n(y), \sigma^n(z)) = \mu^n \tilde{d}_{\text{sum}}(y,z)$ *for all* $n \geq 0$.

(6) *For* $y, z \in V^-(x)$ *we have* $\tilde{d}_{\text{sum}}(\sigma^{-n}(y), \sigma^{-n}(z)) = \mu^n \tilde{d}_{\text{sum}}(y,z)$ *for all* $n \geq 0$.

In the last items we use the parameter $\mu \in (0,1)$ *that we considered in Section 2.1 to define the metrics of* $M = X^{\mathbb{Z}}$.

Proof. To prove that it is mixing take two nonempty open subsets of M. We can assume they contain (topologically) basic open sets U and V with some $N \geq 0$ such that if $x \in U$ (resp. $x \in V$) and $y \in M$ is such that $x_i = y_i$ for all $|i| \geq N$ then $y \in U$ (resp. $y \in V$). Fix $x \in U$ and $y \in V$. Given $|j| > 2N$ take $z \in M$ such that $z_i = x_i$ for $|i| \leq N$ and $z_i = y_{i-j}$ otherwise. In this way, $z \in U$ and $\sigma^j(z) \in V$. This proves that σ is mixing.

If X is finite then we have the shift homeomorphism on a *finite alphabet*, which is well known to be expansive. Suppose that X is infinite. The constant sequences of M are fixed points of σ. As an expansive homeomorphism of a compact metric space can only have finitely many fixed points, we conclude that σ is not expansive if X has infinitely many points.

The shadowing property will be commented in the next subsection and the remaining statements are left to the interested reader as they are direct from the definitions. □

6.1.3. *A shadowing map for the shift homeomorphism.* A proof of the shadowing property of the shift homeomorphism can be found in [3, Theorem 2.3.12]. There, essentially, it is a function $\mathrm{Sh}_\sigma : M^{\mathbb{Z}} \to M$ defined as

$$(\mathrm{Sh}_\sigma(\alpha))_i = (\alpha_i)_0.$$

On $M^{\mathbb{Z}}$ we consider the product topology and the distance $\tilde{\tilde{d}}_s$ defined as follows. For $\alpha, \beta \in M^{\mathbb{Z}}$

$$\tilde{\tilde{d}}_s(\alpha, \beta) = \sum_{i\in\mathbb{Z}} \mu^{|i|}\tilde{d}_{\mathrm{sum}}(\alpha_i, \beta_i) = \sum_{i\in\mathbb{Z}} \mu^{|i|} \sum_{j\in\mathbb{Z}} \mu^{|j|} d((\alpha_i)_j, (\beta_i)_j) = \sum_{i,j\in\mathbb{Z}} \mu^{|i|+|j|} d((\alpha_i)_j, (\beta_i)_j).$$

Notice that

$$\tilde{d}_{\mathrm{sum}}\big(\mathrm{Sh}_\sigma(\alpha), (\mathrm{Sh}_\sigma(\beta))\big) = \sum_{i\in\mathbb{Z}} \mu^{|i|} d\big((\mathrm{Sh}_\sigma(\alpha))_i, (\mathrm{Sh}_\sigma(\beta))_i\big) = \sum_{i\in\mathbb{Z}} \mu^{|i|} d((\alpha_i)_0, (\beta_i)_0).$$

Therefore $\tilde{d}_{\mathrm{sum}}(\mathrm{Sh}_\sigma(\alpha), \mathrm{Sh}_\sigma(\beta)) \le \tilde{\tilde{d}}_s(\alpha, \beta)$. In particular this proves the continuity of Sh_σ. Define $\mathrm{orb}_\sigma : M \to M^{\mathbb{Z}}$ as $(\mathrm{orb}_\sigma(x))_i = \sigma^i(x)$ for all $i \in \mathbb{Z}$. It satisfies $\mathrm{Sh}_\sigma(\mathrm{orb}_\sigma(x)) = x$ for all $x \in M$. Indeed, for $i \in \mathbb{Z}$

$$(\mathrm{Sh}_\sigma(\mathrm{orb}_\sigma(x)))_i = ((\mathrm{orb}_\sigma(x))_i)_0 = (\sigma^i(x))_0 = x_i.$$

Let τ be the translation homeomorphism of $M^{\mathbb{Z}}$. That is, $\tau : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ is defined as

$$(\tau(\alpha))_i = \alpha_{i+1}$$

for all $i \in \mathbb{Z}$ (just as for σ).

Proposition 6.6. *The shadowing map* Sh_σ *is* τ*-equivariant, i.e., the following diagram commutes*:

$$\begin{array}{ccc} M^{\mathbb{Z}} & \xrightarrow{\ \tau\ } & M^{\mathbb{Z}} \\ {\scriptstyle \mathrm{Sh}_\sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{Sh}_\sigma} \\ M & \xrightarrow{\ \sigma\ } & M \end{array}$$

Proof. For $\alpha \in M^{\mathbb{Z}}$ we have

$$(\mathrm{Sh}_\sigma(\tau(\alpha)))_i = ((\tau(\alpha))_i)_0 = (\alpha_{i+1})_0 \quad \text{and} \quad (\sigma(\mathrm{Sh}_\sigma(\alpha)))_i = (\mathrm{Sh}_\sigma(\alpha))_{i+1} = (\alpha_{i+1})_0. \qquad \square$$

Remark 6.7. The space $M^{\mathbb{Z}}$ has two natural shift or translation actions. Indeed, $M^{\mathbb{Z}} = (X^{\mathbb{Z}})^{\mathbb{Z}}$ and this can be regarded as $X^{\mathbb{Z}\times\mathbb{Z}}$; then the shift can act on any coordinate. If $\alpha \in M^{\mathbb{Z}}$ we can write $(\alpha_i)_j = \alpha_{(i,j)}$. On the one hand, the translation τ acts on the first variable by adding 1 to i. On the other hand, the shift acting by adding 1 to j is equivalent to allowing σ to act on each coordinate of $M^{\mathbb{Z}}$. That is, we consider $\tilde{\sigma} : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ as $(\tilde{\sigma}(\alpha))_i = \sigma(\alpha_i)$. We see that

$$((\tilde{\sigma}(\alpha))_i)_j = (\tilde{\sigma}(\alpha_i))_j = (\alpha_i)_{j+1} \quad \text{while} \quad ((\tau(\alpha))_i)_j = (\alpha_{i+1})_j.$$

Remark 6.8. It is natural to ask for the commutativity of the diagram ($\tilde{\sigma}$ in place of τ)

$$\begin{array}{ccc} M^{\mathbb{Z}} & \xrightarrow{\tilde{\sigma}} & M^{\mathbb{Z}} \\ {\scriptstyle \mathrm{Sh}_\sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{Sh}_\sigma} \\ M & \xrightarrow{\sigma} & M \end{array} \tag{24}$$

Notice that this is not the property we call *dynamical equivariance* because we are taking $\tilde{\sigma}$ on $M^{\mathbb{Z}}$ and not on $M^{\mathbb{Z}}(\sigma, \delta)$.

Proposition 6.9. *The diagram (24) commutes if and only if M and X are singletons.*

Proof. Notice that $M = X^{\mathbb{Z}}$ is a singleton if and only if X is a singleton. If X is a singleton all the spaces in the diagram are singletons and the diagram commutes. To see the converse, notice that we already know that $(\sigma(\mathrm{Sh}_\sigma(\alpha)))_i = (\alpha_{i+1})_0$. Also $\mathrm{Sh}_\sigma(\tilde{\sigma}(\alpha))_i = ((\tilde{\sigma}(\alpha))_i)_0 = (\tilde{\sigma}(\alpha_i))_0 = (\alpha_i)_1$. Thus

$$(\alpha_i)_1 = (\alpha_{i+1})_0 \quad \text{for all } i \in \mathbb{Z}. \tag{25}$$

As $\alpha \in M^{\mathbb{Z}}$ is arbitrary, X cannot have two different points. □

To illustrate a result we prove later (Theorem 7.7, the expansivity for homeomorphisms with a Ruelle's bracket), for $\delta > 0$ given define

$$M^{\mathbb{Z}}(\sigma, \delta) = \{\alpha \in M^{\mathbb{Z}} : \tilde{d}_{\mathrm{sum}}(\sigma(\alpha_i), \alpha_{i+1}) \leq \delta \text{ for all } i \in \mathbb{Z}\}.$$

This is the set of δ-orbits of σ.[5]

Proposition 6.10. *The following statements are equivalent*:

(1) *The diagram*

$$\begin{array}{ccc} M^{\mathbb{Z}}(\sigma, \delta) & \xrightarrow{\tilde{\sigma}} & M^{\mathbb{Z}} \\ {\scriptstyle \mathrm{Sh}_\sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{Sh}_\sigma} \\ M & \xrightarrow{\sigma} & M \end{array}$$

commutes for some δ.

(2) *X is finite.*

(3) *σ is expansive.*

Proof. In Proposition 6.5 we have already seen the equivalence of (2) and (3). We have seen in (25) that this diagram commutes for some $\alpha \in M^{\mathbb{Z}}$ if and only if $(\alpha_i)_1 = (\alpha_{i+1})_0$ for all $i \in \mathbb{Z}$. If X is finite we can take δ such that $d_X(p, q) \leq \delta$ for $p, q \in X$ implies $p = q$. If $\alpha \in M^{\mathbb{Z}}(\sigma, \delta)$ then

$$\tilde{d}_{\mathrm{sum}}(\sigma(\alpha_i), \alpha_{i+1}) = \sum_{j \in \mathbb{Z}} \mu^{|j|} d_X\big((\sigma(\alpha_i))_j, (\alpha_{i+1})_j\big) = \sum_{j \in \mathbb{Z}} \mu^{|j|} d_X((\alpha_i)_{j+1}, (\alpha_{i+1})_j) \leq \delta \quad \text{for all } i \in \mathbb{Z}.$$

[5] In general this set is not invariant under $\tilde{\sigma}$, but as $\tilde{\sigma}$ is continuous for all $\varepsilon > 0$ there is $\delta > 0$ such that if α is a δ-orbit then $\tilde{\sigma}(\alpha)$ is an ε-orbit.

For $j = 0$ we have $d_X((\alpha_i)_1, (\alpha_{i+1})_0) \leq \delta$ for all $i \in \mathbb{Z}$. Thus $(\alpha_i)_1 = (\alpha_{i+1})_0$ for all $i \in \mathbb{Z}$ and the diagram commutes.

If X has infinitely many points, given any $\delta > 0$ we can take $p, q \in X$ arbitrarily close such that if we define $(\alpha_i)_j = p$ for all $i, j \in \mathbb{Z}$ except for $(\alpha_1)_0 = q$ then $\alpha \in M^{\mathbb{Z}}(\sigma, \delta)$ but $(\alpha_1)_0 = q \neq p = (\alpha_0)_1$ and the diagram does not commute. □

Remark 6.11. If X has infinitely many points then $\sigma : M \to M$ is not expansive (Proposition 6.10) but has a τ-equivariant shadowing map (Proposition 6.6).

6.1.4. *Mutual inductions of brackets and shadowing maps.* Shadowing maps always induce brackets and some brackets induce shadowing maps (for instance, with Bowen's technique Section 5.1.3). We will illustrate this for the shift homeomorphism (for the shadowing map and the bracket already defined). Define $\mathrm{con}_\sigma : M \times M \to M^{\mathbb{Z}}$ as

$$\mathrm{con}_\sigma(x, y)_i = \begin{cases} \sigma^i(y) & \text{if } i \geq 0, \\ \sigma^i(x) & \text{if } i < 0. \end{cases}$$

For $x, y \in M$ we have

$$[x, y] = \mathrm{Sh}_\sigma(\mathrm{con}_\sigma(x, y))$$

and we see how a shadowing map induces a bracket.

Conversely, we now start with the bracket defined in (23). Take $\alpha \in M^{\mathbb{Z}}$. As in Section 5.1.2 define sequences $x_n, y_n \in M$, for $n \geq 0$ as

$$x_0 = \alpha_0, \qquad x_n = [\sigma(x_{n-1}), \alpha_n] \quad \text{for } n \geq 1, \qquad y_n = \sigma^{-n}(x_n).$$

The next result formally explains the beautiful work of these equations.

Proposition 6.12. *For all $n \geq 0$ and $0 \leq i \leq n$ we have $(y_n)_i = (\alpha_i)_0$.*

Proof. We argue by induction in n. For $n = 0$, $i = 0$ we have $(y_0)_0 = (x_0)_0 = (\alpha_0)_0$. Take $n \geq 1$ and assume that $(y_n)_i = (\alpha_i)_0$ for $0 \leq i \leq n$. If $0 \leq i \leq n$ then

$$(y_{n+1})_i = (\sigma^{-n-1}(x_{n+1}))_i = (x_{n+1})_{i-n-1} = [\sigma(x_n), \alpha_{n+1}]_{i-n-1} = (\sigma(x_n))_{i-n-1} = (x_n)_{i-n}$$

and

$$(\alpha_i)_0 = (y_n)_i = (\sigma^{-n}(x_n))_i = (x_n)_{i-n}.$$

Thus $(y_{n+1})_i = (\alpha_i)_0$ for $0 \leq i \leq n$. For $i = n + 1$

$$(y_{n+1})_{n+1} = (\sigma^{-n-1}(x_{n+1}))_{n+1} = (x_{n+1})_0 = [\sigma(x_n), \alpha_{n+1}]_0 = (\alpha_{n+1})_0.$$ □

From this we see that y_n is a convergent sequence and its limit $z^+ \in M$ satisfies $z^+_n = (\alpha_n)_0$ for all $n \geq 0$. Analogously, with σ^{-1} in place of σ we obtain $z^- \in M$ with $z^-_{-n} = (\alpha_{-n})_0$ for all $n \geq 0$. Finally, the shadowing map is recovered by defining $\mathrm{Sh}_\sigma(\alpha) = [z^-, z^+]$. This way of inducing a shadowing map from a bracket is analogous to Bowen's technique shown in Section 5.1.3.

6.2. ***Topological stability.*** From now on we return to the arbitrary compact metric space (M, d) and a homeomorphism of M. Following Walters [25; 26] we say that $f \in \mathcal{H}(M)$ is *topologically stable* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $d_{C^0}(f, g) < \delta$ then there is a continuous map $h : M \to M$ such that $d(h, I_M) < \varepsilon$ and $fh = hg$ as in the diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ M & \xrightarrow{f} & M \end{array}$$

Remark 6.13. In [22] Z. Nitecki proved that structurally stable C^1 diffeomorphisms are topologically stable. Notice that topological stability is invariant under conjugacy, and thus any homeomorphism conjugate to a structurally stable diffeomorphism is topologically stable. These are the only examples of topologically stable homeomorphisms on closed manifolds known by the author. Walters [26] proved that topologically stable homeomorphisms of closed manifold have the shadowing property.

Proposition 6.14. *If f admits a τ-equivariant shadowing map then it is topologically stable.*

Proof. Let $\mathrm{Sh}_f : M^{\mathbb{Z}}(f, \beta) \to M$ be a shadowing map for f and take $\varepsilon > 0$. By the continuity of Sh_f there is $\gamma > 0$ such that $d(x, y) < \gamma$ implies $d(\mathrm{Sh}_f(x), \mathrm{Sh}_f(y)) < \varepsilon$ for all $x, y \in M^{\mathbb{Z}}(f, \beta)$. By Corollary 2.11 there is $\delta > 0$ such that $d_{C^0}(f, g) < \delta$ implies $d(\mathrm{orb}_g(p), \mathrm{orb}_f(p)) < \gamma$ for all $p \in M$.

Suppose that $d_{C^0}(f, g) < \delta$ and define $h : M \to M$ as $h(p) = \mathrm{Sh}_f(\mathrm{orb}_g(p))$. We have

$$d_{C^0}(h, \mathrm{Id}_M) = \sup_{p \in M} d(h(p), p) = \sup_{p \in M} d(\mathrm{Sh}_f(\mathrm{orb}_g(p)), p) = \sup_{p \in M} d(\mathrm{Sh}_f(\mathrm{orb}_g(p)), \mathrm{Sh}_f(\mathrm{orb}_f(p))) < \varepsilon.$$

We know that h is continuous by the continuity of Sh_f and Corollary 2.11. We have that $fh = hg$ because Sh_f satisfies (22). We proved that f is topologically stable. □

6.3. ***Recurrence.*** Given $p, q \in M$ and $\alpha > 0$ we write $p \overset{\alpha}{\sim} q$ if there are α-orbits of f such that $x_0 = p, x_1, \ldots, x_k = q$ and $q = y_0, y_1, \ldots, y_l = p$. If $p \overset{\alpha}{\sim} q$ for all $\alpha > 0$ we write $p \sim q$. The *chain recurrence set* of f is defined as

$$R(f) = \{p \in M : p \sim p\}.$$

A point $p \in M$ is *wandering* if it has a neighborhood U such that $f^n(U) \cap U = \varnothing$ for all $n \geq 1$. The set of nonwandering points is denoted by $\Omega(f)$. In general $\Omega(f) \subset R(f)$, and equality holds if f has shadowing. A δ-orbit x is *periodic* if there is $n \geq 1$ such that $x_{i+n} = x_i$ for all $i \in \mathbb{Z}$ (i.e., it is periodic by the shift homeomorphism). In [2] N. Aoki proved that if f has the shadowing property then the restriction of f to the nonwandering set has shadowing too.

Proposition 6.15. *If f admits a τ-equivariant shadowing map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f, \delta) \to M$ then*:

(a) $R(f) = \mathrm{clos}(\mathrm{Per}(f))$.

(b) $f|_{\Omega(f)}$ *has a τ-equivariant shadowing map.*

Proof. (a) Take $p \in R(f)$ and $\varepsilon > 0$. Take $\gamma > 0$ such that if $x \in M^{\mathbb{Z}}(f, \gamma)$ then $\mathrm{Sh}_f(x)$ is an ε-shadow of x. As p is chain recurrent, there is a periodic γ-orbit x with $x_0 = p$. If n is the period of x (by the shift) then $\mathrm{Sh}_f(x) = \mathrm{Sh}_f(\tau^n(x)) = f^n(\mathrm{Sh}_f(x))$; thus $\mathrm{Sh}_f(x)$ is a periodic point. Since $q = \mathrm{Sh}_f(x)$ is an ε-shadow of x we have that $d(q, x_0) = d(q, p) < \varepsilon$ and q is periodic for f. We proved that $R(f) \subset \mathrm{clos}(\mathrm{Per}(f))$. As the other inclusion holds for any homeomorphism the proof ends.

(b) For this proof we follow [2] (in fact with our hypothesis the argument is simpler). As explained in that paper, there are finitely many equivalence classes of $\overset{\delta}{\sim}$ and each class is closed and invariant under f. Let $A_1, \ldots, A_k$ be the equivalence classes of $\overset{\delta}{\sim}$. Take $\delta_1 = \inf\{d(A_i, A_j) : 1 \le i < j \le k\} > 0$ and $\alpha > 0$ such that $\alpha < \min\{\delta, \delta_1\}$. Define

$$\Omega^{\mathbb{Z}}(f, \alpha) = \{x \in M^{\mathbb{Z}}(f, \alpha) : x_i \in \Omega(f) \text{ for all } i \in \mathbb{Z}\}.$$

We will show that $\mathrm{Sh}_f(x) \in \Omega(f)$ for all $x \in \Omega^{\mathbb{Z}}(f, \alpha)$. For $x \in \Omega^{\mathbb{Z}}(f, \alpha)$ notice that there is a class A_j containing x_i for all $i \in \mathbb{Z}$. For any $n \ge 1$ we have $x_{-n} \overset{\delta}{\sim} x_n$ and there is a periodic δ-orbit z^n such that $z^n_i = x_i$ for all $|i| \le n$. We have that $\mathrm{Sh}_f(z^n)$ is a periodic point, and thus $\mathrm{Sh}_f(z^n) \in \Omega(f)$. Since $z^n \to x$ as $n \to \infty$, we conclude that $\mathrm{Sh}_f(x) \in \Omega(f)$ and the proof ends. □

6.4. ***Bracket with uniform contraction.*** In Section 5.1 we considered brackets with hyperbolic contraction. In this section we define a weaker condition which is satisfied by the bracket induced by a τ-equivariant shadowing map (Proposition 6.18).

Definition 6.16. An L-bracket has *uniform contraction* if for all $\varepsilon > 0$ there is $m \ge 1$ such that

$$d(f^n([p, q]), f^n(q)) < \varepsilon, \quad d(f^{-n}([p, q]), f^{-n}(p)) < \varepsilon$$

for all $n \ge m$.

Remark 6.17. The brackets with hyperbolic contraction defined in Section 5.1 have uniform contraction.

The next result will be applied in Corollary 7.5 to prove that if f admits a shadowing map which is f-equivariant and τ-equivariant then f is expansive.

Proposition 6.18. *The bracket induced by a τ-equivariant shadowing map has uniform contraction.*

Proof. Let $\mathrm{Sh}_f : M^{\mathbb{Z}}(f, \gamma) \to M$ be a τ-equivariant shadowing map. We will show that the induced bracket $[\,\cdot\,,\cdot\,]_f : \Delta_\gamma \to M$, which according to (2) is defined by

$$[p, q]_f = \mathrm{Sh}_f(\mathrm{con}_f(p, q)),$$

has uniform contraction.

As explained in Proposition 6.5 there is (another) bracket $[\,\cdot\,,\cdot\,]_\tau : M^{\mathbb{Z}} \times M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ which is hyperbolic for $\tau : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ (In this case τ takes the place of the shift σ.) Moreover, for $x, y \in M^{\mathbb{Z}}$, we have

$$\tilde{d}_{\mathrm{sum}}(\tau^n([x, y]_\tau), \tau^n(y)) = \mu^n \tilde{d}_{\mathrm{sum}}(x, y), \quad \tilde{d}_{\mathrm{sum}}(\tau^{-n}([x, y]_\tau), \tau^{-n}(x)) = \mu^n \tilde{d}_{\mathrm{sum}}(x, y)$$

for all $n \ge 0$. Since Sh_f is uniformly continuous, there is $\delta > 0$ such that if $\tilde{d}_{\mathrm{sum}}(x, y) < \delta$ then $d(\mathrm{Sh}_f(x), \mathrm{Sh}_f(y)) < \varepsilon$. Since $\mu \in (0, 1)$ there is $m \ge 1$ such that $\mu^n \,\mathrm{diam}(M^{\mathbb{Z}}) < \delta$ whenever $n \ge m$.

Given $p, q \in M$ with $d(p,q) \leq \gamma$, define $x = \mathrm{orb}_f(p)$ and $y = \mathrm{orb}_f(q)$. Notice that $[x,y]_\tau = \mathrm{con}_f(p,q)$. Then for $n \geq m$ we have

$$\tilde{d}_{\mathrm{sum}}(\tau^n([x,y]_\tau), \tau^n(y)) = \tilde{d}_{\mathrm{sum}}(\tau^n(\mathrm{con}_f(p,q)), \tau^n(\mathrm{orb}_f(q))) \leq \mu^n \operatorname{diam}(M^{\mathbb{Z}}) < \delta$$

and applying Sh_f, which is τ-equivariant, we conclude

$$\begin{aligned}\varepsilon &> d\big(\mathrm{Sh}_f\big(\tau^n(\mathrm{con}_f(p,q))\big)\big), \mathrm{Sh}_f\big(\tau^n(\mathrm{orb}_f(q))\big) = d\big(f^n(\mathrm{Sh}_f(\mathrm{con}_f(p,q))), f^n(\mathrm{Sh}_f(\mathrm{orb}_f(q)))\big) \\ &= d(f^n([p,q]), f^n(q)).\end{aligned}$$

□

7. Topological hyperbolicity

This section is about shadowing maps and brackets for expansive homeomorphisms. In Section 7.1 we consider dynamically equivariant shadowing maps. In Section 7.2 we consider shadowing maps for topologically hyperbolic homeomorphisms. In Section 7.3 we study Ruelle's set of axioms for Smale spaces and topological hyperbolicity.

7.1. ***Dynamically equivariant pseudoorbit maps.*** Let $f : M \to M$ be a homeomorphism and consider $\tilde{f} : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ as $\tilde{f}(x)_i = f(x_i)$ for all $i \in \mathbb{Z}$. By the uniform continuity of f, for all $\varepsilon > 0$ there is $\delta > 0$ such that if x is a δ-orbit then $\tilde{f}(x)$ is an ε-orbit.

Definition 7.1. A map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f,\varepsilon) \to M$ is *f-equivariant* if the following diagram commutes:

$$\begin{array}{ccc} M^{\mathbb{Z}}(f,\delta) & \xrightarrow{\tilde{f}} & M^{\mathbb{Z}}(f,\varepsilon) \\ {\scriptstyle \mathrm{Sh}_f}\downarrow & & \downarrow{\scriptstyle \mathrm{Sh}_f} \\ M & \xrightarrow{f} & M \end{array}$$

Example 7.2. The projection $\mathrm{Sh}_f(x) = x_0$ is f-equivariant. This is because

$$f(\mathrm{Sh}_f(x)) = f(x_0) = \mathrm{Sh}_f(\tilde{f}(x)).$$

Thus the homeomorphisms with h-shadowing (which were considered in Section 3.3) have a dynamically equivariant shadowing map. This also allows us to conclude that there are dynamically equivariant pseudoorbit maps which are not shadowing maps (any homeomorphism without the h-shadowing property).

As we mentioned in Section 1, topologically hyperbolic homeomorphisms have a dynamically equivariant shadowing map (which in addition is τ-equivariant).

7.2. ***Standard expansivity.*** Recall that f is *expansive* if there is $\xi > 0$ such that if $p \neq q$ for $p, q \in M$, then there is $i \in \mathbb{Z}$ such that $d(f^i(p), f^i(q)) > \xi$.

Definition 7.3. A bracket is *f-equivariant* if $f([p,q]) = [f(p), f(q)]$ when both sides are defined.

Proposition 7.4. *If f admits a dynamically equivariant bracket with uniform contraction then f is expansive.*

Proof. Take $\xi > 0$ such that $d(p,q) \leq \xi$ implies $d(p,[p,q]) < \delta$ and $d(p,[q,p]) < \delta$. We will show that ξ is an expansivity constant.

Suppose that $d(f^i(p), f^i(q)) \leq \xi$ for all $i \in \mathbb{Z}$. Let $p_i = f^i(p)$, $q_i = f^i(q)$ and $r_i = [p_i, q_i]$ for all $i \in \mathbb{Z}$. From the f-equivariance we know that $f^i(r_0) = r_i$.

Given $\varepsilon > 0$, consider m from the uniform contraction property. Thus

$$d(f^m(q_i), f^m(r_i)) < \varepsilon, \quad d(f^{-m}(p_i), f^{-m}(r_i)) < \varepsilon$$

for all $i \in \mathbb{Z}$. For $i = -m$ in the first inequality

$$d(q, r_0) = d(q_0, r_0) = d(f^m(q_{-m}), f^m(r_{-m})) < \varepsilon.$$

As ε is arbitrary we conclude $q = r_0$. Analogously, for $i = n$ in the second inequality, we conclude $p = r_0$ and $p = q$. Thus ξ is an expansivity constant for f. □

Corollary 7.5. *If f admits a shadowing map which is f-equivariant and τ-equivariant then f is expansive.*

Proof. By Proposition 6.18 there is a bracket with uniform contraction. Thus the result follows from Proposition 7.4. □

The following result is essentially known.

Theorem 7.6. *The following are equivalent*:

(1) *f is topologically hyperbolic.*

(2) *f admits a shadowing map which is τ-equivariant and f-equivariant.*

Proof. (1) $\Longrightarrow$ (2): Let α be an expansivity constant for f and take δ such that every δ pseudoorbit is $\frac{1}{2}\alpha$-shadowed. From the expansivity, if p and q $\frac{1}{2}\alpha$-shadow a pseudoorbit x then $p = q$. This defines a map $\mathrm{Sh}_f : M^{\mathbb{Z}}(f,\delta) \to M$ as $\mathrm{Sh}_f(x) = p$ where p is the unique point that $\frac{1}{2}\alpha$-shadows x.

We will show that Sh_f is a shadowing map for f by checking the definition. Given $p \in M$ let $q = \mathrm{Sh}_f(\mathrm{orb}_f(p))$. From expansivity we conclude that $p = q$. To prove that Sh_f is equivariant take $x, y \in M^{\mathbb{Z}}(f,\delta)$ such that $y_i = x_{i+1}$ for all $i \in \mathbb{Z}$ and let $p = f(\mathrm{Sh}_f(x))$ and $q = \mathrm{Sh}_f(y)$. Then $d(f^i(f^{-1}(p)), x_i) \leq \frac{1}{2}\alpha$ for all $i \in \mathbb{Z}$ and $d(f^i(q), y_i) = d(f^i(q), x_{i+1}) \leq \frac{1}{2}\alpha$ for all $i \in \mathbb{Z}$. This implies $d(f^i(p), f^i(q)) \leq \alpha$ and $p = q$ from expansivity. Thus Sh_f is equivariant.

It remains to show that Sh_f is continuous. Suppose that $x^n \to y$ in $M^{\mathbb{Z}}(f,\delta)$, $\mathrm{Sh}_f(x^n) = r_n \to p$ and $\mathrm{Sh}_f(y) = q$. Since $\mathrm{Sh}_f(x^n) = r_n$ we have $d(f^i(r_n), x_i^n) \leq \frac{1}{2}\alpha$ and the continuity of f implies $d(f^i(p), y_i) \leq \frac{1}{2}\alpha$ for all $i \in \mathbb{Z}$. Also, since $\mathrm{Sh}_f(y) = q$ we have $d(f^i(q), y_i) \leq \frac{1}{2}\alpha$ for all $i \in \mathbb{Z}$. Thus $d(f^i(p), f^i(q)) \leq \alpha$, and the expansivity of f implies $p = q$. We conclude that Sh_f is continuous. The equivariance properties were explained in Section 1.

(2) $\Longrightarrow$ (1): The expansivity follows by Corollary 7.5. The shadowing property is a consequence of Proposition 3.2. □

7.3. *Ruelle's bracket.* We follow [23, §7.1] concerning what Ruelle called *Smale spaces*. He stated two set axioms that will be recalled in detail. The first set is topological, as it does not involve any dynamics. For $\varepsilon > 0$ given consider a function $[\,\cdot\,,\cdot\,] : \Delta_\varepsilon \to M$ satisfying:

(SS1.1) $[\,\cdot\,,\cdot\,]$ is continuous and $[p, p] = p$ for all $p \in M$.

(SS1.2) $[[p, q], r] = [p, r]$, $[p, [q, r]] = [p, r]$ when the two sides of these relations are defined.

As explained in [23, §7.1] these conditions imply a *local product structure*. Indeed, if for $\delta > 0$ small we define

$$V_p^-(\delta) = \{r = [p, q] \in M : d(p, r) \le \delta,\ d(p, q) \le \varepsilon\},$$
$$V_p^+(\delta) = \{r = [q, p] \in M : d(p, r) \le \delta,\ d(p, q) \le \varepsilon\},$$

then

$$[\,\cdot\,,\cdot\,] : V_p^-(\delta) \times V_p^+(\delta) \to M$$

is a homeomorphism onto a neighborhood of p.[6]

Again, these axioms may not be related to any dynamics. But they can be *dynamically defined* as when considering *canonical coordinates*, i.e., $r = [p, q]$ is the point of intersection of the local stable set of q and the local unstable set of p if f is an Anosov diffeomorphism (or any expansive homeomorphism with shadowing). Another example of a bracket satisfying (SS1) (i.e., (SS1.1) and (SS1.2)) is $[p, q] = p$ for all $(p, q) \in \Delta_\varepsilon$ (the projection on the first coordinate). Also the bracket of the shift satisfies these axioms; recall Section 6.1.1.

Then Ruelle introduces the set of axioms (SS2) with hyperbolic dynamical, meaning:

(SS2.1) *f-equivariance*: $f([p, q]) = [f(p), f(q)]$ when both sides are defined.

(SS2.2) *hyperbolic contraction of distances*:[7] There is $\lambda > 1$ such that

$$d(f^n(p), f^n(q)) \le \lambda^{-n} d(p, q) \quad \text{if } q \in V_p^+(\delta) \text{ and } n \ge 0,$$
$$d(f^{-n}(p), f^{-n}(q)) \le \lambda^{-n} d(p, q) \quad \text{if } q \in V_p^-(\delta) \text{ and } n \ge 0.$$

Ruelle define a *Smale space* as a compact metric space with a bracket and a homeomorphism satisfying (SS1) and (SS2). In [23, §7.3] it is explained that for a Smale space, the homeomorphism is expansive and has shadowing. Thus Ruelle's bracket gives the canonical coordinates explained before. In the mentioned proof of expansivity (SS1.2) is used. Let us show that expansivity and (SS1.2) can be deduced from the other three axioms.

Theorem 7.7. (SS1.1) *and* (SS2) *imply expansivity and* (SS1.2).

[6] Ruelle's definition of the sets $V_p^\pm(\delta)$ is slightly different as he assumes $q = [q, p]$. But assuming (SS1.2) it is easy to see that our definition is equivalent to the original one. We introduce this change in order to reduce the dependence of (SS1.2).

[7] As before, we changed this axiom. The one we give here is equivalent to the original one under the assumption of the remaining axioms.

Proof. The expansivity follows from Proposition 7.4. For the other part we suppose that δ is small, so that 4δ is an expansivity constant. From (SS2.2) we see that $[p, q] \in W^s_\delta(q) \cap W^u_\delta(p)$. Thus $W^u_\delta([p, q]) \subset W^u_{2\delta}(p)$ and

$$[[p, q], r] \in W^s_\delta(r) \cap W^u_{2\delta}(p) \subset W^s_{2\delta}(r) \cap W^u_{2\delta}(p).$$

Also

$$[p, r] \in W^s_\delta(r) \cap W^u_\delta(p) \subset W^s_{2\delta}(r) \cap W^u_{2\delta}(p).$$

Since 4δ is an expansivity constant we conclude that $[[p, q], r] = [p, r]$. Analogously $[p, [q, r]] = [p, r]$ and (SS1.2) is proved. □

Remark 7.8. In order to complete Ruelle's set of axioms we mention that [23, §7.11] adds the condition

(SS3) there is $L > 0$ such that $d(x, [x, y]) \leq Ld(x, y)$.

It is known that every topologically hyperbolic homeomorphism admits a metric satisfying Ruelle's axioms, including (SS3); see [12; 24]. Moreover, it is known that Dovbysh's metric [11] and a self-similar hyperbolic metric [5] also satisfy $L \to 1$ as $d(x, y) \to 0$.

8. Continuous shadowing

Following [18] we say that f is *$\mathcal{T}_0$-continuous shadowing* if for all $\varepsilon > 0$ there are $\delta > 0$ and a continuous map $R_\varepsilon : M^{\mathbb{Z}}(f, \delta) \to M$ such that every δ-orbit x is ε-shadowed by $R(x)$.

Remark 8.1. This is similar to the definition of shadowing map, however the map R_ε depends on ε and it is not required that $R_\varepsilon(\mathrm{orb}_f(p)) = p$ for $p \in M$. It is clear that if Sh_f is a shadowing map then we can consider the maps R_ε as restrictions of Sh_f to $M^{\mathbb{Z}}(f, \delta)$ varying δ. Thus the existence of a shadowing map implies the $\mathcal{T}_0$-continuous shadowing property. In what follows we study the $\mathcal{T}_0$-continuous shadowing property to see whether or not it implies the existence of a shadowing map.

As in this setting the shadowing point of a real orbit may not be the point generating the orbit, it is natural to consider the map $G_\varepsilon : M \to M$ defined as $G_\varepsilon(p) = R_\varepsilon(\mathrm{orb}_f(p))$. We have some direct consequences:

(1) G_ε is continuous (as it is the composition of continuous maps),

(2) by definition, $G_\varepsilon(p) \in \Gamma_\varepsilon(p)$ for all $p \in M$, and

(3) $d_{C^0}(G_\varepsilon, \mathrm{Id}_M) \to 0$ as $\varepsilon \to 0$.

Proposition 8.2. *If G_ε is a homeomorphism for some ε then $R'_\varepsilon = G_\varepsilon^{-1} \circ R_\varepsilon$ satisfies*:

(1) *R'_ε is a pseudoorbit map*:
   (a) *R'_ε is continuous, and*
   (b) *$R'_\varepsilon(\mathrm{orb}_f(p)) = p$ for all $p \in M$.*

(2) *R'_ε 2ε-shadows: $d(f^i(R'_\varepsilon(x)), x_i) \leq 2\varepsilon$ for all $i \in \mathbb{Z}$ and every δ-orbit x.*

Proof. As a composition of continuous functions, R'_ε is continuous. Also

$$R'_\varepsilon(\mathrm{orb}_f(p)) = G_\varepsilon^{-1}(R(\mathrm{orb}_f(p))) = G_\varepsilon^{-1}(G_\varepsilon(p)) = p$$

for all $p \in M$. Finally

$$d(f^i(R'_\varepsilon(x)), x_i) \leq d(f^i(R'_\varepsilon(x)),\, f^i(R_\varepsilon(x))) + d(f^i(R_\varepsilon(x)), x_i),$$

and $d(f^i(R_\varepsilon(x)), x_i) \leq \varepsilon$ as R_ε ε-shadows by definition. Also, since $R_\varepsilon(x) = G_\varepsilon(R'_\varepsilon(x))$ and $G_\varepsilon(R'_\varepsilon(x)) \in \Gamma_\varepsilon(R'_\varepsilon(x))$, we have $d(f^i(R'_\varepsilon(x)),\, f^i(R_\varepsilon(x))) \leq \varepsilon$ for all $i \in \mathbb{Z}$. Thus $d(f^i(R'_\varepsilon(x)), x_i) \leq \varepsilon + \varepsilon$ for all $i \in \mathbb{Z}$. □

Remark 8.3. Proposition 8.2 means that if the maps R_ε induce homeomorphisms G_ε for all $\varepsilon > 0$, then we can assume $R_\varepsilon(\mathrm{orb}_f(p)) = p$ for $p \in M$ (taking R'_ε instead of R_ε).

Next we show some conditions implying that the maps G_ε are homeomorphisms.

Remark 8.4. If ε is an expansivity constant of f then G_ε is the identity of M (in particular it is a homeomorphism). This is because $G_\varepsilon(p) \in \Gamma_\varepsilon(p) = \{p\}$. But, assuming expansivity is too strong, as it is easy to see that the maps R_ε would have to coincide on *small* pseudoorbits. That is, $R_{\varepsilon_1}(x) = R_{\varepsilon_2}(x)$ when both sides are defined.

For $N \geq 1$ we say that f is *N-expansive* if there is $\xi > 0$ such that $\Gamma_\xi(p)$ has at most N points, for all $p \in M$. This definition was introduced by Morales in [21]. For the proof of the next theorem we need a lemma.

Lemma 8.5. *If $T : A \to A$ is a surjective function such that every point is eventually N-periodic — i.e., for all $a \in A$ there are $0 \leq i < j \leq N$ such that $T^i(a) = T^j(a)$ — then every point is periodic and T is injective.*[8]

Proof. First we show that any $d \in M$ is periodic with period at most N. As T is surjective, there is $c \in M$ such that $T^N(c) = d$. Take $0 \leq i < j \leq N$ such that $T^i(c) = T^j(c)$. Applying T^{N-i} we have

$$d = T^N(c) = T^{N-i}(T^i(c)) = T^{N-i}(T^j(c)) = T^{j-i}(T^N(c)) = T^{j-i}(d).$$

To prove that T is injective suppose that $T(a) = T(b)$ and take $n, m \geq 1$ such that $T^n(a) = a$ and $T^m(b) = b$. If $k = mn \geq 1$ then $T^k(a) = a$ and $T^k(b) = b$. We know that $T^{k+1}(a) = T(a) = T(b) = T^{k+1}(b)$. Applying T^{k-1} we obtain $T^{k-1}(T^{k+1}(a)) = T^{k-1}(T^{k+1}(b))$; thus $T^{2k}(a) = T^{2k}(b)$ and $a = b$. □

Theorem 8.6. *If f is N-expansive, has the $\mathcal{T}_0$-continuous shadowing property and M is a closed manifold then f is topologically hyperbolic.*

Proof. In general we know that G_ε is C^0-close to the identity, and as M is a closed manifold, if ε is small we have that G_ε is surjective; see [26].

We know that $G_\varepsilon(p) \in \Gamma_\varepsilon(p)$ for all $p \in M$. Thus $G_\varepsilon^n(p) \in \Gamma_{n\varepsilon}(p)$ for all $n \geq 0$. Suppose that $(N+1)\varepsilon < \xi$. Then the set $\{p, G_\varepsilon(p), G_\varepsilon^2(p), \ldots, G_\varepsilon^N(p)\} \subset \Gamma_\xi(p)$ has at most N points, for any $p \in M$.

[8] In fact $T^{N!} = \mathrm{Id}_A$.

Thus, for some $0 \leq i < j \leq N$, we have $G_\varepsilon^i(p) = G_\varepsilon^j(p)$. This means that every point is eventually N-periodic for G_ε. Applying Lemma 8.5 for $T = G_\varepsilon$ we conclude that G_ε is injective. As it is continuous and bijective on a compact metric space its inverse is continuous and it is a homeomorphism.

This proves that the maps G_ε are homeomorphisms for all ε sufficiently small. By Remark 8.3 we can assume that the maps R_ε satisfy $R_\varepsilon(\mathrm{orb}_f(p)) = p$ for all $p \in M$.

Notice that N-expansivity implies cw-expansivity. If ξ is a cw-expansivity constant we can suppose that $\varepsilon < \frac{1}{2}\xi$ and we have that every δ-orbit x is $\frac{1}{4}\xi$-shadowed by $R_\varepsilon(x)$. This allows us to apply the techniques of the proof of Theorem 3.3 to conclude that f is expansive. □

Theorem 8.7. *If f has the $\mathcal{T}_0$-continuous shadowing property and M is totally disconnected then it admits a shadowing map.*

Proof. Since M is totally disconnected its copy in $M^{\mathbb{Z}}$, namely $\mathrm{orb}_f(M) \subset M^{\mathbb{Z}}$, is totally disconnected too. Take $\varepsilon_n = 1/n$ and for each $n \geq 1$ fix a map R_{ε_n} defined on $M^{\mathbb{Z}}(f, \delta_n)$, with $\delta_n \to 0$.

Define $U_0 = M^{\mathbb{Z}}(f, \delta)$ and for $n \geq 1$ take $U_n \subset U_{n-1} \cap M^{\mathbb{Z}}(f, \delta/(n+1))$ such that U_n is open and closed and contains the copy of M. We will define $\mathrm{Sh}_f : M^{\mathbb{Z}}(f, \delta)$ as follows. For $x \in \mathrm{orb}_f(M)$ we must define $\mathrm{Sh}_f(x) = x_0$ (in order to satisfy $\mathrm{Sh}_f(\mathrm{orb}_f(p)) = p$ for all $p \in M$). For $x \in M^{\mathbb{Z}}(f, \delta) \setminus \mathrm{orb}_f(M)$ there is a maximum $n \geq 0$ such that $x \in U_n$, and we define $\mathrm{Sh}_f(x) = R_n(x)$.

From the construction, clearly Sh_f induces shadowing and is continuous on $M^{\mathbb{Z}}(f, \delta) \setminus \mathrm{orb}_f(M)$. The continuity at $x \in \mathrm{orb}_f(M)$ follows from the fact that $d_{C^0}(G_\varepsilon, \mathrm{Id}_M) \to 0$ as $\varepsilon \to 0$. □

References

[1] M. Achigar, A. Artigue, and J. Vieitez, "New cw-expansive homeomorphisms of surfaces", *Ann. Fac. Sci. Toulouse Math.* (6) **29**:2 (2020), 221–246. MR

[2] N. Aoki, "On homeomorphisms with pseudo-orbit tracing property", *Tokyo J. Math.* **6**:2 (1983), 329–334. MR

[3] N. Aoki and K. Hiraide, *Topological theory of dynamical systems*, North-Holland Mathematical Library **52**, North-Holland, Amsterdam, 1994. MR

[4] A. Artigue, "Dendritations of surfaces", *Ergodic Theory Dynam. Systems* **38**:8 (2018), 2860–2912. MR

[5] A. Artigue, "Self-similar hyperbolicity", *Ergodic Theory Dynam. Systems* **38**:7 (2018), 2422–2446. MR

[6] A. Artigue, B. Carvalho, W. Cordeiro, and J. Vieitez, "Beyond topological hyperbolicity: the L-shadowing property", *J. Differential Equations* **268**:6 (2020), 3057–3080. MR

[7] A. D. Barwell, C. Good, and P. Oprocha, "Shadowing and expansivity in subspaces", *Fund. Math.* **219**:3 (2012), 223–243. MR

[8] A. D. Barwell, C. Good, P. Oprocha, and B. E. Raines, "Characterizations of ω-limit sets in topologically hyperbolic systems", *Discrete Contin. Dyn. Syst.* **33**:5 (2013), 1819–1833. MR

[9] R. Bowen, "ω-limit sets for axiom A diffeomorphisms", *J. Differential Equations* **18**:2 (1975), 333–339. MR

[10] B. Carvalho and W. Cordeiro, "Positively n-expansive homeomorphisms and the L-shadowing property", *J. Dynam. Differential Equations* **31**:2 (2019), 1005–1016. MR

[11] S. A. Dovbysh, "Optimal Lyapunov metrics of expansive homeomorphisms", *Izv. Ross. Akad. Nauk Ser. Mat.* **70**:5 (2006), 31–78. In Russian; translated in *Izv. Math.* **70**:5 (2006), 883–929. MR

[12] D. Fried, "Métriques naturelles sur les espaces de Smale", *C. R. Acad. Sci. Paris Sér. I Math.* **297**:1 (1983), 77–79. MR

[13] C. Good, S. Macías, J. Meddaugh, J. Mitchell, and J. Thomas, "Expansivity and unique shadowing", *Proc. Amer. Math. Soc.* **149**:2 (2021), 671–685. MR

[14] K. Hiraide, "Expansive homeomorphisms of compact surfaces are pseudo-Anosov", *Osaka J. Math.* **27**:1 (1990), 117–162. MR

[15] H. Kato, "Continuum-wise expansive homeomorphisms", *Canad. J. Math.* **45**:3 (1993), 576–598. MR

[16] P. Kloeden, J. Ombach, and A. Pokrovskii, "Continuous and inverse shadowing", *Funct. Differ. Equ.* **6**:1-2 (1999), 137–153. MR

[17] P. Kůrka, *Topological and symbolic dynamics*, Cours Spécialisés **11**, Société Mathématique de France, Paris, 2003. MR

[18] K. Lee, "Continuous inverse shadowing and hyperbolicity", *Bull. Austral. Math. Soc.* **67**:1 (2003), 15–26. MR

[19] J. Lewowicz, "Expansive homeomorphisms of surfaces", *Bol. Soc. Brasil. Mat.* (*N.S.*) **20**:1 (1989), 113–133. MR

[20] R. Mañé, "Expansive homeomorphisms and topological dimension", *Trans. Amer. Math. Soc.* **252** (1979), 313–319. MR

[21] C. Morales, "A generalization of expansivity", *Discrete Contin. Dyn. Syst.* **32**:1 (2012), 293–301. MR

[22] Z. Nitecki, "On semi-stability for diffeomorphisms", *Invent. Math.* **14** (1971), 83–122. MR

[23] D. Ruelle, *Thermodynamic formalism*, 2nd ed., Cambridge Univ. Press, 2004. MR

[24] K. Sakai, "Shadowing properties of $\mathcal{L}$-hyperbolic homeomorphisms", *Topology Appl.* **112**:3 (2001), 229–243. MR

[25] P. Walters, "Anosov diffeomorphisms are topologically stable", *Topology* **9** (1970), 71–78. MR

[26] P. Walters, "On the pseudo-orbit tracing property and its relationship to stability", pp. 231–244 in *The structure of attractors in dynamical systems* (Fargo, ND, 1977), edited by N. G. Markley et al., Lecture Notes in Math. **668**, Springer, 1978. MR

[27] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics **79**, Springer, 1982. MR

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A novel approach through spherical functions in the characterization of invariant functions

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Given a compact subgroup K of the orthogonal group acting on the Euclidean space $\mathbb{R}^n$, Gerald Schwarz proved that every smooth K-invariant function on $\mathbb{R}^n$ can be expressed as a smooth function of a generating set of K-invariant polynomials on n variables. Our goal is to provide an alternative and more straightforward proof of this result, based on Gelfand theory, with a particular focus on spherical functions.

1. Introduction

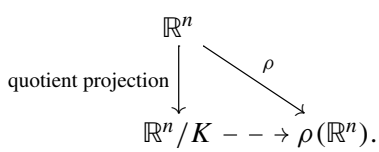
Schwarz's theorem. Given any closed subgroup of rotations and reflections acting naturally on the Euclidean domain $\mathbb{R}^n$, if one considers all the smooth functions invariant under its action, it holds that they can be characterized as smooth functions on the generators of the algebra of invariant polynomials on $\mathbb{R}^n$. Precisely, let K be a compact group acting orthogonally on $\mathbb{R}^n$. From the classical theorem of Hilbert (see [23]), it is known that the algebra of K-invariant polynomials on $\mathbb{R}^n$, denoted by $\mathcal{P}(\mathbb{R}^n)^K$, is finitely generated. In 1975, Gerald Schwarz proved that all smooth K-invariant functions on $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n)^K$, are characterized as smooth functions on an arbitrary system $\{\rho_1, \dots, \rho_l\}$ of generators:

Theorem 1.1 [20, Theorem 1]. *Let K be a compact group acting orthogonally on $\mathbb{R}^n$, and let $\{\rho_1, \dots, \rho_l\}$ be a set of generators of the algebra $\mathcal{P}(\mathbb{R}^n)^K$. Then for every infinitely differentiable K-invariant function $f \in C^\infty(\mathbb{R}^n)^K$, there exists $h \in C^\infty(\mathbb{R}^l)$ such that $f(x) = h(\rho_1(x), \dots, \rho_l(x))$.*

This result is very strong because it relates the differentiable structures of two spaces that in principle were only homeomorphic. Indeed, let $\rho : \mathbb{R}^n \to \mathbb{R}^l$ be given by

$$\rho(x) := (\rho_1(x), \dots, \rho_l(x)),$$

for every $x \in \mathbb{R}^n$. This map induces a homeomorphism between the space of orbits $\mathbb{R}^n/K$ and $\rho(\mathbb{R}^n)$:



As explained in [20], $\mathbb{R}^n/K$ can be given a smooth structure by defining a function on the quotient space $\mathbb{R}^n/K$ to be smooth if, when lifted to a K-invariant function on $\mathbb{R}^n$, it is smooth in the classical sense. At

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the same time, the image set $\rho(\mathbb{R}^n)$, viewed as a closed subset of $\mathbb{R}^l$, has a smooth structure by defining a function on $\rho(\mathbb{R}^n) \subset \mathbb{R}^l$ to be smooth if it is the restriction to $\rho(\mathbb{R}^n)$ of a smooth function on $\mathbb{R}^l$. Thus Theorem 1.1 states that ρ induces a homeomorphism of $\mathbb{R}^n/K$ and $\rho(\mathbb{R}^n)$ together with their smooth structures. That is, by using *pullback* notation, Theorem 1.1 states that the map ρ^* below is onto:

$$\rho^* : C^\infty(\mathbb{R}^l) \to C^\infty(\mathbb{R}^n)^K, \quad (\rho^* h)(x) := h(\rho_1(x), \dots, \rho_l(x)).$$

It is worth mentioning that D. Luna obtained improvements of Schwarz's theorem in [17] in the analytic setting, treating not only the real case but also the complex one, and establishing them in the broader framework of linear actions of reductive groups, rather than only compact orthogonal group actions.

Theorem 1.1 was first conjectured and shown for some particular cases. For example, by using Taylor expansions and analytic extensions, H. Whitney showed in 1943 that even smooth functions f on $\mathbb{R}$ are of the form $f(x) = h(x^2)$ for $h : \mathbb{R} \to \mathbb{R}$ smooth [25], that is, Theorem 1.1 for $n = 1$ and $K = \{\pm 1\}$; see also [7; 24]. G. Glaeser, in 1963, extended Theorem 1.1 for the case of the symmetric group $\boldsymbol{S}_n$ acting on $\mathbb{R}^n$ [12]. Before [20] appeared with the general proof, Theorem 1.1 was deduced for finite groups [4].

Finally, we observe that the image set $\rho^*(C^\infty(\mathbb{R}^l))$ is dense in $C^\infty(\mathbb{R}^n)^K$, as a consequence of the following basic facts:

(1) The space of invariant polynomials $\mathcal{P}(\mathbb{R}^n)^K$ is dense in $C^\infty(\mathbb{R}^n)^K$.[1]

(2) At the polynomial level, the identity $\mathcal{P}(\mathbb{R}^n)^K = \rho^*(\mathcal{P}(\mathbb{R}^l))$ holds.[2]

This suggests that a possible strategy to prove Theorem 1.1 would be to establish the remaining step, namely that $\rho^*(C^\infty(\mathbb{R}^l))$ is closed in $C^\infty(\mathbb{R}^n)^K$. For instance, if the generators $\rho_1, \dots, \rho_l$ are algebraically independent, then $n \geq l$ (roughly speaking, one cannot have more generators than variables), and by a result of Glaeser [12] it follows that $\rho^*(C^\infty(\mathbb{R}^l))$ is closed in $C^\infty(\mathbb{R}^n)^K$. Here however we will adopt a different approach.

Our contributions. Our main objective is to provide an alternative — possibly more elementary — proof of Theorem 1.1 employing *Gelfand theory*.[3] We will leverage the fact that, for any compact subgroup K of the orthogonal group $O(n)$, one has that $(K \ltimes \mathbb{R}^n, K)$ (in short, $(K, \mathbb{R}^n)$) forms a *Gelfand pair*, where $K \ltimes \mathbb{R}^n$ denotes the semidirect product of K and $\mathbb{R}^n$ (known as the n-dimensional connected Euclidean motion group when $K = \mathrm{SO}(n)$, i.e., the group of isometries of $\mathbb{R}^n$). Specifically, by employing *spherical functions*, we aim to explicitly determine, for a given K-invariant function f, a corresponding function h in Theorem 1.1 satisfying $h(\rho(x)) = f(x)$.

[1] Fact 1 follows, on the one hand, from the density of polynomials in $C^\infty(\mathbb{R}^n)$ with respect to the Whitney topology (i.e., uniform convergence of the function and its derivatives on compact sets). On the other hand, because the projection $C^\infty(\mathbb{R}^n) \ni f \mapsto \left(x \mapsto \int_K f(k \cdot x)\, dk\right) \in C^\infty(\mathbb{R}^n)^K$ is linear, continuous, and onto, it is open by the open mapping theorem.

[2] For the second fact, the inclusion $\rho^*(\mathcal{P}(\mathbb{R}^l)) \subseteq \mathcal{P}(\mathbb{R}^n)^K$ is immediate since the polynomials $\rho_1, \dots, \rho_l$ are K-invariant. Conversely, given $p \in \mathcal{P}(\mathbb{R}^n)^K$, we can write it in terms of the generators $\rho_1, \dots, \rho_l$ as a finite sum $p(x) = \sum a_{j,k}(\rho_j(x))^k$, for some coefficients $a_{j,k}$, and defining $h(y_1, \dots, y_l) = \sum a_{j,k}(y_j)^k$ in $\mathbb{R}^l$, we obtain $p(x) = h(\rho_1(x), \dots, \rho_l(x))$, which implies $\rho^*(\mathcal{P}(\mathbb{R}^l)) \supseteq \mathcal{P}(\mathbb{R}^n)^K$.

[3] The original proof by Schwarz relies on *Grothendieck's theory of topological tensor products and nuclear spaces* [13; 19; 21].

Our main contribution is the proof of a version of Theorem 1.1 for integrable K-invariant functions $f:\mathbb{R}^n\to\mathbb{C}$ whose Fourier transform has compact support (see Theorem 1.3 below). Any such function f is, in particular, not only smooth but in fact the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ (by the Paley–Wiener theorem) — i.e., its Taylor series converges everywhere. As a counterpart, we establish stronger regularity for the associated function $h:\mathbb{R}^l\to\mathbb{C}$ than in Theorem 1.1; namely we show that h may be chosen to be the restriction to $\mathbb{R}^l$ of an entire function on $\mathbb{C}^l$.

In addition, the problem addressed by Schwarz in [20] can be reformulated, within the framework of Gelfand theory, as an *extension problem* for the Gelfand transform (see Corollary 1.4 and the description below). The precise definitions and underlying concepts will be introduced and motivated in Section 2. The purpose of the upcoming discussion is simply to provide preliminary connections between this theory and Theorem 1.1.

Schwarz's theorem and Gelfand theory. Given $f:\mathbb{R}^n\to\mathbb{C}$ an integrable K-invariant function, we write $f\in L^1(\mathbb{R}^n)^K$. That is, if $k\cdot x$ denotes the natural action of $k\in K$ on $x\in\mathbb{R}^n$, then $f(k\cdot x)=f(x)$ for all $k\in K$ and $x\in\mathbb{R}^n$. We recall that such natural action $k\cdot x$ can be represented as a matrix-vector multiplication since K is a subgroup of the group of orthogonal $n\times n$ matrices. The (classical) Fourier transform of such function f is well-defined as

$$\hat{f}(\xi):=\int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}\,dx. \tag{1}$$

Since the convolution algebra $L^1(\mathbb{R}^n)^K$ is always commutative (regardless of the choice of K), we have a so-called *Gelfand pair* $(K,\mathbb{R}^n)$. Then the Gelfand theory provides a well-defined framework for the notion of the *spectrum* of the algebra $L^1(\mathbb{R}^n)^K$, along with the so-called *Gelfand transform*. Indeed, the Gelfand transform of f is defined as

$$\mathcal{F}(f)(\varphi_\xi):=\int_{\mathbb{R}^n} f(x)\varphi_\xi(-x)\,dx,\quad \text{where } \varphi_\xi(x):=\int_K e^{i\langle x,k\cdot\xi\rangle}\,dk, \tag{2}$$

where dk denotes the normalized Haar measure on K. Note that $\varphi_\xi=\hat{\mu}_\xi$, where μ_ξ is the orbital measure on $K\cdot\xi$. The set of (bounded) *spherical functions* $\{\varphi_\xi\}$ determines the spectrum Σ of the algebra $L^1(\mathbb{R}^n)^K$. As f is K-invariant, its Fourier transform $\hat{f}$ is also a K-invariant function, and coincides with its Gelfand transform, that is,

$$\mathcal{F}(f)(\varphi_\xi)=\hat{f}(\xi) \tag{3}$$

(think of integrating (1) over K to obtain (2)). Moreover, from the seminal paper [8], the domain Σ of the Gelfand transform can be identified with a closed subset Λ of $\mathbb{C}^l$. This result holds in great generality. In particular, we will show that Λ can be identified with the image of ρ in $\mathbb{R}^l$, allowing us to write

$$\mathcal{F}(f)(\varphi_\xi)=\mathcal{F}(f)(\rho(\xi)). \tag{4}$$

Using this notation within the harmonic analysis framework, we will prove the following proposition and the subsequent version of Theorem 1.1:

Proposition 1.2. *Let $(K, \mathbb{R}^n)$ be a Gelfand pair, let φ_ξ be an associated bounded spherical function, and let $\{\rho_1, \dots, \rho_l\}$ be a set of generators of the algebra $\mathcal{P}(\mathbb{R}^n)^K$. Then there exists a function h_ξ, which is the restriction to $\mathbb{R}^l$ of an entire function on $\mathbb{C}^l$, such that $\varphi_\xi(x) = h_\xi(\rho_1(x), \dots, \rho_l(x))$ for every $x \in \mathbb{R}^n$.*

Theorem 1.3. *Let K be a compact group acting orthogonally on $\mathbb{R}^n$, and let $\{\rho_1, \dots, \rho_l\}$ be a set of generators of the algebra $\mathcal{P}(\mathbb{R}^n)^K$. Then for every K-invariant function $f \in L^1(\mathbb{R}^n)$ with $\hat{f}$ of compact support, there exists a function $h : \mathbb{R}^l \to \mathbb{C}$ that admits an entire extension to $\mathbb{C}^l$, such that $f(x) = h(\rho_1(x), \dots, \rho_l(x))$.*

Closed formulas for choosing the functions h_ξ and h will be provided in Section 3.1 (Definition 3.1) and Section 3.2 (Definition 3.4), respectively.

As a consequence, Schwarz's result may be interpreted as showing that for a smooth K-invariant function g with compact support, $g \in C_c^\infty(\mathbb{R}^n)^K$ (in particular, g is a Schwartz function), the Gelfand transform $\mathcal{F}(g) : \Lambda \to \mathbb{C}$ admits a smooth extension $h : \mathbb{R}^l \to \mathbb{C}$, that is, $\mathcal{F}(g)(\rho(\xi)) = h(\rho(\xi))$. Indeed, for such a function g, consider the function $f = \hat{g}$. Thus $\hat{f}$ is a compactly supported Schwartz function. Applying Theorem 1.3 to $f = \hat{g}$, we obtain that for this $f : \mathbb{R}^n \to \mathbb{C}$, there exists a function $h : \mathbb{R}^l \to \mathbb{C}$, which is the restriction to $\mathbb{R}^l$ of an entire function on $\mathbb{C}^l$ such that

$$h(\rho(x)) = f(x) = \hat{g}(x) = \mathcal{F}(g)(\rho(x)),$$

where we have used (3) and (4). Hence, another contribution of this work can be understood as addressing the extension problem for the Gelfand transform in the setting of "abelian pairs", i.e., Gelfand pairs of the form $(K, \mathbb{R}^n)$. Specifically, our main result, Theorem 1.3, can be also read in terms of the following corollary:

Corollary 1.4. *Given $g \in C^\infty(\mathbb{R}^n)^K$ with compact support, consider $h : \mathbb{R}^l \to \mathbb{C}$ of the form*

$$h(t) = \int_{\mathbb{R}^n} g(-\xi) h_\xi(t)\, d\xi \quad \text{for } h_\xi \text{ given in Proposition 1.2.}$$

Then $h \in C^\infty(\mathbb{R}^l)$ and $\mathcal{F}(g)(\varphi_\xi) = h(\rho(\xi))$.

It is important to mention that in recent years the study of the Gelfand transform for Gelfand pairs arising from semidirect products $K \ltimes N$, where N is a connected simply connected nilpotent Lie group and K acts on N by automorphisms, has received considerable attention. Indeed, several results concerning the extension of the Gelfand transform have already been established in the literature; see, e.g., [1; 2; 9; 10]. Specifically, if $(K \ltimes N, K)$ (or simply, (K, N)) is a Gelfand pair, it can be shown that the associated spectrum Λ can be embedded in a real space $\mathbb{R}^l$. Let $\mathcal{S}(\Lambda)$ denote the space of functions $f : \Lambda \to \mathbb{C}$ that admit a Schwartz-class extension to $\mathbb{R}^l$ (that is, $\mathcal{S}(\Lambda) = \mathcal{S}(\mathbb{R}^l)/\sim$, endowed with the quotient topology, where $f_1 \sim f_2$ if and only if $f_1(\lambda) = f_2(\lambda)$ for all $\lambda \in \Lambda$). In this setting, the Gelfand transform becomes an isomorphism from the space of K-invariant Schwartz functions over N, $\mathcal{S}(N)^K$ onto $\mathcal{S}(\Lambda)$. That is, for every $g \in \mathcal{S}(N)^K$, its Gelfand transform $\mathcal{F}(g)$ admits a Schwartz extension to $\mathbb{R}^l$. Moreover, a control of $\mathcal{F}$ can be established in terms of the seminorms defining the topologies of $\mathcal{S}(N)^K$ and $\mathcal{S}(\mathbb{R}^l)$.

For the particular abelian case $N = \mathbb{R}^n$, in [2, Section 6] the authors prove the extension results for the Gelfand transform between Schwartz spaces by first invoking Theorem 1.1 by Schwarz in [20] and relying on the work [18]. Instead, here our goal is to prove from scratch Theorem 1.1 by appealing to techniques of the Gelfand theory. It is worth remarking that the arguments employed in the aforementioned papers [1; 2; 9; 10] are very deep. The idea of this article is to be as elemental as possible.

Organization. In Section 2, we will elaborate on all the terminology briefly introduced above. In our review of Gelfand's theory, we introduce the notions of Gelfand pairs, spherical functions, and the associated Gelfand transform, with a focus on the Euclidean case. In Section 3, we present our main contributions. We begin in Section 3.1 by considering f as a spherical function ($f = \varphi_\xi$) in the statement of Theorem 1.1, which leads to Definition 3.1 and the proof of Proposition 1.2. Building on these key components, in Section 3.2 we provide the proof of Theorem 1.3, that is, our version of Schwarz's theorem from the perspective of Gelfand theory. Finally, in Section 4 we illustrate our results with examples.

2. Preliminaries

Gelfand's theory is devoted to studying commutative Banach algebras and their *spectra*. One of the most important results from Gelfand's theory establishes that a commutative Banach algebra $\mathcal{A}$ can be mapped through a continuous group homomorphism — the so-called *Gelfand transform* — into an algebra of continuous functions defined over the spectrum of $\mathcal{A}$; see, for example, [6; 11].

As an application to harmonic analysis of the Gelfand theory on commutative Banach algebras, given a locally compact Hausdorff topological group G, natural functional spaces to consider are the *group algebra* $C_c(G)$ of complex-valued continuous functions on G with compact support endowed with the convolution product $*$, or its closure under the L^1-norm, that is, the space of integrable functions $L^1(G)$. To apply Gelfand's theory, the convolution product must be commutative. As this happens if and only if the underlying group G is abelian, there has been a great interest in determining *subalgebras* of $L^1(G)$ that are commutative under the convolution product and invariant under the action of a subgroup of G. This yields to the definition of *Gelfand pairs* (see, e.g., [3; 5; 8; 9; 10; 16; 22]): Given K a compact subgroup of G, we say that (G, K) is a *Gelfand pair*, or that the homogeneous space G/K is *commutative*, if the convolution subalgebra $L^1(G)^K$ of bi-K-invariant integrable functions (i.e., $f(k_1 x k_2) = f(x)$ for all $k_1, k_2 \in K$ and $x \in G$) is commutative. When having (G, K) a Gelfand pair, the Gelfand transform on the commutative Banach algebra $L^1(G)^K$ plays the role of the classical Fourier transform. Indeed, in the context of Fourier analysis on groups, the Gelfand transform is often called the *spherical Fourier transform*.

To define such a transform we need to introduce the spectrum of the algebra $L^1(G)^K$, or the so-called set of spherical functions. We say a function $\varphi \in C(G)^K$ is *spherical* if the associated *character* χ_φ, defined by

$$\chi_\varphi(f) := \int_G f(x)\varphi(-x)\,dx \quad \text{for all } f \in C_c(G)^K,$$

satisfies

$$\chi_\varphi(f*g) = \chi_\varphi(f)\chi_\varphi(g).$$

In this case, the spectrum Σ of the algebra $L^1(G)^K$ can be identified with the set of all bounded spherical functions [15]. Then $\Sigma \subset L^\infty(G)^K$ and it is endowed with the *Gelfand topology*, that is, the weak*-topology (i.e., we say that $\varphi_n \to \varphi$ as $n \to \infty$ if and only if $\chi_{\varphi_n}(f) \to \chi_\varphi(f)$ as $n \to \infty$, for all $f \in L^1(G)^K$) [8]. It can be proven that this topology coincides with the topology of the uniform convergence on compact sets. When G and K are Lie groups, the spherical functions can be characterized by a differential point of view. Finally, the spherical Fourier transform of a function $f \in L^1(G)^K$ is defined as the Gelfand transform associated to the commutative algebra $\mathcal{A} = L^1(G)^K$: as the function $\mathcal{F}(f) : \Sigma \to \mathbb{C}$ given by the formula

$$\mathcal{F}(f)(\varphi) = \int_G f(x)\varphi(-x)\,dx, \tag{5}$$

where $-x$ denotes the inverse of x in the group G (a particular case was introduced in (2)).

A relevant family of study is when G is a semidirect product $G = K \ltimes N$, where N is a connected and simply connected nilpotent Lie group, and K acts by automorphisms on N; see, e.g., [26; 3]. Here $K \ltimes N$ denotes the semidirect product of K and N, that is, the manifold $K \times N$ equipped with the group product

$$(k, x)(k', x') := (kk', x + k \cdot x'),$$

where $k \cdot x$ denotes the action of $k \in K$ on $x \in \mathbb{N}$, and $+$ is the group operation in N (not necessarily abelian) giving it the structure of a Lie group. In this case, the algebra of bi-K-invariant functions $L^1(K \ltimes N)^K$ can be identified with the algebra $L^1(N)^K$ of K-invariant functions

$$L^1(N)^K := \{f \in L^1(N) : f(k \cdot x) = f(x),\ \forall k \in K,\ x \in N\},$$

and for simplicity in the notation (K, N) is commonly used in place of the pair $(K \ltimes N, K)$. To see this, first note that if $f : K \ltimes N \to \mathbb{C}$ is invariant under the right action of K, then

$$f(k, x) = f((\mathrm{Id}, x)(k, 0)) = f(\mathrm{Id}, x),$$

where Id denotes the identity element of K. Therefore $f(k, x)$ can be identified with the function $f(\mathrm{Id}, \cdot) : N \to \mathbb{C}$. Now suppose that f is also invariant under the left action of K. Then

$$f((k, 0)(\mathrm{Id}, x)) = f(k, k \cdot x) = f(\mathrm{Id}, k \cdot x),$$

where the last equality follows from the computation above. This implies that the function $f(\mathrm{Id}, \cdot)$ is invariant under the natural action of K on N.

In these cases, it is well known that the spherical functions are of positive type [3] and can be characterized as the eigenfunctions of the algebra $\mathbb{D}(N)^K$ of all the differentiable operators on N invariant by N-left translations and invariant under the action of K

$$D\varphi = \lambda\varphi \quad \text{for all } D \in \mathbb{D}(N)^K, \tag{6}$$

normalized by taking the value 1 at the group identity. Such an algebra $\mathbb{D}(N)^K$ is finitely generated [14], and we denote by $\{D_1, \ldots, D_l\}$ a system of generators. Moreover, the spherical functions are analytic,

and they are completely determined by the eigenvalues with respect to such an arbitrary set of generators; see [15, page 400]. Then to each spherical function φ we can associate an l-tuple of eigenvalues $(\lambda_1(\varphi), \dots, \lambda_l(\varphi))$ with respect to the differential operators $D_1, \dots, D_l$. Let

$$\Lambda := \{(\lambda_1(\varphi), \dots, \lambda_l(\varphi)) : \varphi \in \Sigma\}. \tag{7}$$

Theorem 2.1 [8]. *Λ is a closed subset of $\mathbb{C}^l$ and the correspondence*

$$\Sigma \ni \varphi \mapsto (\lambda_1(\varphi), \dots, \lambda_l(\varphi)) \in \Lambda$$

is a homeomorphism between Σ, with the Gelfand topology, and Λ, with the relative topology of $\mathbb{C}^l$.

We will consider the abelian or Euclidean case, that is, $N = \mathbb{R}^n$. The classical Fourier transform on the commutative convolution algebra $L^1(\mathbb{R}^n)$ (defined in (1)) is a continuous injection into the space of continuous functions which vanish at infinity $C_0(\mathbb{R}^n)$ (*Riemann–Lebesgue lemma*), which carries convolutions to pointwise products. Indeed, the classical Fourier transform can be identified with the Gelfand transform associated to the Gelfand pair $(K, \mathbb{R}^n)$, where the K is the trivial group $K = \{\mathrm{Id}\}$.

As $L^1(\mathbb{R}^n)$ is commutative with the convolution product, the subalgebra $L^1(\mathbb{R}^n)^K$ is always commutative. Thus $(K, \mathbb{R}^n)$ is a Gelfand pair for any compact subgroup K of the orthogonal group $O(n)$ acting naturally on $\mathbb{R}^n$.

Given $f \in L^1(\mathbb{R}^n)^K$, its Gelfand transform (5) takes the form (2) where the spherical functions, denoted by φ_ξ, can be determined by the integral formula also given in (2). Notice that each function φ_ξ in (2) is infinitely differentiable, K-invariant, bounded by 1, and of positive type. Moreover, $\varphi_\xi = \varphi_{\xi'}$ if and only if $\xi = k \cdot \xi'$ for some $k \in K$. We refer the reader to [26] for a nice study of spherical functions on Euclidean spaces. As a consequence of the definition (2), we highlight a very simple fact, namely, their symmetry:

$$\varphi_\xi(x) = \varphi_x(\xi) \quad \text{for all } x, \xi \in \mathbb{R}^n. \tag{8}$$

Notice that, given $f \in L^1(\mathbb{R}^n)^K$, by using our definitions and the invariance of f under $K \subseteq O(n)$, we can relate the spherical Fourier transform of f and the classical Fourier transform of f recovering (3):

$$\begin{aligned}\mathcal{F}(f)(\varphi_\xi) &= \int_{\mathbb{R}^n} f(x)\varphi_\xi(-x)\,dx = \int_{\mathbb{R}^n} f(x) \int_K e^{-i\langle x, k\cdot\xi\rangle}\,dk\,dx \\ &= \int_{\mathbb{R}^n}\int_K f(k\cdot x)e^{-i\langle x,\xi\rangle}\,dk\,dx = \int_{\mathbb{R}^n}\int_K f(x)e^{-i\langle x,\xi\rangle}\,dx = \hat{f}(\xi).\end{aligned}$$

Hence, under the hypothesis of the *inversion formula* for the classical Fourier transform, we can write

$$f(x) = \int_{\mathbb{R}^n} \mathcal{F}(f)(\varphi_\xi)\varphi_\xi(x)\,d\xi, \tag{9}$$

where $d\xi$ denotes the Lebesgue measure on $\mathbb{R}^n$ multiplied by $1/((2\pi)^n)$. Moreover, there exists a *Plancherel measure* μ, that is, a positive Borel measure on Λ, such that an *inversion formula* of the form

$$f(x) = \int_\Lambda \mathcal{F}(f)(\varphi_\xi)\varphi_\xi(x)\,d\mu(\varphi_\xi)$$

holds under the usual integrability conditions, and where we identify Λ with Σ due to Theorem 2.1.

3. New proof of Schwarz's theorem

As in the statement of Theorem 1.1, let K be a compact group acting orthogonally on $\mathbb{R}^n$, and let $\{\rho_1, \dots, \rho_l\}$ be a set of generators of the algebra $\mathcal{P}(\mathbb{R}^n)^K$ of K-invariant polynomials on $\mathbb{R}^n$. Clearly, we can assume that the polynomials $\rho_1, \dots, \rho_l$ are homogeneous.

Let ∂_{ρ_j} be the differential operator corresponding to the polynomial ρ_j, which, in simple words, is obtained from ρ_j changing the variables x_k by the differential operators ∂_{x_k}, that is,[4]

$$\rho_j(x) = \sum_{I=(i_1,\dots,i_n)} c_I x_1^{i_1} \cdots x_n^{i_n} \implies \partial_{\rho_j} = \sum_{I=(i_1,\dots,i_n)} c_I \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}.$$

Let us consider the following multiple of such a differential operator:

$$D_j := (-i)^{\deg(\rho_j)} \partial_{\rho_j}.$$

We observe that D_j is left-invariant, that is, a differential operator with constant coefficients, and moreover, it is K-invariant (i.e., for every differentiable K-invariant function f, $D_j f$ is also K-invariant). Indeed, $\{D_1, \dots, D_l\}$ forms a system of generators of the algebra $\mathbb{D}(\mathbb{R}^n)^K$ of all K-invariant differentiable operators with constant coefficients.

By the integral expression (2) of spherical functions, we have

$$D_j \varphi_\xi(x) = \int_K \rho_j(k \cdot \xi) e^{i\langle x, k\cdot\xi\rangle} dk = \rho_j(\xi)\varphi_\xi(x) \tag{10}$$

since ρ_j is K-invariant. Thus φ_ξ is an eigenfunction of each D_j corresponding to the eigenvalue

$$\lambda_j(\xi) := \rho_j(\xi). \tag{11}$$

Note that for simplicity in the notation we are writing $\lambda_j(\xi)$ instead of $\lambda_j(\varphi_\xi)$, and that this shows a particular case of Theorem 2.1: Λ can be identified with the image set $\rho(\mathbb{R}^n)$.

We will use the following notation below: Given l- and n-dimensional multi-indexes $J = (j_1, \dots, j_l)$ and $I = (i_1, \dots, i_n)$, we write

$$|J|_{l,\rho} := j_1 \deg(\rho_1) + \cdots + j_l \deg(\rho_l), \quad |I|_n = i_1 + \cdots + i_n. \tag{12}$$

We will also use the convention

$$x^I := x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{for } x = (x_1, \dots, x_n) \quad \text{and} \quad \partial^I := \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n};$$
$$\rho(x)^J := \rho_1(x)^{j_1} \cdots \rho_l(x)^{j_l} \quad \text{for } \rho(x) = (\rho_1(x), \dots, \rho_l(x)) \quad \text{and} \quad D^J := D_1^{j_1} \cdots D_l^{j_l}.$$

3.1. ***Taylor expansions of spherical functions.*** From the definition of φ_ξ in (2) and by expanding $e^{i\langle x, k\cdot\xi\rangle}$ into its power series, it is clear that φ_ξ admits an entire extension. Thus we write

$$\varphi_\xi(x) = \sum_{m=0}^{\infty} p_{m,\xi}(x), \tag{13}$$

[4] This can be generalized to abstract Lie groups through the so-called *symmetrization map*; see [15, Theorem 4.3]. In our case, since $\mathbb{R}^n$ is an abelian group, this map is the identity, that is, $x_k \mapsto \partial_{x_k}$.

where each $p_{m,\xi}$ is a homogeneous polynomial of degree m and where the convergence of the series is absolute and uniform over compacts. Moreover, since the action of K on $\mathbb{R}^n$ is linear, each $p_{m,\xi}$ is K-invariant. Thus by using the generators $\{\rho_1, \ldots, \rho_l\}$ of $\mathcal{P}(\mathbb{R}^n)^K$, one can express each polynomial $p_{m,\xi}$ as

$$p_{m,\xi}(x) = \sum_{J_m : |J_m|_{l,\rho} = m} a_{J_m,\xi}\, \rho(x)^{J_m} = \sum_{I_m : |I_m|_n = m} b_{I_m,\xi}\, x^{I_m}, \tag{14}$$

for appropriate coefficients $a_{J_m,\xi}$ and $b_{I_m,\xi}$, where the first sum runs over multi-indexes $J_m = (j_1, \ldots, j_l)$, and the second sum is the Taylor expansion running over multi-indexes $I_m = (i_1, \ldots, i_n)$. Thus let us write

$$\varphi_\xi(x) = \sum_{J=(j_1,\ldots,j_l)} a_{J,\xi}\, \rho(x)^J = \sum_{I=(i_1,\ldots,i_n)} b_{I,\xi}\, x^I. \tag{15}$$

Then

$$(\partial^I \varphi_\xi)(0) = I! b_{I,\xi}, \quad \text{where } I! := i_1! \cdots i_n!. \tag{16}$$

As in [15, (2) and (3) in the proof of Proposition 2.2, page 400], for each multi-index $I = (i_1, \ldots, i_n)$, we consider the differential operator

$$\partial_0^I := \int_K (\mathrm{Ad}(k)\partial^I)\, dk,$$

where the adjoint action $\mathrm{Ad}(k)$ on the differential operator ∂ is defined by conjugation, that is, for all sufficiently differentiable f on $\mathbb{R}^n$,

$$(\mathrm{Ad}(k)\partial^I f)(x) = (\partial^I g)(k \cdot x), \quad \text{where } g(y) := f(k^{-1} \cdot y).$$

Moreover, for all sufficiently differentiable K-invariant functions f on $\mathbb{R}^n$ it holds that

$$(\partial^I f)(0) = (\partial_0^I f)(0).$$

Therefore $\partial_0^I \in \mathbb{D}(\mathbb{R}^n)^K$ and so it can be written as a polynomial Q_I on the generators $D_1, \ldots, D_l$:

$$\partial_0^I = Q_I(D_1, \ldots, D_l).$$

We note that the order of the differential operator ∂^{I_m} is $|I_m|_n = m$, and so it is the order of differential operator $\partial_0^{I_m}$.

As a consequence, using (6) we have that

$$b_{I,\xi} = \frac{1}{I!}(\partial^I \varphi_\xi)(0) = \frac{1}{I!}(\partial_0^I \varphi_\xi)(0) = \frac{1}{I!} Q_I(\rho_1(\xi), \ldots, \rho_l(\xi)). \tag{17}$$

Besides, from (14), for each l-dimensional multi-index J_m with $|J_m|_{l,\rho} = m$, the coefficient $a_{J_m,\xi}$ depends linearly on $\{b_{I_m,\xi}\}_{I_m : |I_m|_n = m}$. Thus from (17), for each l-dimensional multi-index J, there exists a polynomial q_J in l variables such that

$$a_{J,\xi} = q_J(\rho_1(\xi), \ldots, \rho_l(\xi)). \tag{18}$$

For simplicity, we write $q_J(\rho_1(\xi), \dots, \rho_l(\xi)) = q_J(\rho(\xi))$. Combining this with the series expression (15) for φ_ξ, we can write

$$\varphi_\xi(x) = \sum_J q_J(\rho(\xi))\rho(x)^J. \tag{19}$$

We recall that the series converges absolutely and uniformly over compacts. Indeed due to the symmetry (8), the series (19) converges uniformly as a function of x with fixed ξ, and also with respect to ξ for fixed x.

Definition 3.1. Let $(K, \mathbb{R}^n)$ be a Gelfand pair, and let φ_ξ be an associated bounded spherical function. By using the previous notation, we define the map $h_\xi : \mathbb{R}^l \to \mathbb{C}$ by

$$h_\xi(t) := \sum_J q_J(\rho(\xi))t^J, \tag{20}$$

where $t := (t_1, \dots, t_l)$ and $t^J := {t_1}^{j_1} \cdots {t_l}^{j_l}$ for the multi-index $J = (j_1, \dots, j_l)$.

Notice that for each spherical function φ_ξ we have associated h_ξ given by (20). Now our goal is to prove Proposition 1.2, which can be interpreted as a version of Theorem 1.1 for spherical functions. Indeed, for each $\varphi_\xi : \mathbb{R}^n \to \mathbb{C}$ we will show that the function $h_\xi : \mathbb{R}^l \to \mathbb{C}$ in Definition 3.1 is well defined, satisfies $\varphi_\xi(x) = h_\xi(\rho(x))$, and has certain regularity properties (h_ξ is the restriction to $\mathbb{R}^l$ of an entire function on $\mathbb{C}^l$). This is the most important result of this section and will be crucial in the next section, but we first need the following auxiliary result:

Lemma 3.2. *Let $p_1, p_2, \dots, p_l$ be nonnull homogeneous polynomials in $\mathbb{R}^n$. Given a positive number r, there exists $x_0 \in \mathbb{R}^n$ such that*

$$\min\{|p_1(x_0)|, |p_2(x_0)|, \dots, |p_l(x_0)|\} \geq r. \tag{21}$$

Proof. The zero set of each nonzero polynomial p_j is a proper algebraic subset of $\mathbb{R}^n$, and hence has empty interior. Thus $\bigcup_{j=1}^l \{p_j = 0\}$ is nowhere dense, and there exists $y \in \mathbb{R}^n$ such that $p_j(y) \neq 0$ for every j. Since each p_j is homogeneous, $p_j(sy) = s^{d_j} p_j(y)$ for some $d_j \in \mathbb{N}$ and all $s > 0$. Choosing s large enough and setting $x_0 = sy$, we get (21). □

Proof of Proposition 1.2. Let φ_ξ be a bounded spherical function associated to the Gelfand pair $(K, \mathbb{R}^n)$ and consider the h_ξ as in Definition 3.1.

First, notice that from (19), we have the relation $\varphi_\xi(x) = h_\xi(\rho(x))$ for all $x \in \mathbb{R}^n$. In the remainder of the proof our goal is to prove not only that h_ξ is a well-defined function, but also that it is very regular. In fact, we will show that it admits an entire extension, in particular $h_\xi \in C^\infty(\mathbb{R}^l)$.

By Lemma 3.2, given a positive number r, there exists $x_0 \in \mathbb{R}^n$ such that $r \leq \min\{|\rho_j(x_0)|\}$. Then for $\|t\| \leq r$,

$$\sum_J |q_J(\rho(\xi))|\,|t_1|^{j_1} \cdots |t_l|^{j_l} \leq \sum_J |q_J(\rho(\xi))|\,|\rho_1(x_0)|^{j_1} \cdots |\rho_l(x_0)|^{j_l}. \tag{22}$$

Since the power series (19) of φ_ξ converges absolutely, then the right-hand side of (22) converges. Therefore the series (20) converges absolutely and uniformly for t in the ball of $\mathbb{R}^l$ of radius r centered

at the origin. Since $r > 0$ is arbitrary, the series converges uniformly over compact sets of $\mathbb{R}^l$. Thus $t \mapsto h_\xi(t)$ admits an entire extension to $\mathbb{C}^l$. □

Finally, as a consequence of the symmetry property (8) of the spherical functions, the associated functions defined through the expression (20) satisfy the following property:

Lemma 3.3. *For each fixed $t \in \mathbb{R}^l$, the function $\xi \mapsto h_\xi(t)$ defined by (20) admits an entire extension.*

Proof. The proof follows similar arguments as in Proposition 1.2, swapping the roles of t and ξ and using the symmetry (8) of the spherical functions. Fixing $t \in \mathbb{R}^l$, there exists $x_0 \in \mathbb{R}^n$ such that

$$\sum_J |q_J(\rho(\xi))||t_1|^{j_1} \cdots |t_l|^{j_l} \leq \sum_J |q_J(\rho(\xi))||\rho_1(x_0)|^{j_1} \cdots |\rho_l(x_0)|^{j_l}. \tag{23}$$

Using the symmetry (8), we can say that the power series (19) of φ_{x_0} as a function of ξ converges absolutely and uniformly over compacts. Therefore the right-hand side of (23) converges, because it is bounded above by an absolutely convergent series, and also uniformly over compacts with respect to the variable ξ. Therefore, due to the fact that the series (20) defining $h_\xi(t)$ converges absolutely and uniformly over compacts with respect to the variable ξ (for fixed t), the function $\xi \mapsto h_\xi(t)$ admits an entire extension to $\mathbb{C}^n$. □

Finally, notice that following relations hold true:

$$h_\xi(\rho(x)) = \varphi_\xi(x) = \varphi_x(\xi) = h_x(\rho(\xi)).$$

3.2. ***Main contributions: the proofs of Theorem 1.3 and Corollary 1.4.*** In this section, we prove a version of Schwarz's theorem, stated as Theorem 1.1, using only elements from Gelfand theory in Euclidean spaces.

Definition 3.4. Let $(K, \mathbb{R}^n)$ be a Gelfand pair. For a given $f \in L^1(\mathbb{R}^n)$, K-invariant with $\hat{f}$ of compact support, we define

$$h_f(t) := \int_{\mathbb{R}^n} \hat{f}(\xi) h_\xi(t)\, d\xi \quad \text{for } t = (t_1, \ldots, t_l) \in \mathbb{R}^l, \tag{24}$$

where h_ξ is given by (20).

Proof of Theorem 1.3. Let us consider $f \in L^1(\mathbb{R}^n)$, K-invariant with $\hat{f}$ of compact support. In particular, $f \in C^\infty(\mathbb{R}^n)$. For such f, consider h_f as in (24).

Let us show that the function h_f is well defined. For each fixed t, from Lemma 3.3, $\xi \mapsto h_\xi(t)$ is continuous, and thus for ξ in a compact set $h_\xi(t)$ is bounded by some constant $C(t)$. Therefore

$$\int_{\mathbb{R}^n} |\hat{f}(\xi) h_\xi(t)|\, d\xi = \int_{\mathrm{supp}(\hat{f})} |\hat{f}(\xi) h_\xi(t)|\, d\xi \leq \left(\int_{\mathrm{supp}(\hat{f})} |\hat{f}(\xi)|\, d\xi \right) C(t) < \infty.$$

As a result, $h_f(t)$ is well defined for every $t \in \mathbb{R}^l$.

By the Paley–Wiener theorem, f is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$. Now we will show that h_f is the restriction to $\mathbb{R}^l$ of an entire function on $\mathbb{C}^l$. In order to see that, we will first check

that it is a continuous function in $\mathbb{R}^l$. Consider an arbitrary convergent sequence $(t^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^l$ such that $t^k \to t^*$ for some $t^* \in \mathbb{R}^l$ as $k \to \infty$. By Proposition 1.2, for each ξ, the function $t \mapsto h_\xi(t)$ is continuous in $\mathbb{R}^l$. Then we can apply the Lebesgue dominated convergence theorem:

$$\begin{aligned}\lim_{k\to\infty} h_f(t^k) &= \lim_{k\to\infty} \int_{\operatorname{supp}(\hat{f})} \hat{f}(\xi) h_\xi(t^k)\, d\xi = \int_{\operatorname{supp}(\hat{f})} \hat{f}(\xi) \lim_{k\to\infty} h_\xi(t^k)\, d\xi = \int_{\operatorname{supp}(\hat{f})} \hat{f}(\xi) h_\xi(t^*)\, d\xi \\ &= h_f(t^*).\end{aligned}$$

Finally, we recall that a function in $\mathbb{C}^l$ is analytic if it is analytic on each variable. Let γ be closed piecewise-smooth curve in $\mathbb{C}$, where we understand $\mathbb{C}$ as the first component of $\mathbb{C}^l = \mathbb{C} \times \cdots \times \mathbb{C}$. Then since $h_\xi(z)$ is analytic, we have

$$\oint_\gamma h_\xi(z_1, \ldots, z_l)\, dz_1 = 0.$$

Then

$$\oint_\gamma h_f(z)\, dz_1 = \oint_\gamma \int_{\mathbb{R}^n} \hat{f}(\xi) h_\xi(z)\, d\xi\, dz_1 = \int_{\operatorname{supp}(\hat{f})} \hat{f}(\xi) \underbrace{\oint_\gamma h_\xi(z)\, dz_1}_{=0} d\xi = 0.$$

We were able to interchange the order of integration because we are integrating over compact sets, and so theorems of Fubini and Tonelli hold. Therefore, by Morera's theorem, h_f is analytic on z_1. Since the same argument is valid for all variables, h_f is analytic as a function in $\mathbb{C}^l$. Thus h_f is not only in $C^\infty(\mathbb{R}^l)$, but also admits an entire extension to $\mathbb{C}^l$.

Finally, under the hypothesis of this theorem we can apply the expression (9) for the inversion formula and (3) to obtain $f(x) = h_f(\rho(x))$. Indeed for every $x \in \mathbb{R}^n$ we have

$$f(x) = \int_{\mathbb{R}^n} \mathcal{F}(f)(\varphi_\xi)\varphi_\xi(x)\, d\xi = \int_{\mathbb{R}^n} \mathcal{F}(f)(\varphi_\xi) h_\xi(\rho(x))\, d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi) h_\xi(\rho(x))\, d\xi = h_f(\rho(x)),$$

where the second equality holds from Proposition 1.2. □

Proof of Corollary 1.4. Consider $f = \hat{g}$ and apply Theorem 1.3. Thus if we consider

$$h_f(t) = \int_{\mathbb{R}^n} \hat{f}(\xi) h_\xi(t)\, d\xi = \int_{\mathbb{R}^n} \hat{\hat{g}}(\xi) h_\xi(t)\, d\xi = \int_{\mathbb{R}^n} g(-\xi) h_\xi(x)\, d\xi,$$

it satisfies

$$\mathcal{F}(g)(\varphi_\xi) = \hat{g}(\xi) = f(\xi) = h_f(\rho(\xi)).$$ □

4. Examples

The goal of this section is to provide some more explicit flavor on the computation of the coefficients $a_{J,\xi}$ in (14), expressed in terms of a (nonexplicit) polynomial q_J in the variables $\rho_1(\xi), \ldots, \rho_l(\xi)$ as in (18).

It is easy to check that

$$|J|_{l,\rho} \neq |J'|_{l,\rho} \implies (D^J \rho^{J'})(0) = 0. \tag{25}$$

However, it is difficult to compute $(D^J \rho^{J'})(0)$ for arbitrary l-dimensional multi-indexes J and J'.

Remark 4.1. Let us define $\lambda(\xi) := \Pi_{k=1}^{l} \lambda_k(\xi)$, and so we write $\lambda(\xi)^J := \Pi_{k=1}^{l} \lambda_k(\xi)^{j_k}$ for $J = (j_1, \dots, j_l)$. Under the assumption

$$(D^J \rho^{J'})(0) = 0 \quad \text{if } J \not\equiv J', \tag{26}$$

we would have the following expressions for the coefficients in (15):

$$a_{J,\xi} = \frac{(D^J \varphi_\xi)(0)}{(D^J \rho^J)(0)} = \frac{\lambda(\xi)^J}{(D^J \rho^J)(0)} \underbrace{\varphi_\xi(0)}_{=1} = \frac{\lambda(\xi)^J}{(D^J \rho^J)(0)}. \tag{27}$$

As a result, combining this with (11) in (15), we would obtain

$$\begin{aligned} \varphi_\xi(x) &= \sum_{m=0}^{\infty} \sum_{J_m : |J_m|_{l,\rho} = m} a_{J_m,\xi} \rho(x)^{J_m} = \sum_{m=0}^{\infty} \sum_{J_m : |J_m|_{l,\rho} = m} \frac{\lambda(\xi)^{J_m}}{(D^{J_m} \rho^{J_m})(0)} \rho(x)^{J_m} \\ &= \sum_{m=0}^{\infty} \sum_{J_m : |J_m|_{l,\rho} = m} \frac{i^m \rho(\xi)^{J_m}}{(D^{J_m} \rho^{J_m})(0)} \rho(x)^{J_m}. \end{aligned} \tag{28}$$

This expression is very explicit and only requires the computation of the coefficients $(D^J \rho)(0)$. However, the assumption (26) does not hold in general.

When the algebra $\mathcal{P}$ has a single generator, the relation (26) holds and we have the nice expression of the spherical functions as in (28). We will see this in the first examples.

Example 4.2. Consider the classical theory on Fourier analysis on the real line $\mathbb{R}$. Here K is trivial, and we can consider $\rho(x) := x$ as a single generator of the algebra of polynomials $\mathcal{P}(\mathbb{R})$, and the classical derivative operator in one dimension, ∂_x, as a generator of $\mathbb{D}(\mathbb{R})$. The bounded spherical functions are the complex exponential functions $\varphi_\xi(x) = e^{i\xi x}$ parametrized by $\xi \in \mathbb{R}$. By using their Taylor expansions, we obtain an expression as in (28):

$$\varphi_\xi(x) = e^{i\xi x} = \sum_{k=0}^{\infty} (\partial_x^k \varphi_\xi)(0) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \underbrace{\frac{(i\xi)^k}{k!}}_{a_{k,\xi}} \underbrace{x^k}_{\rho(x)^k} = \sum_{k=0}^{\infty} \frac{i^k \rho(\xi)^k}{(\partial_x^k \rho(x)^k)(0)} \rho(x)^k.$$

In terms of the Schwarz theorem, this is a trivial example, as $\varphi_\xi = h_\xi$, and $f = h_f$ in Theorems 1.1 or 1.3.

Example 4.3. Let us analyze the Gelfand pair $(\mathrm{SO}(n), \mathbb{R}^n)$, where $K = \mathrm{SO}(n)$ is the special orthogonal group.

On the one hand, the algebra of rotational invariant polynomials, $\mathcal{P}(\mathbb{R}^n)^{\mathrm{SO}(n)}$, is generated by a single polynomial, which can be chosen as $\rho(x) := \|x\|^2$. Similarly, $\mathbb{D}(\mathbb{R}^n)^{\mathrm{SO}(n)}$ is generated by the Laplacian operator Δ. Notice that

$$\Delta \|x\|^{2m} = 2m(2(m-1)+n)\|x\|^{2(m-1)} \quad \text{for all } m \in \mathbb{N},$$

and, more generally,

$$\Delta^k \|x\|^{2m} = 2m(2(m-1)+n)\cdots(2(m-k+1))(2(m-k)+n)\|x\|^{2(m-k)} \quad \text{for all } m, k \in \mathbb{N}.$$

In particular,

$$(\Delta^k \|x\|^{2m})(0) = 0 \quad \text{for all } k \neq m, \tag{29}$$

and if $k = m$,

$$\Delta^k \|x\|^{2k} = (2k)!!n(n+2)\cdots(n+2(k-1)), \tag{30}$$

where $2k!! := 2k \cdot 2(k-2) \cdot 2(k-4) \cdots 4 \cdot 2$. Thus in this case, we are under the hypothesis (26).

On the other hand, the algebra $L^1(\mathbb{R}^n)^{\mathrm{SO}(n)}$ is that of all rotational invariant functions in $L^1(\mathbb{R}^n)$, that is, the algebra of integrable radial functions. Its spectrum Λ can be identified with the half-line $\{\|\xi\|e_1 : s \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^n$, for $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$, as we have that for every $\xi \in \mathbb{R}^n$ the bounded spherical functions φ_ξ can be obtained as

$$\varphi_\xi(x) = \int_{\mathrm{SO}(n)} e^{i\langle x, k\cdot\xi\rangle}\, dk = \int_{\mathbb{S}^{n-1}} e^{i\|\xi\|\langle x,\theta\rangle}\, d\theta,$$

where $d\theta$ is the uniform probability measure on the sphere $\mathbb{S}^{n-1}$. We have that $\varphi_{\xi_1} = \varphi_{\xi_2}$ as long as $\|\xi_1\| = \|\xi_2\|$. It holds that

$$(\Delta^k \varphi_\xi)(0) = (i\|\xi\|)^{2k} = i^{2k}\rho(\xi)^k. \tag{31}$$

(Notice that $|k|_{1,\rho} = 2k$.) Moreover, the following series expression is well known for each bounded spherical function (see, e.g., [6]):

$$\varphi_\xi(x) = \sum_k \underbrace{\frac{\Gamma(2/n)}{k!\Gamma(k+n/2)} \frac{(i\|\xi\|)^{2k}}{2^{2k}}}_{a_{k,\xi}} \underbrace{\|x\|^{2k}}_{\rho(x)^k}. \tag{32}$$

Thus we already have an explicit expression for the coefficients $a_{k,\xi}$. The goal now is to show that the formula (32) from the literature coincides with an expression like (28). Indeed, since we are under hypothesis (26), using (27) we have

$$a_{k,\xi} = \frac{(\Delta^k \varphi_\xi)(0)}{(\Delta^k \rho^k)(0)} = \frac{i^{2k}\|\xi\|^k}{(2k)!!n(n+2)\cdots(n+2(k-1))},$$

where we have used (30) and (31). Therefore, to match with (32), one needs to verify the identity

$$\frac{\Gamma(n/2)}{\Gamma(k+n/2)k!2^{2k}} = \frac{1}{(2k)!!n(n+2)\cdots(n+2(k-1))}. \tag{33}$$

It indeed holds true because, on the one hand, it is easy to check that

$$k!2^k = (2k)!!$$

and, on the other hand, by iteratively applying the identity $\Gamma(z+1) = z\Gamma(z)$ for the gamma function, one can reach

$$\Gamma\left(k+\tfrac{1}{2}n\right) = \tfrac{1}{2}n\left(\tfrac{1}{2}n+1\right)\cdots\left(\tfrac{1}{2}n+(k-1)\right)\Gamma\left(\tfrac{1}{2}n\right) = \frac{n(n+2)\cdots n+2(k-1)}{2^k}\Gamma\left(\tfrac{1}{2}n\right).$$

Finally, in this particular example we have the explicit formula

$$h_\xi(t) = \sum_k \frac{\Gamma(2/n)}{k!\Gamma(k+n/2)} \frac{(i\|\xi\|)^{2k}}{2^{2k}} t^k = \sum_k \frac{i^{2k}\|\xi\|^k}{(2k)!!n(n+2)\cdots(n+2(k-1))} t^k.$$

Example 4.4. Consider the Gelfand pair $(\mathbb{Z}_2, \mathbb{R}^2)$. Then $L^1(\mathbb{R}^2)^{\mathbb{Z}_2}$ is the space of integrable even functions. If $\mathcal{P}(\mathbb{R}^2) = \text{span}\{x_1, x_2\}$, then $\mathcal{P}(\mathbb{R}^2)^{\mathbb{Z}_2} = \text{span}\{x_1^2, x_1x_2, x_2^2\}$. Let us define, in this case,

$$\rho_1(x_1, x_2) := x_1^2, \quad \rho_2(x_1, x_2) := x_1x_2, \quad \rho_3(x_1, x_2) := x_2^2.$$

We view this as an interesting example, since on the one hand $n = 2 < 3 = l$, that is, roughly speaking, the number of generators of $\mathcal{P}(\mathbb{R}^2)^{\mathbb{Z}_2}$ is greater than the number of "variables". On the other hand, in this example the generators of $\mathcal{P}(\mathbb{R}^2)^{\mathbb{Z}_2}$ are not algebraically independent; in fact, they satisfy the relation

$$(\rho_2)^2 = \rho_1\rho_3. \tag{34}$$

Let φ_ξ be a bounded spherical function of the pair $(\mathbb{Z}_2, \mathbb{R}^2)$ written as a power series as in (15):

$$\varphi_\xi(x) = \sum_{J=(j_1,j_2,j_3)} a_{J,\xi}\rho_1(x)^{j_1}\rho_2(x)^{j_2}\rho_3(x)^{j_3} \quad \text{for all } x = (x_1, x_2). \tag{35}$$

In order to get a more explicit expression for the coefficients of the series above, we proceed as follows.

(1) First, replace every even power of ρ_2, i.e., every $\rho_2(x)^{2k}$ for $k \in \mathbb{N}$, by the same factor but written in terms of ρ_1 and ρ_3, i.e., $\rho_1(x)^k\rho_3(x)^k$. After this, we get a series expression for (35) in terms of the generators ρ_1, ρ_2, and ρ_3, but "clean" of relations among them. Precisely, given a multi-index $J = (j_1, j_2, j_3)$, we write $j_2 = 2k_J + \omega_J$ for unique $\omega_J \in \{0, 1\}$ and $k_J \in \mathbb{N}$, and consider a second multi-index $I := (i_1, i_2, i_3)$ such that

$$i_1 := k_J + j_1, \quad i_2 := \omega_J, \quad i_3 := k_J + j_3. \tag{36}$$

Then $\rho_1(x)^{j_1}\rho_2(x)^{j_2}\rho_3(x)^{j_3} = \rho_1(x)^{i_1}\rho_2(x)^{i_1}\rho_1(x)^{i_1}$, but we only write the right-hand side expression.

(2) Second, by taking common factors, the terms in (35) associated to the monomials $\rho(x)^J$ and $\rho(x)^I$ are combined in only one term $(a_{J,\xi} + a_{I,\xi})\rho(x)^I$. That is, we can rewrite (35) as

$$\varphi_\xi(x) = \sum_{\substack{I=(u+k,\omega,v+k) \\ \text{such that} \\ \omega\in\{0,1\},u,v,k\in\mathbb{N}}} (a_{(u,2k+\omega,v),\xi} + a_{(u+k,\omega,v+k),\xi}) \underbrace{\rho_1(x)^{u+k}\rho_2(x)^{\omega}\rho_3(x)^{u+k}}_{\rho(x)^I} \quad \text{for all } x = (x_1, x_2), \tag{37}$$

where $\xi = (\xi, \xi_2) \in \mathbb{R}^2$.

(3) Now consider two arbitrary multi-indexes $I = (i_1, i_2, i_3)$ and $I' = (i_1', i_2', i_3')$ that appear in (37), that is, with $i_2, i_2' \in \{0, 1\}$. We will show that

$$(D^I\rho^{I'})(0) \neq 0 \iff I \equiv I'. \tag{38}$$

Indeed, the expression

$$(D^I\rho^{I'})(0) = (\partial_{x_1}^{i_1+i_2}\partial_{x_2}^{i_3+i_2}(x_1)^{i'_1+i'_2}(x_2)^{i'_3+i'_2})(0) \tag{39}$$

is not null if and only if

$$i_1+i_2 = i'_1+i'_2 \quad \text{and} \quad i_3+i_2 = i'_3+i'_2. \tag{40}$$

It is trivial to see that if $I \equiv I'$, then $(D^I\rho^{I'})(0) \neq 0$. Thus only the other direction is left to prove. Arguing by contrapositive, let us assume $I \not\equiv I'$, and show that $(D^I\rho^{I'})(0) = 0$. Since $\deg(\rho_k) = 2$ for every $k = 1, 2, 3$, from (25) we have that if $i_1+i_2+i_3 \neq i'_1+i'_2+i'_3$, then $(D^I\rho^{I'})(0) = 0$. As a result, we can suppose that $i_1+i_2+i_3 = i'_1+i'_2+i'_3$. Let us separate into two cases. First, if $i_2 = i'_2$, since $I \not\equiv I'$, then $i_1 \neq i'_1$ or $i_3 \neq i'_3$, and in either case (39) is not null. Second, if $i_2 \neq i'_2$, since their only two possible values are 0 and 1, we have that (40) holds if and only if i_1 and i'_1 have different parity, as well as i_3 and i'_3 (i.e., one is even and the other is odd). However, this contradicts the equality $i_1+i_2+i_3 = i'_1+i'_2+i'_3$ as we fall under one of the following three cases:

$$\begin{array}{ccc} i_1 & i_2 & i_3 \\ \underbrace{\text{odd}+0+\text{odd}}_{\text{even}} & & \underbrace{\text{even}+1+\text{even}}_{\text{odd}}, \\ \underbrace{\text{even}+0+\text{even}}_{\text{even}} & & \underbrace{\text{odd}+1+\text{odd}}_{\text{odd}}, \\ \underbrace{\text{odd}+0+\text{even}}_{\text{odd}} & & \underbrace{\text{even}+1+\text{odd}}_{\text{even}}. \end{array}$$

Therefore (40) is not satisfied and so $(D^I\rho^{I'})(0) = 0$.

(4) Finally, from (37), for every multi-index $I = (u+k, \omega, v+k)$ with $\omega \in \{0, 1\}$ and $u, v, k \in \mathbb{N}$,

$$\lambda(\xi)^I = D^I\varphi_\xi(0) = (a_{(u,2k+\omega,v),\xi} + a_{(u+k,\omega,v+k),\xi})(D^I\rho^I)(0).$$

Thus using (11),

$$(a_{(u,2k+\omega,v),\xi} + a_{(u+k,\omega,v+k),\xi}) = \frac{i^{2(u+v+2k+\omega)}\rho^I(\xi)}{(D^I\rho^I)(0)} = \frac{-\rho_1(\xi)^{u+k}\rho_2(\xi)^\omega\rho_3(\xi)^{v+k}}{(u+k+\omega)!(v+k+\omega)!}.$$

Hence,

$$\varphi_\xi(x) = \sum_{\substack{I=(u+k,\omega,v+k) \\ \text{such that} \\ \omega\in\{0,1\},u,v,k\in\mathbb{N}}} \frac{-\overbrace{\rho_1(\xi)^{u+k}\rho_2(\xi)^\omega\rho_3(\xi)^{v+k}}^{\rho(\xi)^I}}{(u+k+\omega)!(v+k+\omega)!}\rho_1(x)^{u+k}\rho_2(x)^\omega\rho_3(x)^{u+k},$$

and

$$h_\xi(t_1,t_2,t_3) = \sum_{\substack{I=(u+k,\omega,v+k) \\ \text{such that} \\ \omega\in\{0,1\},u,v,k\in\mathbb{N}}} \frac{-\rho(\xi)^I}{(u+k+\omega)!(v+k+\omega)!}t_1^{u+k}t_2^\omega t_3^{v+k}.$$

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References

[1] F. Astengo, B. Di Blasio, and F. Ricci, "Gelfand transforms of polyradial Schwartz functions on the Heisenberg group", *J. Funct. Anal.* **251**:2 (2007), 772–791. MR

[2] F. Astengo, B. Di Blasio, and F. Ricci, "Gelfand pairs on the Heisenberg group and Schwartz functions", *J. Funct. Anal.* **256**:5 (2009), 1565–1587. MR

[3] C. Benson, J. Jenkins, and G. Ratcliff, "On Gelfand pairs associated with solvable Lie groups", *Trans. Amer. Math. Soc.* **321**:1 (1990), 85–116. MR

[4] E. Bierstone, "Local properties of smooth maps equivariant with respect to finite group actions", *J. Differential Geometry* **10**:4 (1975), 523–540. MR

[5] G. Carcano, "A commutativity condition for algebras of invariant functions", *Boll. Un. Mat. Ital. B* (7) **1**:4 (1987), 1091–1105. MR

[6] G. van Dijk, *Introduction to harmonic analysis and generalized Gelfand pairs*, De Gruyter Studies in Mathematics **36**, De Gruyter, Berlin, 2009. MR

[7] C. L. Fefferman, "A sharp form of Whitney's extension theorem", *Ann. of Math.* (2) **161**:1 (2005), 509–577. MR

[8] F. Ferrari Ruffino, "The topology of the spectrum for Gelfand pairs on Lie groups", *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8) **10**:3 (2007), 569–579. MR

[9] V. Fischer, F. Ricci, and O. Yakimova, "Nilpotent Gelfand pairs and spherical transforms of Schwartz functions, I: Rank-one actions on the centre", *Math. Z.* **271**:1-2 (2012), 221–255. MR

[10] V. Fischer, F. Ricci, and O. Yakimova, "Nilpotent Gelfand pairs and spherical transforms of Schwartz functions, II: Taylor expansions on singular sets", pp. 81–112 in *Lie groups*: *structure*, *actions*, *and representations*, edited by A. Huckleberry et al., Progr. Math. **306**, Springer, 2013. MR

[11] G. B. Folland, *A course in abstract harmonic analysis*, CRC, Boca Raton, FL, 1995. MR

[12] G. Glaeser, "Fonctions composées différentiables", pp. 1–4 in *Séminaire d'Analyse*, edited by P. Lelong, Secrétariat mathématique, Paris, 1963. MR

[13] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**, 1955. MR

[14] S. Helgason, *Differential geometry*, *Lie groups*, *and symmetric spaces*, Pure and Applied Mathematics **80**, Academic, New York, 1978. MR

[15] S. Helgason, *Groups and geometric analysis: integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs **83**, Amer. Math. Soc., Providence, RI, 2000. MR

[16] K. Kikuchi, "On Gelfand pairs associated with nilpotent Lie groups", *J. Math. Kyoto Univ.* **34**:4 (1994), 741–754. MR

[17] D. Luna, "Fonctions différentiables invariantes sous l'opération d'un groupe réductif", *Ann. Inst. Fourier* (*Grenoble*) **26**:1 (1976), 33–49. MR

[18] J. N. Mather, "Differentiable invariants", *Topology* **16**:2 (1977), 145–155. MR

[19] L. Schwartz, "Produit tensoriel topologique d'espaces vectoriels topologiques", *Séminaire Schwartz* **1** (1953-1954), 1–3.

[20] G. W. Schwarz, "Smooth functions invariant under the action of a compact Lie group", *Topology* **14** (1975), 63–68. MR

[21] F. Trèves, *Topological vector spaces*, *distributions and kernels*, Academic, New York, 1967. MR

[22] E. B. Vinberg, "Commutative homogeneous spaces and co-isotropic symplectic actions", *Uspekhi Mat. Nauk* **56**:1(337) (2001), 3–62. In Russian; translated in *Russian Math. Surveys* **56**:1 (2001), 1–60. MR

[23] H. Weyl, *The classical groups: their invariants and representations*, Princeton Univ. Press, 1939. MR

[24] H. Whitney, "Analytic extensions of differentiable functions defined in closed sets", *Trans. Amer. Math. Soc.* **36**:1 (1934), 63–89. MR

[25] H. Whitney, "Differentiable even functions", *Duke Math. J.* **10** (1943), 159–160. MR

[26] J. A. Wolf, "Spherical functions on Euclidean space", *J. Funct. Anal.* **239**:1 (2006), 127–136. MR

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On the difference between perfect powers and integral S-units

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Let $q_1, \dots, q_t$ be distinct prime numbers. Let $a_1, \dots, a_t$ be nonnegative integers. We establish effective lower bounds for $|z^d - q_1^{a_1} \cdots q_t^{a_t}|$ and for its greatest prime factor, which tend to infinity with z^d, where z is a positive integer coprime with $q_1 \cdots q_t$ and $d \geq 2$ is an integer.

1. Introduction

It is by no means obvious that the distance between powers of 2 and powers of 3 tends to infinity. This has been proved by Thue [1909], but his proof does not allow one to determine the set of powers of 2 which differ by a given integer from a power of 3. Gelfond [1935] established an effective lower bound for $|3^m - 2^n|$ which tends to infinity as the maximum of the positive integers m and n tends to infinity. Thue's result was extended by Mahler [1953], who proved that the greatest prime divisor of $|3^m - 2^n|$ tends to infinity with the maximum of the positive integers m and n, but, as for Thue's result, Mahler's proof does not yield any effective lower bound. The interested reader will find further references in the monographs [Shorey and Tijdeman 1986; Ribenboim 1994]. All these results have been considerably generalized, and in an effective way, after the end of the 60s and the development of Alan Baker's theory of linear forms in an arbitrary number of logarithms of algebraic numbers (Gelfond's result solves the case of linear forms in two logarithms); see [Bugeaud 2018]. Here we present a new extension. We replace the sequence of powers of 2 by a sequence of positive integers whose prime divisors belong to a given finite set T, and the sequence of powers of 3 by the sequence of perfect powers coprime with the elements of T.

Let $T := \{q_1, \dots, q_t\}$ be a nonempty set of distinct prime numbers and let $\mathcal{T} = (t_i)_{i \geq 1}$ denote the set of integers greater than 1 whose prime divisors are all in T, ranged in increasing order. A result of Tijdeman [1973] asserts that the difference $t_{i+1} - t_i$ tends to infinity with i and, more precisely, that there exists an effectively computable real number C, depending only on T, such that

$$t_{i+1} - t_i \geq t_i (\log 2t_i)^{-C}, \quad i \geq 1.$$

This result is essentially best possible [Tijdeman 1974].

For an integer $d \geq 2$, let $\mathcal{U}_d$ denote the set composed of the perfect d-th powers z^d, with $z \geq 2$ coprime with $q_1 \cdots q_t$, ranged in increasing order. Let

$$\mathcal{U} = \bigcup_{d \geq 2} \mathcal{U}_d = (u_j)_{j \geq 1}$$

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denote the union over $d \geq 2$ of the sets $\mathcal{U}_d$, ranged in increasing order. While we know by a result of Mihăilescu [2004] that 8 and 9 are the only consecutive perfect powers (Catalan's conjecture), we cannot exclude that there could be infinitely many integers j such that $u_{j+1} = u_j + 2$. We are concerned with the gap between the sets $\mathcal{T}$ and $\mathcal{U}$.

Let $d \geq 2$ be an integer. Let x be an element of $\mathcal{T}$ and u an element of $\mathcal{U}_d$. For $d \geq 3$ it follows from classical results on Thue and Thue–Mahler equations that $|x-u|$ and its greatest prime factor tend to infinity as the maximum of x and u tends to infinity and effectively computable lower bounds have been established; see Section 2. The same holds when $d = 2$, as recently proved in [Bugeaud 2025]. In these results, the exponent d is fixed. Our aim is to give effective lower bounds independent of d for $|x-u|$ and its greatest prime factor.

For an integer z with $|z| \geq 2$, let $P[z]$ denote the greatest prime factor of $|z|$. Set $P[0] = P[-1] = P[1] = 1$. For any positive real number x, we set $\log_* x = \max\{1, \log x\}$.

Theorem 1.1. *We keep the above notation. There exist effectively computable positive real numbers κ_1 and κ_2, depending only on T, such that, for every x in $\mathcal{T}$ and every u in $\mathcal{U}$ we have*

$$|x-u| \geq (\log_* \log X)^{\kappa_1}$$

and

$$P[x-u] \geq \kappa_2 \log_* \log_* \log X \frac{\log_* \log_* \log_* \log X}{\log_* \log_* \log_* \log_* \log X},$$

where we have set $X := \max\{x, u\}$.

For the proof of Theorem 1.1, we simply combine several results already in the literature (or somehow hidden).

Corollary 1.2. *We keep the above notation. Let m be an integer. The Diophantine equation*

$$x - u = m, \quad \text{in } x \in \mathcal{T} \text{ and } u \in \mathcal{U},$$

has only finitely many solutions, and all of them can be effectively determined.

Theorem 1.1 and its corollary illustrate the power of Baker's theory of linear forms in logarithms. They can be extended to arbitrary number fields without any new argument.

Some parts of Corollary 1.2, e.g., when u is in $\mathcal{U}$ with $d \geq 3$, follow from Theorem 3 (see also (12)) of Győry, Pink, and Pintér [Győry et al. 2004]; see also [Győry and Pintér 2008, Corollary 2.4; Shorey and Tijdeman 1986, Theorem 12.2].

Throughout this note, the constants $C_1, C_2, \ldots$ and the constants implied by $\ll$ and $\gg$ are positive, effectively computable, and depend only on T. Furthermore, we let $p_1 = 2$, $p_2 = 3$, $p_3, \ldots$ denote the sequence of prime numbers ranged in increasing order and recall that, by the prime number theorem, $p_s/(s \log s)$ tends to 1 as s tends to infinity.

We keep the notation from Section 1 throughout.

2. The case of a fixed exponent

Let $a_1, \dots, a_t$ be nonnegative integers, not all zero, and set

$$x := q_1^{a_1} \cdots q_t^{a_t}.$$

Let $d \geq 2$ be an integer. Let u_d be in $\mathcal{U}_d$ and write $u_d = z^d$ with $z \geq 2$.

Consider first the case $d = 2$. It follows from [Bugeaud 2025, Theorem 1.4 and Corollary 1.3], that

$$\log|x - u_2| \gg \log X_2 \tag{2-1}$$

and

$$P[x - u_2] \gg \log_* \log X_2 \frac{\log_* \log_* \log X_2}{\log_* \log_* \log_* \log X_2}, \tag{2-2}$$

where we have set $X_2 := \max\{x, u_2\}$.

Assume now that $d \geq 3$. Write

$$\Delta := u_d - x = z^d - q_1^{r_1} \cdots q_t^{r_t} (q_1^{\lfloor a_1/d \rfloor} \cdots q_t^{\lfloor a_t/d \rfloor})^d,$$

where r_i is the remainder in the Euclidean division of a_i by d, for $i = 1, \dots, t$. Thus, we get t^d integer binary forms

$$F_{\underline{r}}(Z, Y) = Z^d - q_1^{r_1} \cdots q_t^{r_t} Y^d, \quad \underline{r} = (r_1, \dots, r_t),$$

of degree d and t^d Thue equations $F_{\underline{r}}(Z, Y) = \Delta$.

It then follows from the explicit bound for the solutions to Thue equations given in [Bugeaud and Győry 1996, Theorem 3] that there exists C_1 such that

$$\log X_d \leq (C_1 d)^{18d} \log|\Delta|, \tag{2-3}$$

where we have set $X_d := \max\{x, u_d\}$.

We can as well derive a lower bound for the greatest prime factor of Δ. Write

$$u_d - x = z^d - q_1^{r_1} \cdots q_t^{r_t} (q_1^{\lfloor a_1/d \rfloor} \cdots q_t^{\lfloor a_t/d \rfloor})^d = \pm p_1^{b_1} \cdots p_s^{b_s},$$

with $b_s \geq 1$, in such a way that $P[x - u_d] = p_s =: P$. Since the degree of the splitting field of the polynomial $F_{\underline{r}}(X, 1)$ is bounded from above by $d(d-1)$, it follows from (the proof of) [Bugeaud and Győry 1996, Theorem 4] that there are C_2 and C_3 such that

$$\log X_d \leq (C_2 d s)^{14ds} P^{d(d-1)} C_3^{2d^2} (\log P)^{ds+2},$$

and thus

$$\log_* \log X_d \ll d^2 P. \tag{2-4}$$

Observe that the slightly better estimate

$$\log_* \log X_d \ll d^2 \frac{P \log_* \log P}{\log P}$$

follows from [Győry and Yu 2006] (see also [Bugeaud et al. 2018, Theorem 2.5]) and is of the same strength as (2-2). In view of the bounds on d obtained in Section 3, this slight improvement on (2-4) has no influence on Theorem 1.1.

3. Completion of the proof of Theorem 1.1

We treat the exponent d as a variable and complete the proof of Theorem 1.1.

Write

$$q_1^{h_1}\cdots q_t^{h_t}(q_1^{\lfloor a_1/2\rfloor}\cdots q_t^{\lfloor a_t/2\rfloor})^2+\Delta=z^d,$$

where h_i is in $\{0,1\}$ for $i=1,\dots,t$. It follows from [Bérczes et al. 2025, Theorem 2.2] (which gives a fully explicit version of the Schinzel–Tijdeman theorem [1976]) that

$$\log d \ll \log|\Delta|.$$

Combined with (2-1) and (2-3), we derive that

$$\log|\Delta| \gg \log_*\log_*\log X,$$

which proves the first assertion of Theorem 1.1.

Recall that

$$\Delta := z^d - q_1^{a_1}\cdots q_t^{a_t} = \pm p_1^{b_1}\cdots p_s^{b_s},$$

with $b_s\ge 1$ and $P=p_s$. We argue as in the proof of [Shorey and Tijdeman 1986, Theorem 9.1]; see also [Bugeaud 2018, Theorem 6.13]. Set

$$B := \max\{a_1,\dots,a_t,b_1,\dots,b_s\}$$

and observe that

$$d\log z \ll B\log P. \tag{3-1}$$

Without any loss of generality, we can assume that d is prime and that d exceeds $q_1,\dots,q_t,p_1,\dots,p_s$. Since d is prime, it is not divisible by p_j for $j=1,\dots,s$, nor by q_i for $i=1,\dots,t$.

Let j be in $\{1,\dots,s\}$ with $b_j\ge 1$. By applying [Bugeaud 2018, Theorem 2.11] (extracted from [Yu 2007]) with

$$\alpha_n=z,\quad b_n=-d,\quad \{\alpha_1,\dots,\alpha_{n-1}\}=\{q_i : a_i\ge 1, 1\le i\le t\},\quad p=p_j,\quad B_n=d,$$

and $\delta=\frac{1}{2}$, we obtain

$$b_j \ll \frac{p_j}{(\log p_j)^2}\max\Big\{(\log z)(\log(p_jd)),\frac{B}{d}\Big\}.$$

Arguing similarly to bound $a_1,\dots,a_t$, we get that there exists C_4 such that

$$\max\{a_1,\dots,a_t\} \ll C_4^s\max\Big\{(\log p_1)\cdots(\log p_s)(\log z)(\log d),\frac{B}{d}\Big\}.$$

This gives

$$d \ll \max\left\{\frac{p_s}{(\log p_s)^2}, C_4^s\right\}$$

or

$$B \ll (\log z)(\log d)\max\left\{\frac{p_s}{\log p_s}, C_4^s(\log p_1)\cdots(\log p_s)\right\}.$$

By (3-1), the latter inequality gives

$$d \ll (\log P)(\log d)\max\left\{\frac{p_s}{\log p_s}, C_4^s(\log p_1)\cdots(\log p_s)\right\}.$$

Consequently, we get

$$d \ll C_4^s \quad \text{or} \quad d \ll (\log d)C_4^s(\log s)^s,$$

and thus (recall that $P = p_s$)

$$\log d \ll s \log_* \log s \ll P\frac{\log_* \log P}{\log P}.$$

Combined with (2-2) and (2-4), this gives

$$\log_* \log_* \log X \ll P\frac{\log_* \log P}{\log P},$$

and the proof of Theorem 1.1 is complete.

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References

[Bérczes et al. 2025] A. Bérczes, Y. Bugeaud, K. Győry, J. Mello, A. Ostafe, and M. Sha, "Explicit bounds for the solutions of superelliptic equations over number fields", *Forum Math.* **37**:1 (2025), 135–158. MR

[Bugeaud 2018] Y. Bugeaud, *Linear forms in logarithms and applications*, IRMA Lectures in Mathematics and Theoretical Physics **28**, European Mathematical Society, Zürich, 2018. MR

[Bugeaud 2025] Y. Bugeaud, "On the difference between squares and integral S-units", *Port. Math.* (online publication June 2025).

[Bugeaud and Győry 1996] Y. Bugeaud and K. Győry, "Bounds for the solutions of Thue–Mahler equations and norm form equations", *Acta Arith.* **74**:3 (1996), 273–292. MR

[Bugeaud et al. 2018] Y. Bugeaud, J.-H. Evertse, and K. Győry, "S-parts of values of univariate polynomials, binary forms and decomposable forms at integral points", *Acta Arith.* **184**:2 (2018), 151–185. MR

[Gelfond 1935] A. Gelfond, "Sur les approximations des nombres transcendants par des nombres algebriques", *C. R. Acad. Sci. URSS* **1935**:2 (1935), 177–179.

[Győry and Pintér 2008] K. Győry and A. Pintér, "Polynomial powers and a common generalization of binomial Thue–Mahler equations and S-unit equations", pp. 103–119 in *Diophantine equations*, edited by N. Saradha, Tata Inst. Fund. Res. Stud. Math. **20**, Tata Inst. Fund. Res., Mumbai, 2008. MR

[Győry and Yu 2006] K. Győry and K. Yu, "Bounds for the solutions of S-unit equations and decomposable form equations", *Acta Arith.* **123**:1 (2006), 9–41. MR

[Győry et al. 2004] K. Győry, I. Pink, and A. Pintér, "Power values of polynomials and binomial Thue–Mahler equations", *Publ. Math. Debrecen* **65**:3-4 (2004), 341–362. MR

[Mahler 1953] K. Mahler, "On the greatest prime factor of $ax^m + by^n$", *Nieuw Arch. Wisk.* (3) **1** (1953), 113–122. MR

[Mihăilescu 2004] P. Mihăilescu, "Primary cyclotomic units and a proof of Catalan's conjecture", *J. Reine Angew. Math.* **572** (2004), 167–195. MR

[Ribenboim 1994] P. Ribenboim, *Catalan's conjecture*: *are* 8 *and* 9 *the only consecutive powers*?, Academic, Boston, MA, 1994. MR

[Schinzel and Tijdeman 1976] A. Schinzel and R. Tijdeman, "On the equation $y^m = P(x)$", *Acta Arith.* **31**:2 (1976), 199–204. MR

[Shorey and Tijdeman 1986] T. N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge Tracts in Mathematics **87**, Cambridge Univ. Press, 1986. MR

[Thue 1909] A. Thue, "Über Annäherungswerte algebraischer Zahlen", *J. Reine Angew. Math.* **135** (1909), 284–305. MR

[Tijdeman 1973] R. Tijdeman, "On integers with many small prime factors", *Compositio Math.* **26** (1973), 319–330. MR

[Tijdeman 1974] R. Tijdeman, "On the maximal distance between integers composed of small primes", *Compositio Math.* **28** (1974), 159–162. MR

[Yu 2007] K. Yu, "p-adic logarithmic forms and group varieties, III", *Forum Math.* **19**:2 (2007), 187–280. MR

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