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SEMICLASSICAL RESOLVENT ESTIMATES FOR SHORT-RANGE L^∞ POTENTIALS

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We prove semiclassical resolvent estimates for real-valued potentials $V \in L^\infty(\mathbb{R}^n)$, $n \geq 3$, satisfying $V(x) = \mathcal{O}(\langle x\rangle^{-\delta})$ with $\delta > 3$.

1. Introduction and statement of results

Our goal in this note is to study the resolvent of the Schrödinger operator

$$P(h) = -h^2\Delta + V(x),$$

where $0 < h \ll 1$ is a semiclassical parameter, Δ is the negative Laplacian in $\mathbb{R}^n$, $n \geq 3$, and $V \in L^\infty(\mathbb{R}^n)$ is a real-valued potential satisfying

$$|V(x)| \leq C\langle x\rangle^{-\delta}, \tag{1-1}$$

with some constants $C > 0$ and $\delta > 3$. More precisely, we are interested in bounding from above the quantity

$$g_s^{\pm}(h,\varepsilon) := \log \|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{L^2\to L^2},$$

where $L^2 := L^2(\mathbb{R}^n)$, $0 < \varepsilon < 1$, $s > \frac{1}{2}$ and $E > 0$ is a fixed energy level independent of h. Such bounds are known in various situations. For example, for long-range real-valued C^1 potentials it is proved in [Datchev 2014] when $n \geq 3$ and in [Shapiro 2019] when $n = 2$ that

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-1}, \tag{1-2}$$

with some constant $C > 0$ independent of h and ε. Previously, the bound (1-2) was proved for smooth potentials in [Burq 2002] and an analog of (1-2) for Hölder potentials was proved in [Vodev 2014b]. A high-frequency analog of (1-2) on more complex Riemannian manifolds was also proved in [Burq 1998; Cardoso and Vodev 2002]. In all these papers the regularity of the potential (and of the perturbation in general) plays an essential role. Without any regularity, the problem of bounding $g_s^{\pm}$ from above by an explicit function of h gets quite tough. Nevertheless, it was recently shown in [Shapiro 2018] that for real-valued compactly supported L^∞ potentials one has the bound

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-4/3}\log(h^{-1}), \tag{1-3}$$

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with some constant $C > 0$ independent of h and ε. The bound (1-3) was also proved in [Klopp and Vogel 2019], still for real-valued compactly supported L^∞ potentials but with the weight $\langle x\rangle^{-s}$ replaced by a cut-off function. When $n = 1$ it was shown in [Dyatlov and Zworski 2019] that we have the better bound (1-2) instead of (1-3). When $n \geq 2$, however, the bound (1-3) seems hard to improve without extra conditions on the potential. The problem of showing that the bound (1-3) is optimal is largely open. In contrast, it is well known that the bound (1-2) cannot be improved in general; e.g., see [Datchev et al. 2015].

In this note we show that the bound (1-3) still holds for noncompactly supported L^∞ potentials when $n \geq 3$. Our main result is the following.

Theorem 1.1. *Under the condition* (1-1), *there exists* $h_0 > 0$ *such that for all* $0 < h \leq h_0$ *the bound* (1-3) *holds true.*

Remark. It is easy to see from the proof, see the inequality (4-2), that the bound (1-3) holds also for a complex-valued potential V satisfying (1-1), provided that its imaginary part satisfies the condition

$$\mp \operatorname{Im} V(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

To prove this theorem we adapt the Carleman estimates proved in [Shapiro 2018] simplifying some key arguments as, for example, the construction of the phase function φ. This is made possible by defining the key function F in Section 3 differently, without involving the second derivative φ''. The consequence is that we do not need to seek φ' as a solution to a differential equation as done in [Shapiro 2018], but it suffices to define it explicitly. Note also that similar (but simpler) Carleman estimates were used in [Vodev 2014a] to prove high-frequency resolvent estimates for the magnetic Schrödinger operator with large L^∞ magnetic potentials.

2. Construction of the phase and weight functions

We will first construct the weight function. We begin by introducing the continuous function

$$\mu(r) = \begin{cases} (r+1)^2 - 1 & \text{for } 0 \leq r \leq a, \\ (a+1)^2 - 1 + (a+1)^{-2s+1} - (r+1)^{-2s+1} & \text{for } r \geq a, \end{cases}$$

where

$$\tfrac{1}{2} < s < \tfrac{1}{2}(\delta - 2) \tag{2-1}$$

and $a = h^{-m}$ with some parameter $m > 0$ to be fixed in the proof of Lemma 2.3 below depending only on δ and s. Clearly, the first derivative (in sense of distributions) of μ satisfies

$$\mu'(r) = \begin{cases} 2(r+1) & \text{for } 0 \leq r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

The main properties of the functions μ and μ' are given in the following.

Lemma 2.1. *For all $r > 0$, $r \neq a$, we have the inequalities*

$$2r^{-1}\mu(r) - \mu'(r) \geq 0, \tag{2-2}$$

$$\mu'(r) \geq C_1(r+1)^{-2s}, \tag{2-3}$$

$$\frac{\mu(r)^2}{\mu'(r)} \leq C_2 a^4 (r+1)^{2s} \tag{2-4}$$

with some constants $C_1, C_2 > 0$.

Proof. For $r < a$ the left-hand side of (2-2) is equal to 2, while for $r > a$ it is bounded from below by

$$2r^{-1}(a^2 + 2a - s) > 2a^2 r^{-1} > 0,$$

provided a is taken large enough. Furthermore, we clearly have (2-3) for $r < a$ with $C_1 = 2$, while for $r > a$ it holds with $C_1 = 2s - 1$. Therefore, (2-3) holds with $C_1 = \min\{2, 2s-1\}$. The bound (2-4) follows with $C_2 = 2C_1^{-1}$ from (2-3) and the observation that $\mu(r)^2 \leq (a+1)^4 \leq 2a^4$ for all r. □

We now turn to the construction of the phase function $\varphi \in C^1([0, +\infty))$ such that $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$. We define the first derivative of φ by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-1} - \tau(a+1)^{-1} & \text{for } 0 \leq r \leq a, \\ 0 & \text{for } r \geq a, \end{cases}$$

where

$$\tau = \tau_0 h^{-1/3}, \tag{2-5}$$

with some parameter $\tau_0 \gg 1$ independent of h to be fixed in Lemma 2.3 below. Clearly, the first derivative of φ' satisfies

$$\varphi''(r) = \begin{cases} -\tau(r+1)^{-2} & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases}$$

Lemma 2.2. *For all $r \geq 0$ we have the bound*

$$h^{-1}\varphi(r) \lesssim h^{-4/3} \log \frac{1}{h}. \tag{2-6}$$

Proof. We have

$$\max \varphi = \int_0^a \varphi'(r)\,dr \leq \tau \int_0^a (r+1)^{-1}\,dr = \tau \log(a+1),$$

which clearly implies (2-6) in view of the choice of τ and a. □

For $r \neq a$, set

$$A(r) = (\mu\varphi'^2)'(r),$$

$$B(r) = \frac{\big(\mu(r)(h^{-1}(r+1)^{-\delta} + |\varphi''(r)|)\big)^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)}.$$

The following lemma will play a crucial role in the proof of the Carleman estimates in the next section.

Lemma 2.3. *Given any $C>0$ independent of the variable r and the parameters h, τ and a, there exist $\tau_0=\tau_0(C)>0$ and $h_0=h_0(C)>0$ so that for τ satisfying* (2-5) *and for all $0<h\le h_0$ we have the inequality*

$$A(r)-CB(r)\ge -\tfrac{1}{2}E\mu'(r) \tag{2-7}$$

for all $r>0,\ r\ne a$.

Proof. For $r<a$ we have

$$\begin{aligned}
A(r)&=-(\varphi'^2)'(r)+\tau^2\partial_r(1-(r+1)(a+1)^{-1})^2\\
&=-2\varphi'(r)\varphi''(r)-2\tau^2(a+1)^{-1}(1-(r+1)(a+1)^{-1})\\
&\ge 2\tau(r+1)^{-2}\varphi'(r)-2\tau^2(a+1)^{-1}\\
&\ge 2\tau(r+1)^{-2}\varphi'(r)-\tau^2a^{-1}\mu'(r)\\
&\ge 2\tau(r+1)^{-2}\varphi'(r)-\mathcal{O}(h^{m-1})\mu'(r),
\end{aligned}$$

where we have used that $\mu'(r)\ge 2$. Taking $m>2$ we get

$$A(r)\ge 2\tau(r+1)^{-2}\varphi'(r)-\mathcal{O}(h)\mu'(r) \tag{2-8}$$

for all $r<a$. We will now bound the function B from above. Let first $0<r\le\frac{1}{2}a$. Since in this case we have

$$\varphi'(r)\ge\tfrac{1}{3}\tau(r+1)^{-1},$$

we obtain

$$\begin{aligned}
B(r)&\lesssim\frac{\mu(r)(h^{-2}(r+1)^{-2\delta}+\varphi''(r)^2)}{h^{-1}\varphi'(r)}\\
&\lesssim(\tau h)^{-1}\frac{\mu(r)(r+1)^{2-2\delta}}{\varphi'(r)^2}\tau(r+1)^{-2}\varphi'(r)+h\frac{\mu(r)\varphi''(r)^2}{\mu'(r)\varphi'(r)}\mu'(r)\\
&\lesssim\tau^{-3}h^{-1}(r+1)^{6-2\delta}\tau(r+1)^{-2}\varphi'(r)+\tau h\mu'(r)\\
&\lesssim\tau_0^{-3}\tau(r+1)^{-2}\varphi'(r)+\tau_0h^{2/3}\mu'(r),
\end{aligned}$$

where we have used that $\delta>3$. This bound, together with (2-8), clearly implies (2-7), provided τ_0^{-1} and h are taken small enough depending on C.

Let now $\frac{1}{2}a<r<a$. Then we have the bound

$$\begin{aligned}
B(r)&\le\left(\frac{\mu(r)}{\mu'(r)}\right)^2(h^{-1}(r+1)^{-\delta}+|\varphi''(r)|)^2\mu'(r)\\
&\lesssim(h^{-2}(r+1)^{2-2\delta}+\tau^2(r+1)^{-2})\mu'(r)\\
&\lesssim(h^{-2}a^{2-2\delta}+\tau^2a^{-2})\mu'(r)\\
&\lesssim(h^{2m(\delta-1)-2}+h^{2m-2/3})\mu'(r)\lesssim h\mu'(r),
\end{aligned}$$

provided m is taken large enough. Again, this bound, together with (2-8), implies (2-7).

It remains to consider the case $r > a$. Using that $\mu = \mathcal{O}(a^2)$, together with (2-3), and taking into account that s satisfies (2-1), we get

$$\begin{aligned} B(r) &= \frac{\left(\mu(r)(h^{-1}(r+1)^{-\delta})\right)^2}{\mu'(r)} \\ &\lesssim h^{-2}a^4(r+1)^{4s-2\delta}\mu'(r) \lesssim h^{-2}a^{4+4s-2\delta}\mu'(r) \\ &\lesssim h^{2m(\delta-2-2s)-2}\mu'(r) \lesssim h\mu'(r), \end{aligned}$$

provided that m is taken large enough. Since in this case $A(r)=0$, the above bound clearly implies (2-7). □

3. Carleman estimates

Our goal in this section is to prove the following:

Theorem 3.1. *Suppose* (1-1) *holds and let s satisfy* (2-1)*. Then, for all functions $f \in H^2(\mathbb{R}^n)$ such that $\langle x\rangle^s(P(h)-E\pm i\varepsilon)f \in L^2$ and for all $0 < h \ll 1$, $0 < \varepsilon \le ha^{-2}$, we have the estimate*

$$\|\langle x\rangle^{-s}e^{\varphi/h}f\|_{L^2} \le Ca^2h^{-1}\|\langle x\rangle^{s}e^{\varphi/h}(P(h)-E\pm i\varepsilon)f\|_{L^2} + Ca\tau(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2}, \tag{3-1}$$

with a constant $C > 0$ independent of h, ε and f.

Proof. We pass to the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, $r = |x|$, $w = x/|x|$, and recall that $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, r^{n-1}drdw)$. In what follows we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the scalar product in $L^2(\mathbb{S}^{n-1})$. We will make use of the identity

$$r^{(n-1)/2}\Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}, \tag{3-2}$$

where $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(n-1)(n-3)$ and Δ_w denotes the negative Laplace–Beltrami operator on $\mathbb{S}^{n-1}$. Set $u = r^{(n-1)/2}e^{\varphi/h}f$ and

$$\begin{aligned} \mathcal{P}^\pm(h) &= r^{(n-1)/2}(P(h)-E\pm i\varepsilon)r^{-(n-1)/2}, \\ \mathcal{P}^\pm_\varphi(h) &= e^{\varphi/h}\mathcal{P}^\pm(h)e^{-\varphi/h}. \end{aligned}$$

Using (3-2) we can write the operator $\mathcal{P}^\pm(h)$ in the coordinates (r, w) as

$$\mathcal{P}^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon + V,$$

where we have put $\mathcal{D}_r = -ih\partial_r$ and $\Lambda_w = -h^2\tilde{\Delta}_w$. Since the function φ depends only on the variable r, this implies

$$\mathcal{P}^\pm_\varphi(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon - \varphi'^2 + h\varphi'' + 2i\varphi'\mathcal{D}_r + V.$$

For $r > 0$, $r \neq a$, introduce the function

$$F(r) = -\langle (r^{-2}\Lambda_w - E - \varphi'(r)^2)u(r,\cdot), u(r,\cdot)\rangle + \|\mathcal{D}_r u(r,\cdot)\|^2$$

and observe that its first derivative is given by

$$F'(r)=\frac{2}{r}\langle r^{-2}\Lambda_w u(r,\cdot),u(r,\cdot)\rangle+((\varphi')^2)'\|u(r,\cdot)\|^2-2h^{-1}\operatorname{Im}\langle\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle$$
$$\pm 2\varepsilon h^{-1}\operatorname{Re}\langle u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle+4h^{-1}\varphi'\|\mathcal{D}_r u(r,\cdot)\|^2+2h^{-1}\operatorname{Im}\langle(V+h\varphi'')u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle.$$

Thus, if μ is the function defined in the previous section, we obtain the identity

$$\mu'F+\mu F'=(2r^{-1}\mu-\mu')\langle r^{-2}\Lambda_w u(r,\cdot),u(r,\cdot)\rangle+(E\mu'+(\mu(\varphi')^2)')\|u(r,\cdot)\|^2$$
$$-2h^{-1}\mu\operatorname{Im}\langle\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle\pm 2\varepsilon h^{-1}\mu\operatorname{Re}\langle u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle$$
$$+(\mu'+4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r,\cdot)\|^2+2h^{-1}\mu\operatorname{Im}\langle(V+h\varphi'')u(r,\cdot),\mathcal{D}_r u(r,\cdot)\rangle.$$

Using that $\Lambda_w\geq 0$, together with (2-2), we get the inequality

$$\begin{aligned}\mu'F+\mu F'&\geq(E\mu'+(\mu(\varphi')^2)')\|u(r,\cdot)\|^2+(\mu'+4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r,\cdot)\|^2\\&\quad-\frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|^2-\tfrac{1}{3}\mu'\|\mathcal{D}_r u(r,\cdot)\|^2-\varepsilon h^{-1}\mu(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2)\\&\quad-3h^{-2}\mu^2(\mu'+4h^{-1}\varphi'\mu)^{-1}\|(V+h\varphi'')u(r,\cdot)\|^2-\tfrac{1}{3}(\mu'+4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r,\cdot)\|^2\\&\geq\big(E\mu'+(\mu(\varphi')^2)'-C\mu^2(\mu'+h^{-1}\varphi'\mu)^{-1}(h^{-1}(r+1)^{-\delta}+|\varphi''|)^2\big)\|u(r,\cdot)\|^2\\&\quad-\frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|^2-\varepsilon h^{-1}\mu(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2),\end{aligned}$$

with some constant $C>0$. Now we use Lemma 2.3 to conclude that

$$\mu'F+\mu F'\geq\tfrac{1}{2}E\mu'\|u(r,\cdot)\|^2-\frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|^2-\varepsilon h^{-1}\mu(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2).$$

We now integrate this inequality with respect to r and use that, since $\mu(0)=0$, we have

$$\int_0^\infty(\mu'F+\mu F')\,dr=0.$$

Thus we obtain the estimate

$$\begin{aligned}&\tfrac{1}{2}E\int_0^\infty\mu'\|u(r,\cdot)\|^2\,dr\\&\qquad\leq 3h^{-2}\int_0^\infty\frac{\mu^2}{\mu'}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|^2\,dr+\varepsilon h^{-1}\int_0^\infty\mu(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2)\,dr.\end{aligned}\tag{3-3}$$

Using that $\mu=\mathcal{O}(a^2)$ together with (2-3) and (2-4) we get from (3-3)

$$\begin{aligned}&\int_0^\infty(r+1)^{-2s}\|u(r,\cdot)\|^2\,dr\\&\quad\leq Ca^4h^{-2}\int_0^\infty(r+1)^{2s}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|^2\,dr+C\varepsilon h^{-1}a^2\int_0^\infty(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2)\,dr,\end{aligned}\tag{3-4}$$

with some constant $C > 0$ independent of h and ε. On the other hand, we have the identity

$$\mathrm{Re}\int_0^\infty \langle 2i\varphi'\mathcal{D}_r u(r,\cdot), u(r,\cdot)\rangle\, dr = \int_0^\infty h\varphi''\|u(r,\cdot)\|^2\, dr$$

and hence

$$\begin{aligned}\mathrm{Re}\int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r,\cdot), u(r,\cdot)\rangle\, dr &= \int_0^\infty \|\mathcal{D}_r u(r,\cdot)\|^2\, dr + \int_0^\infty \langle r^{-2}\Lambda_w u(r,\cdot), u(r,\cdot)\rangle\, dr \\ &\quad - \int_0^\infty (E+\varphi'^2)\|u(r,\cdot)\|^2\, dr + \int_0^\infty \langle Vu(r,\cdot), u(r,\cdot)\rangle\, dr.\end{aligned}$$

This implies

$$\begin{aligned}\int_0^\infty \|\mathcal{D}_r u(r,\cdot)\|^2\, dr &\le \mathcal{O}(\tau^2)\int_0^\infty \|u(r,\cdot)\|^2\, dr \\ &\quad + \gamma\int_0^\infty (r+1)^{-2s}\|u(r,\cdot)\|^2\, dr + \gamma^{-1}\int_0^\infty (r+1)^{2s}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\, dr \qquad (3\text{-}5)\end{aligned}$$

for every $\gamma > 0$. We take now γ small enough, independent of h, and recall that $\varepsilon h^{-1}a^2 \le 1$. Thus, combining the estimates (3-4) and (3-5), we get

$$\begin{aligned}&\int_0^\infty (r+1)^{-2s}\|u(r,\cdot)\|^2\, dr \\ &\quad \le Ca^4h^{-2}\int_0^\infty (r+1)^{2s}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\, dr + C\varepsilon h^{-1}a^2\tau^2\int_0^\infty \|u(r,\cdot)\|^2\, dr, \qquad (3\text{-}6)\end{aligned}$$

with a new constant $C > 0$ independent of h and ε. It is an easy observation now that the estimate (3-6) implies (3-1). □

4. Resolvent estimates

In this section we will derive the bound (1-3) from Theorem 3.1. Indeed, it follows from the estimate (3-1) and Lemma 2.2 that for $0 < h \ll 1$, $0 < \varepsilon \le ha^{-2}$ and s satisfying (2-1) we have

$$\|\langle x\rangle^{-s} f\|_{L^2} \le M\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)f\|_{L^2} + M\varepsilon^{1/2}\|f\|_{L^2}, \qquad (4\text{-}1)$$

where

$$M = \exp(Ch^{-4/3}\log(h^{-1})),$$

with a constant $C > 0$ independent of h and ε. On the other hand, since the operator $P(h)$ is symmetric, we have

$$\begin{aligned}\varepsilon\|f\|_{L^2}^2 &= \pm\,\mathrm{Im}\langle (P(h) - E \pm i\varepsilon)f, f\rangle_{L^2} \\ &\le (2M)^{-2}\|\langle x\rangle^{-s} f\|_{L^2}^2 + (2M)^2\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)f\|_{L^2}^2. \qquad (4\text{-}2)\end{aligned}$$

We rewrite (4-2) in the form

$$M\varepsilon^{1/2}\|f\|_{L^2} \le \tfrac{1}{2}\|\langle x\rangle^{-s} f\|_{L^2} + 2M^2\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)f\|_{L^2}. \qquad (4\text{-}3)$$

We now combine (4-1) and (4-3) to get

$$\|\langle x\rangle^{-s} f\|_{L^2} \le 4M^2 \|\langle x\rangle^{s}(P(h)-E\pm i\varepsilon) f\|_{L^2}. \tag{4-4}$$

It follows from (4-4) that the resolvent estimate

$$\|\langle x\rangle^{-s}(P(h)-E\pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{L^2\to L^2} \le 4M^2 \tag{4-5}$$

holds for all $0 < h \ll 1,\ 0 < \varepsilon \le ha^{-2}$ and s satisfying (2-1). On the other hand, for $\varepsilon \ge ha^{-2}$ the estimate (4-5) holds in a trivial way. Indeed, in this case, since the operator $P(h)$ is symmetric, the norm of the resolvent is bounded above by $\varepsilon^{-1} = \mathcal{O}(h^{-2m-1})$. Finally, observe that if (4-5) holds for s satisfying (2-1), it holds for all $s > \frac{1}{2}$.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O’Neil of the Courant Institute.

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