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ABOUT SMALL EIGENVALUES OF THE WITTEN LAPLACIAN

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We study the low-lying eigenvalues of the semiclassical Witten Laplacian associated to a Morse function φ. Compared to previous works we allow general distributions of critical values of φ, for instance allowing all the local minima to be absolute. The motivation comes from metastable dynamics described by the Kramers–Smoluchowski equation.

1. Introduction

1A. *Motivation.* The Witten Laplacian, Δ_φ was introduced by Witten [1982] to give an analytic proof of Morse inequalities. Its study led to many mathematical developments, most notably the Helffer–Sjöstrand theory [1985] of potential wells in the semiclassical limit. It is defined by twisting the operator d (acting on forms) by a Morse function φ:

$$\Delta_\varphi := d_\varphi^* d_\varphi + d_\varphi d_\varphi^*, \quad d_\varphi := e^{-\varphi/h} h d e^{\varphi/h}. \tag{1-1}$$

It takes a simple form on functions and for the Euclidean metric on $\mathbb{R}^d$ we then have

$$\Delta_\varphi = -h^2\Delta + |\partial_x\varphi|^2 - h\Delta\varphi. \tag{1-2}$$

Even in that case using the action on 1-forms is highly beneficial — see [Michel and Zworski 2018] for an introduction in the simple one-dimensional setting.

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More recently the Witten Laplacian appeared in quantitative studies of metastability for kinetic equations — see for instance [Hérau and Nier 2004; Helffer, Klein and Nier 2004; Hérau, Hitrik and Sjöstrand 2011; Di Gesù, Lelièvre, Le Peutrec and Nectoux 2017].

Other interesting developments also include connecting the "Arrhenius rates" (exponential widths S of small eigenvalues λ_h in (1-8)) with barcodes of the Morse–Barannikov complex in [Le Peutrec, Nier and Viterbo 2013] and showing that (in the case of compact manifolds) the eigenvalues of the Witten Laplacian converge, as $h \to 0$, to the Ruelle resonances of the gradient flow of φ in [Dang and Rivière 2017].

This paper continues the study of the Witten Laplacian by considering functions φ with *general* distributions on critical values, in particular functions with several equal minima and equal values at saddle points. (In works related to Morse theory it is natural to assume that all critical values are distinct.) As emphasized in [Michel and Zworski 2018] such functions lead to interesting effective dynamics for the Kramers–Smoluchowski equation (1-6) — see Figure 2 and Section 1B. This, and more general situations in which equal critical values are allowed (see Figure 4 for a schematic illustration of an allowed landscape), leads to new subtle difficulties.

To explain metastable dynamics consider a particle evolving in an energy landscape φ and submitted to random forces. The position X_t of such a particle at time t satisfies the over-damped Langevin equation

$$\dot{X}_t = -2\nabla\varphi(X_t) + \sqrt{2h}\,\dot{B}_t, \tag{1-3}$$

where h is the temperature of the system and B_t is a Brownian force. This equation appears for instance in physics to describe the microscopic evolution of a charged gas assuming the mass of the particles is negligible.

Assuming that the potential φ has several wells, a particle starting at a local minimum of the function φ can, due to the presence of the random force, move over a saddle point and reach another energy well — see Figure 1 for a schematic illustration.

The celebrated Eyring–Kramers law describes the average time it takes to escape from a well, in the regime of low temperature, $h \to 0$. In his pioneering work, Kramers [1940] considered a one-dimensional model, see Figure 1, and predicted that the average transition time, τ_φ, from a local minimum A to the

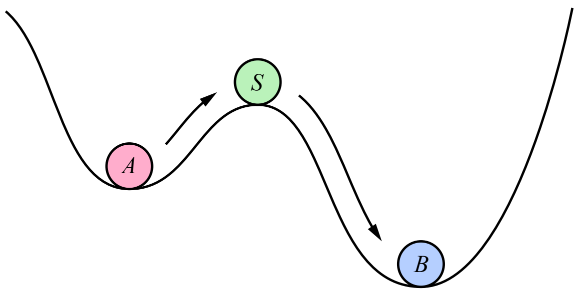

Figure 1. Metastable dynamics: random force allows a state localized near one minimum A to reach another minimum B passing a saddle point (a local maximum in dimension 1).

nearest saddle point S is exponentially large with respect to h^{-1}:

$$\tau_\varphi \simeq a_\varphi e^{\kappa_\varphi/h}, \quad \kappa_\varphi = \varphi(S) - \varphi(A), \quad a_\varphi = 2\pi|\varphi''(A)\varphi''(B)|^{-1/2}. \tag{1-4}$$

Hence, for h small this average transition time is large and this explains the terminology of A being a *metastable state*. (Once we get past S, the transition time to B is bounded and hence τ_φ is effectively the transition time from the state A to the state B.)

The Eyring–Kramers law has important applications in which the trajectory (1-3) is used to implement computational algorithms. Roughly speaking it proceeds as follows: in order to compute some thermodynamical quantities

$$\mathbb{E}_\mu(f) = \int_{\mathbb{R}^d} f(x)\, d\mu(x) \tag{1-5}$$

associated with a measure μ and an observable f, we introduce a random dynamics X_t which is ergodic with respect to μ. We then use the Monte Carlo method to approximate $\mathbb{E}_\mu(f)$ by the long-time average of f along any trajectory — see [Lelièvre, Rousset and Stoltz 2010] for an introduction. In many situations $d\mu(x) = Z_h e^{-\varphi(x)/h}$ for some potential φ and the over-damped Langevin dynamics (1-3) can be used as X_t. The time needed for the process X_t to explore the whole space $\mathbb{R}^d$ (which ensures the validity of the Monte Carlo approximation method) is directly linked to the metastable properties discussed previously. Understanding this metastable behavior is then of interest if, for instance, we need to evaluate the stopping time or to accelerate the convergence.

The mathematical proof of Eyring–Kramers law in a generic setting was first obtained by a potential-theory approach in [Bovier, Gayrard and Klein 2005] and then by semiclassical methods in [Helffer, Klein and Nier 2004]. The semiclassical point of view and connection to the Witten Laplacian can be seen by considering the Langevin equation (1-3) at the macroscopic level. In that case statistical distributions $\rho(t,x)$ of particles are governed by the Kramers–Smoluchowski equation

$$\partial_t \rho - h\Delta\rho - 2\,\mathrm{div}(\rho\nabla\varphi) = 0. \tag{1-6}$$

This is equivalent to

$$h\partial_t\tilde{\rho} + \Delta_\varphi\tilde{\rho} = 0, \quad \tilde{\rho} := e^{\varphi/h}\rho,$$

where Δ_φ is the Witten Laplacian (1-2) associated to φ. In view of (1-1), Δ_φ is nonnegative and under a confining assumption on the function φ, it has a nontrivial kernel corresponding to the global equilibrium of (1-6). (Confining assumption means that φ grows fast enough so that $e^{-\varphi/h} \in L^2$.) As a consequence, the behavior of $\tilde{\rho}$ when $t \to \infty$ is determined by the small eigenvalues of Δ_φ. In particular, any state associated to a small eigenvalue is stable for exponentially long times. These are the metastable states, and the inverses of the corresponding eigenvalues yield their lifetimes. Helffer, Klein and Nier [2004] obtained a full description of the small eigenvalues of the Witten Laplacian in a general setting. For the Kramers–Smoluchowski equation, their result implies that if the initial probability distribution ρ_0 belongs to $L^2(e^{2\varphi/h}\,dx)$, then the solution ρ of (1-6) converges exponentially fast to the equilibrium probability distribution $c_h^{-2}e^{-2\varphi/h}$ (where c_h is a normalizing factor)

$$\|\rho(t)-c_h^{-2}e^{-2\varphi/h}\|_{L^2(e^{2\varphi/h}\,dx)} \le e^{-\lambda_h t/h}\|\rho_0\|_{L^2(e^{2\varphi/h}\,dx)}. \tag{1-7}$$

Moreover, the rate of convergence

$$\lambda_h/h = b(h)e^{-2S/h}, \quad \lambda_h := \min \sigma(\Delta_\varphi)\setminus\{0\}, \tag{1-8}$$

is described by the Eyring–Kramers law, that is:

- S is the biggest height a particle has to pass in order to reach the unique global minimum.
- The prefactor $b(h)$ has an asymptotic expansion with respect to the parameter h, $b(h)\sim\sum_k b_k h^k$ and its leading term is given by an explicit formula in terms of the Hessian of φ.

More precisely, the assumptions made in [Helffer, Klein and Nier 2004] imply that there exist a unique minimum m and a unique saddle point s of φ such that $S=\varphi(s)-\varphi(m)$. Then, the leading term of $b(h)$ is

$$b_0 = \frac{|\mu_1(s)|}{\pi}\sqrt{\frac{\det \operatorname{Hess}(\varphi)(m)}{|\det \operatorname{Hess}(\varphi)(s)|}}, \tag{1-9}$$

where $\mu_1(s)$ denotes the negative eigenvalue of $\operatorname{Hess}(\varphi)(s)$. In the case of a double well, this formula is exactly the one predicted by Kramers [1940]. In view of (1-7) the transmission time is approximately the inverse of λ_h of (1-8). Hence the result of [Helffer, Klein and Nier 2004] is in agreement with (1-4). (Note that in dimension 1, $\varphi''(s)=\mu_1(s)$.)

The method developed in [Helffer, Klein and Nier 2004] to compute the small eigenvalues of the Witten Laplacian was successfully used on bounded domains in [Helffer and Nier 2006; Le Peutrec 2010] and in the study of semiclassical random walks [Bony, Hérau and Michel 2015].

The range of potential φ covered by these papers does not include many cases which are important in practice. Roughly speaking, Helffer, Klein and Nier [2004] make an assumption on the relative position of minima and saddle points that ensures that the small eigenvalues are all of different size. Among the limitations of this assumption is the fact that the potential φ cannot have saddle points or minima with the same value. In many physical applications the energy landscape may not satisfy that assumption. Also, the energy potential may have symmetries which again are not allowed by the assumptions in [Helffer, Klein and Nier 2004]. For instance this is the case of some homogeneous systems such as Lennard-Jones clusters — see [Wales 2006] for an example and a discussion.

The aim of this paper is to study the spectral properties of Δ_φ in the case where φ is a general Morse function without restrictions on the relative positions of the critical values.

1B. *An example.* A motivating example is given by $\varphi:\mathbb{R}^d\to\mathbb{R}$ which has n_0 minima all at the same level and n_1 saddle points all at the same level — see Figure 2, where the x represent minima and the o local maxima. Denote by $S=\varphi(s)-\varphi(m)$ the difference of the value at the saddle points and at the minima. To simplify the setting further, we assume also that the function $\operatorname{Hess}(\varphi)(x)$ has eigenvalues ±1 when x belongs to the set of minima and saddle points.

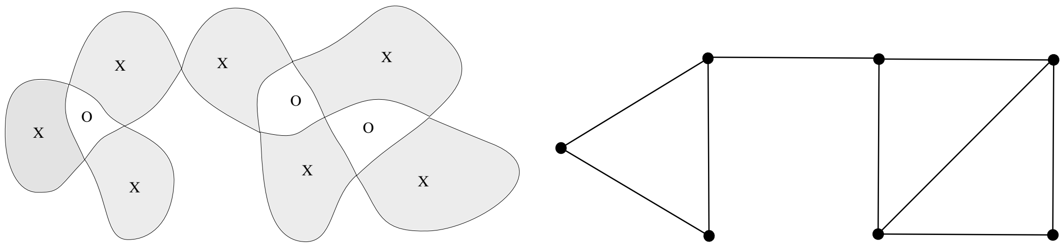

Figure 2. Left: the sublevel set $\{\varphi < \sigma\}$ (shaded region) associated to a potential φ having a unique saddle value σ. The x's represent local minima, the o's, local maxima. Right: the graph associated to the potential on the left.

This case is not allowed under the assumptions of [Helffer, Klein and Nier 2004] yet it displays some interesting phenomena. More precisely, in the very simplified case discussed in this section, a consequence of Theorem 7.1 below is the following:

Theorem 1.1. *Under the assumptions of this subsection, there exist $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, Δ_φ has exactly n_0 eigenvalues λ_k, $k = 1, \dots, n_0$, in the interval $[0, \epsilon_0 h]$. The lowest eigenvalue is $\lambda_1 = 0$ and*

$$\lambda_k = h b_k(h) e^{-2S/h}, \quad k = 2, \dots, n_0.$$

The prefactors $b_k(h)$ satisfy $b_k(h) \sim \sum_{j=0}^{\infty} h^j b_{k,j}$ and the terms $b_{k,0}$ are given by the nonzero eigenvalues of the graph Laplacian for the graph $\mathcal{G}$ whose vertices are the minima of φ and whose edges are the saddle points joining two minima (see Figure 2).

In terms of the Kramers–Smoluchowski equation (1-6), Theorem 1.1 exhibits metastable states whose lifetimes (given by the inverse of the above eigenvalues) are described by the graph $\mathcal{G}$. At the level of particles, these new rules of computation can be understood as follows. Since all the minima are at the same level, the equilibrium state is equidistributed among all the minima. Moreover, since all the saddle points are at the same level, an ergodic trajectory of (1-3) will visit all the minima in the same time scale, by traveling along the edges of the graph $\mathcal{G}$. Hence, the effective long-time dynamics of the Kramers–Smoluchowski equation is given by the heat equation for the graph Laplacian of $\mathcal{G}$ — see [Michel and Zworski 2018, Theorem 3].

Earlier results, in dimension 1 and for finite times, on effective dynamics were obtained in [Peletier, Savaré and Veneroni 2012] using Γ-convergence, in [Herrmann and Niethammer 2011] using Wasserstein gradient flows and in [Evans and Tabrizian 2016]. We also remark that the same graph Laplacian was constructed in [Landim, Misturini and Tsunoda 2015] in a discrete setting.

Under our special assumptions the coefficients b_k do not depend on the second derivative of φ as in the usual case. In the more general case of arbitrary Hessians, $\mathcal{G}$ has to be replaced by a weighted graph with weights depending on the Hessians in an explicit way — see Theorem 7.1.

To motivate objects introduced in the next section, we now discuss what happens if we modify the potential φ in the following way: suppose that φ has the structure shown in Figure 2 but one of the minimal

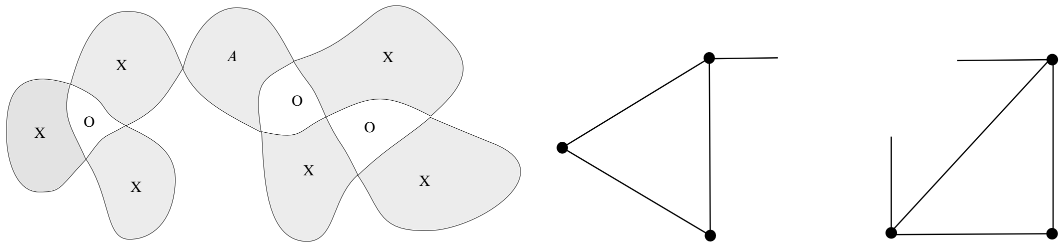

Figure 3. Left: the sublevel set $\{\varphi < \sigma\}$ (shaded region) associated to a potential φ having a unique saddle value σ. The x's represent local minima, the o's, local maxima. Right: the two hypergraphs associated to the potential on the left (the missing vertex corresponds to the minimum A).

values is made higher or lower. In Figure 3, the modified minimum is denoted by A. Then, we can associate to this potential the two hypergraphs corresponding to minima at the same level and linked by a saddle value (see Figure 3). If A is an absolute minimum, then equilibrium distribution is concentrated in A and the prefactor $b_k(h)$ will be given by the smallest nonzero eigenvalue of the two hypergraphs introduced above (roughly speaking this represents the maximum time needed to reach A). In the opposite case, A is no longer a global minimum and the equilibrium state is uniformly distributed among all the absolute minima. In order to visit each site of the equilibrium state, a particle will necessarily pass through the point A. This heuristic explains why the computation of the prefactor $b_k(h)$ will involve a more complicated procedure describing the interaction between the two hypergraphs via the well A.

The main contribution of this paper is to describe these phenomena in a quantitative way.

2. Framework and results

Let X be either $\mathbb{R}^d$ or a compact manifold of dimension d without boundary and let $\varphi : X \to \mathbb{R}$ be a smooth Morse function. Consider the semiclassical Witten Laplacian associated to φ:

$$\Delta_\varphi = -h^2\Delta + |\nabla\varphi|^2 - h\Delta\varphi, \tag{2-1}$$

where $h \in]0, 1]$ denotes the semiclassical parameter.

If X is a compact manifold, the operator Δ_φ is selfadjoint with domain $H^2(X)$ and its resolvent is compact. In the case $X = \mathbb{R}^d$ we make the additional assumption that there exist $C > 0$ and a compact $K \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d \setminus K$, we have

$$|\nabla\varphi(x)| \geq \frac{1}{C}, \quad |\mathrm{Hess}(\varphi(x))| \leq C|\nabla\varphi|^2 \quad \text{and} \quad \varphi(x) \geq C|x|. \tag{2-2}$$

Then, Δ_φ is essentially selfadjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and thanks to (2-2), there exist $h_0 > 0$ and $c_0 > 0$ such that for all $h \in]0, h_0]$, we have

$$\sigma_{\mathrm{ess}}(\Delta_\varphi) \subset [c_0, \infty[.$$

In both situations X compact or $X = \mathbb{R}^d$, it is well known that Δ_φ is nonnegative. Hence $\sigma(\Delta_\varphi) \subset [0, \infty[$ and it follows from the above remarks that $\sigma(\Delta_\varphi) \cap [0, c_0[$ is made of eigenvalues with no accumulation point except maybe c_0. Moreover $e^{-\varphi/h}$ is clearly in the kernel of Δ_φ and belongs to $L^2(\mathbb{R}^d)$ thanks to (2-2), so that the lowest eigenvalue of Δ_φ is clearly 0.

Since φ is a Morse function (and thanks to assumption (2-2) in the case $X = \mathbb{R}^d$), the set $\mathcal{U}$ of critical points is finite. In the following, for $p = 0, \dots, d$, we will denote by $\mathcal{U}^{(p)}$ the set of critical points of φ of index p. Hence, $\mathcal{U}^{(0)}$ is the set of minima and $\mathcal{U}^{(1)}$ the set of saddle points of φ. Throughout the paper, we will write $n_j = \#\mathcal{U}^{(j)}$.

From the pioneering work [Witten 1982], it is well known that for small h, there is a correspondence between the small eigenvalues of Δ_φ and the critical points of φ. More precisely, by standard localization arguments one can show that there exists $\epsilon_0 > 0$ such that for $h > 0$ small enough, Δ_φ has exactly n_0 eigenvalues in the interval $[0, \epsilon_0 h]$, which we denote by $0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_{n_0}$. This result is easily proved in [Cycon, Froese, Kirsch and Simon 1987] with $\epsilon_0 h$ replaced by $h^{3/2}$. The proof with $\epsilon_0 h$ can be found in [Helffer and Sjöstrand 1985, Proposition 1.7] (see also [Michel and Zworski 2018, Proposition 1] for a self-contained proof). Moreover, these eigenvalues are actually exponentially small; that is, they live in an interval $[0, e^{-C/h}]$ for some $C > 0$ (see [Helffer 1988] for a proof). From a topological point of view, this information (together with the equivalent estimates for the Witten Laplacian $\Delta_\varphi^{(p)}$ acting on p-forms) is sufficient to establish a correspondence between the small eigenvalues of $\Delta_\varphi^{(p)}$ and the critical points of φ of index p (this was the key point in the Witten's proof of Morse inequalities). However, for applications to the description of metastable dynamics, it is important to get some accurate description of the λ_j. Our main theorem will give some asymptotic of these eigenvalues for any Morse function φ, without any assumption on the relative position of minimal and saddle values of φ.

Before going further, we introduce notation used in this paper. For $x_0 \in X$ and $r > 0$, introduce the geodesic ball $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$.

Throughout, we will say that $\boldsymbol{s}$ is a saddle point if it is a critical point of index 1.

Given $a(h), b(h) > 0$, two functions of the semiclassical parameter, we say that $a(h) \asymp b(h)$ if there exists some constant $c_1, c_2 > 0$ such that for all $h > 0$ small we have $c_1 b(h) \le a(h) \le c_2 b(h)$. We say that a family of vectors $(a(h))_{h \in]0,1]}$ in a normed vector space V admits a classical expansion if there exists a sequence of vectors $(a_n)_{n \in \mathbb{N}}$ independent of h and such that for all $N \in \mathbb{N}$, there exists some constant $C_N > 0$ such that

$$\Big\| a(h) - \sum_{n=0}^{N} h^n a_n \Big\|_V \le C_N h^{N+1} \quad \text{for all } h \in]0, 1].$$

We set $a(h) \sim \sum_{n=0}^{\infty} h^n a_n$.

As we shall see later, we will have to analyze carefully some finite-dimensional matrices which are strongly related to the critical points of φ. Given any subsets $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{U}$, it will be convenient to introduce the finite-dimensional vector space $\mathscr{F}(\mathcal{B}_j)$ of real-valued functions on $\mathcal{B}_j$. We shall then denote by $\mathscr{M}(\mathcal{B}_1, \mathcal{B}_2)$ the vector space of linear operators from $\mathscr{F}(\mathcal{B}_1)$ into $\mathscr{F}(\mathcal{B}_2)$.

2A. ***Labeling of minima.*** Let us now recall the general labeling of minima introduced in [Helffer, Klein and Nier 2004] and generalized in [Hérau, Hitrik and Sjöstrand 2011]. The main ingredient is the notion of separating saddle point, which is defined as follows. Given a saddle point $\boldsymbol{s}$ of φ, and $r > 0$ small

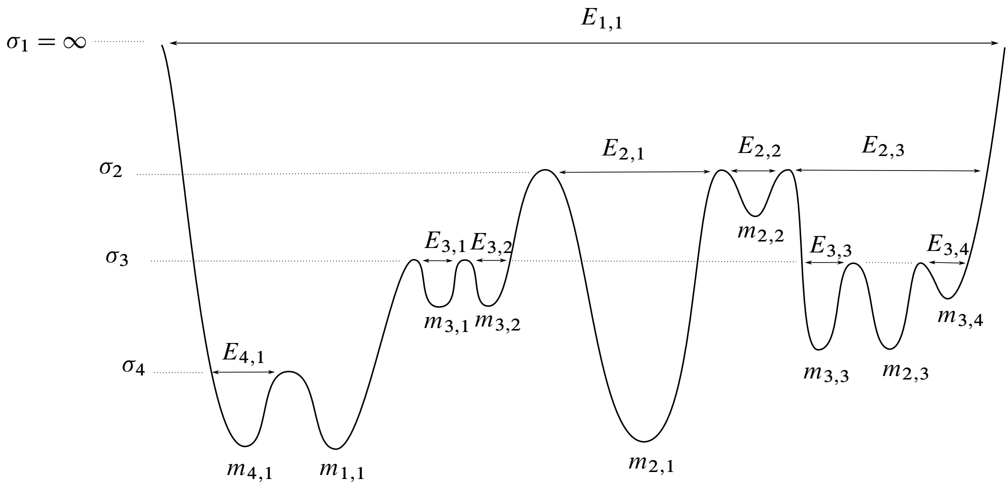

Figure 4. Labeling procedure.

enough, the set

$$\{x \in B(\boldsymbol{s}, r) : \varphi(x) < \varphi(\boldsymbol{s})\}$$

has exactly two connected components $C_j(\boldsymbol{s}, r)$, $j = 1, 2$. The following definition is taken from [Hérau, Hitrik and Sjöstrand 2011, Definition 4.1].

Definition 2.1. We say that $\boldsymbol{s} \in X$ is a separating saddle point (ssp) if it is a saddle point and if $C_1(\boldsymbol{s}, r)$ and $C_2(\boldsymbol{s}, r)$ are contained in two different connected components of $\{x \in X : \varphi(x) < \varphi(\boldsymbol{s})\}$. We will denote by $\mathcal{V}^{(1)}$ the set of separating saddle points.

We say that $\sigma \in \mathbb{R}$ is a separating saddle value (ssv) if it is of the form $\sigma = \varphi(\boldsymbol{s})$ with $\boldsymbol{s} \in \mathcal{V}^{(1)}$. We denote by $\underline{\Sigma} = \varphi(\mathcal{V}^{(1)})$ the set of separating saddle values.

We say that $E \subset X$ is a critical component if there exists $\sigma \in \underline{\Sigma}$ such that E is a connected component of $\{\varphi < \sigma\}$ and if $\partial E \cap \mathcal{V}^{(1)} \neq \varnothing$. We denote by $\mathscr{C}$ the set of critical components.

Let us now describe the labeling procedure of [Hérau, Hitrik and Sjöstrand 2011]. Since φ is a Morse function, it has finitely many critical points and so $\underline{\Sigma}$ is finite. We denote by $\sigma_2 > \sigma_3 > \cdots > \sigma_N$ its elements and for convenience we also introduce a fictive infinite saddle value $\sigma_1 = +\infty$ and write $\Sigma = \underline{\Sigma} \cup \{\sigma_1\}$. Starting from σ_1, we will recursively associate to each σ_i a finite family of local minima $(\boldsymbol{m}_{i,j})_j$ and a finite family of critical components $(E_{i,j})_j$ (see Figure 4):

- Let $X_{\sigma_1} = \{x \in X : \varphi(x) < \sigma_1 = \infty\} = X$. We let $\boldsymbol{m}_{1,1}$ be any global minimum of φ (not necessarily unique) and $E_{1,1} = X$.

- Next we consider $X_{\sigma_2} = \{x \in X : \varphi(x) < \sigma_2\}$. This is the union of its finitely many connected components. Exactly one of these components contains $\boldsymbol{m}_{1,1}$ and the other components are denoted by $E_{2,1}, \ldots, E_{2,N_2}$. In each component $E_{2,j}$, we pick up a point $\boldsymbol{m}_{2,j}$ which is a global minimum of $\varphi_{|E_{2,j}}$.

- Suppose now that the families $(\boldsymbol{m}_{k,j})_j$ and $(E_{k,j})_j$ have been constructed until rank $k = i - 1$. The set $X_{\sigma_i} = \{x \in X : \varphi(x) < \sigma_i\}$ has again finitely many connected components and we label by $E_{i,j}$,

$j = 1, \dots, N_i$, those that do not contain any $\boldsymbol{m}_{k,l}$ with $k < i$. In each $E_{i,j}$ we pick a point $\boldsymbol{m}_{i,j}$ which is a global minimum of $\varphi_{|E_{i,j}}$. Observe that for all $i \geq 2$, the components $E_{i,j}$ are all critical.

We run the procedure until all the minima have been labeled.

Remark 2.2. The above labeling satisfies the following property. For any $\sigma_i \in \Sigma$ and any connected component A_i of $\{\varphi < \sigma_i\}$, there exists a unique (k, l) such that $k \leq i$ and $\boldsymbol{m}_{k,l} \in A_i$.

Proof. Let us start with the existence part of the result. If A_i is one of the $E_{i,j}$ for some j, then take $k = i$ and $l = j$. Otherwise, this means that in the labeling procedure, A_i already contained a minimum $\boldsymbol{m}_{k,l}$ with $k < i$.

Let us prove the uniqueness part. Assume that $\boldsymbol{m}_{k,l}, \boldsymbol{m}_{k',l'} \in A_i$ with $k \leq k' \leq i$. Then $A_i \cap E_{k',l'} \neq \varnothing$ and since A_i is a connected component of $\{\varphi < \sigma_i\}$ with $\sigma_i \leq \sigma_{k'}$ it follows that $A_i \subset E_{k',l'}$. Since $\boldsymbol{m}_{k,l} \in A_i$, it follows that $\boldsymbol{m}_{k,l} \in E_{k',l'}$ which is impossible unless $(k, l) = (k', l')$. □

Using the above labeling, Hérau, Hitrik and Sjöstrand [2011] made some significant progress (in the more general situation of Kramers–Fokker–Planck operators, but this applies to Witten Laplacian). First, they showed in Theorem 7.1 of that paper that the exponentially small eigenvalues $(\lambda_{\boldsymbol{m}}(h))_{\boldsymbol{m} \in \mathcal{U}^{(0)}}$ of Δ_φ (indexed by the sequence of local minima) satisfy $\lambda_{\boldsymbol{m}}(h) \asymp h e^{-2S(\boldsymbol{m})/h}$ for the sequence of Arrhenius numbers $(S(\boldsymbol{m}))_{\boldsymbol{m} \in \mathcal{U}^{(0)}}$ defined by $S(\boldsymbol{m}_{i,j}) = \sigma_i - f(\boldsymbol{m}_{i,j})$ with the above notation. However, their method does not work to prove that $h^{-1} \lambda_{\boldsymbol{m}}(h) e^{2S(\boldsymbol{m})/h}$ admits a limit when $h \to 0$. In order to compute the asymptotic expansion of the eigenvalues $\lambda_{\boldsymbol{m}}(h)$, they need to make some additional assumption on the interaction between minima and saddle points (see Assumption 5.1 in [Hérau, Hitrik and Sjöstrand 2011]). This hypothesis, which is a generalization of the one made in [Helffer, Klein and Nier 2004], can be formulated as follows with the notation of the preceding section:

Generic Assumption. For all $i = 1, \dots, N$, $j = 1, \dots, N_i$, the following hold true:

(i) $\boldsymbol{m}_{i,j}$ is the unique global minimum of the application $\varphi_{|E_{i,j}}$.

(ii) If E is a connected component of $\{\varphi < \sigma_i\}$ such that $E \cap \mathcal{V}^{(1)} \neq \varnothing$, there exists a unique $\boldsymbol{s} \in \mathcal{V}^{(1)}$ such that $\varphi(\boldsymbol{s}) = \sup \varphi(E \cap \mathcal{V}^{(1)})$. In particular, $\varphi^{-1}(]-\infty, \varphi(\boldsymbol{s})[) \cap E$ is the union of exactly two different connected components.

Throughout the paper, we denote this assumption by (GA).

Under this assumption, there exists a bijection between $\mathcal{U}^{(0)}$ and $\mathcal{V}^{(1)} \cup \{\boldsymbol{s}_1\}$, where $\boldsymbol{s}_1$ is a fictive saddle point associated to $\sigma_1 = \infty$ and for which by convention $\varphi(\boldsymbol{s}_1) = \infty$. Using this one-to-one correspondence, the authors exhibit some labeling $\mathcal{U}^{(0)} = \{\boldsymbol{m}_1, \dots, \boldsymbol{m}_{n_0}\}$ and $\mathcal{V}^{(1)} \cup \{\boldsymbol{s}_1\} = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_{n_0}\}$ such that the small eigenvalues $\lambda_i(h)$ are of the form $h b_i(h) e^{-2S_i/h}$ with $S_i = \varphi(\boldsymbol{s}_i) - \varphi(\boldsymbol{m}_i)$. Moreover, they prove that the $b_i(h)$ have a classical expansion and compute the leading term of this expansion; see [Hérau, Hitrik and Sjöstrand 2011, Theorem 5.10].

As it is stated above, (GA) is not exactly Assumption 5.1 stated in [Hérau, Hitrik and Sjöstrand 2011]. Indeed, it is supposed in that paper that (ii) holds true only for E being a critical component. However, as indicated by the anonymous referee, we can easily construct some function φ satisfying this assumption for which there is no bijection between $\mathcal{U}^{(0)}$ and $\mathcal{V}^{(1)}$. To see this, first consider in dimension 1 a potential φ with four minima m_j, $j = 1, \dots, 4$, and three saddle points s_j, $j = 1, \dots, 3$, such that $m_1 < s_1 < m_2 < s_2 < m_3 < s_3 < m_4$ and such that $\varphi(m_1) < \varphi(m_4) < \varphi(m_2) = \varphi(m_3)$ and $\varphi(s_1) = \varphi(s_2) < \varphi(s_3)$. Since the component of $\{\varphi < \varphi(s_3)\}$ containing m_1 is not critical, this function satisfies Assumption 5.1 in [Hérau, Hitrik and Sjöstrand 2011]. It doesn't satisfy (GA) as stated above. In higher dimensions, one can easily generalize this construction to obtain potentials satisfying Assumption 5.1 in [Hérau, Hitrik and Sjöstrand 2011], with a fixed number of minima and an arbitrarily large number of separating saddle points (think for instance of many saddle points between the well containing m_1 and the well containing m_2). This shows that Assumption 5.1 is not sufficient to ensure a bijection between minima and separating saddle points.

Let us emphasize that the above remark doesn't affect the rest of the work done in [Hérau, Hitrik and Sjöstrand 2011], where we can easily use the above corrected version of Assumption 5.1.

Let us observe that the Generic Assumption allows some degeneracy in the sequence (S_j); that is, there may exist j such that $S_j = S_{j+1}$. However, (GA) remains restrictive for the following reasons:

- It permits only potentials φ for which $\mathcal{U}^{(0)}$ and $\mathcal{V}^{(1)} \cup \{\boldsymbol{s}_1\}$ have the same cardinality.
- The eventual degenerate heights are associated to weakly interacting eigenstates in the following sense. Assume for instance that $S_j = S_{j+1}$ for some $j = 1, \dots, n_0 - 1$ and modify slightly the function φ near the minimum $\boldsymbol{m}_j$. Then the coefficient b_j is modified, whereas the classical expansion of b_{j+1} remains unchanged.

Figures 6 and 7 below present some examples of potentials where (GA) is not satisfied. These examples, as well as an example in higher dimensions, are discussed in detail in Section 7C.

In the present paper, we obtain an asymptotic expansion for the $\lambda_i(h)$ for general Morse functions φ without any additional assumptions on the relative position of minima and ssp's.

2B. *Main result.* In order to state our main result, we introduce some notation that will be used throughout the paper. First, using the above labeling, we define $\boldsymbol{\sigma} : \mathcal{U}^{(0)} \to \Sigma$ by $\boldsymbol{\sigma}(\boldsymbol{m}_{i,j}) = \sigma_i$ and $S : \mathcal{U}^{(0)} \to]0, +\infty]$ by $S(\boldsymbol{m}) = \boldsymbol{\sigma}(\boldsymbol{m}) - \varphi(\boldsymbol{m})$. We let $\mathcal{S} = S(\mathcal{U}^{(0)})$; then with the notation of the preceding section, we have

$$\mathcal{S} = \{\sigma_i - \varphi(\boldsymbol{m}_{i,j}) : i = 1, \dots, N,\ j = 1, \dots, N_i\}. \tag{2-3}$$

Throughout the paper, we denote by $\underline{\boldsymbol{m}} = \boldsymbol{m}_{1,1}$ the (not necessarily unique) absolute minimum of φ that was chosen at the first step of the labeling procedure, and we let

$$\underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0)} \setminus \{\underline{\boldsymbol{m}}\}. \tag{2-4}$$

Using again the above labeling, we can associate a critical component to any local minimum. More precisely, we define

$$E : \mathcal{U}^{(0)} \to \mathscr{C} \cup \{X\} \tag{2-5}$$

by $E(\boldsymbol{m}_{i,j}) = E_{i,j}$. Observe that by definition, this application is injective. Using this map, we can associate to each minimum $\boldsymbol{m} \in \mathcal{U}^{(0)}$ a boundary set given by $\Gamma(\boldsymbol{m}) = \partial E(\boldsymbol{m})$. Thanks to the fact that φ is a smooth Morse function, for any $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$, the set $\Gamma(\boldsymbol{m})$ is a finite union of compact submanifolds of X of dimension $d-1$ with conic singularities at the saddle points. For our construction of quasimodes, we also need to introduce the set

$$H(\boldsymbol{m}) := \{\boldsymbol{m}' \in E(\boldsymbol{m}) \cap \mathcal{U}^{(0)} : \varphi(\boldsymbol{m}') = \varphi(\boldsymbol{m})\}. \tag{2-6}$$

Given $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$, we have $\boldsymbol{\sigma}(\boldsymbol{m}) = \sigma_i$ for some $i \geq 2$. Moreover, since $\sigma_{i-1} > \sigma_i$, there exists a unique connected component of $\{\varphi < \sigma_{i-1}\}$ that contains $\boldsymbol{m}$ (observe that this component is not necessarily critical). We denote that component by $E_-(\boldsymbol{m})$, and by

$$E_- : \underline{\mathcal{U}}^{(0)} \to \Omega(X) \tag{2-7}$$

the corresponding application, where $\Omega(X)$ is the collection of connected open subsets of X. Thanks to Remark 2.2, we know that for any $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$, there exists a unique $\boldsymbol{m}' \in E_-(\boldsymbol{m}) \cap \mathcal{U}^{(0)}$, denoted by $\hat{\boldsymbol{m}}(\boldsymbol{m})$, such that $\boldsymbol{\sigma}(\boldsymbol{m}') > \boldsymbol{\sigma}(\boldsymbol{m})$. In particular,

$$\text{for all } \boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}, \quad \varphi(\hat{\boldsymbol{m}}(\boldsymbol{m})) \leq \varphi(\boldsymbol{m}), \tag{2-8}$$

and we denote by $\widehat{E}(\boldsymbol{m})$ the connected component of $\{\varphi < \sigma(\boldsymbol{m})\}$ containing $\hat{\boldsymbol{m}}(\boldsymbol{m})$. It holds additionally $\widehat{E}(\boldsymbol{m}) \subset E_-(\boldsymbol{m})$ and we can easily see that $\widehat{E}(\boldsymbol{m})$ is always a critical component. Throughout, we denote by

$$\widehat{E} : \underline{\mathcal{U}}^{(0)} \to \mathscr{C}, \tag{2-9}$$

$$\hat{\boldsymbol{m}} : \underline{\mathcal{U}}^{(0)} \to \mathcal{U}^{(0)} \tag{2-10}$$

the corresponding applications. The fact that the inequality in (2-8) is large or strict plays an important role in our analysis.

Definition 2.3. Let $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$. We say that $\boldsymbol{m}$ is of type I if $\varphi(\hat{\boldsymbol{m}}(\boldsymbol{m})) < \varphi(\boldsymbol{m})$. If $\varphi(\hat{\boldsymbol{m}}(\boldsymbol{m})) = \varphi(\boldsymbol{m})$, we say that $\boldsymbol{m}$ is of type II. We define

$$\mathcal{U}^{(0),\mathrm{I}} = \{\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)} : \boldsymbol{m} \text{ is of type I}\},$$
$$\mathcal{U}^{(0),\mathrm{II}} = \{\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)} : \boldsymbol{m} \text{ is of type II}\}.$$

We have clearly the disjoint union $\underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0),\mathrm{I}} \cup \mathcal{U}^{(0),\mathrm{II}}$.

Example 2.4. Let us compute the preceding object in the case of the potential φ represented in Figure 4. The results are presented in Figure 5.

- Let us start with the object associated to σ_2. By definition, $\widehat{E}(\boldsymbol{m}_{2,1}) = \widehat{E}(\boldsymbol{m}_{2,2}) = \widehat{E}(\boldsymbol{m}_{2,3}) = \widetilde{E}_2$, where $\widetilde{E}_2$ is the connected component of $\{\varphi < \sigma_2\}$ that contains $\boldsymbol{m}_{1,1}$. Then we have $\hat{\boldsymbol{m}}(\boldsymbol{m}_{2,1}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,2}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,3}) = \boldsymbol{m}_{1,1}$.

  Since $\varphi(\boldsymbol{m}_{1,1}) = \varphi(\boldsymbol{m}_{2,1}) < \varphi(\boldsymbol{m}_{2,3}) < \varphi(\boldsymbol{m}_{2,2})$, we know $\boldsymbol{m}_{2,1}$ is of type II, whereas $\boldsymbol{m}_{2,2}$ and $\boldsymbol{m}_{2,3}$ are of type I.

- Consider now the level σ_3. We have $E_-(\boldsymbol{m}_{3,1}) = E_-(\boldsymbol{m}_{3,2}) = \widetilde{E}_2$ and $E_-(\boldsymbol{m}_{3,3}) = E_-(\boldsymbol{m}_{3,4}) = E_{2,3}$. Therefore, $\widehat{E}(\boldsymbol{m}_{3,1}) = \widehat{E}(\boldsymbol{m}_{3,2}) = \widetilde{E}_3$, where $\widetilde{E}_3$ is the connected component of $\{\varphi < \sigma_3\}$ that contains $m_{1,1}$. Similarly, we have $\widehat{E}(\boldsymbol{m}_{3,3}) = \widehat{E}(\boldsymbol{m}_{3,4}) = \widetilde{E}_3'$, where $\widetilde{E}_3'$ is the connected component of $\{\varphi < \sigma_3\}$ that contains $m_{2,3}$. From these computations, it follows that $\hat{\boldsymbol{m}}(\boldsymbol{m}_{3,1}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{3,2}) = \boldsymbol{m}_{1,1}$ and

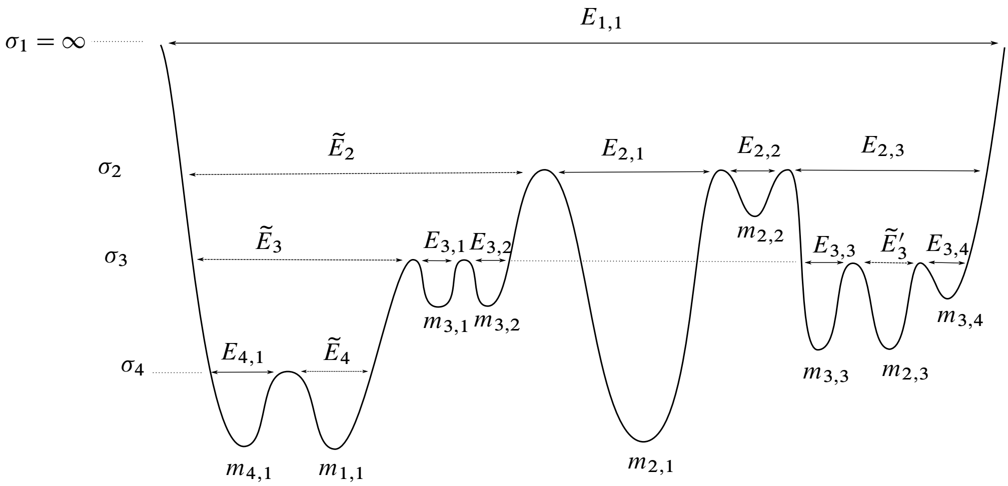

Figure 5. Computations of Example 2.4.

since $\varphi(\boldsymbol{m}_{1,1}) < \varphi(\boldsymbol{m}_{3,1}) = \varphi(\boldsymbol{m}_{3,2})$ it follows that $\boldsymbol{m}_{3,1}$ and $\boldsymbol{m}_{3,2}$ are both of type I. On the other hand, $\hat{\boldsymbol{m}}(\boldsymbol{m}_{3,3}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{3,4}) = \boldsymbol{m}_{2,3}$ and since $\varphi(\boldsymbol{m}_{2,3}) = \varphi(\boldsymbol{m}_{3,3}) < \varphi(\boldsymbol{m}_{3,4})$ it follows that $\boldsymbol{m}_{3,3}$ is of type II and $\boldsymbol{m}_{3,4}$ is of type I.

- Finally, $E_-(\boldsymbol{m}_{4,1}) = \widetilde{E}_3$, $\widehat{E}(\boldsymbol{m}_{4,1}) = \widetilde{E}_4$ as represented on Figure 5 and $\hat{\boldsymbol{m}}(\boldsymbol{m}_{4,1}) = \boldsymbol{m}_{1,1}$. Since $\varphi(\boldsymbol{m}_{1,1}) = \varphi(\boldsymbol{m}_{4,1})$, it follows that $\boldsymbol{m}_{4,1}$ is of type II.

The points of type II play an important role in our analysis. Given $\sigma \in \Sigma$, let $\Omega_\sigma = \Omega_\sigma^0 \cup \widehat{\Omega}_\sigma$, with

$$\Omega_\sigma^0 = \{E(\boldsymbol{m}) : \boldsymbol{m} \in \boldsymbol{\sigma}^{-1}(\sigma)\} \tag{2-11}$$

and $\widehat{\Omega}_\sigma$ be defined by $\widehat{\Omega}_\sigma = \varnothing$ if $\sigma = \sigma_1$ and

$$\widehat{\Omega}_\sigma = \{\widehat{E}(\boldsymbol{m}) : \boldsymbol{m} \in \boldsymbol{\sigma}^{-1}(\sigma) \cap \mathcal{U}^{(0),\mathrm{II}}\} \tag{2-12}$$

if $\sigma \in \underline{\Sigma}$.

Definition 2.5. We define an equivalence relation $\mathcal{R}$ on $\mathcal{U}^{(0)}$ by $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$ if and only if

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{m}) = \boldsymbol{\sigma}(\boldsymbol{m}') = \sigma, \\ \exists \omega_1, \dots, \omega_K \in \Omega_\sigma \text{ such that } \boldsymbol{m} \in \omega_1, \boldsymbol{m}' \in \omega_K \text{ and } \forall k = 1, \dots, K-1,\ \bar{\omega}_k \cap \bar{\omega}_{k+1} \neq \varnothing. \end{cases} \tag{2-13}$$

Throughout the paper, we denote by $\mathrm{Cl}(\boldsymbol{m})$ the equivalence class of $\boldsymbol{m}$ for the relation $\mathcal{R}$. Observe that since $\underline{\boldsymbol{m}}$ is the only minimum such that $\boldsymbol{\sigma}(\boldsymbol{m}) = \infty$, we have $\mathrm{Cl}(\underline{\boldsymbol{m}}) = \{\underline{\boldsymbol{m}}\}$.

Let us denote by $(\mathcal{U}_\alpha^{(0)})_{\alpha \in \mathcal{A}}$ the equivalence classes of $\mathcal{R}$ with $\mathcal{A}$ a finite set. We have evidently

$$\mathcal{U}^{(0)} = \bigsqcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha^{(0)}. \tag{2-14}$$

We need also to consider the set $\underline{\mathcal{A}}$ defined by $\underline{\mathcal{A}} = \mathcal{A} \setminus \{\underline{\alpha}\}$, where $\mathcal{U}_{\underline{\alpha}}^{(0)} = \{\underline{\boldsymbol{m}}\}$ is the equivalence class of the absolute minimum chosen for φ. Throughout, we will write $q_\alpha = \#\mathcal{U}_\alpha^{(0)}$. We will also use the following partition of $\mathcal{U}_\alpha^{(0)}$ for any $\alpha \in \underline{\mathcal{A}}$:

$$\mathcal{U}_\alpha^{(0),\mathrm{I}} := \mathcal{U}_\alpha^{(0)} \cap \mathcal{U}^{(0),\mathrm{I}}, \quad \mathcal{U}_\alpha^{(0),\mathrm{II}} := \mathcal{U}_\alpha^{(0)} \cap \mathcal{U}^{(0),\mathrm{II}}. \tag{2-15}$$

Proposition 2.6. *Let $\alpha \in \underline{\mathcal{A}}$. The applications $\boldsymbol{\sigma}$, E_-, $\widehat{E}$ and $\hat{\boldsymbol{m}}$ are constant on $\mathcal{U}_\alpha^{(0)}$.*

Proof. For $\boldsymbol{\sigma}$, it is a direct consequence of the definition. Suppose now that $\boldsymbol{m}, \boldsymbol{m}' \in \underline{\mathcal{U}}^{(0)}$ satisfy $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$ and $\boldsymbol{m} \neq \boldsymbol{m}'$. Then, $\boldsymbol{m}$ and $\boldsymbol{m}'$ belong to the same connected component of $\{\varphi \leq \sigma(\boldsymbol{m})\}$. Hence, the uniqueness part in the definition of E_- shows that $E_-(\boldsymbol{m}) = E_-(\boldsymbol{m}')$. Since $E_-(\boldsymbol{m}) = E_-(\boldsymbol{m}')$, the identity $\hat{\boldsymbol{m}}(\boldsymbol{m}) = \hat{\boldsymbol{m}}(\boldsymbol{m}')$ follows directly from the definition of $\hat{\boldsymbol{m}}$. This implies automatically $\widehat{E}(\boldsymbol{m}) = \widehat{E}(\boldsymbol{m}')$. $\square$

Thanks to the above proposition, given $\alpha \in \underline{\mathcal{A}}$, we will write respectively $\boldsymbol{\sigma}(\alpha)$, $E_-(\alpha)$, $\widehat{E}(\alpha)$ and $\hat{\boldsymbol{m}}(\alpha)$ instead of $\boldsymbol{\sigma}(\boldsymbol{m})$, $E_-(\boldsymbol{m})$, $\widehat{E}(\boldsymbol{m})$, $\hat{\boldsymbol{m}}(\boldsymbol{m})$ for some $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$.

Definition 2.7. We say that

- α is of type I if $\varphi(\hat{\boldsymbol{m}}(\alpha)) < \varphi(\boldsymbol{m})$ for all $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$,
- α is of type II if there exists $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ such that $\varphi(\hat{\boldsymbol{m}}(\alpha)) = \varphi(\boldsymbol{m})$.

Recall that the height function $S : \mathcal{U}^{(0)} \to \mathbb{R}$ and the set of heights $\mathcal{S} = S(\mathcal{U}^{(0)})$ were defined by (2-3) and above. For any $\alpha \in \mathcal{A}$, we let

$$\mathcal{S}_\alpha = S(\mathcal{U}_\alpha^{(0)}) \quad \text{and} \quad p(\alpha) = \#\mathcal{S}_\alpha. \tag{2-16}$$

There exist some integers $\nu_1^\alpha < \nu_2^\alpha < \cdots < \nu_{p(\alpha)}^\alpha$ such that

$$\mathcal{S}_\alpha = \{S_{\nu_1^\alpha}, \dots, S_{\nu_{p(\alpha)}^\alpha}\}.$$

In the theorem below, proved in Sections 5 and 6, we sum up in a rather vague way the description of the small eigenvalues that we obtained.

Theorem 2.8. *There exist $c > 0$ and some symmetric positive definite matrices $\mathcal{M}^\alpha$, $\alpha \in \mathcal{A}$, such that counted with multiplicity, we have $\sigma(\Delta_\varphi) \setminus \{0\} = \bigcup_{\alpha \in \mathcal{A}} \sigma(\mathcal{M}^\alpha)(1 + \mathcal{O}(e^{-c/h}))$, with*

$$\sigma(\mathcal{M}^\alpha) = \bigcup_{j=1}^{p(\alpha)} h e^{-2h^{-1} S_{\nu_j^\alpha}} \sigma(M^{\alpha,j})$$

for some symmetric positive definite matrices $M^{\alpha,j}$ having a classical expansion with invertible leading term given in Theorem 5.8. Moreover 0 is a multiplicity-1 eigenvalue.

Let us make a few comments on this theorem.

First, observe that since $M^{\alpha,j}$ has a classical expansion with invertible leading term $M_0^{\alpha,j}$, its eigenvalues $\zeta_r^{\alpha,j}$, $r = 1, \dots, r^{\alpha,j}$, have a classical expansion

$$\zeta_r^{\alpha,j}(h) \sim \sum_k h^k \zeta_{r,k}^{\alpha,j},$$

with $\zeta_{r,0}^{\alpha,j}$ eigenvalue of the matrix $M_0^{\alpha,j}$.

Compared to previous results obtained under the Generic Assumption, the main difference is that the prefactors $\zeta_{r,k}^{\alpha,j}$ are more difficult to compute since they are obtained as the eigenvalues of the matrices $M^{\alpha,j}$. When (GA) is satisfied, the $M^{\alpha,j}$ are 1×1 matrices whose spectrum is direct to obtain. In the general case, this is not true anymore and the construction of the matrices $M^{\alpha,j}$ is more involved. In particular, it depends dramatically on the number $p(\alpha) = \# S(\mathcal{U}_\alpha^{(0)})$. Observe that this number is also equal to the number of different values taken by φ on the equivalence class $\mathcal{U}_\alpha^{(0)}$.

If $p(\alpha) = 1$, the coefficients of $M^{\alpha,j}$ depend only on the pairs $(\boldsymbol{m}, \boldsymbol{s})$ for which $\varphi(\boldsymbol{s}) - \varphi(\boldsymbol{m}) = S_{\nu_j^\alpha}$. Except for the fact that the different eigenvalues $\zeta_r^{\alpha,j}$, $r = 1, \dots, r^{\alpha,j}$, are linked together, the situation is similar to that encountered in the generic case. Actually, we prove in Appendix B that if (GA) is satisfied then $\mathrm{Cl}(\boldsymbol{m})$ is reduced to one point for any $\boldsymbol{m}$, and in particular $p(\alpha) = 1$ for all α.

In the case where $p(\alpha) \geq 2$, the matrix is more difficult to compute. It comes from an application of the Schur complement method and it depends on some pairs $(\boldsymbol{m}, \boldsymbol{s})$ for which the height $\varphi(\boldsymbol{s}) - \varphi(\boldsymbol{m})$ is smaller than $S_{\nu_j^\alpha}$. In other words, the lifetime of the metastable state $\boldsymbol{m}$ is not entirely described by the height that is needed to jump over in order to reach the nearest lower-energy position. It depends also on some interactions with some higher-energy states that are not present in the classical Eyring–Kramers formula. To our knowledge, this is the first time that such a phenomena is exhibited.

Let us now compute $p(\alpha)$ on explicit examples. Let us fix $n = 2$ and consider the potentials φ given respectively by Figures 6 and 7. In both cases, $\hat{\boldsymbol{m}}(\boldsymbol{m}_{2,1}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,2}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,3}) = \boldsymbol{m}_{1,1}$, which we denote by $\hat{\boldsymbol{m}}$ for short. Since $\varphi(\hat{\boldsymbol{m}}) < \varphi(\boldsymbol{m}_{2,j})$ for all j, there is no point of type II, $\mathcal{U}^{(0),\mathrm{II}} = \varnothing$ and hence $\Omega_{\sigma_2} = \{E_{2,1}, E_{2,2}, E_{2,3}\}$. Therefore, we can compute easily the equivalence classes of $\mathcal{R}$ in both cases:

- In the case of Figure 6, we have three equivalence classes: $c_1 = \{\boldsymbol{m}_{1,1}\}$, $c_2 = \{\boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}\}$ and $c_3 = \{\boldsymbol{m}_{2,3}\}$. The potential φ is constant on each equivalence class, and hence $p(c_1) = p(c_2) = p(c_3) = 1$.

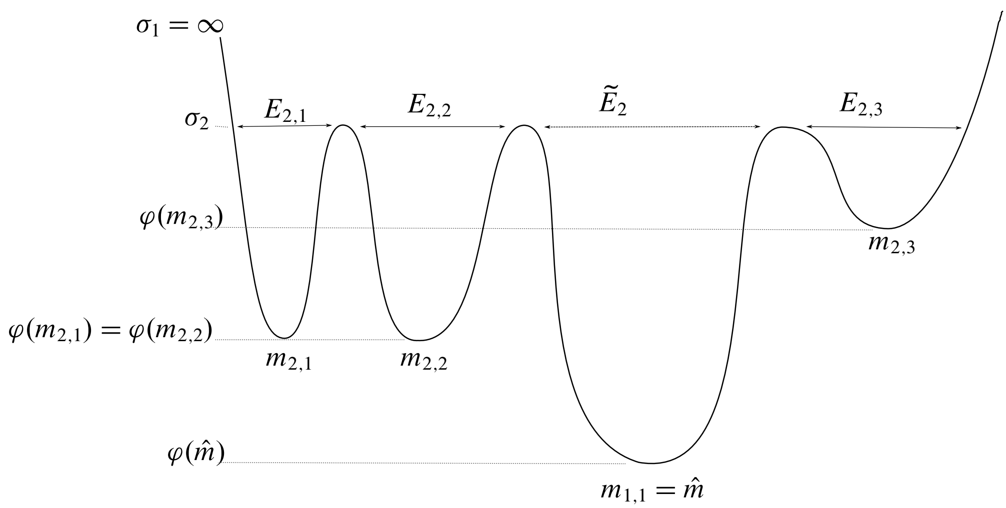

Figure 6. A potential with $p(\alpha) = 1$.

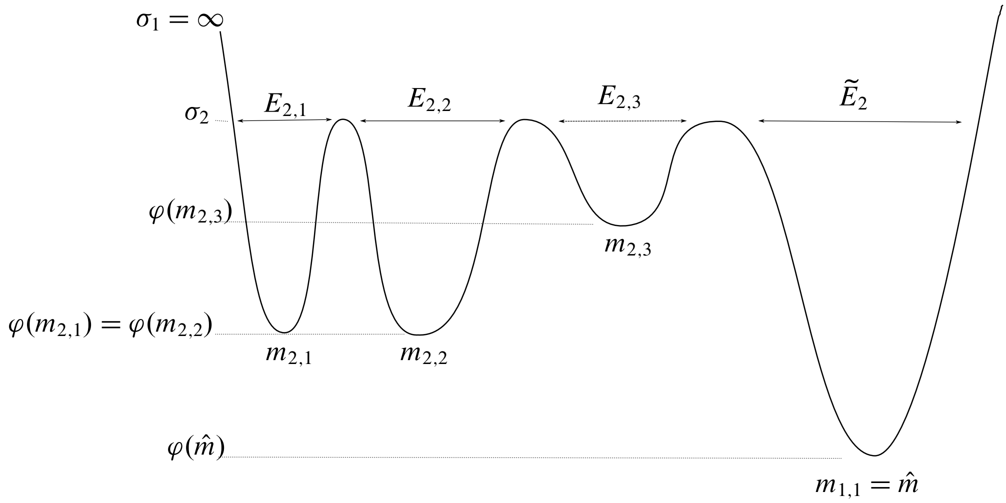

Figure 7. A potential with $p(\alpha) = 2$.

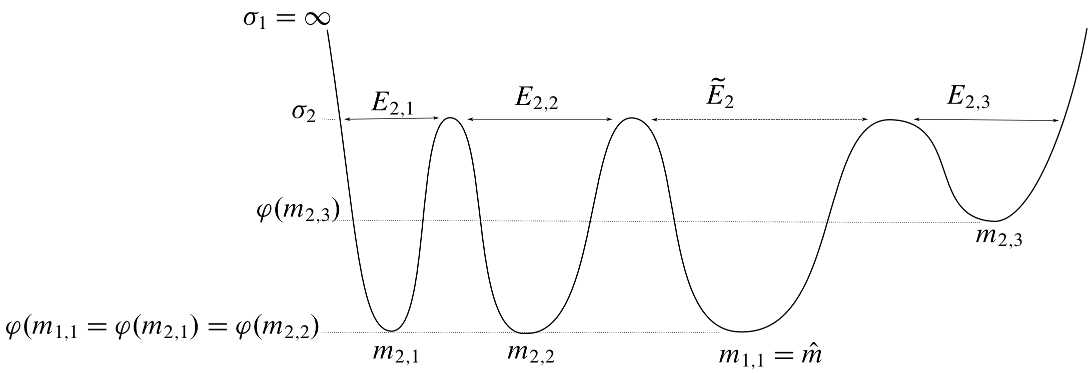

Figure 8. An example with points of type II.

- In the case of Figure 7, we have two equivalence classes: $c_1 = \{\boldsymbol{m}_{1,1}\}$ and $c_2 = \{\boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}, \boldsymbol{m}_{2,3}\}$. The potential φ takes two different values on c_2: $p(c_2) = 2$.

We will come back to these examples at the end of the paper and compute explicitly the spectrum of Δ_φ in both cases.

Let us finish this discussion with an example where $\mathcal{U}^{(0),\mathrm{II}} \neq \varnothing$. Consider the potential given by Figure 8. In that case $\hat{\boldsymbol{m}}(\boldsymbol{m}_{2,1}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,2}) = \hat{\boldsymbol{m}}(\boldsymbol{m}_{2,2}) = \boldsymbol{m}_{1,1}$, which we denote by $\hat{\boldsymbol{m}}$ for short. Since $\varphi(\hat{\boldsymbol{m}}) = \varphi(\boldsymbol{m}_{2,1}) = \varphi(\boldsymbol{m}_{2,2}) < \varphi(\boldsymbol{m}_{2,3})$, we know $\boldsymbol{m}_{2,1}$ and $\boldsymbol{m}_{2,2}$ are of type II and $\boldsymbol{m}_{2,3}$ is of type I. We still have $\Omega^{\boldsymbol{0}}_{\sigma_2} = \{E_{2,1}, E_{2,2}, E_{2,3}\}$ but contrary to the previous case $\widehat{\Omega}_{\sigma_2} = \{\widetilde{E}_2\}$ is nonempty. It follows that $\Omega_{\sigma_2} = \{E_{2,1}, E_{2,2}, E_{2,3}, \widetilde{E}_2\}$ and $\mathcal{R}$ admits two equivalence classes: $c_1 = \{\boldsymbol{m}_{1,1}\}$ and $c_2 = \{\boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}, \boldsymbol{m}_{2,3}\}$. The potential φ takes two different values on c_2 and hence $p(c_2) = 2$.

2C. ***General strategy of the proof.*** Let us recall the general strategy followed in [Helffer, Klein and Nier 2004]. The starting point is to use the supersymmetric structure of the Witten Laplacian. For $0 \le k \le n$, let $\Omega^k(X) = \mathcal{C}^\infty(X, \Lambda^k T^*X)$ be the space of k-differential forms and denote by $d : \Omega^k(X) \to \Omega^{k+1}(X)$ the exterior derivative and by $d^* : \Omega^k(X) \to \Omega^{k-1}(X)$ its adjoint for the natural pairing. The Witten complex associated to the function φ is defined by the semiclassical weighted de Rham differentiation

$$d_{\varphi,h} = e^{-\varphi/h} \circ hd \circ e^{\varphi/h} = hd + d\varphi^\wedge$$

and its adjoint

$$d^*_{\varphi,h} = e^{\varphi/h} \circ hd^* \circ e^{-\varphi/h} = hd^* + d\varphi^\lrcorner.$$

Then the semiclassical Witten Laplacian is defined on the forms of any degree by

$$\Delta_\varphi = d^*_{\varphi,h} \circ d_{\varphi,h} + d_{\varphi,h} \circ d^*_{\varphi,h}. \tag{2-17}$$

When restricted to the space of p-forms we denote this operator by $\Delta_\varphi^{(p)}$ (observe that in the case $p = 0$, the above formula yields easily (2-1)). Then, we have the intertwining relation

$$d_{\varphi,h}\Delta_\varphi^{(p)} = \Delta_\varphi^{(p+1)} d_{\varphi,h} \tag{2-18}$$

and its analogue for the coderivative

$$d^*_{\varphi,h}\Delta_\varphi^{(p+1)} = \Delta_\varphi^{(p)} d^*_{\varphi,h}. \tag{2-19}$$

For any $p = 0, \dots, d$, it follows from (2-2) that $\Delta_\varphi^{(p)}$ (as an unbounded operator on L^2) is essentially self-adjoint on the space of compactly supported smooth forms. We still denote by $\Delta_\varphi^{(p)}$ its unique self-adjoint extension. Then $\Delta_\varphi^{(p)}$ is nonnegative and thanks to (2-2), there exists $c_0 > 0$ such that $\sigma_{\text{ess}}(\Delta_\varphi^{(p)}) \subset [c_0, +\infty[$ for any $h > 0$ small enough (in the case where X is a compact manifold, $\Delta_\varphi^{(p)}$ has actually compact resolvent). Moreover, there exists $\epsilon_p > 0$ such that for $h > 0$ small enough, it has exactly n_p eigenvalues in the interval $[0, \epsilon_p h]$, where n_p denotes the number of critical points of index p of φ. We shall denote by $E^{(p)}$ the spectral subspace associated to these small eigenvalues of $\Delta_\varphi^{(p)}$. Then $\dim E^{(p)} = n_p$ and relations (2-18), (2-19) show that

$$d_{\varphi,h}(E^{(p)}) \subset E^{(p+1)} \quad \text{and} \quad d^*_{\varphi,h}(E^{(p+1)}) \subset E^{(p)}. \tag{2-20}$$

This shows in particular that $d_{\varphi,h}$ acts from $E^{(0)}$ into $E^{(1)}$ and we shall denote by $\mathcal{L}$ this operator. Similarly $\Delta_\varphi^{(0)}$ acts on $E^{(0)}$ and we denote by $\mathcal{M}$ this operator. By (2-17), we get

$$\mathcal{M} = \mathcal{L}^*\mathcal{L}.$$

The general strategy used in [Helffer, Klein and Nier 2004] (that we will follow in the present work), is to construct appropriate bases of $E^{(0)}$ and $E^{(1)}$ in which one can compute handily the singular values of $\mathcal{L}$. The main idea to construct such bases is to build accurate quasimodes for Δ_φ and to project them on the spaces $E^{(j)}$. The construction of the quasimodes is performed in Section 3. The quasimodes for 1-forms are the ones constructed in [Helffer and Sjöstrand 1985]. The main properties of these quasimodes will be recalled in Section 3C. Concerning the quasimodes on 0-forms, we cannot use the ones constructed in

[Helffer, Klein and Nier 2004] since many important properties that are required for our analysis fail to be true in the present situation (for instance, the quasiorthogonality). In Section 3B, we use the partition of $\mathcal{U}^{(0)}$ into equivalence classes of $\mathcal{R}$ to construct a family of quasimodes on 0-forms adapted to our setting. Each quasimode will be associated to a minimum $\boldsymbol{m} \in \mathcal{U}^{(0)}$.

In Section 4, we compute the matrix $\mathcal{L}$ in the above basis. One arrives at a block diagonal matrix $\operatorname{diag}(\mathcal{L}^{\alpha}, \alpha \in \mathcal{A})$ whose singular values are the singular values of each block.

Section 5 is devoted to the computation of singular values of the above blocks. The main difficulty is that given two minima $\boldsymbol{m}, \boldsymbol{m}'$ in the same equivalence class, we do not necessarily have $S(\boldsymbol{m}) = S(\boldsymbol{m}')$. For equivalence classes satisfying this property (that is, $p(\alpha) = 1$), each block $\mathcal{L}^{\alpha}$ of the matrix $\mathcal{L}$ has a typical size $e^{-S(\alpha)/h}$ and the situation could be handled quite easily. But more complicated cases may arise where quasimodes yielding different heights $S(\boldsymbol{m})$ are interacting. In order to treat the full general case, we use the Schur complement method combined with an induction on $p(\alpha)$. Running the induction step requires exhibiting a specific structure of the matrices under consideration (see Sections 5A and 5B). In Section 5C, we prove a general result for such matrices, which we use to conclude in Section 5D.

In Section 6, we prove Theorem 2.8.

In the Appendices, we collect several results linear algebra. We also provide a list of notation used in the paper.

3. Construction of adapted quasimodes

3A. ***Gathering minima by equivalence class.*** Let us start this section with a proposition collecting some elementary facts about E, E_- and $\widehat{E}$.

Proposition 3.1. *Let $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{U}^{(0)}$ such that $\boldsymbol{m} \neq \boldsymbol{m}'$. Then, we have the following*:

(i) *If $\sigma(\boldsymbol{m}) = \sigma(\boldsymbol{m}')$, then*

  (i.a) $E(\boldsymbol{m}) \cap E(\boldsymbol{m}') = \varnothing$,

  (i.b) *either $E_-(\boldsymbol{m}) = E_-(\boldsymbol{m}')$ or $E_-(\boldsymbol{m}) \cap E_-(\boldsymbol{m}') = \varnothing$*,

  (i.c) *if $E_-(\boldsymbol{m}) = E_-(\boldsymbol{m}')$ then $\widehat{E}(\boldsymbol{m}) = \widehat{E}(\boldsymbol{m}')$; otherwise $\widehat{E}(\boldsymbol{m}) \cap \widehat{E}(\boldsymbol{m}') = \varnothing$.*

(ii) *If $\sigma(\boldsymbol{m}) > \sigma(\boldsymbol{m}')$, then*

  (ii.a) *either $E(\boldsymbol{m}) \cap E(\boldsymbol{m}') = \varnothing$ or $E_-(\boldsymbol{m}') \subset E(\boldsymbol{m})$*,

  (ii.b) *either $E_-(\boldsymbol{m}) \cap E_-(\boldsymbol{m}') = \varnothing$ or $E_-(\boldsymbol{m}') \subset E_-(\boldsymbol{m})$.*

Proof. Let $\boldsymbol{m} \neq \boldsymbol{m}'$ be two minima. Assume first that $\sigma(\boldsymbol{m}) = \sigma(\boldsymbol{m}') = \sigma$. Since $\boldsymbol{m} \neq \boldsymbol{m}'$ and $\boldsymbol{\sigma}^{-1}(\infty) = \{\underline{\boldsymbol{m}}\}$, we have necessarily $\boldsymbol{m}, \boldsymbol{m}' \in \underline{\mathcal{U}}^{(0)}$. In particular, $E_-(\nu)$, $\widehat{E}(\nu)$, $\nu = \boldsymbol{m}, \boldsymbol{m}'$, are well-defined. Moreover, since $E(\boldsymbol{m})$ and $E(\boldsymbol{m}')$ are two connected components of $\{\varphi < \sigma\}$, we have either $E(\boldsymbol{m}) = E(\boldsymbol{m}')$ or $E(\boldsymbol{m}) \cap E(\boldsymbol{m}') = \varnothing$. Since $\boldsymbol{m} \neq \boldsymbol{m}'$ and E is injective, $E(\boldsymbol{m}) \cap E(\boldsymbol{m}') = \varnothing$, which proves (i.a).

Since $E_-(\boldsymbol{m})$ and $E_-(\boldsymbol{m}')$ are two connected component of the same set $\{\varphi < \tau\}$ for some $\tau > \sigma(\boldsymbol{m})$, (i.b) is obvious.

Suppose now that $E_-(\boldsymbol{m}) = E_-(\boldsymbol{m}')$. Since $\sigma(\boldsymbol{m}) = \sigma(\boldsymbol{m}')$, we have $\hat{\boldsymbol{m}}(\boldsymbol{m}) = \hat{\boldsymbol{m}}(\boldsymbol{m}')$. Moreover, since $\widehat{E}(\boldsymbol{m})$ is the unique connected component of $\{\varphi < \sigma(\boldsymbol{m})\}$ containing $\hat{\boldsymbol{m}}(\boldsymbol{m})$, we get $\widehat{E}(\boldsymbol{m}) = \widehat{E}(\boldsymbol{m}')$.

If $E_-(\boldsymbol{m})$ and $E_-(\boldsymbol{m}')$ are disjoint, then $\widehat{E}(\boldsymbol{m})$ and $\widehat{E}(\boldsymbol{m}')$ are also disjoint since $\widehat{E}(\boldsymbol{m}) \subset E_-(\boldsymbol{m})$ and $\widehat{E}(\boldsymbol{m}') \subset E_-(\boldsymbol{m}')$. This completes the proof of (i.c).

Let us now prove (ii) and assume that $\sigma(\boldsymbol{m}) > \sigma(\boldsymbol{m}')$. Once again, since $\boldsymbol{\sigma}^{-1}(\infty) = \{\underline{\boldsymbol{m}}\}$, we have $\boldsymbol{m}' \in \underline{\mathcal{U}}^{(0)}$. If $E(\boldsymbol{m}') \cap E(\boldsymbol{m}) \neq \varnothing$, then $E_-(\boldsymbol{m}') \cap E(\boldsymbol{m}) \neq \varnothing$. Moreover, $E_-(\boldsymbol{m}')$ is a connected component of $\{\varphi < \tau\}$ for some $\tau \leq \sigma(\boldsymbol{m})$. Since $E(\boldsymbol{m})$ is a connected component of $\{\varphi < \sigma(\boldsymbol{m})\} \supset \{\varphi < \tau\}$, we have $E_-(\boldsymbol{m}') \subset E(\boldsymbol{m})$ which proves (ii.a).

The point (ii.b) is proved by similar arguments. □

Let us now decompose the set of separating saddle points according to the equivalence classes. Given $\alpha \in \underline{\mathcal{A}}$, introduce the closed set

$$G(\alpha) = \bigcup_{\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}} \overline{E(\boldsymbol{m})} \tag{3-1}$$

and for any $\alpha \in \underline{\mathcal{A}}$ let

$$\mathcal{V}_\alpha^{(1)} = \{\boldsymbol{s} \in \mathcal{V}^{(1)} : \varphi(\boldsymbol{s}) = \boldsymbol{\sigma}(\alpha)\} \cap G(\alpha). \tag{3-2}$$

For any $\alpha \in \underline{\mathcal{A}}$, let

$$\widehat{\mathcal{U}}_\alpha^{(0)} = \mathcal{U}_\alpha^{(0)} \cup \{\hat{\boldsymbol{m}}(\alpha)\} \tag{3-3}$$

and define an application Γ_α from $\widehat{\mathcal{U}}_\alpha^{(0)}$ into the closed subsets of X by

$$\begin{cases} \Gamma_\alpha(\boldsymbol{m}) = \Gamma(\boldsymbol{m}) & \text{if } \boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}, \\ \Gamma_\alpha(\hat{\boldsymbol{m}}(\alpha)) = \partial \widehat{E}(\alpha), \end{cases} \tag{3-4}$$

where Γ is defined below (2-5).

Remark 3.2. Since $\widehat{E}(\boldsymbol{m}) \subsetneq E(\hat{\boldsymbol{m}})$, the application Γ_α is slightly different from the application Γ defined in below (2-5). Observe also that for all $\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$, $\Gamma_\alpha(\boldsymbol{m})$ is the boundary of the connected component of $\{\varphi < \varphi(\boldsymbol{s})\}$ that contains $\boldsymbol{m}$.

Lemma 3.3. *The collection $(\mathcal{V}_\alpha^{(1)})_{\alpha \in \underline{\mathcal{A}}}$ is a partition of $\mathcal{V}^{(1)}$. Moreover, for all $\alpha \in \underline{\mathcal{A}}$ and $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)}$, there exists $\boldsymbol{m}_1(\boldsymbol{s}) \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_\alpha^{(0)}$ such that*

$$\boldsymbol{s} \in \Gamma_\alpha(\boldsymbol{m}_1) \cap \Gamma_\alpha(\boldsymbol{m}_2). \tag{3-5}$$

One can chose $\boldsymbol{m}_1, \boldsymbol{m}_2$ in order that $S(\boldsymbol{m}_1) \leq S(\boldsymbol{m}_2)$ (that is, $\varphi(\boldsymbol{m}_1) \geq \varphi(\boldsymbol{m}_2)$). Up to permutation, the pair $(\boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}))$ is unique.

Proof. Let $\boldsymbol{s} \in \mathcal{V}^{(1)}$; then $\varphi(\boldsymbol{s}) \in \underline{\Sigma}$ and there exists $k \geq 2$ such that $\varphi(\boldsymbol{s}) = \sigma_k$. By definition, there exist two different connected components E_1, E_2 of $\{\varphi < \sigma_k\}$ such that $\boldsymbol{s} \in \overline{E}_1 \cap \overline{E}_2$. From the existence part of Remark 2.2 there exist $\boldsymbol{m}_{l,i} \in E_1$ and $\boldsymbol{m}_{l',i'} \in E_2$ with $l' \leq l \leq k$. Moreover, one has necessarily $l = k$. Otherwise $\boldsymbol{\sigma}(\boldsymbol{m}_{l,i}) > \sigma_k$ and since $\overline{E}_1 \cap \overline{E}_2 \neq \varnothing$, this would imply that $\boldsymbol{m}_{l',i'} \in E(\boldsymbol{m}_{l,i})$, which is impossible since $l' \leq l$. Hence we have $l = k$. Therefore E_1 is equal to $E(\boldsymbol{m}_{l,i})$ with $\boldsymbol{m}_{l,i} \in \mathcal{U}_\alpha^{(0)}$, which proves that $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)}$. Moreover, E_2 is either of the form $E_2 = E(\boldsymbol{m}_{l',i'})$ with $\boldsymbol{m}_{l',i'} \in \mathcal{U}_\alpha^{(0)}$ (if $l' = k$) or $E_2 = \widehat{E}(\boldsymbol{m}_{l,i})$ (if $l' < k$). Setting $\boldsymbol{m}_1(\boldsymbol{s}) = \boldsymbol{m}_{l,i} \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{m}_2(\boldsymbol{s}) = \boldsymbol{m}_{l',i'} \in \widehat{\mathcal{U}}_\alpha^{(0)}$, one has $\boldsymbol{s} \in \Gamma_\alpha(\boldsymbol{m}_1) \cap \Gamma_\alpha(\boldsymbol{m}_2)$ and since $l \geq l'$ one has also $\varphi(\boldsymbol{m}_1) \geq \varphi(\boldsymbol{m}_2)$.

Let us now prove that the union of the $\mathcal{V}_\alpha^{(1)}$ for $\alpha \in \underline{\mathcal{A}}$ is disjoint. Suppose that $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)} \cap \mathcal{V}_\beta^{(1)}$. Then $\sigma(\alpha) = \varphi(\boldsymbol{s}) = \sigma(\beta)$. Moreover, there exist $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{m}' \in \mathcal{U}_\beta^{(0)}$ such that $\boldsymbol{s} \in \overline{E(\boldsymbol{m})} \cap \overline{E(\boldsymbol{m}')}$. This proves that $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$ and hence $\alpha = \beta$.

The uniqueness of $(\boldsymbol{m}_1, \boldsymbol{m}_2)$ up to permutation is obvious. □

Let us now introduce an extra partition that will be useful in the sequel.

Lemma 3.4. *For all $\alpha \in \mathcal{A}$ there exists a partition $\mathcal{V}_\alpha^{(1)} = \mathcal{V}_\alpha^{(1),\boldsymbol{b}} \sqcup \mathcal{V}_\alpha^{(1),\boldsymbol{i}}$ such that the following hold true*:

(i) *For any $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1),\boldsymbol{i}}$, $\boldsymbol{m}_1(\boldsymbol{s})$ and $\boldsymbol{m}_2(\boldsymbol{s})$ belong to $\mathcal{U}_\alpha^{(0)}$.*

(ii) *The set $\mathcal{V}_\alpha^{(1),b}$ is nonempty and for all $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1),b}$ one has $\boldsymbol{m}_1(\boldsymbol{s}) \in \mathcal{U}_\alpha^{(0)}$, $\boldsymbol{m}_2(\boldsymbol{s}) = \hat{\boldsymbol{m}}(\alpha)$ and*

$$\boldsymbol{s} \in \Gamma_\alpha(\boldsymbol{m}_1(\boldsymbol{s})) \cap \Gamma_\alpha(\hat{\boldsymbol{m}}(\alpha)).$$

Proof. Define $\mathcal{V}_\alpha^{(1),\boldsymbol{i}} = \{\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)} : \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}) \in \mathcal{U}_\alpha^{(0)}\}$. Then (i) is true by definition. Moreover, defining $\mathcal{V}_\alpha^{(1),\boldsymbol{b}} = \mathcal{V}_\alpha^{(1)} \setminus \mathcal{V}_\alpha^{(1),\boldsymbol{i}}$, one has automatically the partition property and it remains to prove (ii).

Since $\alpha \in \underline{\mathcal{A}}$, the set $\overline{\widehat{E}(\alpha)} \cap \big(\bigcup_{\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}} \overline{E(\boldsymbol{m})}\big)$ is nonempty and contained in $\mathcal{V}_\alpha^{(1),\boldsymbol{b}}$. This proves that $\mathcal{V}_\alpha^{(1),\boldsymbol{b}}$ is not empty. Suppose now that $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1),b}$. It follows from Lemma 3.3 that $\boldsymbol{m}_1(\boldsymbol{s}) \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_\alpha^{(0)}$. But by the definition of $\mathcal{V}_\alpha^{(1),\boldsymbol{b}}$, $\boldsymbol{m}_2(\boldsymbol{s})$ cannot belong to $\mathcal{U}_\alpha^{(0)}$, which implies by definition that $\boldsymbol{m}_2(\boldsymbol{s}) = \hat{\boldsymbol{m}}(\alpha)$. This completes the proof of (ii). □

3B. ***Quasimodes for* 0*-forms.*** In this section we construct a family of quasimodes of $\Delta_\varphi^{(0)}$ associated to the minima of φ. Each of these quasimodes will be of the form $x \mapsto h^{-d/4}\chi_{\boldsymbol{m}}(x)e^{-(\varphi(x)-\varphi(\boldsymbol{m}))/h}$ with some suitable cut-off functions $\chi_{\boldsymbol{m}}$ associated to a minimum $\boldsymbol{m} \in \mathcal{U}^{(0)}$.

Following [Helffer, Klein and Nier 2004], we can associate to each minimum $\boldsymbol{m} \in \mathcal{U}^{(0)}$ a cut-off function $\chi_{\boldsymbol{m}}$ in the following way. For $\boldsymbol{m} = \underline{\boldsymbol{m}}$, we simply take $\chi_{\underline{\boldsymbol{m}}} = 1$. For $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$ we introduce some small parameters $\epsilon, \tilde{\epsilon}, \delta > 0$ with $\tilde{\epsilon} < \epsilon$ and we define

$$E_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m}) = \bigg(\bigg(E(\boldsymbol{m}) \setminus \bigcup_{\boldsymbol{s} \in \mathcal{V}^{(1)} \cap \Gamma(\boldsymbol{m})} B(\boldsymbol{s}, \epsilon)\bigg) + B(0, \tilde{\epsilon})\bigg) \cup \bigg(\bigcup_{\boldsymbol{s} \in (\mathcal{U}^{(1)} \setminus \mathcal{V}^{(1)}) \cap \Gamma(\boldsymbol{m})} B(\boldsymbol{s}, \delta)\bigg). \tag{3-6}$$

Proposition 3.5. *Let $\chi_{\boldsymbol{m}}$ be any function in $\mathscr{C}_c^\infty(E_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}))$ such that $\chi_{\boldsymbol{m}} = 1$ on $E_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})$. There exist $\epsilon_0 > 0$, $\delta_0 > 0$ and $C > 0$ such that for all $0 < \delta < \delta_0$, all $0 < \epsilon < \epsilon_0$ and all $0 < \tilde{\epsilon} < \epsilon/4$, the following hold true*:

(a) *If $x \in \operatorname{supp}(\chi_{\boldsymbol{m}})$ and $\varphi(x) < \sigma(\boldsymbol{m})$, then $x \in E(\boldsymbol{m})$.*

(b) *There exists $c_\epsilon > 0$ such that for all $x \in \operatorname{supp}(\nabla\chi_{\boldsymbol{m}})$, we have*

- *either $x \notin \bigcup_{\boldsymbol{s} \in \mathcal{V}^{(1)} \cap \Gamma(\boldsymbol{m})} B(\boldsymbol{s}, \epsilon)$ and*

$$\sigma(\boldsymbol{m}) + c_\epsilon^{-1} < \varphi(x) < \sigma(\boldsymbol{m}) + c_\epsilon,$$

- *or $x \in \bigcup_{\boldsymbol{s} \in \mathcal{V}^{(1)} \cap \Gamma(\boldsymbol{m})} B(\boldsymbol{s}, \epsilon)$ and*

$$|\varphi(x) - \sigma(\boldsymbol{m})| \leq C\epsilon.$$

(c) *For all* $\boldsymbol{s}\in\mathcal{U}^{(1)}\setminus(\mathcal{V}^{(1)}\cap\Gamma(\boldsymbol{m}))$, *one has* $\mathrm{dist}(\boldsymbol{s},\mathrm{supp}\,\nabla\chi_{\boldsymbol{m}})\geq\delta$. *If moreover* $\boldsymbol{s}\in\mathrm{supp}(\chi_{\boldsymbol{m}})$ *then* $\boldsymbol{s}\in\overline{E(\boldsymbol{m})}$ *and* $\chi_{\boldsymbol{m}}=1$ *near* $\boldsymbol{s}$.

(d) *Suppose that* $\boldsymbol{m}\in\mathcal{U}^{(0)}_\alpha$, $\alpha\in\mathcal{A}$, *and let* $\boldsymbol{s}\in\mathcal{V}^{(1)}\cap\mathrm{supp}(\chi_{\boldsymbol{m}})$. *Then, there exists* $\beta\in\underline{\mathcal{A}}$ *such that* $\boldsymbol{\sigma}(\beta)<\boldsymbol{\sigma}(\alpha)$, $\boldsymbol{s}\in\mathcal{V}^{(1)}_\beta$ *and* $\bigcup_{\boldsymbol{m}'\in\mathcal{U}^{(0)}_\beta}E(\boldsymbol{m}')\subset\{x\in X:\chi_{\boldsymbol{m}}(x)=1\}$.

Proof. Observe that the construction of the cut-off functions $\chi_{\boldsymbol{m}}$ is slightly different to that of the $\chi_{k,\epsilon}$ in Proposition 4.2 in [Helffer, Klein and Nier 2004] (in particular because there can exist more than one separating saddle point on $\partial E(\boldsymbol{m})$).

Let $\delta_1=\min\{|\boldsymbol{s}-\boldsymbol{s}'|:\boldsymbol{s},\boldsymbol{s}'\in\mathcal{U}^{(1)},\ \boldsymbol{s}\neq\boldsymbol{s}'\}$ and $\delta_2=\min\{\mathrm{dist}(\boldsymbol{s},\Gamma(\boldsymbol{m})):\boldsymbol{s}\in E(\boldsymbol{m})\cap\mathcal{U}^{(1)}\}$. Let $0<\delta<\frac{1}{4}\min(\delta_1,\delta_2)$ and $\epsilon_0>0$ such that there exists $C>0$ such that for all $0<\epsilon<\epsilon_0$ and all $\boldsymbol{s}\in\mathcal{V}^{(1)}$, one has

$$|\varphi(x)-\varphi(\boldsymbol{s})|<C\epsilon\quad\text{for all } x\in B(\boldsymbol{s},\epsilon).$$

This is possible since φ is a smooth function. Then (a) and (b) above can be proved much as Proposition 4.2 in [Helffer, Klein and Nier 2004] and (c) is a direct consequence of our choice of δ.

Let us now prove (d). By definition, if $\boldsymbol{s}\in\mathcal{V}^{(1)}\cap\mathrm{supp}(\chi_{\boldsymbol{m}})$, then $\boldsymbol{s}\in E(\boldsymbol{m})$ (here we use the condition $0<\tilde{\epsilon}<\epsilon/4$). Hence, there exists $\beta\neq\alpha$ such that $\boldsymbol{s}\in\mathcal{V}^{(1)}_\beta$ and one has additionally $\boldsymbol{\sigma}(\beta)<\boldsymbol{\sigma}(\alpha)$. By definition of the sets $E(\boldsymbol{m})$, this implies that

$$\bigcup_{\boldsymbol{m}'\in\mathcal{U}^{(0)}_\beta}E(\boldsymbol{m}')\subset E(\boldsymbol{m})\setminus\bigcup_{\boldsymbol{s}'\in\mathcal{V}^{(1)}\cap\Gamma(\boldsymbol{m})}B(\boldsymbol{s}',\epsilon)$$

for any $\epsilon\in]0,\epsilon_0[$ with $\epsilon_0>0$ small enough independent of δ. This implies the results. ☐

We are now in position to define the quasimodes in a recursive way on the values of $\boldsymbol{\sigma}(\alpha)$.

- We start with the quasimode associated to $\underline{\boldsymbol{m}}$. We set

$$f^{(0)}_{\underline{\boldsymbol{m}}}(x)=c(\underline{\boldsymbol{m}},h)h^{-d/4}e^{(\varphi(\underline{\boldsymbol{m}})-\varphi(x))/h},\tag{3-7}$$

where $c(\underline{\boldsymbol{m}},h)$ is a normalizing constant such that $\|f_{\underline{\boldsymbol{m}}}\|_{L^2}=1$. Due to the fact that φ may have several global minima, the function $f^{(0)}_{\underline{\boldsymbol{m}}}$ does not concentrate only on $\underline{\boldsymbol{m}}$ but on the reunion of all global minima. Hence the normalizing factor $c(\underline{\boldsymbol{m}},h)$ is computed by adding the contributions coming from each of these minima via quadratic approximation. More precisely, it follows from the Laplace method that $c(\underline{\boldsymbol{m}},h)\sim\sum_{k=0}^\infty h^k\gamma_k(\underline{\boldsymbol{m}})$ with the function γ_0 given by

$$\gamma_0(\boldsymbol{m})^{-2}=\pi^{d/2}\sum_{\boldsymbol{m}'\in H(\boldsymbol{m})}|\det\mathrm{Hess}\,\varphi(\boldsymbol{m}')|^{-1/2},\tag{3-8}$$

where by definition (2-6) one has

$$H(\boldsymbol{m}):=\{\boldsymbol{m}'\in E(\boldsymbol{m})\cap\mathcal{U}^{(0)}:\varphi(\boldsymbol{m}')=\varphi(\boldsymbol{m})\}.$$

Finally, observe that $f^{(0)}_{\underline{\boldsymbol{m}}}$ is an exact quasimode: $\Delta_\varphi f^{(0)}_{\underline{\boldsymbol{m}}}=0$.

• Suppose now that $k \in \{2,\dots,K\}$ and that the quasimodes $f_{\boldsymbol{m}}^{(0)}$ have been constructed for $\boldsymbol{m} \in \bigcup_{\alpha' \in \underline{\mathcal{A}}, \boldsymbol{\sigma}(\alpha') \le \sigma_{k-1}} \mathcal{U}_\alpha^{(0)}$, and let us define $f_{\boldsymbol{m}}^{(0)}$ for $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ with $\boldsymbol{\sigma}(\alpha) = \sigma_k$. The form of the quasimode associated to $\boldsymbol{m}$ depends on the type of $\boldsymbol{m}$ as introduced in Definition 2.3.

– If $\boldsymbol{m}$ is of type I, then we define $f_{\boldsymbol{m}}^{(0)}$ as in [Helffer, Klein and Nier 2004] by

$$f_{\boldsymbol{m}}^{(0)}(x) = c(\boldsymbol{m},h) h^{-d/4} \chi_{\boldsymbol{m}}(x) e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}, \tag{3-9}$$

where $\chi_{\boldsymbol{m}}$ is the cut-off function associated to $\boldsymbol{m}$ defined in Proposition 3.5 and $c(\boldsymbol{m},h)$ is again a normalizing constant such that $\| f_{\boldsymbol{m}}^{(0)} \|_{L^2} = 1$. As before, we have to add all the contributions of minima in $E(\boldsymbol{m})$ at the same height as $\boldsymbol{m}$. We get $c(\boldsymbol{m},h) \sim \sum_{k=0}^{\infty} h^k \gamma_k(\boldsymbol{m})$ with $\gamma_0(\boldsymbol{m})$ given by (3-8).

– Let us now construct quasimodes associated to minima $\boldsymbol{m}$ of type II. We assume here that $\mathcal{U}^{(0),\mathrm{II}} \neq \varnothing$ and we define

$$\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}} = \mathcal{U}_\alpha^{(0),\mathrm{II}} \cup \{\hat{\boldsymbol{m}}\}, \tag{3-10}$$

where for short, we write $\hat{\boldsymbol{m}} = \hat{\boldsymbol{m}}(\alpha)$ and $q_\alpha^{\mathrm{II}} = \# \mathcal{U}_\alpha^{(0),\mathrm{II}}$.

Let us introduce an additional cut-off function around $\hat{\boldsymbol{m}}$ that we define as follows. Recall that $\widehat{E}(\alpha)$ denotes the connected component of $\{x \in E_-(\boldsymbol{m}) : \varphi(x) < \sigma(\boldsymbol{m})\}$ that contains $\hat{\boldsymbol{m}}$. As before, we introduce some parameters $\epsilon, \tilde{\epsilon}, \delta > 0$ with $\tilde{\epsilon} < \epsilon$ and we define

$$\widehat{E}_{\epsilon,\tilde{\epsilon},\delta}(\alpha) = \Bigg(\bigg(\widehat{E}(\alpha) \setminus \bigcup_{\boldsymbol{s} \in \mathcal{V}^{(1)} \cap \partial \widehat{E}(\alpha)} B(\boldsymbol{s},\epsilon) \bigg) + B(0,\tilde{\epsilon}) \Bigg) \cup \Bigg(\bigcup_{\boldsymbol{s} \in (\mathcal{U}^{(1)} \setminus \mathcal{V}^{(1)}) \cap \partial \widehat{E}(\alpha)} B(\boldsymbol{s},\delta) \Bigg).$$

Then, we let $\hat{\chi}_{\hat{\boldsymbol{m}}}$ be any function in $\mathscr{C}_c^\infty(\widehat{E}_{\epsilon,2\tilde{\epsilon},2\delta}(\alpha))$ such that $\hat{\chi}_{\hat{\boldsymbol{m}}} = 1$ on $\widehat{E}_{\epsilon,\tilde{\epsilon},\delta}(\alpha)$. For $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0),\mathrm{II}}$, we let $\hat{\chi}_{\boldsymbol{m}} = \chi_{\boldsymbol{m}}$, with $\chi_{\boldsymbol{m}}$ defined in Proposition 3.5. We want to construct the quasimode as a linear combination of the $\hat{\chi}_{\boldsymbol{m}} e^{-\varphi/h}$, $\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$. In order to chose the coefficients, let us introduce $\mathcal{F}_\alpha = \mathscr{F}(\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}})$, the finite vector space of functions from $\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$ into $\mathbb{R}$. This space has dimension $q_\alpha^{\mathrm{II}} + 1$ and is endowed with the usual Euclidean structure

$$\langle \theta, \theta' \rangle_{\mathcal{F}_\alpha} = \sum_{\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}} \theta(\boldsymbol{m}) \theta'(\boldsymbol{m}).$$

We denote by N the associated norm. Eventually, we define $\theta_0^\alpha \in \mathcal{F}_\alpha$ by

$$\theta_0^\alpha(\boldsymbol{m}) = \frac{c_0^\alpha(h)}{c(\boldsymbol{m},h)}, \tag{3-11}$$

where $c(\boldsymbol{m},h)$ is the unique positive constant such that the function

$$\tilde{f}_{\boldsymbol{m}} := c(\boldsymbol{m},h) h^{-d/4} \hat{\chi}_{\boldsymbol{m}} e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}$$

satisfies $\| \tilde{f}_{\boldsymbol{m}} \|_{L^2} = 1$ and $c_0^\alpha(h)$ is defined by $N(\theta_0^\alpha) = 1$. Let us now extend the definition of the set $H(\boldsymbol{m})$ in the following way. Given $\alpha \in \mathcal{A}$ and $\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$ we define

$$\widehat{H}_\alpha(\boldsymbol{m}) = \begin{cases} H(\boldsymbol{m}) & \text{if } \boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}, \\ \{\boldsymbol{m}' \in \widehat{E}(\alpha) \cap \mathcal{U}^{(0)} : \varphi(\boldsymbol{m}') = \varphi(\hat{\boldsymbol{m}})\} & \text{if } \boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}} \setminus \mathcal{U}_\alpha^{(0)}. \end{cases} \tag{3-12}$$

Observe that if α is of type II, since $E(\hat{\boldsymbol{m}}(\alpha))$ is larger than $\widehat{E}(\alpha)$, the sets $H(\hat{\boldsymbol{m}}(\alpha))$ and $\widehat{H}_\alpha(\hat{\boldsymbol{m}}(\alpha))$ may be different. From the preceding definition, it follows that for all $\boldsymbol{m}\in\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$, the normalizing factor $c(\boldsymbol{m},h)$ admits a classical expansion $c(\boldsymbol{m},h)=\sum_k h^k\gamma_k(\boldsymbol{m})$ with

$$\gamma_0(\boldsymbol{m})^{-2}=\pi^{d/2}\sum_{\boldsymbol{m}'\in\widehat{H}_\alpha(\boldsymbol{m})}|\det\operatorname{Hess}\varphi(\boldsymbol{m}')|^{-1/2}. \tag{3-13}$$

Therefore, we can compute the constant $c_0^\alpha(h)$, and we get

$$c_0^\alpha(h)=\pi^{-d/4}\biggl(\sum_{\nu\in\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}}\sum_{\boldsymbol{m}'\in\widehat{H}_\alpha(\nu)}|\det\operatorname{Hess}\varphi(\boldsymbol{m}')|^{-1/2}\biggr)^{-1/2}+\mathcal{O}(h).$$

Here the index α is used to indicate that the function is associated to $\mathcal{U}_\alpha^{(0),\mathrm{II}}$.

Lemma 3.6. *There exist some functions $\theta_1^\alpha,\dots,\theta_{q_\alpha^{\mathrm{II}}}^\alpha\in\mathcal{F}_\alpha$ such that the following hold true*:

(i) *$\{\theta_j^\alpha,\ j=0,\dots,q_\alpha^{\mathrm{II}}\}$ is an orthonormal basis of $\mathcal{F}_\alpha$.*

(ii) *The functions θ_j^α admit a classical expansion*

$$\theta_j^\alpha=\sum_{k\ge 0}h^k\theta_j^{\alpha,k}$$

and for all $j\ge 1$, the leading terms $\theta_j^{\alpha,0}$ are orthogonal to the function $\theta_0^{\alpha,0}(\boldsymbol{m})=c_0^\alpha(0)/\gamma_0(\boldsymbol{m})$.

Proof. First observe that θ_0^α admits a classical expansion $\theta_0^\alpha\sim\sum_{j\ge0}h^j\theta_0^{\alpha,j}$ with $\theta_0^{\alpha,0}(\boldsymbol{m})=c_0^\alpha(0)/\gamma_0(\boldsymbol{m})$. Since $(\theta_0^{\alpha,0})^\perp$ is a q_α^{II}-dimensional subspace of $\mathcal{F}_\alpha$, it admits an orthonormal basis $(\tilde{\theta}_j^{\alpha,0})$ independent of h. Then the functions $\tilde{\theta}_j^\alpha$ defined by

$$\tilde{\theta}_j^\alpha:=\tilde{\theta}_j^{\alpha,0}-\langle\tilde{\theta}_j^{\alpha,0},\theta_0^\alpha\rangle\theta_0^\alpha$$

form a basis of $(\theta_0^\alpha)^\perp$. Moreover, the $\tilde{\theta}_j^\alpha$ admit a classical expansion and since $\langle\tilde{\theta}_j^{\alpha,0},\theta_0^\alpha\rangle=\mathcal{O}(h)$ for any j, they satisfy

$$\langle\tilde{\theta}_j^\alpha,\tilde{\theta}_k^\alpha\rangle=\delta_{jk}+\mathcal{O}(h^2).$$

Defining the (θ_j^α) as the Gram–Schmidt orthonormalization of the $(\tilde{\theta}_j^\alpha)$, we get the desired result. ☐

Observe that since $\mathcal{U}_\alpha^{(0),\mathrm{II}}$ has q_α^{II} elements, the functions $\theta_1^\alpha,\dots,\theta_{q_\alpha^{\mathrm{II}}}^\alpha$ can also be indexed by $\mathcal{U}_\alpha^{(0),\mathrm{II}}$ using any arbitrary bijection. We end up with a family of functions $(\theta_{\boldsymbol{m}}^\alpha)_{\boldsymbol{m}\in\mathcal{U}_\alpha^{(0),\mathrm{II}}}$ and for convenience we will also write $\theta_{\hat{\boldsymbol{m}}}^\alpha=\theta_0^\alpha$. Then, we define the q_α^{II} quasimodes associated to the $\boldsymbol{m}\in\mathcal{U}_\alpha^{(0),\mathrm{II}}$ by

$$f_{\boldsymbol{m}}^{(0)}(x)=h^{-d/4}\sum_{\boldsymbol{m}'\in\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}}\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}')c(\boldsymbol{m}',h)\hat{\chi}_{\boldsymbol{m}'}(x)e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}, \tag{3-14}$$

where the normalization factor $c(\boldsymbol{m}',h)$ is defined above and ensures that

$$\|c(\boldsymbol{m}',h)h^{-d/4}\hat{\chi}_{\boldsymbol{m}'}(x)e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}\|_{L^2}=1.$$

Before going further and as a preparation for the final analysis we would like to write the quasimode given by (3-9) and (3-14) in the same fashion. For this purpose, we define $\widehat{\mathcal{U}}_\alpha^{(0)}$ by

$$\widehat{\mathcal{U}}_\alpha^{(0)} = \mathcal{U}_\alpha^{(0),\mathrm{I}} \cup \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}, \tag{3-15}$$

with the convention that $\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}} = \varnothing$ if $\mathcal{U}_\alpha^{(0),\mathrm{II}} = \varnothing$ (observe that $\widehat{\mathcal{U}}_\alpha^{(0)}$ is equal to the set $\widehat{\mathcal{U}}_\alpha^{(0)}$ defined in (3-3) if and only if $\mathcal{U}_\alpha^{(0),\mathrm{II}} \neq \varnothing$). Then, we define $\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}')$ for any $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$, $\boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0)}$) in the following way:

- If $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0),\mathrm{II}}$ and $\boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$, we keep the above definition.
- Otherwise, we set

$$\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}') = \delta_{\boldsymbol{m},\boldsymbol{m}'}. \tag{3-16}$$

Then the formulas in (3-9) and (3-14) can be summarized in

$$f_{\boldsymbol{m}}^{(0)}(x) = h^{-d/4} \sum_{\boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}') c(\boldsymbol{m}', h) \chi_{\boldsymbol{m}'}(x) e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}, \tag{3-17}$$

with $\widehat{\mathcal{U}}_\alpha^{(0)}$ and θ^α as above.

Definition 3.7. For any $\alpha \in \underline{\mathcal{A}}$, let us denote by $\mathscr{T}^\alpha \in \mathscr{M}(\mathcal{U}_\alpha^{(0)}, \widehat{\mathcal{U}}_\alpha^{(0)})$ the matrix given by

$$\mathscr{T}^\alpha = (\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}'))_{\boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0)}, \boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}}$$

Let us remark that if all points of $\mathcal{U}_\alpha^{(0)}$ are of type I, then $\mathscr{T}^\alpha$ is just the $q_\alpha \times q_\alpha$ identity matrix, whereas if $\mathcal{U}_\alpha^{(0),\mathrm{II}} \neq \varnothing$, it is a $(q_\alpha + 1) \times q_\alpha$ matrix. Observe also that the partitions $\mathcal{U}_\alpha^{(0)} = \mathcal{U}_\alpha^{(0),\mathrm{I}} \sqcup \mathcal{U}_\alpha^{(0),\mathrm{II}}$ and $\widehat{\mathcal{U}}_\alpha^{(0)} = \mathcal{U}_\alpha^{(0),\mathrm{I}} \sqcup \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$ induce decompositions of the corresponding vector spaces:

$$\mathscr{F}(\mathcal{U}_\alpha^{(0)}) = \mathscr{F}(\mathcal{U}_\alpha^{(0),\mathrm{I}}) \oplus \mathscr{F}(\mathcal{U}_\alpha^{(0),\mathrm{II}}), \tag{3-18}$$

$$\mathscr{F}(\widehat{\mathcal{U}}_\alpha^{(0)}) = \mathscr{F}(\mathcal{U}_\alpha^{(0),\mathrm{I}}) \oplus \mathscr{F}(\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}). \tag{3-19}$$

From the above construction, one deduces that in a suitable basis the matrix $\mathscr{T}^\alpha$ is block diagonal with Id on the upper-left corner and a certain orthogonal matrix in the lower-right corner. More precisely, there exists an orthogonal matrix $\widehat{\mathscr{T}}^\alpha \in \mathscr{M}(\mathcal{U}_\alpha^{(0),\mathrm{II}}, \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}})$ such that for any $f = f^{\mathrm{I}} + f^{\mathrm{II}}$ with $f^{\mathrm{I}} \in \mathscr{F}(\mathcal{U}_\alpha^{(0),\mathrm{I}})$ and $f^{\mathrm{II}} \in \mathscr{F}(\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}})$, one has

$$\mathscr{T} f(\boldsymbol{m}) = f^{\mathrm{I}}(\boldsymbol{m}) + (\widehat{\mathscr{T}}^\alpha f^{\mathrm{II}})(\boldsymbol{m}). \tag{3-20}$$

Moreover, the matrix $\widehat{\mathscr{T}}^\alpha$ is just the matrix $(\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}'))_{\boldsymbol{m} \in \mathcal{U}_\alpha^{(0),\mathrm{II}}, \boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}}$ whose coefficients are given by Lemma 3.6. In particular, $\mathrm{Ran}\, \widehat{\mathscr{T}}^\alpha = (\mathbb{R}\theta_0^\alpha)^\perp$, where θ_0^α is defined by (3-11).

For any $\boldsymbol{m} \in \mathcal{U}^{(0)}$, let us introduce the set $F(\boldsymbol{m})$, defined as follows. If $\boldsymbol{m} = \underline{\boldsymbol{m}}$, let $F(\underline{\boldsymbol{m}}) = X$. If $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0),\mathrm{I}} := \underline{\mathcal{U}}^{(0)} \cap \mathcal{U}^{(0),\mathrm{I}}$, let $F(\boldsymbol{m}) = \overline{E(\boldsymbol{m})}$ and if $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0),\mathrm{II}} := \underline{\mathcal{U}}^{(0)} \cap \mathcal{U}^{(0),\mathrm{II}}$, let

$$F(\boldsymbol{m}) = \Bigg(\bigcup_{\boldsymbol{m}' \in \mathcal{U}_\alpha^{(0),\mathrm{II}}} \overline{E(\boldsymbol{m}')} \Bigg) \cup \overline{\widehat{E}(\boldsymbol{m})}, \tag{3-21}$$

where α is such that $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$. Observe that we always have $\overline{E(\boldsymbol{m})} \subset F(\boldsymbol{m})$.

Proposition 3.8. *Let $\boldsymbol{m},\boldsymbol{m}'\in\mathcal{U}^{(0)}$ be such that $\boldsymbol{m}\neq\boldsymbol{m}'$. The following hold true:*

(i) *If $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$ then*

(i.a) *if $\boldsymbol{m}$ or $\boldsymbol{m}'$ is of type I, then $F(\boldsymbol{m})\cap F(\boldsymbol{m}')\subset\mathcal{V}^{(1)}$,*

(i.b) *if $\boldsymbol{m}$ and $\boldsymbol{m}'$ are both of type II, then $F(\boldsymbol{m})=F(\boldsymbol{m}')$.*

(ii) *If $\boldsymbol{m}'\notin\mathrm{Cl}(\boldsymbol{m})$, then*

(ii.a) *if $\sigma(\boldsymbol{m})=\sigma(\boldsymbol{m}')$, then $F(\boldsymbol{m})\cap F(\boldsymbol{m}')=\varnothing$,*

(ii.b) *if $\sigma(\boldsymbol{m})>\sigma(\boldsymbol{m}')$, then either $F(\boldsymbol{m})\cap F(\boldsymbol{m}')=\varnothing$ or $F(\boldsymbol{m}')\subset\mathring{F}(\boldsymbol{m})$.*

Proof. Let $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$ with $\boldsymbol{m}\neq\boldsymbol{m}'$. As in the proof of Proposition 3.1, one has necessarily $\boldsymbol{m},\boldsymbol{m}'\neq\underline{\boldsymbol{m}}$. Assume first that $\boldsymbol{m}$ is of type I. Then $F(\boldsymbol{m})=\overline{E(\boldsymbol{m})}$. If $\boldsymbol{m}'$ is also of type I, then $F(\boldsymbol{m}')=\overline{E(\boldsymbol{m}')}$. Moreover since $\boldsymbol{m}\neq\boldsymbol{m}'$, it follows from (i.a) of Proposition 3.1 that $E(\boldsymbol{m})\cap E(\boldsymbol{m}')=\varnothing$. Therefore, $F(\boldsymbol{m})\cap F(\boldsymbol{m}')$ is either empty or is reduced to a union of saddle points which are separating by definition. If $\boldsymbol{m}'$ is of type II, the same proof works. This completes the proof of (i.a).

Suppose now that $\boldsymbol{m}$ and $\boldsymbol{m}'$ are both of type II. Since $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$, it follows that $\widehat{E}(\boldsymbol{m})=\widehat{E}(\boldsymbol{m}')$ and hence $F(\boldsymbol{m})=F(\boldsymbol{m}')$ which shows (i.b).

Suppose now that $\boldsymbol{m}'\notin\mathrm{Cl}(\boldsymbol{m})$. Consider first the case where $\sigma(\boldsymbol{m})=\sigma(\boldsymbol{m}')$. Then, one has necessarily $F(\boldsymbol{m})\cap F(\boldsymbol{m}')=\varnothing$; otherwise we would have $\boldsymbol{m}\mathcal{R}\boldsymbol{m}'$.

Suppose now that $\sigma(\boldsymbol{m})>\sigma(\boldsymbol{m}')$ and that $F(\boldsymbol{m})\cap F(\boldsymbol{m}')\neq\varnothing$. If $\boldsymbol{m}=\underline{\boldsymbol{m}}$, then $F(\boldsymbol{m})=X$ and the conclusion is obvious. Suppose now that $\boldsymbol{m}\in\underline{\mathcal{U}}^{(0)}$ and consider first the case where $\boldsymbol{m}$ and $\boldsymbol{m}'$ are of type I. Then $F(\boldsymbol{m})=\overline{E(\boldsymbol{m})}$ and $F(\boldsymbol{m}')=\overline{E(\boldsymbol{m}')}$ and since $\sigma(\boldsymbol{m})>\sigma(\boldsymbol{m}')$, it follows that $E(\boldsymbol{m})\cap E(\boldsymbol{m}')\neq\varnothing$. Hence (ii.a) of Proposition 3.1 shows that $E_-(\boldsymbol{m}')\subset E(\boldsymbol{m})$ which yields $F(\boldsymbol{m}')\subset E(\boldsymbol{m})=\mathring{F}(\boldsymbol{m})$. If $\boldsymbol{m}$ is of type I and $\boldsymbol{m}'$ is of type II, then one has $E(\boldsymbol{m})\cap\widetilde{E}\neq\varnothing$ with either $\widetilde{E}=E(\boldsymbol{m}'')$ for some $\boldsymbol{m}''\in\mathrm{Cl}(\boldsymbol{m}')$ or $\widetilde{E}=\widehat{E}(\boldsymbol{m}')$. As before, $E(\boldsymbol{m})$ contains the connected component of $\{\varphi<\boldsymbol{\sigma}(\boldsymbol{m})\}$ that contains $\widetilde{E}$ and the same proof works.

Let us now suppose that $\boldsymbol{m}$ is of type II and $\boldsymbol{m}'$ is of type I. Then $E(\boldsymbol{m}')\cap\widetilde{E}\neq\varnothing$ with either $\widetilde{E}=E(\boldsymbol{m}'')$ for some $\boldsymbol{m}''\in\mathrm{Cl}(\boldsymbol{m})$ or $\widetilde{E}=\widehat{E}(\boldsymbol{m})$. In both cases one sees easily that $E_-(\boldsymbol{m}')\subset\widetilde{E}$, which proves the result.

The case where both $\boldsymbol{m}$ and $\boldsymbol{m}'$ are of type II is left to the reader. ☐

Let us now give some information on the support of the quasimodes. For $\boldsymbol{m}\in\underline{\mathcal{U}}^{(0)}$, let us introduce the set

$$F_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})=\Bigg(\bigg(F(\boldsymbol{m})\setminus\bigcup_{\boldsymbol{s}\in\mathcal{V}^{(1)}\cap\partial F(\boldsymbol{m})}B(\boldsymbol{s},\epsilon)\bigg)+\overline{B(0,\tilde{\epsilon})}\Bigg)\cup\Bigg(\bigcup_{\boldsymbol{s}\in(\mathcal{U}^{(1)}\setminus\mathcal{V}^{(1)})\cap\partial F(\boldsymbol{m})}\overline{B(\boldsymbol{s},\delta)}\Bigg).\tag{3-22}$$

If $\boldsymbol{m}$ is of type I, it is clear that $F_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})=\overline{E_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})}$ and if $\boldsymbol{m}$ is of type II, one has

$$F_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})=\overline{\widehat{E}_{\epsilon,\tilde{\epsilon},\delta}(\alpha)}\cup\Bigg(\bigcup_{\boldsymbol{m}'\in\mathcal{U}_\alpha^{(0),\,\mathrm{II}}}\overline{E_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m})}\Bigg).$$

From the above construction one deduces the following proposition.

Proposition 3.9. *There exists $\epsilon_0, \delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and all $0 < \tilde{\epsilon} < \epsilon/4 < \epsilon_0/4$, the following hold true*:

(i) *For any $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{U}^{(0)}$*

$$F(\boldsymbol{m}) \cap F(\boldsymbol{m}') = \varnothing \quad \Longrightarrow \quad F_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m}) \cap F_{\epsilon,\tilde{\epsilon},\delta}(\boldsymbol{m}') = \varnothing.$$

(ii) *For any $\alpha \in \underline{A}$ and $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$, one has* $\operatorname{supp}(f_{\boldsymbol{m}}^{(0)}) \subset F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m})$ *and*

$$\text{for all } \boldsymbol{s} \in \mathcal{U}^{(1)} \setminus (\mathcal{V}^{(1)} \cap \partial F(\boldsymbol{m})), \quad d_\varphi f_{\boldsymbol{m}}^{(0)} = 0 \quad \text{in } B(\boldsymbol{s}, \delta).$$

Proof. Observe that

$$F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}) \subset F(\boldsymbol{m}) + \overline{B(0, 2\max(\delta, \tilde{\epsilon}))}.$$

Since $F(\boldsymbol{m})$ and $F(\boldsymbol{m}')$ are compact, the first point of the proposition immediately follows. The second point of the proposition is a direct consequence of (c) of Proposition 3.5. ☐

Recall that the functions $f_{\boldsymbol{m}}^{(0)}$, $\boldsymbol{m} \in \mathcal{U}^{(0)}$, depend on $\epsilon, \tilde{\epsilon}, \delta$ via the definition of the cut-off function $\chi_{\boldsymbol{m}}$. This family is quasiorthonormal in the following sense.

Proposition 3.10. *There exist $\epsilon_0, \delta_0, \beta > 0$ such that for all $0 < \delta < \delta_0$, $0 < \tilde{\epsilon} < \epsilon/4 < \epsilon_0/4$ and all $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{U}^{(0)}$, one has*

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = \delta_{\boldsymbol{m},\boldsymbol{m}'} + \mathcal{O}(e^{-\beta/h}).$$

Proof. Throughout the proof, we assume that $0 < \delta < \delta_0$ and $0 < \tilde{\epsilon} < \epsilon/4 < \epsilon_0/4$ as in Proposition 3.5 and we decrease ϵ_0, δ_0 if necessary.

Let $\boldsymbol{m}, \boldsymbol{m}'$ be two minima.

- Consider first the case where $\boldsymbol{m} \mathcal{R} \boldsymbol{m}'$. If $\boldsymbol{m} = \underline{\boldsymbol{m}}$, one has necessarily $\boldsymbol{m}' = \boldsymbol{m}$ and hence

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = \| f_{\boldsymbol{m}}^{(0)} \|^2 = 1$$

by construction. Consider now the case where $\boldsymbol{m}, \boldsymbol{m}' \neq \underline{\boldsymbol{m}}$ and suppose first that $\boldsymbol{m}$ or $\boldsymbol{m}'$ is of type I. If $\boldsymbol{m} = \boldsymbol{m}'$, the definition of $c(\boldsymbol{m}, h)$ shows that $\| f_{\boldsymbol{m}} \| = 1$. If $\boldsymbol{m} \neq \boldsymbol{m}'$, it follows from (ii) of Proposition 3.9 that $f_{\boldsymbol{m}}^{(0)}$ and $f_{\boldsymbol{m}'}^{(0)}$ are supported in $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m})$ and $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}')$ respectively. Moreover, thanks to (i) of Proposition 3.8, one has $F(\boldsymbol{m}) \cap F(\boldsymbol{m}') \subset \mathcal{V}^{(1)} \cap \partial F(\boldsymbol{m})$. Hence, one can choose ϵ_0 sufficiently small, so that $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}) \cap F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}') = \varnothing$. Therefore, $\operatorname{supp}(f_{\boldsymbol{m}}^{(0)}) \cap \operatorname{supp}(f_{\boldsymbol{m}'}^{(0)}) = \varnothing$ and hence $f_{\boldsymbol{m}}^{(0)}$ and $f_{\boldsymbol{m}'}^{(0)}$ are orthogonal.

Suppose now that $\boldsymbol{m}$ and $\boldsymbol{m}'$ are both of type II. Then, we can write

$$f_{\boldsymbol{m}}^{(0)}(x) = h^{-d/4} \sum_{\nu_1 \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_{\boldsymbol{m}}(\nu_1) c(\nu_1, h) \hat{\chi}_{\nu_1}(x) e^{(\varphi(\hat{\boldsymbol{m}}(\boldsymbol{m})) - \varphi(x))/h},$$

$$f_{\boldsymbol{m}'}^{(0)}(x) = h^{-d/4} \sum_{\nu_2 \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_{\boldsymbol{m}'}(\nu_2) c(\nu_2, h) \hat{\chi}_{\nu_2}(x) e^{(\varphi(\hat{\boldsymbol{m}}(\boldsymbol{m})) - \varphi(x))/h}.$$

Since, for $\nu_2 \neq \nu_1$, $\hat{\chi}_{\nu_1}$ and $\hat{\chi}_{\nu_2}$ have again disjoint support for $\epsilon_0, \delta_0 > 0$ small enough, we get

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = h^{-d/2} \sum_{\nu \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_{\boldsymbol{m}}(\nu)\theta_{\boldsymbol{m}'}(\nu)|c(\nu,h)|^2 \int_X |\hat{\chi}_\nu(x)|^2 e^{2(\varphi(\hat{\boldsymbol{m}}(\boldsymbol{m}))-\varphi(x))/h}\, dx$$
$$= \langle \theta_{\boldsymbol{m}}, \theta_{\boldsymbol{m}'} \rangle_{\mathcal{F}_\alpha} = \delta_{\boldsymbol{m},\boldsymbol{m}'}.$$

This shows in particular that $\| f_{\boldsymbol{m}}^{(0)} \|_{L^2} = 1$ for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$.

• Suppose now, that $\boldsymbol{m}' \notin \mathrm{Cl}(\boldsymbol{m})$ (in particular $\boldsymbol{m} \neq \boldsymbol{m}'$). If $\sigma(\boldsymbol{m}') = \sigma(\boldsymbol{m})$ then $F(\boldsymbol{m}) \cap F(\boldsymbol{m}') = \varnothing$ thanks to (ii.a) of Proposition 3.8, and (i) of Proposition 3.9 implies that $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}) \cap F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}') = \varnothing$. Then, the first part of (ii) of Proposition 3.9 proves that $f_{\boldsymbol{m}}^{(0)}$ and $f_{\boldsymbol{m}'}^{(0)}$ are orthogonal.

Consider now the case where $\sigma(\boldsymbol{m}) \neq \sigma(\boldsymbol{m}')$; say, $\sigma(\boldsymbol{m}) > \sigma(\boldsymbol{m}')$. From (ii.b) of Proposition 3.8, we know that either $F(\boldsymbol{m}')$ is disjoint from $F(\boldsymbol{m})$ or $F(\boldsymbol{m}') \subset \mathring{F}(\boldsymbol{m})$. In the first case, we get immediately $\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = 0$ by the same argument as before. Suppose we are in the second situation, that is $F(\boldsymbol{m}') \subset \mathring{F}(\boldsymbol{m})$. By definition, we have $\varphi(\boldsymbol{m}) \le \varphi(\boldsymbol{m}')$, and by taking $\epsilon_0, \delta_0 > 0$ small enough we can ensure that $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}') \subset \mathring{F}_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m})$.

Suppose first that $\varphi(\boldsymbol{m}) < \varphi(\boldsymbol{m}')$. A priori we don't know if $\boldsymbol{m}, \boldsymbol{m}'$ are of type I or II. However, since $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}') \subset \mathring{F}_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m})$,

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = \int_{F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}')} f_{\boldsymbol{m}}^{(0)}(x) f_{\boldsymbol{m}'}^{(0)}(x)\, dx$$

and

$$(f_{\boldsymbol{m}}^{(0)})_{|F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}')} = \tilde{c}(\boldsymbol{m},h) h^{-d/4} e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}, \tag{3-23}$$

where the constant $\tilde{c}(\boldsymbol{m},h)$ is uniformly bounded with respect to h. This is clear if $\boldsymbol{m}$ is of type I. If $\boldsymbol{m}$ is of type II and, say, $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$, then $F(\boldsymbol{m}') \subset \mathring{F}(\boldsymbol{m})$ implies that there exists $\nu \in \widehat{\mathcal{U}}_\alpha^{(0)}$ such that $F(\boldsymbol{m}') \subset E(\nu)$ (or $\widehat{E}(\nu)$). Then the general formula (3-14) shows (3-23). Moreover, by construction, there exists a cut-off function $\psi \in \mathscr{C}_c^\infty(\mathring{F}_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}))$ independent of h such that $\inf_{\operatorname{supp}\psi} \varphi = \varphi(\boldsymbol{m}')$ and

$$|f_{\boldsymbol{m}'}^{(0)}(x)| \le h^{-d/4} \psi(x) e^{(\varphi(\boldsymbol{m}')-\varphi(x))/h}$$

and it follows that

$$|\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle| \le C h^{-d/2} \int \psi(x) e^{(\varphi(\boldsymbol{m}')+\varphi(\boldsymbol{m})-2\varphi(x))/h}\, dx \le C' h^{-d/2} e^{(\varphi(\boldsymbol{m})-\varphi(\boldsymbol{m}'))/h}.$$

Since $\varphi(\boldsymbol{m}') > \varphi(\boldsymbol{m})$, this proves the result.

It remains to study the case where $\varphi(\boldsymbol{m}) = \varphi(\boldsymbol{m}')$. Let $\alpha, \alpha' \in \mathcal{A}$ be such that $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{m}' \in \mathcal{U}_{\alpha'}^{(0)}$. From the above assumption, we also have $\sigma(\boldsymbol{m}) > \sigma(\boldsymbol{m}')$ and $F(\boldsymbol{m}') \subset \mathring{F}(\boldsymbol{m})$. Since $\sigma(\boldsymbol{m}) > \sigma(\boldsymbol{m}')$ and $\varphi(\boldsymbol{m}) = \varphi(\boldsymbol{m}')$, we know $f_{\boldsymbol{m}'}^{(0)}$ is necessarily of type II. It has the form (3-14) and since $F_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m}') \subset \mathring{F}_{\epsilon,2\tilde{\epsilon},2\delta}(\boldsymbol{m})$, (3-23) still holds true. Hence, we get

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)} \rangle = \tilde{c}(\boldsymbol{m},h) h^{-d/2} \sum_{\nu \in \widehat{\mathcal{U}}_{\alpha'}^{(0),\mathrm{II}}} \theta_{\boldsymbol{m}'}(\nu) c(\nu,h) \int \hat{\chi}_\nu(x) e^{2(\varphi(x)-\varphi(\boldsymbol{m}))/h}\, dx. \tag{3-24}$$

On the other hand, by a standard argument based on the Laplace method, we know that there exist $r > 0$ and $\beta > 0$ such that for all $\nu \in \widehat{\mathcal{U}}_{\alpha'}^{(0),\mathrm{II}}$, we have

$$\begin{aligned} h^{-d/2}c(\nu,h)\int \hat{\chi}_\nu(x)e^{2(\varphi(x)-\varphi(\boldsymbol{m}))/h}\,dx &= h^{-d/2}c(\nu,h)\sum_{\nu'\in H(\nu)}\int_{B(\nu',r)} e^{2(\varphi(x)-\varphi(\boldsymbol{m}))/h}\,dx+\mathcal{O}(e^{-\beta/h}) \\ &= h^{-d/2}c(\nu,h)\int |\hat{\chi}_\nu(x)|^2 e^{2(\varphi(x)-\varphi(\boldsymbol{m}))/h}\,dx+\mathcal{O}(e^{-\beta/h}) \\ &= \frac{1}{c(\nu,h)}+\mathcal{O}(e^{-\beta/h}). \end{aligned}$$

Plugging this in (3-24), we get

$$\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)}\rangle = \tilde{c}(\boldsymbol{m},h)\sum_{\nu\in\widehat{\mathcal{U}}_{\alpha'}^{(0),\mathrm{II}}}\theta_{\boldsymbol{m}'}(\nu)\frac{1}{c(\nu,h)}+\mathcal{O}(e^{-\beta/h}) = \frac{\tilde{c}(\boldsymbol{m},h)}{c_0^{\alpha'}(h)}\langle\theta_{\boldsymbol{m}'},\theta_0^{\alpha'}\rangle_{\mathcal{F}_{\alpha'}}+\mathcal{O}(e^{-\beta/h}). \tag{3-25}$$

Since $\theta_{\boldsymbol{m}'}$ is orthogonal to $\theta_0^{\alpha'}$ by construction, the first term of the right-hand side above vanishes and we get $\langle f_{\boldsymbol{m}}^{(0)}, f_{\boldsymbol{m}'}^{(0)}\rangle = \mathcal{O}(e^{-\beta/h})$. □

We end this section by giving an exponential estimate of the action of $d_{\varphi,h}$ on the quasimodes.

Lemma 3.11. *There exists $C > 0$ such that for all $\epsilon > 0$ small enough, we have*

$$d_{\varphi,h} f_{\boldsymbol{m}}^{(0)} = \mathcal{O}(e^{-(S(\boldsymbol{m})-C\epsilon)/h})$$

for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$.

Proof. The result is classical, but since the quasimodes $f_{\boldsymbol{m}}^{(0)}$ are slightly different from the usual ones, we have to check the estimates. Let $\boldsymbol{m} \in \mathcal{U}^{(0)}$ and let us compute $d_{\varphi,h} f_{\boldsymbol{m}}^{(0)}$.

If $\boldsymbol{m} = \underline{\boldsymbol{m}}$, then $d_{\varphi,h} f_{\boldsymbol{m}}^{(0)} = 0$ and there is nothing to do.

Suppose now that $\boldsymbol{m} \neq \underline{\boldsymbol{m}}$. From (3-17), one has

$$d_{\varphi,h} f_{\boldsymbol{m}}^{(0)}(x) = h^{1-d/4}\sum_{\boldsymbol{m}'\in\widehat{\mathcal{U}}_\alpha^{(0)}}\theta_{\boldsymbol{m}}(\boldsymbol{m}')c(\boldsymbol{m}',h)\nabla\hat{\chi}_{\boldsymbol{m}'}(x)e^{(\varphi(\boldsymbol{m})-\varphi(x))/h}.$$

All terms of the above sum corresponding to $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ are $\mathcal{O}(e^{-(S(\boldsymbol{m})-C\epsilon)/h})$ by (b) of Proposition 3.5. The only new term is the one corresponding to $\hat{\boldsymbol{m}}(\boldsymbol{m})$. Since $\hat{\chi}_{\hat{\boldsymbol{m}}} \in C_c^\infty(\widehat{E}_{\epsilon,2\tilde{\epsilon},2\delta})$ and is equal to 1 on $\widehat{E}_{\epsilon,\tilde{\epsilon},\delta}$, we have again

$$\varphi(x)-\varphi(\hat{\boldsymbol{m}}) = \varphi(x)-\varphi(\boldsymbol{m}) \geq S(\boldsymbol{m})-C\epsilon$$

on $\operatorname{supp}(\nabla\hat{\chi}_{\hat{\boldsymbol{m}}})$ and the proof is complete. □

3C. ***Quasimodes for 1-forms.*** This section is devoted to the quasimodes associated to low-lying eigenvalues of $\Delta_\varphi^{(1)}$. The construction of these quasimodes was done in [Helffer and Sjöstrand 1985] and we refer to that paper for all the proofs. Here, we just describe the main properties of these functions. In this section $\omega_{\boldsymbol{s}}$ denotes a small neighborhood of $\boldsymbol{s} \in \mathcal{U}^{(1)}$ that may be chosen as small as needed independently of ϵ_0 fixed in previous sections.

Given any saddle point $s \in \mathcal{U}^{(1)}$, and any appropriate open neighborhood ω_s of s, let $P_{\varphi,s}$ denote the operator $\Delta_\varphi^{(1)}$ restricted to ω_s with Dirichlet boundary conditions. Let u_s denote a normalized fundamental state of $P_{\varphi,s}$. The quasimodes $f_s^{(1)}$ are then defined by

$$f_s^{(1)}(x) := \epsilon_0 \|\psi_s u_s\|^{-1} \psi_s(x) u_s(x), \tag{3-26}$$

where ψ_s is a well-chosen C_0^∞ localization function supported in ω_s and equal to 1 near s and $\epsilon_0 = \pm 1$ will be fixed later. By taking ω_s sufficiently small, we can ensure that the $f_s^{(1)}$ have disjoint supports, and thanks to (c) of Proposition 3.5, we can also shrink ω_s so that

$$\text{for all } s \in \mathcal{U}^{(1)} \setminus \mathcal{V}^{(1)}, \ \text{ for all } m \in \mathcal{U}^{(0)}, \quad (s \in \operatorname{supp}(\chi_m) \implies \chi_m = 1 \text{ on } \omega_s). \tag{3-27}$$

Observe that this choice of ω_s depends on δ_0 but not on ϵ_0. From this construction, we immediately deduce that

$$\langle f_s^{(1)}, f_{s'}^{(1)} \rangle = \delta_{s,s'}, \tag{3-28}$$

and hence the family $\{f_s^{(1)} : s \in \mathcal{U}^{(1)}\}$ is a free family of 1-forms. From [Helffer 1988, Proposition 5.2.6], one knows that the eigenvalues of $P_{\varphi,s}$ are exponentially small. Using Agmon estimates, it follows that there exists $\beta > 0$ independent of ϵ such that

$$\Delta_\varphi^{(1)} f_s^{(1)} = \mathcal{O}(e^{-\beta/h}). \tag{3-29}$$

Combined with the spectral theorem, this proves that the n_1 eigenvalues of $\Delta_\varphi^{(1)}$ in $[0, \epsilon_1 h]$ are actually $\mathcal{O}(e^{-\beta/h})$ (see [Helffer 1988, Proposition 5.2.5] for details).

Furthermore, Theorem 2.5 of [Helffer and Sjöstrand 1985] implies that these quasimodes have a WKB expansion given by

$$f_s^{(1)}(x) = \epsilon_0 h^{-d/4} \psi_s(x) b_s^{(1)}(x,h) e^{-\varphi_{+,s}(x)/h}, \tag{3-30}$$

where $b_s^{(1)}(x,h)$ is a 1-form having a semiclassical asymptotic, and $\varphi_{+,s}$ is the phase generating the outgoing manifold of $|\xi|^2 - |\nabla_x \varphi(x)|^2$ at $(s,0)$ (see [Dimassi and Sjöstrand 1999, Chapter 3] for details on such constructions). In particular, the phase function $\varphi_{+,s}$ satisfies the eikonal equation $|\nabla_x \varphi_{+,s}|^2 = |\nabla_x \varphi|^2$ and $\varphi_{+,s}(x) \asymp |x-s|^2$ near s (the notation $\asymp$ was defined in the paragraph before Section 2A). For other properties of $\varphi_{+,s}$ we refer to [Helffer and Sjöstrand 1985].

3D. ***Projection onto the eigenspaces.*** The next step in our analysis is to project the preceding quasimodes onto the generalized eigenspaces associated to exponentially small eigenvalues. Recall that we have built in the preceding section quasimodes $f_m^{(0)}$, $m \in \mathcal{U}^{(0)}$, with good orthogonality properties. To each of these quasimodes we will associate a function in $E^{(0)}$, the eigenspace associated to $o(h)$ eigenvalues. For this, we first define the spectral projector

$$\Pi^{(0)} = \frac{1}{2\pi i} \int_\gamma (z - \Delta_\varphi^{(0)})^{-1} \, dz, \tag{3-31}$$

where $\gamma = \partial B(0, \epsilon_0 h)$ and $\epsilon_0 > 0$ is such that $\sigma(\Delta_\varphi) \cap [0, 2\epsilon_0 h] \subset [0, e^{-C/h}]$. From the fact that $\Delta_\varphi^{(0)}$ is selfadjoint, we get that

$$\|\Pi^{(0)}\| = 1.$$

We now introduce the projection of the quasimodes constructed above, $e_{\boldsymbol{m}}^{(0)} = \Pi^{(0)}(f_{\boldsymbol{m}}^{(0)})$. We have the following:

Lemma 3.12. *The system $(e_{\boldsymbol{m}}^{(0)})_{\boldsymbol{m}\in\mathcal{U}^{(0)}}$ is free and spans $E^{(0)}$. Additionally, there exists $\beta > 0$ independent of ϵ_0 such that for all $0 < \tilde{\epsilon} < \epsilon/4 < \epsilon_0/4$, one has*

$$e_{\boldsymbol{m}}^{(0)} = f_{\boldsymbol{m}}^{(0)} + \mathcal{O}(e^{-\beta/h}) \quad \text{and} \quad \langle e_{\boldsymbol{m}}^{(0)}, e_{\boldsymbol{m}'}^{(0)} \rangle = \delta_{\boldsymbol{m},\boldsymbol{m}'} + \mathcal{O}(e^{-\beta/h})$$

for all $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{U}^{(0)}$.

Proof. The argument is very classical. We recall it for reader's convenience. One has

$$\begin{aligned} e_{\boldsymbol{m}}^{(0)} - f_{\boldsymbol{m}}^{(0)} = (\Pi^{(0)} - \mathrm{Id}) f_{\boldsymbol{m}}^{(0)} &= \frac{1}{2\pi i} \int_\gamma ((z - \Delta_\varphi^{(0)})^{-1} - z^{-1}) f_{\boldsymbol{m}}^{(0)}\, dz \\ &= \frac{1}{2\pi i} \int_\gamma (z - \Delta_\varphi^{(0)})^{-1} z^{-1} \Delta_\varphi^{(0)} f_{\boldsymbol{m}}^{(0)}\, dz. \end{aligned} \tag{3-32}$$

Since $(z - \Delta_\varphi^{(0)})^{-1} = \mathcal{O}(h^{-1})$ on γ, it follows from Lemma 3.11 that $e_{\boldsymbol{m}}^{(0)} - f_{\boldsymbol{m}}^{(0)} = \mathcal{O}(e^{-\beta/h})$ for some $\beta > 0$. This proves the first point. Combining this information with Proposition 3.10 we get immediately the second point. ☐

We can do a similar study for $\Delta_\varphi^{(1)}$, for which we know that the n_1 eigenvalues lying in $[0, \epsilon_1 h]$ are actually $\mathcal{O}(e^{-\alpha'/h})$. To the family of quasimodes $(f_{\boldsymbol{s}}^{(1)})_{\boldsymbol{s}\in\mathcal{U}^{(1)}}$, we now associate a family of functions in $E^{(1)}$, the eigenspace associated to eigenvalues of $\Delta_\varphi^{(1)}$ in $[0, \epsilon_1 h]$. Thanks to the spectral properties of the selfadjoint operator $\Delta_\varphi^{(1)}$, its spectral projector onto $E^{(1)}$ is given by

$$\Pi^{(1)} = \frac{1}{2\pi i} \int_\gamma (z - \Delta_\varphi^{(1)})^{-1}\, dz, \tag{3-33}$$

where $\gamma = \partial B(0, \epsilon_1 h)$ with ϵ_1 defined above. In the sequel, we write $e_{\boldsymbol{s}}^{(1)} = \Pi^{(1)}(f_{\boldsymbol{s}}^{(1)})$. The family $(e_{\boldsymbol{s}}^{(1)})_{\boldsymbol{s}}$ satisfies the following estimates.

Lemma 3.13. *The system $(e_{\boldsymbol{s}}^{(1)})_{\boldsymbol{s}\in\mathcal{U}^{(1)}}$ is free and spans $E^{(1)}$. Additionally, we have*

$$e_{\boldsymbol{s}}^{(1)} = f_{\boldsymbol{s}}^{(1)} + \mathcal{O}(e^{-\beta'/h}) \quad \text{and} \quad \langle e_{\boldsymbol{s}}^{(1)}, e_{\boldsymbol{s}'}^{(1)} \rangle = \delta_{\boldsymbol{s},\boldsymbol{s}'} + \mathcal{O}(e^{-\beta'/h}),$$

with $\beta' > 0$ independent of ϵ.

Proof. Using the orthonormality of the $f_j^{(1)}$ and (3-29), the proof is the same as that of Lemma 3.12. ☐

4. Preliminaries for singular values analysis

This section is a preparation for the study of the singular values of the operator $\mathcal{L} : E^{(0)} \to E^{(1)}$ defined below (2-20). We simplify the forthcoming study by several reductions and changes of basis. Let us denote by $\mathcal{L}^{\pi}$ the $n_1 \times n_0$ matrix given by

$$\mathcal{L}^{\pi}_{\boldsymbol{s},\boldsymbol{m}} = \langle e^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} e^{(0)}_{\boldsymbol{m}} \rangle \quad \text{for all } \boldsymbol{s} \in \mathcal{U}^{(1)},\ \boldsymbol{m} \in \mathcal{U}^{(0)}, \tag{4-1}$$

with $e^{(1)}_{\boldsymbol{s}}$, $e^{(0)}_{\boldsymbol{m}}$ defined in the preceding section. Since $(e^{(0)}_{\boldsymbol{m}})$ and $(e^{(1)}_{\boldsymbol{s}})$ are almost orthonormal bases (thanks to Lemmas 3.12 and 3.13), this matrix is close to the matrix of the operator $\mathcal{L}$ in these bases. We first work on the matrix $\mathcal{L}^{\pi}$.

Recall that $\underline{\boldsymbol{m}}$ denotes the absolute minimum of φ associated to the connected component $E(\underline{\boldsymbol{m}}) = X$. Since $\Delta^{(0)}_{\varphi} e_{\underline{\boldsymbol{m}}} = 0$, the nonzero singular values of $\mathcal{L}^{\pi}$ are exactly the singular values of the reduced matrix $\mathcal{L}^{\pi,\prime}$ defined by $\mathcal{L}^{\pi,\prime}_{\boldsymbol{s},\boldsymbol{m}} = \mathcal{L}^{\pi}_{\boldsymbol{s},\boldsymbol{m}}$ for all $\boldsymbol{s} \in \mathcal{U}^{(1)}$, $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$ with $\underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0)} \setminus \{\underline{\boldsymbol{m}}\}$.

Lemma 4.1. *There exists $\beta'' > 0$ such that for $\epsilon > 0$ sufficiently small, one has*

$$\mathcal{L}^{\pi,\prime}_{\boldsymbol{s},\boldsymbol{m}} = \langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle + \mathcal{O}(e^{-(S(\boldsymbol{m})+\beta'')/h})$$

for all $\boldsymbol{s} \in \mathcal{U}^{(1)}$, $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$.

Proof. The trick to get the good error estimate above is now well-known (see for instance the proof of Proposition 5.8 in [Hérau, Hitrik and Sjöstrand 2011]) but we recall the proof for reader's convenience. Let $\boldsymbol{s} \in \mathcal{U}^{(1)}$, $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$; then thanks to (2-18) we have

$$\begin{aligned}\langle e^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} e^{(0)}_{\boldsymbol{m}} \rangle &= \langle e^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} \Pi^{(0)} f^{(0)}_{\boldsymbol{m}} \rangle = \langle e^{(1)}_{\boldsymbol{s}}, \Pi^{(1)} d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle = \langle e^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle \\ &= \langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle + \langle e^{(1)}_{\boldsymbol{s}} - f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle.\end{aligned}$$

But from Lemmas 3.11 and 3.13 and the Cauchy–Schwarz inequality one gets

$$|\langle e^{(1)}_{\boldsymbol{s}} - f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle| \le C e^{-(\beta' + S(\boldsymbol{m}) - C\epsilon)/h}.$$

Since β' is independent of ϵ, one can conclude by taking ϵ small enough and $\beta'' = \beta'/2$. □

Let us denote by $\mathcal{L}^{\mathrm{bkw}} \in \mathscr{M}(\mathcal{U}^{(0)}, \mathcal{U}^{(1)})$ the matrix defined by

$$\mathcal{L}^{\mathrm{bkw}}_{\boldsymbol{s},\boldsymbol{m}} = \langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle \quad \text{for all } \boldsymbol{s} \in \mathcal{U}^{(1)},\ \boldsymbol{m} \in \mathcal{U}^{(0)}. \tag{4-2}$$

Of course, the first column of this matrix is identically zero and it is more interesting to consider the matrix $\mathcal{L}^{\mathrm{bkw},\prime} \in \mathscr{M}(\underline{\mathcal{U}}^{(0)}, \mathcal{U}^{(1)})$ defined by

$$\mathcal{L}^{\mathrm{bkw},\prime}_{\boldsymbol{s},\boldsymbol{m}} = \langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle \quad \text{for all } \boldsymbol{s} \in \mathcal{U}^{(1)},\ \boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}. \tag{4-3}$$

As we shall see later, the singular values of $\mathcal{L}^{\pi,\prime}$ and $\mathcal{L}^{\mathrm{bkw},\prime}$ are exponentially close and it is natural to study the matrix $\mathcal{L}^{\mathrm{bkw},\prime}$. For $\boldsymbol{s} \in \mathcal{U}^{(1)} \setminus \mathcal{V}^{(1)}$ and $\boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}$, thanks to (ii) of Proposition 3.9 one has $d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} = 0$ near $\boldsymbol{s}$, and hence

$$\langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle = 0. \tag{4-4}$$

Therefore, the singular values of $\mathcal{L}^{\mathrm{bkw},\prime}$ are equal to the singular values of the reduced matrix $\mathcal{L}^{\mathrm{bkw},\prime\prime} \in \mathscr{M}(\underline{\mathcal{U}}^{(0)}, \mathcal{V}^{(1)})$ defined by

$$\mathcal{L}^{\mathrm{bkw},\prime\prime}_{\boldsymbol{s},\boldsymbol{m}} = \langle f^{(1)}_{\boldsymbol{s}}, d_{\varphi,h} f^{(0)}_{\boldsymbol{m}} \rangle \quad \text{for all } \boldsymbol{s} \in \mathcal{V}^{(1)},\ \boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}. \tag{4-5}$$

In order to study this matrix, we need to introduce a new enumeration of critical points. Let us start with some abstract notation. Assume that $(\mathcal{I}, \leq)$ and $(\mathcal{J}, \leq)$ are two totally ordered sets and let $A = (a_{ij})_{i\in\mathcal{I}, j\in\mathcal{J}}$ be the associated matrix (with i, j enumerated in increasing order). Assume that we have partitions $\mathcal{P}_{\mathcal{I}}$ and $\mathcal{P}_{\mathcal{J}}$ of $\mathcal{I}$ and $\mathcal{J}$ respectively

$$\mathcal{P}_{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_{N_{\mathcal{I}}}) \quad \text{and} \quad \mathcal{P}_{\mathcal{J}} = (\mathcal{J}_1, \dots, \mathcal{J}_{N_{\mathcal{J}}}).$$

Assume that each partition admits a total order $\preceq$ (that is, we can compare the subsets $\mathcal{I}_i$). Then we get a total order $\preceq$ on $\mathcal{I}$ (resp. $\mathcal{J}$) by using the associated lexicographical order:

$$i \preceq j \quad \Longleftrightarrow \quad (\exists \mathcal{I}_\alpha \preceq \mathcal{I}_\beta,\ i \in \mathcal{I}_\alpha \text{ and } j \in \mathcal{I}_\beta) \text{ or } (\exists \mathcal{I}_\alpha,\ i, j \in \mathcal{I}_\alpha \text{ and } i \leq j).$$

Hence, there exists a unique $\alpha : (\mathcal{I}, \leq) \to (\mathcal{I}, \preceq)$ which is strictly increasing (and hence bijective). Similarly, there is a unique $\beta : (\mathcal{J}, \leq) \to (\mathcal{J}, \preceq)$ which is strictly increasing. We denote by $A_{\mathcal{P}_{\mathcal{I}}, \mathcal{P}_{\mathcal{J}}}$ the matrix $(a_{\alpha(i),\beta(j)})_{i\in\mathcal{I}, j\in\mathcal{J}}$. This matrix is obtained from A by intertwining the basis vector; hence it has exactly the same singular values.

Let us go back to the matrix $\mathcal{L}^{\text{bkw},\prime\prime}$. Consider the partitions of $\underline{\mathcal{U}}^{(0)}$ and $\mathcal{V}^{(1)}$ given by

$$\mathcal{P}^{(0)} = \{\mathcal{U}_\alpha^{(0)},\ \alpha \in \underline{\mathcal{A}}\} \quad \text{and} \quad \mathcal{P}^{(1)} = \{\mathcal{V}_\beta^{(1)},\ \beta \in \underline{\mathcal{A}}\}.$$

At this stage of our analysis, we do not need any specific choice of order on these partitions. We just endow $\underline{\mathcal{A}}$ with any total order and for all $\alpha, \beta \in \underline{\mathcal{A}}$ we choose any arbitrary total order on $\mathcal{U}_\alpha^{(0)}$ and $\mathcal{V}_\beta^{(1)}$. This gives an order on the above partitions and we denote by $\mathscr{L} = (\mathscr{L}^{\alpha,\beta})_{\alpha,\beta\in\underline{\mathcal{A}}}$ the matrix $\mathcal{L}^{\text{bkw},\prime\prime}$ associated to these partitions. Observe here that each $\mathscr{L}^{\alpha,\beta}$ is itself a matrix $\mathscr{L}^{\alpha,\beta} = (\mathscr{L}^{\alpha,\beta}_{\boldsymbol{s},\boldsymbol{m}})_{\boldsymbol{s}\in\mathcal{V}_\beta^{(1)}, \boldsymbol{m}\in\mathcal{U}_\alpha^{(0)}}$.

Lemma 4.2. *For all $\alpha \neq \beta$, we have $\mathscr{L}^{\alpha,\beta} = 0$.*

Proof. Let $\alpha, \beta \in \underline{\mathcal{A}}$ such that $\alpha \neq \beta$ and let $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ and $\boldsymbol{s} \in \mathcal{V}_\beta^{(1)}$. If $\boldsymbol{\sigma}(\alpha) = \boldsymbol{\sigma}(\beta)$ then $\alpha \neq \beta$ implies that $\boldsymbol{s} \notin F(\boldsymbol{m})$. Shrinking if necessary (by taking $\epsilon_0, \delta_0 > 0$ small enough) the support of $f_{\boldsymbol{m}}^{(0)}$ and $f_{\boldsymbol{s}}^{(1)}$, it follows that these functions have disjoint supports so that their scalar product vanishes.

If $\boldsymbol{\sigma}(\alpha) \neq \boldsymbol{\sigma}(\beta)$, then by construction $d_{\varphi,h} f_{\boldsymbol{m}}^{(0)}$ is supported near $\{\varphi = \boldsymbol{\sigma}(\alpha)\}$ whereas $e_{\boldsymbol{s}}^{(1)}$ is supported near $\{\varphi = \boldsymbol{\sigma}(\beta)\}$. Since this two sets are disjoint we get $\langle f_{\boldsymbol{s}}^{(1)}, d_{\varphi,h} e_{\boldsymbol{m}}^{(0)} \rangle = 0$ and the proof is complete. □

From this lemma we deduce that the matrix $\mathscr{L}$ admits a block-diagonal structure

$$\mathscr{L} = \operatorname{diag}(\mathscr{L}^\alpha, \alpha \in \underline{\mathcal{A}}), \tag{4-6}$$

with $\mathscr{L}^\alpha := \mathscr{L}^{\alpha,\alpha}$. Recall from Definition 3.7 that for any $\alpha \in \underline{\mathcal{A}}$, the matrix $\mathscr{T}^\alpha \in \mathscr{M}(\mathcal{U}_\alpha^{(0)}, \widehat{\mathcal{U}}_\alpha^{(0)})$ is given by $\mathscr{T}^\alpha = (\theta_{\boldsymbol{m}}^\alpha(\boldsymbol{m}'))_{\boldsymbol{m}'\in\widehat{\mathcal{U}}_\alpha^{(0)}, \boldsymbol{m}\in\mathcal{U}_\alpha^{(0)}}$. We have the following factorization result on $\mathscr{L}^\alpha$.

Lemma 4.3. *We have $\mathscr{L}^\alpha = \widehat{\mathscr{L}}^\alpha \mathscr{T}^\alpha$, where the matrix $\widehat{\mathscr{L}}^\alpha = (\hat{\ell}^\alpha_{\boldsymbol{s},\boldsymbol{m}'})_{\boldsymbol{s},\boldsymbol{m}'} \in \mathscr{M}(\widehat{\mathcal{U}}_\alpha^{(0)}, \mathcal{V}_\alpha^{(1)})$ is given by*

$$\hat{\ell}^\alpha_{\boldsymbol{s},\boldsymbol{m}'} = \langle f_{\boldsymbol{s}}^{(1)}, d_{\varphi,h} g_{\boldsymbol{m}'}^{(0)} \rangle \quad \textit{for all } \boldsymbol{s} \in \mathcal{V}_\alpha^{(1)},\ \boldsymbol{m}' \in \widehat{\mathcal{U}}_\alpha^{(0)},$$

with $g_{\boldsymbol{m}'}^{(0)}(x) = h^{-d/4} c(\boldsymbol{m}', h) \hat{\chi}_{\boldsymbol{m}'}(x) e^{\varphi(\boldsymbol{m}')-\varphi(x)/h}$.

Proof. Let $s \in \mathcal{V}_\alpha^{(1)}$, $m \in \mathcal{U}_\alpha^{(0)}$. From (3-17), one has

$$\langle f_s^{(1)}, d_{\varphi,h} f_m^{(0)} \rangle = h^{-d/4} \sum_{m' \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_m^\alpha(m') c(m',h) \langle f_s^{(1)}, h\, d\, \hat{\chi}_{m'}(x) e^{(\varphi(m)-\varphi(x))/h} \rangle.$$

Moreover, the function φ being constant on $\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$, we can replace $\varphi(m)$ by $\varphi(m')$ in the above identity and it follows that

$$\langle f_s^{(1)}, d_{\varphi,h} f_m^{(0)} \rangle = \sum_{m' \in \widehat{\mathcal{U}}_\alpha^{(0)}} \theta_m^\alpha(m') \langle f_s^{(1)}, d_{\varphi,h} g_{m'}^{(0)} \rangle,$$

which is exactly the result to be proved. □

One of the crucial points of our analysis is to compute the coefficient $\hat{\ell}_{s,m}^\alpha$. Given $m \in \widehat{\mathcal{U}}_\alpha^{(0)}$, we define

$$h_\varphi(m) = \Bigg(\sum_{m' \in \widehat{H}_\alpha(m)} |\det \operatorname{Hess} \varphi(m)|^{-1/2} \Bigg)^{-1/2}, \tag{4-7}$$

with $\widehat{H}_\alpha(m)$ defined in (3-12). One has clearly $h_\varphi(m) = \pi^{d/4} \gamma_0(m)$, with γ_0 given by (3-13). Moreover, in the case where $H(m) = \{m\}$, one has $h_\varphi(m) = |\det \operatorname{Hess} \varphi(m)|^{1/4}$. Given $s \in \mathcal{V}^{(1)}$, we denote by $\hat{\lambda}_1(s)$ the unique negative eigenvalue of $\operatorname{Hess} \varphi(s)$. In order to keep uniform notation, we also extend the definition (4-7) to saddle points by

$$h_\varphi(s) = |\det \operatorname{Hess} \varphi(s)|^{1/4}.$$

Eventually, we introduce the diagonal matrix $\widehat{\Omega}^\alpha \in \mathscr{M}(\widehat{\mathcal{U}}_\alpha^{(0)}, \widehat{\mathcal{U}}_\alpha^{(0)})$ defined by

$$\widehat{\Omega}^\alpha f(m) = e^{-\widehat{S}(m)/h} f(m) \quad \text{for all } m \in \widehat{\mathcal{U}}_\alpha^{(0)}, \tag{4-8}$$

with $\widehat{S}(m) = \sigma(\alpha) - \varphi(m)$. For $m \in \mathcal{U}_\alpha^{(0)}$, one has of course $\sigma(\alpha) = \sigma(m)$ and hence $\widehat{S}(m) = S(m)$ but this fails to be true for $m = \hat{m}(\alpha)$. We then define the rescaled matrix $\widetilde{\mathscr{L}}^\alpha = (\tilde{\ell}_{s,m}^\alpha) \in \mathscr{M}(\widehat{\mathcal{U}}_\alpha^{(0)}, \mathcal{V}_\alpha^{(1)})$ by

$$\widehat{\mathscr{L}}^\alpha = \widetilde{\mathscr{L}}^\alpha \widehat{\Omega}^\alpha;$$

i.e.,

$$\tilde{\ell}_{s,m}^\alpha = e^{\widehat{S}(m)/h} \hat{\ell}_{s,m}^\alpha \quad \text{for all } s \in \mathcal{V}_\alpha^{(1)},\ m \in \widehat{\mathcal{U}}_\alpha^{(0)}. \tag{4-9}$$

Going back to the matrix $\mathscr{L}^\alpha$, one has

$$\mathscr{L}^\alpha = \widetilde{\mathscr{L}}^\alpha \widehat{\Omega}^\alpha \mathscr{T}^\alpha.$$

Moreover, as already noticed below Definition 3.7, one has $\mathscr{T}^\alpha f(m) = f(m)$ for any f supported on $\mathcal{U}_\alpha^{(0),\mathrm{I}}$. Hence we get

$$\mathscr{L}^\alpha = \widetilde{\mathscr{L}}^\alpha \mathscr{T}^\alpha \Omega^\alpha, \tag{4-10}$$

with $\Omega^\alpha \in \mathscr{M}(\mathcal{U}_\alpha^{(0)}, \mathcal{U}_\alpha^{(0)})$ defined by $\Omega^\alpha f(m) = e^{-S(m)/h} f(m)$. The following lemma gives an asymptotic expansion of the matrix $\widetilde{\mathscr{L}}^\alpha$. We recall that $m_1(s)$ and $m_2(s)$ were defined in Lemma 3.3.

Lemma 4.4. *Let $\alpha \in \underline{A}$ and $s \in \mathcal{V}_\alpha^{(1)}$, $m \in \widehat{\mathcal{U}}_\alpha^{(0)}$. The following hold true*:

(i) *If $m \notin \{m_1(s), m_2(s)\}$, then $\tilde{\ell}_{s,m}^\alpha = 0$.*

(ii) *The coefficients* $\tilde{\ell}^{\alpha}_{s,m}$ *admit a classical expansion* $\tilde{\ell}^{\alpha}_{s,m} \sim h^{1/2}\sum_{k\geq 0} h^k \tilde{\ell}^{\alpha,k}_{s,m}$. *Moreover, one can choose* $\epsilon_0 = \pm 1$ *in* (3-26) *so that the leading terms satisfy*

$$\tilde{\ell}^{\alpha,0}_{s,m_1(s)} = \pi^{-1/2}|\hat{\lambda}_1(s)|^{1/2}\frac{h_\varphi(m_1(s))}{h_\varphi(s)}, \tag{4-11}$$

and in the case where $m_2(s) \in \widehat{\mathcal{U}}^{(0)}_\alpha$,

$$\tilde{\ell}^{\alpha,0}_{s,m_2(s)} = -\pi^{-1/2}|\hat{\lambda}_1(s)|^{1/2}\frac{h_\varphi(m_2(s))}{h_\varphi(s)}. \tag{4-12}$$

In particular, if $m_2(s) \in \widehat{\mathcal{U}}^{(0)}_\alpha$, *one has*

$$\frac{\tilde{\ell}^{\alpha,0}_{s,m_1(s)}}{h_\varphi(m_1(s))} = -\frac{\tilde{\ell}^{\alpha,0}_{s,m_2(s)}}{h_\varphi(m_2(s))} \tag{4-13}$$

for all $s \in \mathcal{V}^{(1)}_\alpha$.

Proof. Suppose first that $m \neq m_1(s), m_2(s)$. Then, $\operatorname{supp}(d_{\varphi,h} g^{(0)}_m) = \operatorname{supp}(d\hat{\chi}_m)$ is contained in a small neighborhood ω of $\Gamma(m)$. Since $m \neq m_1(s), m_2(s)$ it follows from Lemma 3.3 that $s \notin \omega$ and hence $\tilde{\ell}^{\alpha}_{s,m} = 0$ which proves (i).

Let us now compute the coefficients $\tilde{\ell}_{s,m}$ for $m \in \{m_1(s), m_2(s)\} \cap \widehat{\mathcal{U}}^{(0)}_\alpha$ (observe that this set may be reduced to $m_1(s)$). We compute these coefficients in the case where $m_2(s) \in \widehat{\mathcal{U}}^{(0)}_\alpha$. If it is not the case, the only nonzero coefficient is $\tilde{\ell}_{s,m_1(s)}$, which is computed in the same way. Recall from (3-30), that the quasimodes on 1-forms are given by

$$f^{(1)}_s = \epsilon_0 h^{-d/4}\psi_s(x) b^{(1)}_s(x,h) e^{-\varphi_{+,s}(x)/h}.$$

Summing up the construction of [Helffer, Klein and Nier 2004, Section 4.2], there exists an open neighborhood V_s of s on which one can find a system of local Morse coordinates $(y,z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ in which s is the origin and such that the following properties hold true:

(1) In the above coordinate system one has

$$\varphi = \varphi(s) + \frac{1}{2}\Big(\hat{\lambda}^1(s)y^2 + \sum_{j=2}^{d}\hat{\lambda}_j(s)z_j^2\Big),$$

$$\varphi_+ = \frac{1}{2}\Big(-\hat{\lambda}^1(s)y^2 + \sum_{j=2}^{d}\hat{\lambda}_j(s)z_j^2\Big),$$

where $(\hat{\lambda}_j(s))_{j=1,\dots,d}$ are the eigenvalues of $\operatorname{Hess}(\varphi)$ at point s.

(2) The amplitude $b^{(1)}_s(x,h)$ admits a classical expansion

$$b^{(1)}_s \sim \sum_{k=0}^{\infty} h^k w_{s,k} \tag{4-14}$$

with

$$w_{s,0} = (-1)^{d-1}\frac{|\det \operatorname{Hess}\varphi(s)|^{1/4}}{\pi^{d/4}}dy \quad \text{on } \{z=0\}. \tag{4-15}$$

(3) One can chose the orientation of the y-axis so that

$$E(\boldsymbol{m}_1(\boldsymbol{s}))\cap V_{\boldsymbol{s}}\subset\{y<0\}\cap V_{\boldsymbol{s}}\quad\text{and}\quad E(\boldsymbol{m}_2(\boldsymbol{s}))\cap V_{\boldsymbol{s}}\subset\{y>0\}\cap V_{\boldsymbol{s}}.$$

Moreover, the cut-off function $\chi_{\boldsymbol{m}}$ can be constructed so that:

(4) In $V_{\boldsymbol{s}}$ the functions $\hat{\chi}_{\boldsymbol{m}_j}$, $j=1,2$, depend only on the variable y.

Additionally, one can shrink $\omega_{\boldsymbol{s}}$ so that:

(5) $\operatorname{supp}(f_{\boldsymbol{s}}^{(1)})$ is contained in $V_{\boldsymbol{s}}$.

Observe that the only minor (but important) difference with [Helffer, Klein and Nier 2004] is the property (3), saying that each $\chi_{\boldsymbol{m}_j}$, $j=1,2$, is supported in one of the two different half-planes $\{y\lessgtr 0\}$. Let us now compute the first coefficient in the asymptotic expansion of $\tilde{\ell}^{p,\alpha}_{\boldsymbol{s},\boldsymbol{m}}$. Using the above properties, Proposition 3.5 and following the computations of [Helffer, Klein and Nier 2004, Section 6] we get

$$\begin{aligned}\hat{\ell}^{\alpha}_{\boldsymbol{s},\boldsymbol{m}}&=\langle f_{\boldsymbol{s}}^{(1)},d_{\varphi,h}g_{\boldsymbol{m}}^{(0)}\rangle\\&=h^{1-d/2}c(\boldsymbol{m},h)e(\boldsymbol{s},h)\int_{B(\boldsymbol{s},\epsilon)}e^{-(\varphi_+(x)+\varphi(x)-\varphi(\boldsymbol{m}))/h}(\hat{\chi}'_{\boldsymbol{m}}(y)+\mathcal{O}(h))\,dy\wedge dz_2\wedge\cdots\wedge dz_d\\&\qquad\qquad+\mathcal{O}_\epsilon(e^{-(\varphi(\boldsymbol{s})-\varphi(\boldsymbol{m})+c_\epsilon)/h}),\end{aligned}$$

with

$$e(\boldsymbol{s},h)=\epsilon_0(-1)^{d-1}\frac{|\det\operatorname{Hess}\varphi(\boldsymbol{s})|^{1/4}}{\pi^{d/4}}+\mathcal{O}(h)=\epsilon_0(-1)^{d-1}\pi^{-d/4}h_\varphi(\boldsymbol{s})+\mathcal{O}(h).$$

Using the local form of φ and φ_+, we get

$$\begin{aligned}\hat{\ell}^{\alpha}_{\boldsymbol{s},\boldsymbol{m}}=h^{1-d/2}c(\boldsymbol{m},h)e(\boldsymbol{s},h)e^{-(\varphi(\boldsymbol{s})-\varphi(\boldsymbol{m}))/h}\int_{B(\boldsymbol{s},\epsilon)}e^{-g_-(z)/h}(\hat{\chi}'_{\boldsymbol{m}}(y)+\mathcal{O}(h))\,dy\wedge dz_2\wedge\cdots\wedge dz_d\\+\mathcal{O}_\epsilon(e^{-(\varphi(\boldsymbol{s})-\varphi(\boldsymbol{m})+c_\epsilon)/h}),\end{aligned}$$

with $g_-(z)=\sum_{j=2}^d\hat{\lambda}_j(\boldsymbol{s})z_j^2$. Since $\hat{\chi}_{\boldsymbol{m}}$ depends only on y and $g_-\geq c\nu^2$ on $|z|_\infty\geq\nu$, the integration domain $B(\boldsymbol{s},\epsilon)$ can be replaced by a smaller one $W_{\boldsymbol{s}}=\{|y|<\epsilon,\ |z|_\infty\leq\nu_\epsilon\}$ modulo exponentially small error terms. Using also the identity $\widehat{S}(\boldsymbol{m})=\varphi(\boldsymbol{s})-\varphi(\boldsymbol{m})$, we get

$$\hat{\ell}^{\alpha}_{\boldsymbol{s},\boldsymbol{m}}=I_\epsilon(h)e^{-\widehat{S}(\boldsymbol{m})/h}+\mathcal{O}_\epsilon(e^{-(\widehat{S}(\boldsymbol{m})+c_\epsilon)/h}),$$

with

$$I_\epsilon(h)=h^{1-d/2}c(\boldsymbol{m},h)e(\boldsymbol{s},h)\int_{W_{\boldsymbol{s}}}e^{-g_-(z)/h}(\hat{\chi}'_{\boldsymbol{m}}(y)+\mathcal{O}(h))\,dy\wedge dz_2\wedge\cdots\wedge dz_d.$$

The integral on the right-hand side can be easily computed by means of Stoke's formula and the Laplace method. We get

$$\begin{aligned}I_\epsilon(h)&=h^{1-d/2}c(\boldsymbol{m},h)e(\boldsymbol{s},h)([\hat{\chi}_{\boldsymbol{m}}]^{\epsilon}_{-\epsilon}+\mathcal{O}(h))\int_{|z|_\infty\leq\nu_\epsilon}e^{-g_-(z)/h}\,dz_2\wedge\cdots\wedge dz_d\\&=h^{1/2}c(\boldsymbol{m},h)e(\boldsymbol{s},h)([\hat{\chi}_{\boldsymbol{m}}]^{\epsilon}_{-\epsilon}+\mathcal{O}(h))\left(\frac{\pi^{(d-1)/2}}{|\hat{\lambda}_2(\boldsymbol{s})\cdots\hat{\lambda}_d(\boldsymbol{s})|^{1/2}}\right).\end{aligned}$$

Combining this with the expressions of $c(\boldsymbol{m},h)$ and $e(\boldsymbol{s},h)$, we obtain

$$\tilde{\ell}^{\alpha,0}_{\boldsymbol{s},\boldsymbol{m}} = \epsilon_0(-1)^{d-1}[\hat{\chi}_{\boldsymbol{m}}]^{\epsilon}_{-\epsilon}\pi^{-1/2}|\hat{\lambda}_1(\boldsymbol{s})|^{1/2}\frac{h_\varphi(\boldsymbol{m})}{h_\varphi(\boldsymbol{s})}.$$

We now remark that with our choice of $\hat{\chi}_m$, one has $[\hat{\chi}_{\boldsymbol{m}_1}]^{\epsilon}_{-\epsilon} = -1$ and $[\hat{\chi}_{\boldsymbol{m}_2}]^{\epsilon}_{-\epsilon} = 1$. Taking $\epsilon_0 = (-1)^d$, we get immediately the formula of (ii). ☐

5. Computation of the approximated singular values

From Lemma A.2, we know that the singular values of a block-diagonal matrix are given by the singular values of each block. Hence, in view of the results of the preceding section, we study the matrices $\mathscr{L}^\alpha$. The first step in the analysis is to prove that $\mathscr{L}^\alpha$ is injective except for $\alpha = \underline{\alpha}$.

5A. *Injectivity of the matrix $\mathscr{L}^\alpha$.* We first compute the kernel of the matrix $\widetilde{\mathscr{L}}^\alpha$.

Lemma 5.1. *Let $\alpha \in \underline{\mathcal{A}}$. Then*:

- *If α is of type I (that is, $\mathcal{U}^{(0),\mathrm{II}}_\alpha = \varnothing$), then $\widetilde{\mathscr{L}}^{\alpha,0}$ is injective.*
- *If α is of type II, then* $\mathrm{Ker}(\widetilde{\mathscr{L}}^{\alpha,0}) = \mathbb{R}\xi_0$, *where $\xi_0 \in \mathbb{R}^{\hat{q}_\alpha} \simeq \mathcal{F}_\alpha$ is defined by*

$$\xi_0(\boldsymbol{m}) = h_\varphi(\boldsymbol{m})^{-1}$$

for all $\boldsymbol{m} \in \widehat{\mathcal{U}}^{(0)}_\alpha$.

Proof. Suppose first that α is of type II. Let $x \in \mathscr{F}_\alpha = \mathscr{F}(\widehat{\mathcal{U}}^{(0)}_\alpha)$ be such that $\widetilde{\mathscr{L}}^{\alpha,0}x = 0$. Then

$$\sum_{\boldsymbol{m}\in\widehat{\mathcal{U}}^{(0)}_\alpha} \tilde{\ell}^{\alpha,0}_{\boldsymbol{s},\boldsymbol{m}} x_{\boldsymbol{m}} = 0 \quad \text{for all } \boldsymbol{s} \in \mathcal{V}^{(1)}_\alpha. \tag{5-1}$$

From (i) of Lemma 4.4 it follows that

$$\tilde{\ell}^{\alpha,0}_{\boldsymbol{s},\boldsymbol{m}_1(\boldsymbol{s})} x_{\boldsymbol{m}_1(\boldsymbol{s})} = -\tilde{\ell}^{\alpha,0}_{\boldsymbol{s},\boldsymbol{m}_2(\boldsymbol{s})} x_{\boldsymbol{m}_2(\boldsymbol{s})} \quad \text{for all } \boldsymbol{s} \in \mathcal{V}^{(1)}_\alpha.$$

Moreover, since α is of type II, we know $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}^{(0)}_\alpha$ for any $\boldsymbol{s} \in \mathcal{V}^{(1)}_\alpha$ and thanks to (4-13) we get

$$x_{\boldsymbol{m}_1(\boldsymbol{s})} h_\varphi(\boldsymbol{m}_1(\boldsymbol{s})) = x_{\boldsymbol{m}_2(\boldsymbol{s})} h_\varphi(\boldsymbol{m}_2(\boldsymbol{s})) \quad \text{for all } \boldsymbol{s} \in \mathcal{V}^{(1)}_\alpha. \tag{5-2}$$

Now, we recall that for any $\boldsymbol{s} \in \mathcal{V}^{(1)}_\alpha$, $\boldsymbol{m}_1(\boldsymbol{s})$ and $\boldsymbol{m}_2(\boldsymbol{s})$ are exactly the two minima such that $\boldsymbol{s} = \Gamma_\alpha(\boldsymbol{m}_1) \cap \Gamma_\alpha(\boldsymbol{m}_2)$. Therefore, we deduce from (5-2) that

$$\text{for all } \boldsymbol{m},\boldsymbol{m}' \in \widehat{\mathcal{U}}^{(0)}_\alpha, \quad (\Gamma_\alpha(\boldsymbol{m}) \cap \Gamma_\alpha(\boldsymbol{m}') \neq \varnothing \implies h_\varphi(\boldsymbol{m})x_{\boldsymbol{m}} = h_\varphi(\boldsymbol{m}')x_{\boldsymbol{m}'}).$$

By the definition of the equivalence relation $\mathcal{R}$, this implies that $x_{\boldsymbol{m}}h_\varphi(\boldsymbol{m})$ is constant on $\widehat{\mathcal{U}}^{(0)}_\alpha$, which means exactly that $x \in \mathbb{R}\xi_0$.

Suppose now that α is of type I and let $x \in \mathscr{F}(\mathcal{U}^{(0)}_\alpha)$ such that $\widetilde{\mathscr{L}}^{\alpha,0}x = 0$. As before, one shows that there exists a constant c such that for all $\boldsymbol{m} \in \mathcal{U}^{(0)}_\alpha$, $h_\varphi(\boldsymbol{m})x_{\boldsymbol{m}} = c$. Recall that the nonempty set $\mathcal{V}^{(1),b}_\alpha$

was defined in Lemma 3.4. Given $s_b \in \mathcal{V}_\alpha^{(1),\boldsymbol{b}}$, since $\boldsymbol{m}_2(s_b) = \hat{\boldsymbol{m}}(\alpha) \notin \widehat{\mathcal{U}}_\alpha^{(0)}$, one has $\tilde{\ell}^{\alpha,0}_{s_b,\boldsymbol{m}} = 0$ for any $\boldsymbol{m} \neq \boldsymbol{m}_1(s_b)$ and

$$\tilde{\ell}^{\alpha,0}_{s_b,\boldsymbol{m}_1(s_b)} = \pi^{-1/2} |\hat{\lambda}_1(s_b)|^{1/2} \neq 0.$$

Combined with (5-1) this shows that $x_{\boldsymbol{m}_1(s_b)} = 0$ and hence $c = 0$, which proves that $\mathrm{Ker}(\widetilde{\mathscr{L}}^{\alpha,0}) = 0$. □

Proposition 5.2. *Let $\alpha \in \underline{\mathcal{A}}$, then the matrix $\widehat{\mathscr{L}}^\alpha := \widetilde{\mathscr{L}}^\alpha \mathscr{T}^\alpha$ admits a classical expansion $\widehat{\mathscr{L}}^\alpha \sim h^{1/2} \sum_j h^j \widehat{\mathscr{L}}^{\alpha,j}$ and the matrix $\widehat{\mathscr{L}}^{\alpha,0}$ is injective.*

Proof. By Lemmas 3.6 and 4.4 the matrices $\widetilde{\mathscr{L}}^\alpha$ and $\mathscr{T}^\alpha$ admit classical expansions $\widetilde{\mathscr{L}}^\alpha \sim h^{1/2} \sum h^j \widetilde{\mathscr{L}}^{\alpha,j}$ and $\mathscr{T}^\alpha \sim \sum h^j \mathscr{T}^{\alpha,j}$. Therefore, $\widehat{\mathscr{L}}^\alpha$ admits a classical expansion $\widehat{\mathscr{L}}^\alpha \sim h^{1/2} \sum_j h^j \widehat{\mathscr{L}}^{\alpha,j}$ with $\widehat{\mathscr{L}}^{\alpha,0} = \widetilde{\mathscr{L}}^{\alpha,0} \mathscr{T}^{\alpha,0}$.

Let us now prove that $\widehat{\mathscr{L}}^{\alpha,0}$ is injective.

Suppose first that α is of type I. Then $\mathscr{T}^\alpha = \mathscr{T}^{\alpha,0} = \mathrm{Id}$ and the result follows immediately from the first part of Lemma 5.1.

Suppose now that α is of type II and let $x \in \mathscr{F}(\mathcal{U}^{(0)})$ be such that $\widetilde{\mathscr{L}}^{\alpha,0} \mathscr{T}^{\alpha,0} x = 0$. We have the decomposition $x = x^{\mathrm{I}} + x^{\mathrm{II}}$, with $x^\bullet$ supported in $\widehat{\mathcal{U}}^{(0),\bullet}$. Thanks to (3-20), we have

$$\mathscr{T}^{\alpha,0} x(\boldsymbol{m}) = x^{\mathrm{I}}(\boldsymbol{m}) + (\widehat{\mathscr{T}}^{\alpha,0} x^{\mathrm{II}})(\boldsymbol{m}),$$

with $\widehat{\mathscr{T}}^{\alpha,0} : \mathscr{F}(\mathcal{U}^{(0),\mathrm{II}}) \to \mathscr{F}(\widehat{\mathcal{U}}^{(0),\mathrm{II}})$ such that $\mathrm{Ran}\, \widehat{\mathscr{T}}^{\alpha,0} = (\mathbb{R}\theta_0^\alpha)^\perp$, where the function θ_0^α is defined by (3-11). On the other hand, we have $\ker \widetilde{\mathscr{L}}^{\alpha,0} = \mathbb{R}\xi_0$ and we have the decomposition $\xi_0 = \xi_0^{\mathrm{I}} + \xi_0^{\mathrm{II}}$, with $\xi_0^{\mathrm{II}} = \theta^{\alpha,0}$. The equation $\widetilde{\mathscr{L}}^{\alpha,0} \mathscr{T}^{\alpha,0} x = 0$ implies that there exists $\lambda \in \mathbb{R}$ such that $\mathscr{T}^{\alpha,0} x = \lambda \xi_0$ and hence $\widehat{\mathscr{T}}^{\alpha,0} x^{\mathrm{II}} = \lambda \xi_0^{\mathrm{II}}$. On the other hand, by construction, $\mathrm{Ran}\, \widehat{\mathscr{T}}^{\alpha,0} = (\xi_0^{\mathrm{II}})^\perp$. This implies that $\lambda = 0$ and proves the result. □

Corollary 5.3. *For all $\alpha \in \underline{\mathcal{A}}$ the matrix $\mathscr{L}^\alpha$ is injective.*

Proof. This follows directly from the above proposition and the fact that

$$\mathscr{L}^\alpha = \widehat{\mathscr{L}}^\alpha \mathscr{T}^\alpha = \widetilde{\mathscr{L}}^\alpha \widehat{\Omega}^\alpha \mathscr{T}^\alpha = \widehat{\mathscr{L}}^\alpha \Omega^\alpha, \tag{5-3}$$

with Ω^α defined below (4-10) which is invertible. □

5B. *Graded structure of the matrices $\mathscr{L}^\alpha$.* Throughout this section, we assume that $\alpha \in \mathcal{A}$ is fixed. Recall that we defined $\mathcal{S}_\alpha = S(\mathcal{U}_\alpha^{(0)}),\ p(\alpha) = \#\mathcal{S}_\alpha$ and some integers $\nu_1^\alpha < \cdots < \nu_{p(\alpha)}^\alpha$ such that

$$\mathcal{S}_\alpha = \{S_{\nu_1^\alpha}, \ldots, S_{\nu_{p(\alpha)}^\alpha}\},$$

with the convention $S_{\nu_1^\alpha} > \cdots > S_{\nu_{p(\alpha)}^\alpha}$. In order to lighten the notation we will drop the indices α and write from now $p = p(\alpha),\ \nu_j = \nu_j^\alpha$. To the set of heights $\mathcal{S}_\alpha$, we can associate a natural partition

$$\widehat{\mathcal{U}}_\alpha^{(0)} = \bigsqcup_{n=1}^{p} \widehat{\mathcal{U}}_{\alpha,n}^{(0)} \tag{5-4}$$

with $\widehat{\mathcal{U}}_{\alpha,n}^{(0)} = \{\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0)} : \varphi(\boldsymbol{m}) = \boldsymbol{\sigma}(\alpha) - S_{\nu_n}\}$. We order this partition by deciding that $\widehat{\mathcal{U}}_{\alpha,n+1}^{(0)} \prec \widehat{\mathcal{U}}_{\alpha,n}^{(0)}$. On the other hand, we recall that $\mathscr{L}^\alpha = \widehat{\mathscr{L}}^\alpha \Omega^\alpha$ with $\widehat{\mathscr{L}}^\alpha = \widetilde{\mathscr{L}}^\alpha \mathscr{T}^\alpha$. Let us compute the matrices $\widehat{\mathscr{L}}^\alpha$

and Ω^α in the basis given by the above partition of $\widehat{\mathcal{U}}_\alpha^{(0)}$. With a slight abuse of notation we still denote by $\widehat{\mathscr{L}}^\alpha$ and Ω^α the resulting matrices. Since $\widehat{S}(\boldsymbol{m}) = \boldsymbol{\sigma}(\alpha) - S_{\nu_k}$ on $\mathcal{U}_{\alpha,k}^{(0)}$, it follows from (4-8) that in the above partition, the matrix Ω^α can be written

$$\Omega^\alpha = \begin{pmatrix} e^{-S_{\nu_p}/h} I_{r_p} & 0 & \cdots & \cdots & 0 \\ 0 & e^{-S_{\nu_{p-1}}/h} I_{r_{p-1}} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{-S_{\nu_1}/h} I_{r_1} \end{pmatrix}, \tag{5-5}$$

where the $r_j = \#\mathcal{U}_{\alpha,j}^{(0)}$ are such that $r_1 + \cdots + r_p = \#\mathcal{U}_\alpha^{(0)}$. Factorizing by $e^{-S_{\nu_p}/h}$, we get $\Omega^\alpha = e^{-S_{\nu_p}/h}\widehat{\Omega}^\alpha(\tau)$, with

$$\widehat{\Omega}^\alpha(\tau) = \begin{pmatrix} I_{r_p} & 0 & \cdots & \cdots & 0 \\ 0 & \tau_2 I_{r_{p-1}} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \tau_2\tau_3\cdots\tau_p I_{r_1} \end{pmatrix}, \tag{5-6}$$

where $\tau = (\tau_2, \dots, \tau_p) \in (\mathbb{R}_+^*)^p$ is defined by $\tau_j = e^{(S_{\nu_{p-(j-2)}} - S_{\nu_{p-(j-1)}})/h}$ for any $j = 2, \dots, p$. With this new notation, one deduces from (5-3), that $\mathscr{L}^{\alpha,*}\mathscr{L}^\alpha = he^{-2S_{\nu_p}/h}\widehat{\mathcal{M}}^\alpha(\tau)$, with

$$\widehat{\mathcal{M}}^\alpha(\tau) = \widehat{\Omega}^\alpha(\tau)(h^{-1}\widehat{\mathscr{L}}^{\alpha,*}\widehat{\mathscr{L}}^\alpha)\widehat{\Omega}^\alpha(\tau). \tag{5-7}$$

It turns out that such matrices can be described in a slightly more general setting that is useful to compute their spectrum. We introduce this setting now. Throughout, we denote by $\mathscr{S}^+(E)$ the set of symmetric positive definite matrices on a vector space E. We will denote by $\mathscr{S}_{\mathrm{cl}}^+(E)$ the set of h-dependent matrices $M(h) \in \mathscr{S}^+(E)$ admitting a classical expansion $M(h) \sim \sum_j h^j M_j$ with $M_0 \in \mathscr{S}^+(E)$. We will sometimes drop E and write for short $\mathscr{S}^+$, $\mathscr{S}_{\mathrm{cl}}^+$.

Definition 5.4. Let $\mathscr{E} = (E_j)_{j=1,\dots,p}$ be a sequence of finite-dimensional vector spaces E_j of dimension $r_j > 0$, let $E = \bigoplus_{j=1,\dots,p} E_j$ and let $\tau = (\tau_2, \dots, \tau_p) \in (\mathbb{R}_+^*)^{p-1}$. Suppose that $\tau \mapsto \mathcal{M}(\tau)$ is a smooth map from $(\mathbb{R}_+^*)^{p-1}$ into the set of matrices $\mathscr{M}(E)$:

- We say that $\mathcal{M}(\tau)$ is an $(\mathscr{E}, \tau)$-graded matrix if there exists $\mathcal{M}' \in \mathscr{S}^+(E)$ independent of τ such that $\mathcal{M}(\tau) = \Omega(\tau)\mathcal{M}'\Omega(\tau)$, with $\Omega(\tau) \in \mathscr{M}(E)$ of the form (5-6); that is, $\Omega = \operatorname{diag}(\epsilon_j(\tau) I_{r_j},\ j = 1, \dots, p)$, where $\epsilon_1(\tau) = 1$ and $\epsilon_j(\tau) = \left(\prod_{k=2}^{j} \tau_k\right)$ for all $j \geq 2$.
- We say that a family of $(\mathscr{E}, \tau)$-graded matrices $\mathcal{M}_h(\tau),\ h \in]0, h_0]$ is classical if one has $\mathcal{M}_h(\tau) = \Omega(\tau)\mathcal{M}'(h)\Omega(\tau)$ for some matrix $\mathcal{M}'(h) \in \mathscr{S}_{\mathrm{cl}}^+(E)$.

Throughout, we denote by $\mathscr{G}(\mathscr{E}, \tau)$ the set of $(\mathscr{E}, \tau)$-graded matrices and by $\mathscr{G}_{\mathrm{cl}}(\mathscr{E}, \tau)$ the set of classical $(\mathscr{E}, \tau)$-graded matrices.

Let us remark that for $p = 1$, a graded matrix is simply a τ-independent symmetric positive definite matrix.

Lemma 5.5. *Suppose that $\mathcal{M}_h(\tau)$ is a classical $(\mathscr{E},\tau)$-graded family of matrices and that $p \geq 2$. Then one has*

$$\mathcal{M}_h(\tau) = \begin{pmatrix} J(h) & \tau_2 B_h(\tau')^* \\ \tau_2 B_h(\tau') & \tau_2^2 \mathcal{N}_h(\tau') \end{pmatrix}, \tag{5-8}$$

with

- $J(h) \in \mathscr{S}_{\mathrm{cl}}^{+}(E_1)$,
- $\mathcal{N}_h(\tau') \in \mathscr{G}_{\mathrm{cl}}(\mathscr{E}',\tau')$, *with* $\tau' = (\tau_3,\dots,\tau_p)$ *and* $\mathscr{E}' = (E_j)_{j=2,\dots,p}$,
- $B_h(\tau') \in \mathscr{M}\big(E_1, \bigoplus_{j=2}^{p} E_j\big)$ *satisfying*

$$B_h(\tau')^* = (b_2(h)^*, \tau_3 b_3(h)^*, \tau_3\tau_4 b_4(h)^*, \dots, \tau_3\cdots\tau_p b_p(h)^*),$$

  with $b_j(h) : E_1 \to E_j$ *independent of* τ *admitting a classical expansion.*

Moreover, the matrix $\mathcal{N}_h(\tau') - B_h(\tau')J(h)^{-1}B_h(\tau')^*$ *belongs to* $\mathscr{G}_{\mathrm{cl}}(\mathscr{E}',\tau')$.

Proof. Assume that $\mathcal{M}_h(\tau) = \Omega(\tau)\mathcal{M}'(h)\Omega(\tau)$, with $\Omega(\tau)$ of the form (5-6). First observe that

$$\Omega(\tau) = \begin{pmatrix} I_{r_p} & 0 \\ 0 & \tau_2\Omega'(\tau'), \end{pmatrix},$$

with

$$\Omega'(\tau') = \begin{pmatrix} I_{r_{p-1}} & 0 & \cdots & \cdots & 0 \\ 0 & \tau_3 I_{r_{p-2}} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \tau_3\tau_4\cdots\tau_p I_{r_1} \end{pmatrix}.$$

On the other hand, we can write

$$\mathcal{M}'(h) = \begin{pmatrix} J(h) & B'(h)^* \\ B'(h) & \mathcal{N}'(h) \end{pmatrix},$$

with $J(h), \mathcal{N}'(h) \in \mathscr{S}_{\mathrm{cl}}^{+}$ and $B'(h)$ admitting a classical expansion. Therefore,

$$\Omega(\tau)\mathcal{M}'_h\Omega(\tau) = \begin{pmatrix} J(h) & \tau_2 B'(h)^*\Omega'(\tau') \\ \tau_2\Omega'(\tau')B'(h) & \tau_2^2\Omega'(\tau')\mathcal{N}'(h)\Omega'(\tau') \end{pmatrix},$$

which has exactly the form (5-8) with $B_h(\tau') = \Omega'(\tau')B'(h)$ and $\mathcal{N}_h(\tau') = \Omega'(\tau')\mathcal{N}'(h)\Omega'(\tau')$. By construction, $\mathcal{N}_h(\tau')$ belongs to $\mathscr{G}_{\mathrm{cl}}(\mathscr{E}',\tau')$ and $B_h(\tau')$ has the required form.

It remains to prove that

$$\mathcal{R}_h := \mathcal{N}_h(\tau') - B_h(\tau')J(h)^{-1}B_h(\tau')^*$$

belongs to $\mathscr{G}_{\mathrm{cl}}(\mathscr{E}',\tau')$. First observe that since $J(h)$ is symmetric positive definite, this quantity is well-defined. Moreover, one has by construction

$$\begin{aligned} \mathcal{R}_h &= \Omega'(\tau')\mathcal{N}'(h)\Omega'(\tau') - \Omega'(\tau')B'(h)J(h)^{-1}B'(h)^*\Omega'(\tau') \\ &= \Omega'(\tau')\mathcal{R}'(h)\Omega'(\tau'), \end{aligned}$$

with $\mathcal{R}'(h)=\mathcal{N}'(h)-B'(h)J(h)^{-1}B'(h)^*$. Since $J(h)\in\mathscr{S}^+_{\mathrm{cl}}$, we have $J(h)^{-1}\in\mathscr{S}^+_{\mathrm{cl}}$ and $\mathcal{R}'(h)$ admits a classical expansion $\mathcal{R}'(h)\sim\sum_j h^j\mathcal{R}'_j$ with

$$\mathcal{R}'_0=J_0-B'_0J_0^{-1}(B'_0)^*.$$

Moreover, since $\mathcal{M}'(h)\in\mathscr{S}^+_{\mathrm{cl}}$, the matrix

$$\mathcal{M}'_0=\begin{pmatrix}J_0 & (B'_0)^*\\ B'_0 & \mathcal{N}'_0\end{pmatrix}$$

is symmetric definite positive. Hence, it follows directly from Lemma A.5 that $\mathcal{R}'_0\in\mathscr{S}^+$. □

5C. *The spectrum of graded matrices.* Using Lemma 5.5, we define an application $\mathcal{R}:\mathscr{G}_{\mathrm{cl}}(\mathscr{E},\tau)\to\mathscr{G}_{\mathrm{cl}}(\mathscr{E}',\tau')$, with $\tau'=(\tau_3,\dots,\tau_p)$ and $\mathscr{E}'=\bigoplus_{j=2}^p E_j$, by

$$\mathcal{R}(\mathcal{M}_h(\tau))=\mathcal{N}_h(\tau')-B_h(\tau')J(h)^{-1}B_h^*(\tau') \tag{5-9}$$

for any $\mathcal{M}_h(\tau)\in\mathscr{G}_{\mathrm{cl}}(\mathscr{E},\tau)$. Of course, the map $\mathcal{R}$ depends on $\mathscr{E}$ and τ, but we omit this dependence since the set on which $\mathcal{R}$ is acting will be obvious in the sequel. By a slight abuse of notation we will write $\mathcal{R}^k=\mathcal{R}\circ\cdots\circ\mathcal{R}$ (k times). Obviously, $\mathcal{R}^k$ acts from $\mathscr{G}(\mathscr{E},\tau)$ into $\mathscr{G}(\mathscr{E}^{(k)},\tau^{(k)})$ with $\mathscr{E}^{(k)}=\bigoplus_{j=k+1}^p E_j$ and $\tau^{(k)}=(\tau_{k+2},\dots,\tau_p)$. In the same way, we defined $\mathcal{R}$, we can define a map $\mathcal{J}:\mathscr{G}_{\mathrm{cl}}(\mathscr{E},\tau)\to\mathscr{S}^+_{\mathrm{cl}}(E_1)$ by $\mathcal{J}(\mathcal{M}_h(\tau))=\mathcal{M}_h$ if $p=1$ and $\mathcal{J}(\mathcal{M}_h(\tau))=J(h)$ for any $\mathcal{M}_h(\tau)$ having the form (5-8) if $p\geq 2$.

Theorem 5.6. *Let $\mathscr{E}=(E_j)_{j=1,\dots,p}$ be a finite sequence of vector spaces E_j of finite dimension $n_j=\dim E_j$ and let $\tau=(\tau_2,\dots,\tau_p)\in(\mathbb{R}^*_+)^{p-1}$. Suppose that $\mathcal{M}_h(\tau)$ is classical $(\mathscr{E},\tau)$-graded. There exists $h_0>0$ and $\delta>0$ such that uniformly with respect to $h\in\,]0,h_0]$ and $|\tau|_\infty<\delta$, one has*

$$\sigma(\mathcal{M}_h(\tau))=\bigsqcup_{j=1}^p \epsilon_j\sigma(\mathcal{J}\circ\mathcal{R}^{j-1}(\mathcal{M}_h(\tau)))(1+\mathcal{O}(|\tau|_\infty^2)), \tag{5-10}$$

with $\epsilon_j=\epsilon_j(\tau)$ given in Definition 5.4.

Remark 5.7. In the above theorem, the matrix $\mathcal{J}\circ\mathcal{R}^{j-1}(\mathcal{M}_h(\tau))$ is always independent of the parameter τ. Let us define $\{\lambda_1^j\leq\cdots\leq\lambda_{n_j}^j\}=\sigma(\mathcal{J}\circ\mathcal{R}^{j-1}(\mathcal{M}_h(\tau)))$. The identity (5-10) means that there exists $a,b>0$ independent of τ,h such that

$$\sigma(\mathcal{M}_h(\tau))\subset\bigsqcup_{j=1}^p\epsilon_j[a,b]$$

and that, for all $j=1,\dots,p$, $\mathcal{M}_h(\tau)$ has exactly n_j eigenvalues $\mu_1^j\leq\cdots\leq\mu_{n_j}^j$ in $\epsilon_j[a,b]$ and

$$\mu_n^j=\epsilon_j(\lambda_n^j+\mathcal{O}(|\tau|_\infty^2)).$$

Proof. We prove the theorem by induction on p. Throughout the proof the notation $\mathcal{O}(\cdot)$ is uniform with respect to the parameters h and τ. For $p=1$, $\mathcal{M}_h(\tau)=\mathcal{M}_h\in\mathscr{S}^+_{\mathrm{cl}}(E_1)$ is independent of τ and $\mathcal{J}\mathcal{R}^0(\mathcal{M}_h(\tau))=\mathcal{J}\mathcal{M}_h(\tau)=\mathcal{M}_h$, which proves the statement.

Suppose now that $p \geq 2$ and let $\mathcal{M}_h(\tau) \in \mathscr{G}_{\mathrm{cl}}(\mathscr{E}, \tau)$. We have

$$\mathcal{M}_h(\tau) = \begin{pmatrix} J(h) & \tau_2 B_h(\tau')^* \\ \tau_2 B_h(\tau') & \tau_2^2 \mathcal{N}_h(\tau') \end{pmatrix},$$

with $J(h)$, $B_h(\tau')$ and $\mathcal{N}_h(\tau')$ as in Lemma 5.5. In order to lighten the notation we will drop the variables τ, τ' in the proof below. For $\lambda \in \mathbb{C}$, let

$$\mathcal{P}(\lambda) := \mathcal{M}_h - \lambda = \begin{pmatrix} J(h) - \lambda & \tau_2 B_h^* \\ \tau_2 B_h & \tau_2^2 \mathcal{N}_h - \lambda \end{pmatrix}. \tag{5-11}$$

This is an holomorphic function, and since it is nontrivial, its inverse is well-defined except for a finite number of values of λ which are exactly the eigenvalues of $\mathcal{M}_h$. Moreover $\lambda \in \mathbb{C} \mapsto \mathcal{P}(\lambda)^{-1}$ is meromorphic with poles in $\sigma(\mathcal{M}_h)$ and for any μ in $\sigma(\mathcal{M}_h)$, the rank of the residue of $\mathcal{P}(\lambda)^{-1}$ at μ is exactly the multiplicity of μ as an eigenvalue.

Let us first prove that $\mathcal{M}_h$ admits at least n_1 eigenvalues of size 1. Let $\lambda_n^1 = \lambda_n^1(h)$, $n = 1, \dots, n_1$, denote the increasing sequence of eigenvalues of the positive definite matrix $J(h)$. Since $J(h) = J_0 + \mathcal{O}(h)$ with $J_0 \in \mathscr{S}^+$, the $\lambda_n^1(h)$ satisfy $\lambda_n^1(h) = \lambda_{n,0}^1 + \mathcal{O}(h)$, with $\lambda_{n,0}^1$ an eigenvalue of J_0. In particular $\lambda_{n,0}^1 > 0$ for all $n = 1, \dots, n_1$ and hence there exists $c_1, d_1 > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ and all $n = 1, \dots, n_1$, one has $\lambda_n^1(h) \in [c_1, d_1]$. Let $n \in \{1, \dots, n_1\}$ be fixed and consider $D_n = D_n(h, \tau_2) = \{z \in \mathbb{C} : |z - \lambda_n^1| \leq M\tau_2^2\}$ for some $M > 0$ that will be chosen large enough later and $\widetilde{D}_n = \{z \in \mathbb{C} : |z - \lambda_n^1| \leq 2M\tau_2^2\}$. Observe that for $h, \tau_2 > 0$ small enough, the disks $\widetilde{D}_n$ are disjoint. By definition, one has $\mathcal{N}_h(\tau') = \mathcal{O}(1)$ and since $\lambda_n^1 \geq c_1 > 0$, this implies that for $\tau_2 > 0$ small enough with respect to c and $\lambda \in \widetilde{D}_n$, the matrix $\tau_2^2 \mathcal{N}_h(\tau') - \lambda$ is invertible, and $(\tau_2^2 \mathcal{N}_h(\tau') - \lambda)^{-1} = \mathcal{O}(1)$. Moreover, for $\lambda \in \widetilde{D}_n \setminus D_n$, $J(h) - \lambda$ is invertible and $(J(h) - \lambda)^{-1} = \mathcal{O}(\tau_2^{-2} M^{-1})$. This implies that for $M > 0$ large enough, $J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h$ is invertible with

$$\begin{aligned} (J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h)^{-1} &= (J(h) - \lambda)^{-1} \bigl(I - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h (J(h) - \lambda)^{-1}\bigr)^{-1} \\ &= (J(h) - \lambda)^{-1} (1 + \mathcal{O}(M^{-1})). \end{aligned} \tag{5-12}$$

Hence, the standard Schur complement procedure shows that for $\lambda \in \widetilde{D}_n \setminus D_n$, $\mathcal{P}(\lambda)$ is invertible with inverse $\mathcal{E}(\lambda)$ given by

$$\mathcal{E}(\lambda) = \begin{pmatrix} E(\lambda) & -\tau_2 E(\lambda) B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} \\ -\tau_2 (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h E(\lambda) & E_0(\lambda) \end{pmatrix}, \tag{5-13}$$

with

$$\begin{aligned} E(\lambda) &= (J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h)^{-1}, \\ E_0(\lambda) &= (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} + \tau_2^2 (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h E(\lambda) B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1}. \end{aligned}$$

By functional calculus and the Cauchy formula, the number of eigenvalues of $\mathcal{M}_h$ (counted with multiplicity) in D_n is equal to the rank of the projector

$$\Pi_n = \frac{1}{2i\pi} \int_{\partial D_n} \mathcal{E}(\lambda)\, d\lambda.$$

One has $\mathrm{rk}(\pi_n) \geq \mathrm{rk}(\widetilde{\Pi}_n)$, where we set

$$\widetilde{\Pi}_n = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \Pi_n \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, an elementary computation shows that

$$\widetilde{\Pi}_n = \frac{1}{2i\pi} \int_{\partial D_n} \begin{pmatrix} E(\lambda) & 0 \\ 0 & 0 \end{pmatrix} d\lambda = \begin{pmatrix} E_n & 0 \\ 0 & 0 \end{pmatrix},$$

with

$$E_n = \frac{1}{2i\pi} \int_{\partial D_n} (J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h)^{-1} \, d\lambda.$$

As a consequence we get $\mathrm{rk}(\Pi_n) \geq \mathrm{rk}(E_n)$. Moreover, for M large enough independent of (h, τ), the matrix $(I - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h (J(h) - \lambda)^{-1})^{-1}$ is holomorphic in $\widetilde{D}_n$. It follows from (5-12) that the rank of E_n is exactly the multiplicity of λ_n^1 and hence the rank of Π_n is bounded from below by the multiplicity of λ_n^1. Therefore, $\mathcal{M}_h$ admits at least n_1 eigenvalues $\mu_1^1 \leq \cdots \leq \mu_{n_1}^1$ in the interval $[c_1 - M\tau_2^2, d_1 + M\tau_2^2]$ and these eigenvalues satisfy

$$\mu_n^1 = \lambda_n^1 + \mathcal{O}(\tau_2^2) \quad \text{for all } n = 1, \ldots, n_1. \tag{5-14}$$

Let us now study the eigenvalues below τ_2^2. Throughout the proof, we let $t = |\tau'|_\infty$. Thanks to the last part of Lemma 5.5, the matrix $\mathcal{Z}_h(\tau') := \mathcal{R}(\mathcal{M}_h(\tau)) = \mathcal{N}_h - B_h J(h)^{-1} B_h^*$ is classical $(\mathscr{E}', \tau')$-graded. Hence, it follows from the induction hypothesis that uniformly with respect to h, one has

$$\sigma(\mathcal{Z}_h(\tau')) = \bigsqcup_{j=2}^{p} \tilde{\epsilon}_j \sigma(\mathcal{J} \circ \mathcal{R}^{j-2}(\mathcal{Z}_h(\tau')))(1 + \mathcal{O}(|\tau'|_\infty^2)), \tag{5-15}$$

with $\tilde{\epsilon}_j = \left(\prod_{l=3}^{j} \tau_l\right)^2$ for $j \geq 3$ and $\tilde{\epsilon}_2 = 1$. Moreover, by definition, one has $\mathcal{Z}_h = \mathcal{R}(\mathcal{M}_h(\tau))$; hence (5-15) can be rewritten as

$$\sigma(\mathcal{Z}_h(\tau')) = \bigsqcup_{j=2}^{p} \tilde{\epsilon}_j \sigma(\mathcal{J} \circ \mathcal{R}^{j-1}(\mathcal{M}_h(\tau)))(1 + \mathcal{O}(|\tau'|_\infty^2)). \tag{5-16}$$

Since $\mathcal{M}_h(\tau') \in \mathscr{G}_{\mathrm{cl}}(\mathscr{E}, \tau)$, for all $j = 2, \ldots, p$ the matrix $\mathcal{J} \circ \mathcal{R}^{j-1}(\mathcal{M}_h(\tau))$ belongs to $\mathscr{S}_{\mathrm{cl}}^{+}(E_j)$. For $j = 2, \ldots, p$, let $\lambda_1^j(h) \leq \cdots \leq \lambda_{n_j}^j(h)$ denote the eigenvalues of the symmetric matrix $\mathcal{J} \circ \mathcal{R}^{j-1}(\mathcal{M}_h(\tau))$. As above, this implies that there exist $c_j, d_j > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$ the eigenvalues $\lambda_n^j(h)$ satisfy $\lambda_n^j(h) \in [c_j, d_j]$ for all $n = 1, \ldots, n_j$. Suppose now that $j \in \{2, \ldots, p\}$ and $n \in \{1, \ldots, n_j\}$ are fixed and consider $D'_{j,n} = \{z \in \mathbb{C} : |z - \epsilon_j \lambda_n^j| \leq M t^2 \epsilon_j\}$ for some $M > 0$ to be chosen large enough and $\widetilde{D}'_{j,n} = \{z \in \mathbb{C} : |z - \epsilon_j \lambda_n^j| \leq 2M t^2 \epsilon_j\}$. As above, we introduce also the corresponding projector

$$\Pi'_{j,n} = \frac{1}{2i\pi} \int_{\partial D'_{j,n}} \mathcal{E}(\lambda) \, d\lambda.$$

Since J_0 is invertible, we know that for λ in $\widetilde{D}'_{j,n}$ and h,t small enough, $J(h)-\lambda$ is invertible and once again the Schur complement formula permits us to write the inverse of $\mathcal{P}(\lambda)$,

$$\mathcal{E}(\lambda)=\begin{pmatrix} E_0(\lambda) & -\tau_2(J(h)-\lambda)^{-1}B_h^*E(\lambda) \\ -\tau_2 E(\lambda)B_h(J(h)-\lambda)^{-1} & E(\lambda)\end{pmatrix}, \tag{5-17}$$

with

$$E(\lambda)=(\tau_2^2\mathcal{N}_h-\lambda-\tau_2^2B_h(J(h)-\lambda)^{-1}B_h^*)^{-1},$$
$$E_0(\lambda)=(J(h)-\lambda)^{-1}+\tau_2^2(J(h)-\lambda)^{-1}B_h^*E(\lambda)B_h(J(h)-\lambda)^{-1}.$$

Setting $\lambda=\tau_2^2 z$, we get (using the relation $\epsilon_j=\tau_2^2\tilde{\epsilon}_j$)

$$\Pi'_{j,n}=\frac{\tau_2^2}{2i\pi}\int_{\partial\widehat{D}'_{j,n}}\mathcal{E}(\tau_2^2 z)\,dz,$$

with $\widehat{D}'_n=\{z\in\mathbb{C}:|z-\tilde{\epsilon}_j\lambda_n^j|\le Mt^2\tilde{\epsilon}_j\}$. Moreover, for $|z-\tilde{\epsilon}_j\lambda_n^j|=Mt^2\tilde{\epsilon}_j$, the matrix $J(h)$ is invertible with $J(h)^{-1}=\mathcal{O}(1)$; hence we have

$$\begin{aligned}E(\tau_2^2 z)&=\tau_2^{-2}(\mathcal{N}_h-z-B_h(J(h)-\tau_2^2z)^{-1}B_h^*)^{-1}\\&=\tau_2^{-2}(\mathcal{Z}_h-z+\mathcal{O}(\tau_2^2|z|))^{-1}\\&=\tau_2^{-2}(\mathcal{Z}_h-z)^{-1}(I+\mathcal{O}(\tau_2^2\tilde{\epsilon}_j\|(\mathcal{Z}_h-z)^{-1}\|)).\end{aligned}$$

Moreover, by the definition of $\widehat{D}'_{j,n}$ and thanks to (5-15), one has $\operatorname{dist}(z,\sigma(\mathcal{Z}_h))\ge\frac{1}{2}Mt^2\tilde{\epsilon}_j$ for any $z\in\partial\widehat{D}'_{j,n}$. Hence $\|(\mathcal{Z}_h-z)^{-1}\|\le 2(Mt^2\tilde{\epsilon}_j)^{-1}$ and since $t\ge\tau_2$, it follows that

$$E(\tau_2^2z)=\tau_2^{-2}(\mathcal{Z}_h-z)^{-1}(I+\mathcal{O}(M^{-1})).$$

Integrating along $\partial\widetilde{D}'_{j,n}$ and working as above, we get

$$\Pi'_{j,n}=\frac{1}{2i\pi}\int_{\partial D'_{j,n}}\begin{pmatrix}E_0(\lambda) & R_{\tau_2}^{\dagger}E(\lambda)\\ E(\lambda)R_{\tau_2} & E(\lambda)\end{pmatrix}d\lambda,$$

with $R_{\tau_2}(\lambda)=-\tau_2(\tau_2^2\mathcal{N}_h-\lambda)^{-1}B_h$ and $R_{\tau_2}^{\dagger}(\lambda)=-\tau_2B_h^*(\tau_2^2\mathcal{N}_h-\lambda)^{-1}$. The same argument as above shows that $\operatorname{rk}(\Pi_n)\ge\operatorname{rk}(E'_n)$ with

$$E'_n=\frac{\tau_2^2}{2i\pi}\int_{\partial\widehat{D}'_{j,n}}E(\tau_2^2z)\,dz=\frac{1}{2i\pi}\int_{\partial\widehat{D}'_{j,n}}(\mathcal{Z}_h-z)^{-1}(I+\mathcal{O}(M^{-1}))^{-1}\,dz.$$

By the induction hypothesis, this shows that the rank of E'_n is exactly the multiplicity of λ_n^j and hence the rank of $\Pi'_{j,n}$ is bounded from below by this multiplicity. Therefore, for any $j=2,\dots,p$, $\mathcal{M}_h$ admits at least n_j eigenvalues $\mu_1^1\le\cdots\le\mu_{n_1}^1$ in the interval $\epsilon_j[c_j-Mt^2,d_j+Mt^2]$ and these eigenvalues satisfy

$$\mu_n^j=\epsilon_j(\lambda_n^j+\mathcal{O}(|\tau|_\infty^2))\quad\text{for all } n=1,\dots,n_j. \tag{5-18}$$

Combining this estimate with (5-14) and using the fact the $\dim(E)=\sum_{j=1}^p r_j$, we obtain the μ_n^j are the only eigenvalues of $\mathcal{M}_h$. ☐

5D. ***The singular values of*** $\mathscr{L}^{\alpha}$**.** Given $\boldsymbol{m}, \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{U}^{(0)}$ and $\boldsymbol{s} \in \mathcal{V}^{(1)}$, we define

$$\upsilon_2(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1, \boldsymbol{m}_2) = \pi^{-1/2}|\hat{\lambda}_1(\boldsymbol{s})|^{1/2}\left(\frac{h_\varphi(\boldsymbol{m}_1)}{h_\varphi(\boldsymbol{s})}\delta_{\boldsymbol{m},\boldsymbol{m}_1} - \frac{h_\varphi(\boldsymbol{m}_2)}{h_\varphi(\boldsymbol{s})}\delta_{\boldsymbol{m},\boldsymbol{m}_2}\right), \tag{5-19}$$

$$\upsilon_1(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1) = \pi^{-1/2}|\hat{\lambda}_1(\boldsymbol{s})|^{1/2}\frac{h_\varphi(\boldsymbol{m}_1)}{h_\varphi(\boldsymbol{s})}\delta_{\boldsymbol{m},\boldsymbol{m}_1}. \tag{5-20}$$

Let us define the matrix $\Upsilon^{\alpha} \in \mathscr{M}(\widehat{\mathcal{U}}_{\alpha}^{(0)}, \mathcal{V}_{\alpha}^{(1)})$ by

$$\Upsilon^{\alpha}(\boldsymbol{s}, \boldsymbol{m}) = \begin{cases} \upsilon_2(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s})) & \text{if } \boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha}^{(0)}, \\ \upsilon_1(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1(\boldsymbol{s})) & \text{if } \boldsymbol{m}_2(\boldsymbol{s}) \notin \widehat{\mathcal{U}}_{\alpha}^{(0)}, \end{cases} \tag{5-21}$$

where the indices $\boldsymbol{m}, \boldsymbol{s}$ are enumerated according to the partitions of Section 5B. Observe that with this notation, the conclusion of Lemma 4.4 can be written as $\widetilde{\mathscr{L}}^{\alpha,0} = \Upsilon$. Moreover, the above expression can be simplified according to the type of α. More precisely,

- if α is of type I, then $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha}^{(0)}$ if and only if $\boldsymbol{s} \in \mathcal{V}_{\alpha}^{(1),i}$,
- if α is of type II, then $\boldsymbol{m}_2(\boldsymbol{s})$ is always in $\widehat{\mathcal{U}}_{\alpha}^{(0)}$.

Theorem 5.8. *Let $\mathcal{M}^{\alpha} = \mathscr{L}^{\alpha,*}\mathscr{L}^{\alpha}$. There exists $c > 0$ such that, counted with multiplicity, one has*

$$\sigma(\mathcal{M}^{\alpha}) = \bigsqcup_{j=1}^{p(\alpha)} he^{-2h^{-1}S_{\nu_j^{\alpha}}}\sigma(M^{\alpha,j})(1 + \mathcal{O}(e^{-c/h})),$$

where the matrices $M^{\alpha,j}$ have a classical expansion $M^{\alpha,j} \sim \sum h^k M_k^{\alpha,j}$ whose leading term is given by

$$M_0^{\alpha,j} = \mathcal{J}\mathcal{R}^{j-1}(\mathcal{Z}^{\alpha}),$$

where $\mathcal{Z}^{\alpha} = \Omega^{\alpha}\mathscr{T}^{\alpha,0}\Upsilon^{\alpha,*}\Upsilon^{\alpha}\mathscr{T}^{\alpha,0}\Omega^{\alpha}$ belongs to $\mathscr{G}(\mathscr{E}, \tau)$ with $\mathscr{E} = (\mathscr{F}(\widehat{\mathcal{U}}_{\alpha,j}^{(0)}))_{j=1,\dots,p}$ and $\tau = (\tau_j)_{j=1,\dots,p}$, with $\tau_j = e^{(S_{\nu_{p-(j-2)}} - S_{\nu_{p-(j-1)}})/h}$.

Proof. One has

$$\mathcal{M}^{\alpha} = \mathscr{L}^{\alpha,*}\mathscr{L}^{\alpha} = he^{-2S_{p_1}/h}\widehat{\mathcal{M}}^{\alpha},$$

with $\widehat{\mathcal{M}}^{\alpha}$ given by (5-7),

$$\widehat{\mathcal{M}}^{\alpha}(\tau) = \widehat{\Omega}^{\alpha}(\tau)^*\widehat{\mathcal{M}}^{\alpha,\prime}\widehat{\Omega}^{\alpha}(\tau),$$

with $\widehat{\mathcal{M}}^{\alpha,\prime} = (h^{-1}\widehat{\mathscr{L}}^{\alpha,*}\widehat{\mathscr{L}}^{\alpha})$. Of course, this matrix is symmetric positive and thanks to Proposition 5.2, it admits a classical expansion

$$\widehat{\mathcal{M}}^{\alpha,\prime} \sim \sum_k h^k\widehat{\mathcal{M}}_k^{\alpha,\prime}$$

with $\widehat{\mathcal{M}}_0^{\alpha,\prime} = (\widehat{\mathscr{L}}^{\alpha,0})^*\widehat{\mathscr{L}}^{\alpha,0} = \mathscr{T}^{\alpha,0}\Upsilon^{\alpha,*}\Upsilon^{\alpha}\mathscr{T}^{\alpha,0} \in \mathscr{S}^+$. This shows that $\widehat{\mathcal{M}}^{\alpha,\prime}$ belongs to $\mathscr{S}_{\mathrm{cl}}^+$. Hence $\widehat{\mathcal{M}}^{\alpha}$ is classical $(\mathscr{E}, \tau)$-graded with $\mathscr{E} = (\mathscr{F}(\widehat{\mathcal{U}}_{\alpha,j}^{(0)}))_{j=1,\dots,p}$ and $\tau = (\tau_2, \dots, \tau_p)$, with $\tau_j = e^{(S_{\nu_{p-(j-2)}} - S_{\nu_{p-(j-1)}})/h}$ and the conclusion follows directly from Theorem 5.6. ∎

6. Proof of the main theorem

In this section we explain how one can deduce Theorem 2.8 from Theorem 5.8. As in [Helffer, Klein and Nier 2004], the general idea is to compare the singular values of the successive reduced matrix by a mean of Fan inequalities. As preparation, we shall compare the matrices $\mathcal{L}^{\pi,\prime}$ and $\mathcal{L}^{\mathrm{bkw},\prime}$ defined in Section 3. First, observe that thanks to (4-4), (4-5), (4-10), one has

$$\mathcal{L}^{\mathrm{bkw},\prime} = \mathscr{J}\mathcal{L}^{\mathrm{bkw},\prime\prime} = \mathscr{J}\mathscr{L} = \mathscr{J}\tilde{\mathscr{L}}\mathscr{T}\Omega, \tag{6-1}$$

with $\mathscr{J} : \mathscr{F}(\mathcal{V}^{(1)}) \to \mathscr{F}(\mathcal{U}^{(1)})$ defined by $\mathscr{J}_{\boldsymbol{s},\boldsymbol{s}'} = \delta_{\boldsymbol{s},\boldsymbol{s}'}$, $\tilde{\mathscr{L}} = \mathrm{diag}(\tilde{\mathscr{L}}^{\alpha}, \alpha \in \underline{\mathcal{A}})$, $\mathscr{T} = \mathrm{diag}(\mathscr{T}^{\alpha}, \alpha \in \underline{\mathcal{A}})$ and $\Omega = \mathrm{diag}(\Omega^{\alpha}, \alpha \in \underline{\mathcal{A}})$.

Lemma 6.1. *There exists* $\gamma > 0$ *such that*

$$\mathcal{L}^{\pi,\prime} = (\mathscr{J} + \mathcal{O}(e^{-\gamma/h}))\mathscr{L}.$$

Proof. First observe that thanks to Lemma 4.1, one has

$$\mathcal{L}^{\pi,\prime} = \mathcal{L}^{\mathrm{bkw},\prime} + \mathcal{R}, \tag{6-2}$$

with $\mathcal{R} : \mathscr{F}(\underline{\mathcal{U}}^{(0)}) \to \mathscr{F}(\mathcal{U}^{(1)})$ satisfying

$$\mathcal{R}_{\boldsymbol{s},\boldsymbol{m}} = \mathcal{O}(e^{-(S(\boldsymbol{m})+\gamma)/h}) \quad \text{for all } \boldsymbol{m} \in \underline{\mathcal{U}}^{(0)}, \tag{6-3}$$

for some $\gamma > 0$. Using (6-1), we get

$$\mathcal{L}^{\pi,\prime} = \mathscr{J}\tilde{\mathscr{L}}\mathscr{T}\Omega + \tilde{\mathcal{R}}\Omega,$$

with $\tilde{\mathcal{R}} = \mathcal{O}(e^{-\gamma/h})$. Hence, we have to prove that there exists $\overline{\mathcal{R}} : \mathscr{F}(\mathcal{V}^{(1)}) \to \mathscr{F}(\mathcal{U}^{(1)})$ such that $\tilde{\mathcal{R}} = \overline{\mathcal{R}}\tilde{\mathscr{L}}\mathscr{T}$ and $\overline{\mathcal{R}} = \mathcal{O}(e^{-\gamma/h})$. From Proposition 5.2, we know that the matrix $\mathscr{W} := (\tilde{\mathscr{L}}\mathscr{T})^*\tilde{\mathscr{L}}\mathscr{T}$ is invertible with inverse uniformly bounded with respect to h. This allows us to define $\overline{\mathcal{R}} := \tilde{\mathcal{R}}\mathscr{W}^{-1}(\tilde{\mathscr{L}}\mathscr{T})^*$. Thanks to the above remarks, we have $\overline{\mathcal{R}} = \mathcal{O}(e^{-\gamma/h})$ and by construction

$$\overline{\mathcal{R}}\tilde{\mathscr{L}}\mathscr{T} = \tilde{\mathcal{R}}\mathscr{W}^{-1}(\tilde{\mathscr{L}}\mathscr{T})^*\tilde{\mathscr{L}}\mathscr{T} = \tilde{\mathcal{R}},$$

which completes the proof. □

We are now ready to prove Theorem 2.8. Until the end of this section, $\gamma > 0$ denotes a constant independent of h that may change from line to line. We shall also denote by $\mathrm{SV}(M)$ the singular values of any matrix M.

From Section 2C, we know that the n_0 exponentially small eigenvalues of $\Delta_{\varphi}^{(0)}$ are the square of the singular values of the matrix $\mathcal{L}$. Thanks to Lemmas 3.12 and 3.13, we have

$$\mathcal{L} = (\mathrm{Id} + \mathcal{O}(e^{-\gamma/h}))\mathcal{L}^{\pi}(\mathrm{Id} + \mathcal{O}(e^{-\gamma/h}))$$

and it follows from the Fan inequality (Lemma A.1) that

$$\mathrm{SV}(\mathcal{L}) = \mathrm{SV}(\mathcal{L}^{\pi})(1 + \mathcal{O}(e^{-\gamma/h})).$$

Hence, we are reduced to computing the singular values of $\mathcal{L}^\pi$. Since the first column of $\mathcal{L}^\pi$ is the null vector, it follows that the nonzero singular values of $\mathcal{L}^\pi$ are the singular values of $\mathcal{L}^{\pi,\prime}$. From Lemma 6.1, we know that

$$\mathcal{L}^{\pi,\prime} = (\mathscr{J} + \mathcal{O}(e^{-\gamma/h}))\mathscr{L}, \tag{6-4}$$

and since $\mathscr{J}^*\mathscr{J} = \mathrm{Id}$ this implies for h small enough

$$\mathscr{L} = (\mathscr{J}^* + \mathcal{O}(e^{-\gamma/h}))\mathcal{L}^{\pi,\prime}. \tag{6-5}$$

Using the fact that $\|\mathscr{J}\| = \|\mathscr{J}^*\| = 1$, (6-4) and (6-5) combined with Lemma A.1 show that

$$\mathrm{SV}(\mathcal{L}^{\pi,\prime}) = (1 + \mathcal{O}(e^{-\gamma/h}))\,\mathrm{SV}(\mathscr{L}).$$

Combined with Theorem 5.8 this proves Theorem 2.8.

7. Some particular cases and examples

In this section, we rephrase Theorem 5.8 in the particular situations $p(\alpha) = 1$ and $p(\alpha) = 2$.

7A. *The case* $p(\alpha) = 1$. In this section we assume that $p(\alpha) = 1$. Then, the set $\mathcal{S}_\alpha$ is reduced to a singleton $\mathcal{S}_\alpha = \{S_{\nu_1^\alpha}\}$. Moreover, the points of $\mathcal{U}_\alpha^{(0)}$ are either all of type I, or all of type II.

7A1. *The case where α is of type II.* We first assume that α is of type II. Then all the points $\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)}$ are of type II and Theorem 5.8 takes the following form.

Theorem 7.1. *Let $\alpha \in \mathcal{A}$ be such that $p(\alpha) = 1$ and all the points of $\mathcal{U}_\alpha^{(0)}$ are of type II. Then the matrix $\mathscr{L}^\alpha$ has exactly $q_\alpha = \#\mathcal{U}_\alpha^{(0)}$ singular values counted with multiplicity $\rho_{\alpha,\mu}(h)$, $\mu = 1, \dots, q_\alpha$. They have the form*

$$\rho_{\alpha,\mu}(h) = h^{1/2}\zeta_{\alpha,\mu}(h)e^{-S_{\nu_1^\alpha}/h},$$

where $\zeta_{\alpha,\mu} \sim \sum_{r=0}^\infty h^r \zeta_{\alpha,\mu,r}$ is a classical symbol such that the $\zeta_{\alpha,\mu,0}$, $\mu = 1, \dots, q_\alpha$, are the nonzero singular values of the matrix $\Upsilon^\alpha \in \mathscr{M}(\widehat{\mathcal{U}}_\alpha^{(0)}, \mathcal{V}_\alpha^{(1)})$ given by

$$\Upsilon^\alpha_{\boldsymbol{s},\boldsymbol{m}} = \pi^{-1/2}|\hat{\lambda}_1(\boldsymbol{s})|^{1/2}\left(\frac{h_\varphi(\boldsymbol{m}_1(\boldsymbol{s}))}{h_\varphi(\boldsymbol{s})}\delta_{\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s})} - \frac{h_\varphi(\boldsymbol{m}_2(\boldsymbol{s}))}{h_\varphi(\boldsymbol{s})}\delta_{\boldsymbol{m},\boldsymbol{m}_2(\boldsymbol{s})}\right) \quad \text{for all } \boldsymbol{s} \in \mathcal{V}_\alpha^{(1)},\ \boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0)},$$

with $\boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s})$ defined in Lemma 3.3.

Observe that the description of the approximated small eigenvalues of Δ_φ in the above theorem is very close in spirit to that obtained in nondegenerate situations. Though, the different eigenvalues $\rho_{\alpha,\mu}$ are linked to one another, the only minima involved in the computation of the prefactors $\zeta_{\alpha,\mu}$ are associated to the typical height $S_{\nu_1^\alpha}$. In that sense, we can say that the above formula is a generalized Eyring–Kramers formula.

As already mentioned in the Introduction, the matrix Υ^α enjoys a nice interpretation in terms of graph theory. In order to simplify, suppose that the function φ is such that the coefficients of Υ^α are either 1 or -1. Define a graph $\mathcal{G}_\alpha$ associated to the equivalence class α in the following way. The vertices of the graph are the minima $\boldsymbol{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$ and the edges are the saddle points $\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)}$. The two vertices associated to the edge $s \in \mathcal{V}_\alpha^{(1)}$ are just $m_1(s)$ and $m_2(s)$. With this definition it turns out that the matrix Υ^α is the transpose of the incidence matrix of a certain oriented version of the graph $\mathcal{G}_a$. As a consequence, the $|\zeta_{\alpha,\mu,0}|^2$ are the eigenvalues of the corresponding graph Laplacian $\Delta_{\mathcal{G}} = (\delta_{m,m'})_{m,m'\in\widehat{\mathcal{U}}^{(0)}}$ defined by

$$\delta_{m,m'} = \begin{cases} \mathrm{d}(m) & \text{if } m = m', \\ -1 & \text{if } m \neq m' \text{ and there is an edge between } m \text{ and } m', \\ 0 & \text{otherwise,} \end{cases} \tag{7-1}$$

where the degree $\mathrm{d}(m)$ is the number of edges incident to the vertex m.

Figure 2 in the Introduction presents an example of a potential φ having one unique saddle value σ and such that all local minima are absolute minima. We represent also in Figure 2 the graph associated to the nontrivial equivalence class (that is, the one which is not reduced to one element).

In the case where the coefficients of Υ^α are not necessarily equal to ± 1, the same interpretation is available with weighted graphs. We refer to [Cvetković, Doob and Sachs 1995] for definitions and standard results on graph theory.

7A2. *The case where α is of type I.* In this section, we compute explicitly the singular values of $\mathscr{L}^\alpha$, when α is of type I.

Theorem 7.2. *Let $\alpha \in \mathcal{A}$ be such that $p(\alpha) = 1$ and all the points of $\mathcal{U}_\alpha^{(0)}$ are of type I. Then, the matrix $\mathscr{L}^\alpha$ has exactly $q_\alpha := \#\mathcal{U}_\alpha^{(0)}$ singular values counted with multiplicity. These singular values $\rho_{\alpha,\mu}(h)$, $\mu = 1, \dots, q_\alpha$, have the form*

$$\rho_{\alpha,\mu}(h) = \zeta_{\alpha,\mu}(h) e^{-S_{\nu_1^\alpha}/h},$$

where $\zeta_{\alpha,\mu} \sim h^{1/2} \sum_{r=0}^\infty h^r \zeta_{\alpha,\mu,r}$ has a classical expansion such that $\zeta_{\alpha,\mu,0}$ are the q_α singular values of the matrix Υ^α given by

$$\Upsilon_{s,m} = \pi^{-1/2} |\hat{\lambda}_1(s)|^{1/2} \left(\frac{h_\varphi(m_1(s))}{h_\varphi(s)} \delta_{m,m_1(s)} - \frac{h_\varphi(m_2(s))}{h_\varphi(s)} \delta_{m,m_2(s)} \right)$$

if $s \in \mathcal{V}_\alpha^{(1),i}$ and

$$\Upsilon_{s,m} = \pi^{-1/2} |\hat{\lambda}_1(s)|^{1/2} \frac{h_\varphi(m_1(s))}{h_\varphi(s)} \delta_{m,m_1(s)}$$

if $s \in \mathcal{V}_\alpha^{(1),b}$. Moreover, these singular values are nonzero.

As in the case of points of type II we can interpret the matrix $\widetilde{\mathscr{L}}^{\alpha,0}$ in terms of graphs. However, some saddle points are now associated to only one minimum. In terms of the graph, this leads to some edges having only one vertex, which means that we are dealing with hypergraphs.

7B. ***The case $p(\alpha) = 2$.*** Throughout this section we assume that $p(\alpha) = 2$. Then φ takes two different values $\varphi_- < \varphi_+$ on $\mathcal{U}_\alpha^{(0)}$. One has $\mathcal{S}_\alpha = \{S_{\nu_+^\alpha} < S_{\nu_-^\alpha}\}$ with $S_{\nu_\pm^\alpha} = \sigma(\alpha) - \varphi_\pm$.

7B1. *The case where α is of type II.* The partition (5-4) takes the form $\widehat{\mathcal{U}}_\alpha^{(0)} = \widehat{\mathcal{U}}_{\alpha,+}^{(0)} \sqcup \widehat{\mathcal{U}}_{\alpha,-}^{(0)}$ with $\widehat{\mathcal{U}}_{\alpha,\pm}^{(0)} = \{\boldsymbol{m} \in \mathcal{U}_\alpha^{(0)} : \varphi(\boldsymbol{m}) = \varphi_\pm\}$. Since α is of type II, $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_\alpha^{(0)}$ for all $\boldsymbol{s}$. It is then convenient to introduce the partition of $\mathcal{V}_\alpha^{(1)}$ given by

$$\mathcal{V}_\alpha^{(1)} = \mathcal{V}_{\alpha,+}^{(1)} \sqcup \mathcal{V}_{\alpha,+-}^{(1)} \cup \mathcal{V}_{\alpha,-}^{(1)}, \tag{7-2}$$

with $\mathcal{V}_{\alpha,+}^{(1)} = \{\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)} : \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha,+}^{(0)}\}$ and $\mathcal{V}_{\alpha,-}^{(1)} = \{\boldsymbol{s} \in \mathcal{V}_\alpha^{(1)} : \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha,-}^{(0)}\}$, where the functions $\boldsymbol{m}_1, \boldsymbol{m}_2$ are defined by Lemma 3.3. In the case $\boldsymbol{s} \in \mathcal{V}_{\alpha,+-}^{(1)}$, it follows from the choice of Lemma 3.3 that $\boldsymbol{m}_1(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha,+}^{(0)}$ and $\boldsymbol{m}_2(\boldsymbol{s}) \in \widehat{\mathcal{U}}_{\alpha,-}^{(0)}$. We order the above partitions by deciding $\widehat{\mathcal{U}}_{\alpha,+}^{(0)} \prec \widehat{\mathcal{U}}_{\alpha,-}^{(0)}$ and $\mathcal{V}_{\alpha,+}^{(1)} \prec \mathcal{V}_{\alpha,+-}^{(1)} \prec \mathcal{V}_{\alpha,-}^{(1)}$. Then, the matrix $\mathscr{Y}^\alpha := h^{-1/2} e^{-S_{\nu_+^\alpha}/h} \widehat{\mathscr{L}}^\alpha$ has the form

$$\mathscr{Y}^\alpha = \begin{pmatrix} \iota & 0 \\ b_{+-} & \tau b_{-+} \\ 0 & \tau a \end{pmatrix},$$

where $\tau = e^{(S_{\nu_+^\alpha} - S_{\nu_-^\alpha})/h}$ and the matrices ι, b_{+-}, b_{-+} admit a classical expansion whose principal terms are given by the formula

- for all $\boldsymbol{s} \in \mathcal{V}_{\alpha,+}^{(1)}$ and $\boldsymbol{m} \in \widehat{\mathcal{U}}_{\alpha,+}^{(0)}$ one has $\iota_{\boldsymbol{s},\boldsymbol{m}}^0 = \upsilon_2(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}))$,
- for all $\boldsymbol{s} \in \mathcal{V}_{\alpha,-}^{(1)}$ and $\boldsymbol{m} \in \widehat{\mathcal{U}}_{\alpha,-}^{(0)}$ one has $a_{\boldsymbol{s},\boldsymbol{m}}^0 = \upsilon_2(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}))$,
- for all $\boldsymbol{s} \in \mathcal{V}_{\alpha,+-}^{(1)}$, $\boldsymbol{m} \in \widehat{\mathcal{U}}_{\alpha,+}^{(0)}$ and $\boldsymbol{m}' \in \widehat{\mathcal{U}}_{\alpha,-}^{(0)}$ one has $(b_{+-}^0)_{\boldsymbol{s},\boldsymbol{m}} = \upsilon_1(\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{m}_1(\boldsymbol{s}))$ and $(b_{-+}^0)_{\boldsymbol{s},\boldsymbol{m}'} = -\upsilon_1(\boldsymbol{s}, \boldsymbol{m}', \boldsymbol{m}_2(\boldsymbol{s}))$,

with υ_2, υ_1 given by (5-19), (5-20). By a standard block-matrix computation one has

$$(\mathscr{Y}^\alpha)^* \mathscr{Y}^\alpha = \begin{pmatrix} J & \tau \widehat{B} \\ \tau \widehat{B}^* & \tau^2 \widehat{A} \end{pmatrix}, \tag{7-3}$$

with $J = \iota^* \iota + b_{+-}^* b_{+-}$, $\widehat{B} = b_{+-}^* b_{-+}$ and $\widehat{A} = a^* a + b_{-+}^* b_{-+}$. All these matrices admit a classical expansion, $\widehat{A} \simeq \sum_{k \geq 0} h^k \widehat{A}^k$, $\widehat{B} \simeq \sum_{k \geq 0} h^k \widehat{B}^k$, $J = \sum_{k \geq 0} h^k J^k$ and one has $J^0 = \iota^{0,*} \iota^0 + b_{+-}^{0,*} b_{+-}^0$, $\widehat{B}^0 = b_{+-}^{0,*} b_{-+}^0$ and $\widehat{A}^0 = a^{0,*} a^0 + b_{-+}^{0,*} b_{-+}^0$, where we use the notation $(c^j)^* = c^{j,*}$.

Theorem 7.3. *The matrix $\mathscr{L}^\alpha$ has exactly $q_{\alpha,\pm} = \#\mathcal{U}_{\alpha,\pm}^{(0)}$ singular values $\lambda_{\alpha,\mu}^\pm(h)$, $\mu = 1, \dots, q_{\alpha,\pm}$, counted with multiplicity which are of order $h^{1/2} e^{-S_{\nu_\pm^\alpha}/h}$. These singular values have the form*

$$\lambda_{\alpha,\mu}^\pm(h) = \zeta_{\alpha,\mu}^\pm(h) e^{-S_{\nu_\pm^\alpha}/h},$$

where

$$\zeta_{\alpha,\mu}^\pm \sim h^{1/2} \sum_k h^k \zeta_{\alpha,\mu,k}^\pm$$

is a classical symbol such that $(\zeta_{\alpha,\mu,0}^\pm)^2$ are the $q_{\alpha,\pm}$ nonzero eigenvalues of the matrices $G^\pm$ given by $G^+ = J^0$ and

$$G^- = \widehat{A}^0 - (\widehat{B}^0)^* (J^0)^{-1} \widehat{B}^0,$$

where $\widehat{A}^0$, J^0 and $\widehat{B}^0$ are defined below (7-3).

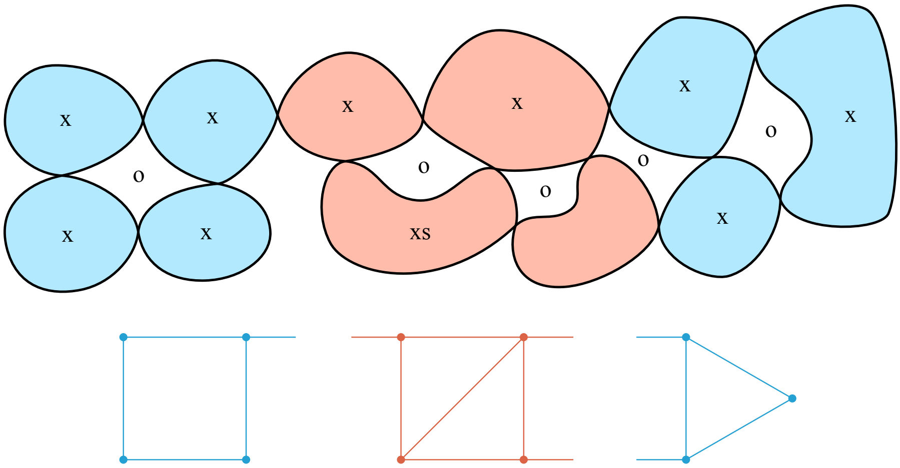

Figure 9. Top: the sublevel set $\{\varphi < \sigma\}$ associated to a potential φ having a unique saddle value and two minimal values. Bottom: the associated hypergraphs.

Let us make a few comments on this theorem. First, observe that the prefactor $\zeta^\pm = \zeta^\pm_{\alpha,\mu}$ obeys two different laws whether we are in the "+" or "−" case. In the "+" case, ζ^+ is determined by the matrix J^0 which depends only on the minima $\boldsymbol{m} \in \mathcal{U}^{(0)}_\alpha$ such that $S(\boldsymbol{m}) = S_{\nu_+}$. In that sense, the behavior of ζ^+ obeys a law similar to the generalized Eyring–Kramers law of Theorem 7.1. In the "−" case, the situation is different since the matrix G^- involves values of φ on all minima and not only those for which $S(\boldsymbol{m}) = S_{\nu_-}$. Hence the term $(\hat{B}^0)^*(J^0)^{-1}\hat{B}^0$ in the definition of G^- can be understood as a tunneling term between minima associated to both heights.

This interpretation is confirmed by the following example. Suppose that φ has two distinct minimal values and one saddle value. Figure 9 below represents such a potential. The blue wells correspond to the absolute minimal value and the red one to the other minimal value. All the saddle points are supposed to be at the same level. Then, the matrices $\hat{A}^0$ and J^0 can be viewed as the Laplacians of the hypergraphs built as follows. First we consider the graph $\mathcal{G}$ associated to all the minima whose vertex are the minima and edges are the saddle points between two minima (without distinction on the level of the minima). The blue and red hypergraphs $\mathcal{G}_b$ and $\mathcal{G}_r$ are obtained by cutting the graph $\mathcal{G}$ on edges between a blue and a red minimum. Eventually, the matrix B links blue and red minima.

7B2. *The case where α is of type I.* In this section we assume that α is of type I. The partition (5-4) takes the form $\mathcal{U}^{(0)}_\alpha = \mathcal{U}^{(0)}_{\alpha,-} \sqcup \mathcal{U}^{(0)}_{\alpha,+}$ with

$$\mathcal{U}^{(0)}_{\alpha,\pm} = \{\boldsymbol{m} \in \mathcal{U}^{(0)}_\alpha : \varphi(\boldsymbol{m}) = \varphi_\pm\}.$$

We order the two elements of $\mathscr{P}^{(0)}_\alpha$ by deciding $\mathcal{U}^{(0)}_{\alpha,+} \prec \mathcal{U}^{(0)}_{\alpha,-}$. In order to deal with the saddle points, we introduce the partition $\mathscr{P}^{(1)}_\alpha$ which is a mix of partitions used in Lemma 3.4 and Section 7B1:

$$\mathcal{V}^{(1)}_\alpha = \mathcal{V}^{(1)}_{\alpha,+,b} \sqcup \mathcal{V}^{(1)}_{\alpha,+,i} \sqcup \mathcal{V}^{(1)}_{\alpha,+-} \sqcup \mathcal{V}^{(1)}_{\alpha,-,b} \sqcup \mathcal{V}^{(1)}_{\alpha,-,i}$$

with

$$\begin{aligned}
\mathcal{V}^{(1)}_{\alpha,+,-} &= \{\boldsymbol{s} \in \mathcal{V}^{(1),i}_{\alpha} : \boldsymbol{m}_1(\boldsymbol{s}) \in \mathcal{U}^{(0)}_{\alpha,+}, \boldsymbol{m}_2(\boldsymbol{s}) \in \mathcal{U}^{(0)}_{\alpha,-}\},\\
\mathcal{V}^{(1)}_{\alpha,+,i} &= \{\boldsymbol{s} \in \mathcal{V}^{(1),i}_{\alpha} : \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}) \in \mathcal{U}^{(0)}_{\alpha,+}\},\\
\mathcal{V}^{(1)}_{\alpha,+,b} &= \{s \in \mathcal{V}^{(1),b}_{\alpha} : \boldsymbol{m}_1(s) \in \mathcal{U}^{(0)}_{\alpha,+}\},\\
\mathcal{V}^{(1)}_{\alpha,-,i} &= \{\boldsymbol{s} \in \mathcal{V}^{(1),i}_{\alpha} : \boldsymbol{m}_1(\boldsymbol{s}), \boldsymbol{m}_2(\boldsymbol{s}) \in \mathcal{U}^{(0)}_{\alpha,-}\},\\
\mathcal{V}^{(1)}_{\alpha,-,b} &= \{s \in \mathcal{V}^{(1),b}_{\alpha} : \boldsymbol{m}_1(s) \in \mathcal{U}^{(0)}_{\alpha,-}\}.
\end{aligned} \tag{7-4}$$

Here the functions $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ are defined by Lemma 3.3. One has the following

Theorem 7.4. *Assume that $p(\alpha) = 2$ and α is of type I. The matrix $\mathscr{L}^{\alpha}$ has exactly $q_{\alpha,\pm} = \#\mathcal{U}^{(0)}_{\alpha,\pm}$ singular values $\lambda^{\pm}_{\alpha,\mu}(h)$, $\mu = 1, \dots, q_{\alpha,\pm}$, counted with multiplicity which are of order $h^{1/2}e^{-S_{\nu^{\alpha}_{\pm}}/h}$. These singular values have the form*

$$\lambda^{\pm}_{\alpha,\mu}(h) = \zeta^{\pm}_{\alpha,\mu}(h)e^{-S_{\nu^{\alpha}_{\pm}}/h}$$

where $\zeta^{\pm}_{\alpha,\mu} \sim h^{1/2}\sum_k h^k \zeta^{\pm}_{\alpha,\mu,k}$ is a classical symbol such that $(\zeta^{\pm}_{\alpha,\mu,0})^2$ are the $q_{\alpha,\pm}$ eigenvalues (which are nonzero) of the matrices $G^{\pm}$ given by $G^{+} = J^0$ and $G^{-} = A^0 - (B^0)^*(J^0)^{-1}B^0$, where A^0, B^0 and J^0 are defined by

$$J^0 = \iota^{0,*}\iota^0 + b^{0,*}_{+-}b^0_{+-}, \quad B^0 = b^{0,*}_{+-}b^0_{-+}, \quad A^0 = a^{0,*}a^0 + b^{0,*}_{-+}b^0_{-+},$$

with the matrices a^0, b^0_{+-}, b^0_{-+} and ι^0 defined by

- *for all $\boldsymbol{s} \in \mathcal{V}^{(1)}_{\alpha,+,i}$ and $\boldsymbol{m} \in \mathcal{U}^{(0)}_{\alpha,+}$ one has $\iota^0_{\boldsymbol{s},\boldsymbol{m}} = \Upsilon_2(\boldsymbol{s},\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s}),\boldsymbol{m}_2(\boldsymbol{s}))$,*
- *for all $\boldsymbol{s} \in \mathcal{V}^{(1)}_{\alpha,+,b}$ and $\boldsymbol{m} \in \mathcal{U}^{(0)}_{\alpha,+}$ one has $\iota^0_{\boldsymbol{s},\boldsymbol{m}} = \Upsilon_1(\boldsymbol{s},\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s}))$,*
- *for all $\boldsymbol{s} \in \mathcal{V}^{(1)}_{\alpha,-,i}$ and $\boldsymbol{m} \in \mathcal{U}^{(0)}_{\alpha,-}$ one has $a^0_{\boldsymbol{s},\boldsymbol{m}} = \Upsilon_2(\boldsymbol{s},\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s}),\boldsymbol{m}_2(\boldsymbol{s}))$,*
- *for all $\boldsymbol{s} \in \mathcal{V}^{(1)}_{\alpha,-,b}$ and $\boldsymbol{m} \in \mathcal{U}^{(0)}_{\alpha,-}$ one has $a^0_{\boldsymbol{s},\boldsymbol{m}} = \Upsilon_1(\boldsymbol{s},\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s}))$,*
- *for all $\boldsymbol{s} \in \mathcal{V}^{(1)}_{\alpha,+-}$, $\boldsymbol{m} \in \mathcal{U}^{(0)}_{\alpha,+}$ and $\boldsymbol{m}' \in \mathcal{U}^{(0)}_{\alpha,-}$ one has $(b^0_{+-})_{\boldsymbol{s},\boldsymbol{m}} = \Upsilon_1(\boldsymbol{s},\boldsymbol{m},\boldsymbol{m}_1(\boldsymbol{s}))$ and $(b^0_{-+})_{\boldsymbol{s},\boldsymbol{m}'} = -\Upsilon_1(\boldsymbol{s},\boldsymbol{m}',\boldsymbol{m}_2(\boldsymbol{s}))$.*

7C. *Some examples.*

7C1. *Computations in dimension 1 with $p(\alpha) = 1$.* Let us compute the small eigenvalues of the potential φ represented in Figure 10.

As already noticed in the discussion below Theorem 2.8, there are exactly three equivalence classes for $\mathcal{R}$ in that case: $\mathcal{U}^{(0)}_1 = \{\boldsymbol{m}_{1,1}\}$, $\mathcal{U}^{(0)}_2 = \{\boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}\}$ and $\mathcal{U}^{(0)}_3 = \{\boldsymbol{m}_{2,3}\}$. Let us denote by $\boldsymbol{s}_1$ the saddle point between $\boldsymbol{m}_{2,1}$ and $\boldsymbol{m}_{2,2}$, by $\boldsymbol{s}_2$ the saddle point between $\boldsymbol{m}_{2,2}$ and $\boldsymbol{m}_{1,1}$ and by $\boldsymbol{s}_3$ the saddle point between $\boldsymbol{m}_{1,1}$ and $\boldsymbol{m}_{2,3}$. Define also $S_2 = \varphi(\boldsymbol{s}_1) - \varphi(\boldsymbol{m}_{2,1}) = \varphi(\boldsymbol{s}_1) - \varphi(\boldsymbol{m}_{2,2})$ and $S_3 = \varphi(\boldsymbol{s}_3) - \varphi(\boldsymbol{m}_{2,3})$. Observe also that for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$, one has $H(\boldsymbol{m}) = \{\boldsymbol{m}\}$. Then the matrix $\mathcal{L}^{\mathrm{bkw}}$ defined by (4-2), admits the form

$$\mathcal{L}^{\mathrm{bkw}} = \left(\frac{h}{\pi}\right)^{1/2} \begin{pmatrix} 0 & d^2_{1,1} & d^2_{1,2} & 0\\ 0 & d^2_{2,1} & d^2_{2,2} & 0\\ 0 & 0 & 0 & d^3 \end{pmatrix},$$

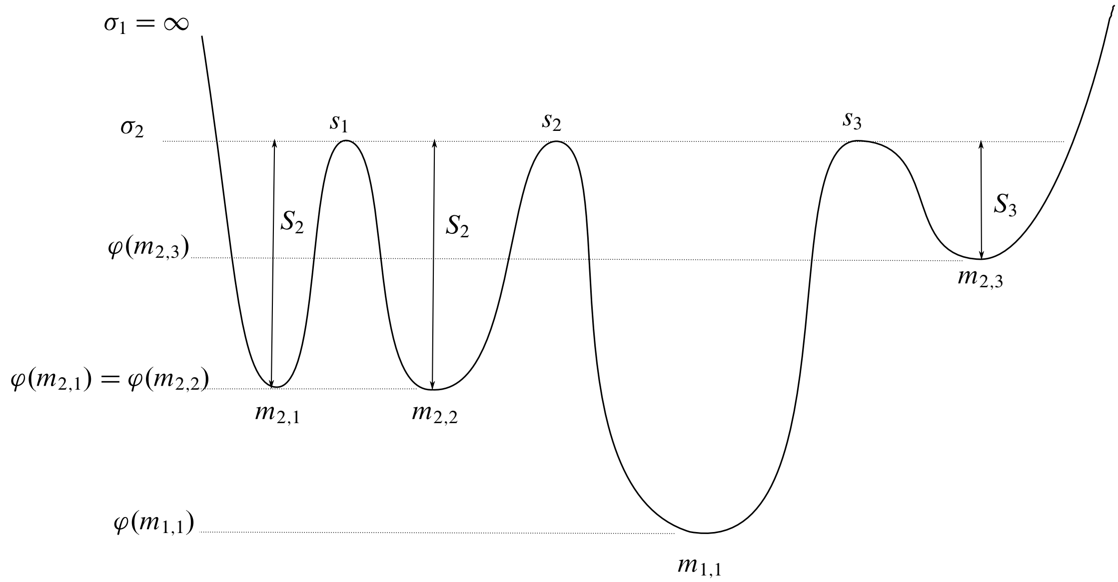

Figure 10. A potential with $p(\alpha) = 1$ for all α.

with the coefficients given by

$$d^2_{1,1} = (|\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,1})|^{1/4} + \mathcal{O}(h))e^{-S_2/h}, \quad d^2_{1,2} = -(|\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,2})|^{1/4} + \mathcal{O}(h))e^{-S_2/h},$$

$$d^2_{2,1} = 0, \quad d^2_{2,2} = (|\varphi''(\boldsymbol{s}_2)\varphi''(\boldsymbol{m}_{2,2})|^{1/4} + \mathcal{O}(h))e^{-S_2/h}, \quad d^3 = (|\varphi''(\boldsymbol{s}_3)\varphi''(\boldsymbol{m}_{2,3})|^{1/4} + \mathcal{O}(h))e^{-S_3/h}.$$

The corresponding squares of singular values are then

$$\lambda_0 = 0, \quad \lambda_3 = \frac{h}{\pi}(|\varphi''(\boldsymbol{s}_3)\varphi''(\boldsymbol{m}_{2,3})|^{1/2} + \mathcal{O}(h))e^{-2S_3/h} \quad \text{and} \quad \lambda_2^{\pm} = \frac{h}{\pi}(\mu_2^{\pm} + \mathcal{O}(h))e^{-2S_2/h},$$

where $\mu_2^{\pm}$ are the squares of the singular values of the matrix

$$\widetilde{\mathcal{D}}^2 = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

with $a = |\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,1})|^{1/4}$, $b = |\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,2})|^{1/4}$ and $c = |\varphi''(\boldsymbol{s}_2)\varphi''(\boldsymbol{m}_{2,2})|^{1/4}$. It follows that

$$(\widetilde{\mathcal{D}}^2)^*\widetilde{\mathcal{D}}^2 = \begin{pmatrix} a^2 & -ab \\ -ab & b^2+c^2 \end{pmatrix},$$

whose eigenvalues can be computed handily. For instance, if $|\varphi''(\boldsymbol{s})| = |\varphi''(\boldsymbol{m})| = 1$ for all $\boldsymbol{s} \in \mathcal{U}^{(1)}$ and $\boldsymbol{m} \in \mathcal{U}^{(0)}$, one has

$$(\widetilde{\mathcal{D}}^2)^*\widetilde{\mathcal{D}}^2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

whose eigenvalues are $\mu_2^{\pm} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$.

We would like to conclude this example by noticing that one has necessarily $\mu_2^+ \neq \mu_2^-$. Indeed, if one computes the characteristic polynomial of the above matrix, one finds $P(x) = x^2 - (a^2+b^2+c^2)x + a^2c^2$, whose discriminant is given by

$$\Delta = (a^2+b^2+c^2)^2 - 4a^2c^2 = ((a-c)^2+b^2)((a+c)^2+b^2).$$

Since φ is a Morse function, one has $b \neq 0$ and hence $\Delta > 0$.

7C2. *Computations in dimension 1 with $p(\alpha) = 2$.* Suppose now that the potential φ is as represented in Figure 7. As already noticed there are exactly two equivalence classes for $\mathcal{R}$ in that case, $\mathcal{U}_1^{(0)} = \{\boldsymbol{m}_{1,1}\}$ and $\mathcal{U}_2^{(0)} = \{\boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}, \boldsymbol{m}_{2,3}\}$, and again, one has $H(\boldsymbol{m}) = \{\boldsymbol{m}\}$ for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$. Let us denote by $\boldsymbol{s}_1$ the saddle point between $\boldsymbol{m}_{2,1}$ and $\boldsymbol{m}_{2,2}$, by $\boldsymbol{s}_2$ the saddle point between $\boldsymbol{m}_{2,2}$ and $\boldsymbol{m}_{2,3}$ and by $\boldsymbol{s}_3$ the saddle point between $\boldsymbol{m}_{2,3}$ and $\boldsymbol{m}_{1,1}$. Define also $S_2 = \varphi(\boldsymbol{s}_1) - \varphi(\boldsymbol{m}_{2,1}) = \varphi(\boldsymbol{s}_1) - \varphi(\boldsymbol{m}_{2,2})$ and $S_3 = \varphi(\boldsymbol{s}_2) - \varphi(\boldsymbol{m}_{2,3})$. Then the matrix $\mathcal{L}^{\mathrm{bkw},\prime\prime}$ admits the following form in the basis $(f^{(0)}_{\boldsymbol{m}_{2,3}}, f^{(0)}_{\boldsymbol{m}_{2,1}}, f^{(0)}_{\boldsymbol{m}_{2,2}})$ and $(f^{(1)}_{\boldsymbol{s}_3}, f^{(1)}_{\boldsymbol{s}_2}, f^{(1)}_{\boldsymbol{s}_1})$:

$$\mathcal{L}^{\mathrm{bkw},\prime\prime} = \left(\frac{h}{\pi}\right)^{1/2} e^{-S_3/h} \begin{pmatrix} \iota & 0 & 0 \\ b_1 & 0 & b_2 e^{-(S_2-S_3)/h} \\ 0 & a_1 e^{-(S_2-S_3)/h} & a_2 e^{-(S_2-S_3)/h} \end{pmatrix},$$

with the leading terms of the coefficients given by

$$\iota^0 = -|\varphi''(\boldsymbol{s}_3)\varphi''(\boldsymbol{m}_{2,3})|^{1/4}, \quad b_1^0 = |\varphi''(\boldsymbol{s}_2)\varphi''(\boldsymbol{m}_{2,3})|^{1/4}, \quad b_2^0 = |\varphi''(\boldsymbol{s}_2)\varphi''(\boldsymbol{m}_{2,2})|^{1/4}$$

and

$$a_1^0 = |\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,1})|^{1/4}, \quad a_2^0 = -|\varphi''(\boldsymbol{s}_1)\varphi''(\boldsymbol{m}_{2,2})|^{1/4}.$$

In order to simplify the computation, assume that $\varphi''(\boldsymbol{m}) = 1$ for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$ and $\varphi''(\boldsymbol{s}_1) = \varphi''(\boldsymbol{s}_2) = -1$. Define $\theta = |\varphi''(\boldsymbol{s}_3)|$ and $\tau = e^{-(S_2-S_3)/h}$. Then

$$\mathcal{L}^{\mathrm{bkw},\prime\prime} = \left(\frac{h}{\pi}\right)^{1/2} e^{-S_3/h} \left(\begin{pmatrix} -\theta & 0 & 0 \\ 1 & 0 & \tau \\ 0 & \tau & -\tau \end{pmatrix} + \mathcal{O}(h) \right).$$

Hence, we can apply Theorem 7.4 with

$$a^0 = (1\ -1), \quad \iota_0 = -\theta, \quad b^0_{+-} = 1, \quad b^0_{-+} = (0\ 1).$$

It follows that the singular values of order $e^{-S_2/h}$ are

$$\mu_\pm(h) = \left(\frac{h}{\pi}\right)^{1/2} e^{-S_2/h} (\sqrt{\lambda_\pm} + \mathcal{O}(h)),$$

with $\lambda_\pm$ eigenvalues of $M^0 := A^0 - (B^0)^*(J^0)^{-1}B^0$, with

$$A^0 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad B^0 = (0\ -1), \quad J^0 = 1 + \theta^2.$$

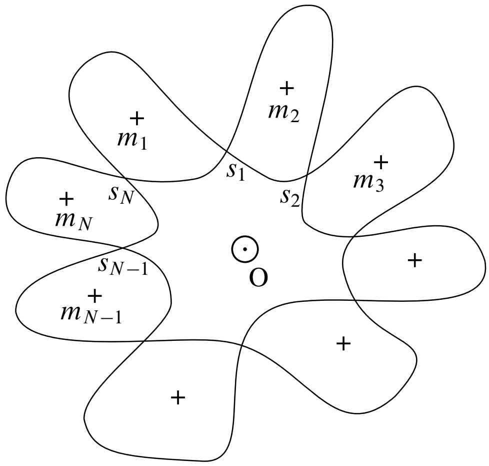

Figure 11. N wells in dimension 2.

Hence

$$M^0 = \begin{pmatrix} 1 & -1 \\ -1 & 2-\nu \end{pmatrix},$$

with $\nu = 1/(1+\theta^2) \in]0,1[$. The eigenvalues of this matrix are

$$\lambda_\pm = \frac{3-\nu}{2} \pm \frac{\sqrt{(3-\nu)^2 - 4(1-\nu)}}{2}.$$

This can be seen as perturbations by the well of height S_3 of the eigenvalues $\lambda_\pm$ computed in the previous example (obtained by taking $\nu = 0$ in the above formula).

7C3. *Computations in higher dimensions.* Consider the case of the potential φ having $N \geq 3$ minima $\boldsymbol{m}_1, \dots, \boldsymbol{m}_N$ and one local maximum at the origin as presented in Figure 11. Assume also that there are exactly N saddle points $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$, all at the same height $\varphi(\boldsymbol{s}_j) = \sigma_2$ and that the set $\{\varphi < \sigma_2\}$ has exactly N connected components $E_1, \dots, E_N$, each E_j containing the minimum m_j, and that for all $j = 1, \dots, N$, $\{\boldsymbol{s}_j\} = \overline{E}_j \cap \overline{E}_{j+1}$ with the convention $E_{N+1} = E_1$. Assume in addition that all the $\varphi(\boldsymbol{m}_j)$ are equal and write $S = \sigma_2 - \varphi(\boldsymbol{m}_1)$. Let us choose $\boldsymbol{m}_1$ as the global minimum associated to $\sigma_1 = \infty$. Then all the other minima are associated to the saddle value σ_2. It is clear that they all belong to the same equivalence class and that they are all of type II. Moreover, for all $\boldsymbol{m} \in \mathcal{U}^{(0)} \setminus \{\boldsymbol{m}_1\}$, one has $H(\boldsymbol{m}) = \{\boldsymbol{m}\}$. Then, we can apply Theorem 7.1 to get the spectrum of the Witten Laplacian associated to φ. It follows that the eigenvalues are given by $\lambda_1 = 0$ and for all $n = 2, \dots, N$

$$\lambda_n(h) = b_n(h) e^{-2S/h}(1 + \mathcal{O}(e^{-\alpha/h})), \tag{7-5}$$

where b_n admits a classical expansion

$$b_n(h) \simeq \frac{h}{\pi} \sum_{k \geq 0} b_{n,k} h^k.$$

Moreover, one has $b_{n,0} = \mu_n^2$, where the $\mu_n,\ n = 2,\dots,N$, are the nonzero singular values of the matrix

$$\mathcal{L} := \begin{pmatrix} \alpha_1\beta_1 & -\alpha_2\beta_1 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2\beta_2 & -\alpha_3\beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3\beta_3 & \ddots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & \alpha_{N-1}\beta_{N-1} & -\alpha_N\beta_{N-1} \\ -\alpha_1\beta_N & 0 & \cdots & \cdots & 0 & \alpha_N\beta_N \end{pmatrix},$$

where we set $\alpha_j = \varphi''(\boldsymbol{m}_j)^{1/4}$ and $\beta_j = (-\varphi''(\boldsymbol{s}_j))^{1/4}$.

If one assumes additionally that α_j and β_j are independent of j, let say $\alpha_j = \alpha$ and $\beta_j = \beta$, then $\mathcal{L} = \alpha\beta\mathcal{A}$ with

$$\mathscr{A} = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The singular values of $\mathscr{A}$ are the square roots of the eigenvalues of

$$\mathscr{A}^*\mathscr{A} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

which are known to be $\nu_k = 2(1 - \cos(2k\pi/N)),\ k = 0,\dots,N-1$. In particular, for all $2 \le k < N/2$, ν_k has multiplicity 2 since $\nu_k = \nu_{N-k}$.

Suppose now that the potential φ is invariant by a rotation of angle $2\pi/N$; then (7-5) still holds true with $b_n(h)$ being the singular values of a matrix of the form

$$\mathscr{A} = \theta(h) \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix},$$

with $\theta(h) \simeq \sum_{k\ge 0} h^k\theta_k$. Hence, the above computation is still valid and it follows that for $2 \le k < N/2$, $b_k(h) = b_{N-k}(h)$. This permits us to recover the results of [Hérau, Hitrik and Sjöstrand 2011, Section 7.4].

Appendix A: Some results in linear algebra

We collect here some helpful results from linear algebra.

Lemma A.1 (Fan inequalities). *Let A, B be two matrices and denote by $\mu_n(X)$ the singular values of X. Then*

$$\mu_n(AB) \le \|B\|\mu_n(A),$$
$$\mu_n(AB) \le \|A\|\mu_n(B),$$

where $\|C\|$ denotes the norm of $C : \mathbb{R}^p \to \mathbb{R}^q$ with $\mathbb{R}^\bullet$ endowed with ℓ^2 norms.

Proof. See [Simon 1979]. □

Lemma A.2. *Let $A = \operatorname{diag}(A_1, \dots, A_N)$ be a block diagonal matrix. Then the singular values of A are the singular values of the A_n counted with multiplicities.*

Proof. It is straightforward, since $A^*A = \operatorname{diag}(A_1^*A_1, \dots, A_N^*A_N)$. □

Lemma A.3. *Let E, F be two finite-dimensional vector spaces and $A(h) : E \to F$ be a family of linear operators depending on a parameter $h \in]0,1]$. Assume that $A(h)$ admits a classical expansion $A(h) \sim \sum_{k\ge 0} h^k A_k$ and that the matrix A_0 has nonzero singular values. Then, for $h > 0$ small enough the singular values $\mu_n(h)$ of $A(h)$ admit a classical expansion*

$$\mu_n(h) \sim \sum_{k\ge 0} h^k \mu_n^k,$$

where the μ_n^0 are the singular values of A_0.

Proof. Since the singular values of $A(h)$ are the eigenvalues of A^*A, which is selfadjoint, the result follows easily from Kato's perturbation theory of analytic families of selfadjoint operators [Kato 1966, Chapter 2, Section 1] applied to the expansion of A^*A in h powers cut at finite rank. □

Lemma A.4. *Let A be a $p \times (q+1)$ matrix and T a $(q+1)\times q$ matrix. Assume that $T^*T = \mathrm{Id}$ and that $\ker A = \operatorname{Ran}(T)^\perp$. Then the singular values of A are $\{0, z_1, \dots, z_q\}$, where $z_1, \dots, z_q$ are the singular values of AT.*

Proof. First observe that since $\ker A = \operatorname{Ran}(T)^\perp$, 0 is a singular value of multiplicity 1 of A. Let us denote by $\tilde{\xi}_0$ a unit vector such that $\ker A = \mathbb{R}\tilde{\xi}_0$. By definition, there exists an orthonormal basis $\xi_1, \dots, \xi_q$ of $\mathbb{R}^q$ such that

$$T^*A^*AT\xi_k = z_k^2\xi_k \tag{A-1}$$

for all $k = 1, \dots, q$. Let us set $\tilde{\xi}_k = T\xi_k$. Since $T^*T = \mathrm{Id}$; then the set of $\tilde{\xi}_k$ is an orthonormal family of $\mathbb{R}^{q+1}$. Moreover, since $\ker A = \operatorname{Ran}(T)^\perp$, we have $\Xi = \{\tilde{\xi}_0, \dots, \tilde{\xi}_q\}$ is an orthonormal basis of $\mathbb{R}^{q+1}$. Moreover, for all $k = 1, \dots, q$, it follows from (A-1) that

$$|A\tilde{\xi}_k|^2 = |AT\xi_k|^2 = z_k^2.$$

This shows that the matrix A^*A in the basis Ξ is exactly $\operatorname{diag}(0, z_1^2, \dots, z_q^2)$ and proves the result. □

Lemma A.5. *Let $\mathcal{M}$ be a real matrix. Assume that $\mathcal{M}$ is symmetric definite positive and that it admits a block decomposition*

$$\mathcal{M} = \begin{pmatrix} J & B^* \\ B & N \end{pmatrix}.$$

Then J and $N - B^* J^{-1} B$ are symmetric definite positive.

Proof. This is quite standard, but we recall the proof for the reader's convenience. Of course J and $N - B^* J^{-1} B$ are symmetric. Moreover, since $\mathcal{M}$ is positive definite,

$$\langle Jx, x \rangle = \left\langle \mathcal{M} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \right\rangle \geq c|x|^2$$

for some $c > 0$. This shows that J is definite positive. On the other hand, setting

$$\Omega = \begin{pmatrix} I & -J^{-1} B^* \\ 0 & I \end{pmatrix},$$

one has

$$\Omega^* \mathcal{M} \Omega = \begin{pmatrix} J & 0 \\ 0 & N - B J^{-1} B^* \end{pmatrix}.$$

Since $\mathcal{M}$ is positive definite, this implies that $N - B J^{-1} B^*$ is positive definite. □

Appendix B: Link between $\mathcal{R}$ and the Generic Assumption

Proposition B.1. *Suppose that the Generic Assumption is satisfied; that is, for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$ one has the following:*

- *$\varphi_{|E(\boldsymbol{m})}$ has a unique minimum point.*
- *If E is a connected component of $\{\varphi < \sigma(\boldsymbol{m})\}$ such that $E \cap \mathcal{V}^{(1)} \neq \emptyset$, there exists a unique $\boldsymbol{s} \in \mathcal{V}^{(1)}$ such that $\varphi(\boldsymbol{s}) = \sup E \cap \mathcal{V}^{(1)}$. In particular, $E \cap \varphi^{-1}(]-\infty, \varphi(\boldsymbol{s})[)$ is the union of exactly two different connected components.*

Then for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$, $\mathrm{Cl}(\boldsymbol{m})$ is reduced to $\{\boldsymbol{m}\}$.

Proof. If $\boldsymbol{m} = \underline{\boldsymbol{m}}$ there is nothing to prove. Suppose that $\boldsymbol{m} \in \underline{\mathcal{U}}^{0)}$ and apply assumption (ii) to $E_-(\boldsymbol{m})$. One has evidently $\mathcal{V}^{(1)} \cap E_-(\boldsymbol{m}) \neq \emptyset$ since it contains $\overline{E(\boldsymbol{m})} \subset E_-(\boldsymbol{m})$ and $E(\boldsymbol{m})$ is a critical component. Hence, $E_-(\boldsymbol{m}) \cap \{\varphi < \sigma(\boldsymbol{m})\}$ has exactly two connected components which are necessarily $\widehat{E}(\boldsymbol{m})$ and $E(\boldsymbol{m})$. Suppose now that $\boldsymbol{m}' \mathcal{R} \boldsymbol{m}$. Then $\sigma(\boldsymbol{m}') = \sigma(\boldsymbol{m})$ and hence $\boldsymbol{m}' \notin \widehat{E}(\boldsymbol{m})$. Therefore $\boldsymbol{m}' \in E(\boldsymbol{m})$, which implies $\boldsymbol{m} = \boldsymbol{m}'$. □

Remark B.2. There exist functions φ such that $\mathrm{Cl}(\boldsymbol{m}) = \{\boldsymbol{m}\}$ for all $\boldsymbol{m} \in \mathcal{U}^{(0)}$ and that do not satisfy the Generic Assumption. Take for instance $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with two minima $\boldsymbol{m}_1, \boldsymbol{m}_2$ and two saddle points $\boldsymbol{s}_1, \boldsymbol{s}_2$ such that

$$\varphi(\boldsymbol{m}_1) < \varphi(\boldsymbol{m}_2) < \varphi(\boldsymbol{s}_1) = \varphi(\boldsymbol{s}_2).$$

Then, of course $\mathrm{Cl}(\boldsymbol{m}_j) = \{\boldsymbol{m}_j\}$ for $j = 1, 2$. On the other hand, since $\boldsymbol{s}_1, \boldsymbol{s}_2$ are two saddle points at the same height (which turns out to be the maximal height of saddle points), (ii) of (GA) is not satisfied.

Appendix C: List of symbols

We list the notation used in the paper and give the first place each appears:

$\mathcal{U}^{(0)}, \mathcal{U}^{(1)}$	page 155
n_0, n_1	page 155
$\mathscr{F}(\cdot)$	page 155
$\mathcal{V}^{(1)}$	Definition 2.1
$\mathscr{C}, \underline{\Sigma}, \Sigma$	Definition 2.1
$S, \boldsymbol{\sigma}$	above (2-3)
$\mathcal{S}$	(2-3)
$\underline{\mathcal{U}}^{(0)}$	(2-4)
E	(2-5)
$\Gamma(\boldsymbol{m})$	below (2-5)
$H(\boldsymbol{m})$	(2-6)
E_-	(2-7)
$\widehat{E}$	(2-9)
$\hat{\boldsymbol{m}}$	(2-10)
$\mathcal{U}^{(0),\mathrm{I}}, \mathcal{U}^{(0),\mathrm{II}}$	Definition 2.3
$\mathcal{R}$	Definition 2.5
$\mathcal{U}_\alpha^{(0)}$	(2-14)
$\mathcal{A}, \underline{\mathcal{A}}$	below (2-14)
q_α	below (2-14)
$\mathcal{U}_\alpha^{(0),\mathrm{I}}, \mathcal{U}_\alpha^{(0),\mathrm{II}}$	below (2-14)
$\mathcal{S}_\alpha$	(2-16)
$p(\alpha)$	(2-16)
ν_j^α	below (2-16)
$\mathcal{V}_\alpha^{(1)}$	(3-2)
$\widehat{\mathcal{U}}_\alpha^{(0)}$	(3-3)
Γ_α	(3-4)
$\mathcal{V}_\alpha^{(1),b}, \mathcal{V}_\alpha^{(1),i}$	Lemma 3.4
$\widehat{\mathcal{U}}_\alpha^{(0),\mathrm{II}}$	(3-10)
$\theta_0^\alpha(\boldsymbol{m})$	(3-11)
$\widehat{H}_\alpha(\boldsymbol{m})$	(3-12)
$\widehat{\mathcal{U}}_\alpha^{(0)}$	(3-15)
$\mathscr{T}^\alpha$	Definition 3.7
$\mathcal{L}^\pi$	(4-1)
$\mathcal{L}^{\pi,\prime}$	below (4-1)
$\mathcal{L}^{\mathrm{bkw}}$	(4-2)
$\mathcal{L}^{\mathrm{bkw},\prime}$	(4-3)
$\mathcal{L}^{\mathrm{bkw},\prime\prime}$	(4-5)
$\mathscr{L}$	above Lemma 4.2
$\mathscr{L}^\alpha$	(4-6)
$\widehat{\mathscr{L}}^\alpha$	Lemma 4.3
$h_\varphi(\boldsymbol{m})$	(4-7)
$\widetilde{\mathscr{L}}^\alpha$	(4-9)
$\widehat{\mathscr{L}}^\alpha$	(5-3)
$\mathscr{S}^+, \mathscr{S}_{\mathrm{cl}}^+$	above Definition 5.4
$\mathscr{G}(\mathscr{E},\tau), \mathscr{G}_{\mathrm{cl}}(\mathscr{E},\tau)$	Definition 5.4
υ_2	(5-19)
υ_1	(5-20)

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