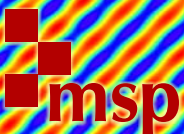


# PURE and APPLIED ANALYSIS

# PAA

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**SEMICLASSICAL RESOLVENT ESTIMATES  
FOR SHORT-RANGE  $L^\infty$  POTENTIALS**



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## SEMICLASSICAL RESOLVENT ESTIMATES FOR SHORT-RANGE $L^\infty$ POTENTIALS

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We prove semiclassical resolvent estimates for real-valued potentials  $V \in L^\infty(\mathbb{R}^n)$ ,  $n \geq 3$ , satisfying  $V(x) = \mathcal{O}(\langle x \rangle^{-\delta})$  with  $\delta > 3$ .

### 1. Introduction and statement of results

Our goal in this note is to study the resolvent of the Schrödinger operator

$$P(h) = -h^2 \Delta + V(x),$$

where  $0 < h \ll 1$  is a semiclassical parameter,  $\Delta$  is the negative Laplacian in  $\mathbb{R}^n$ ,  $n \geq 3$ , and  $V \in L^\infty(\mathbb{R}^n)$  is a real-valued potential satisfying

$$|V(x)| \leq C \langle x \rangle^{-\delta}, \tag{1-1}$$

with some constants  $C > 0$  and  $\delta > 3$ . More precisely, we are interested in bounding from above the quantity

$$g_s^\pm(h, \varepsilon) := \log \|\langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2},$$

where  $L^2 := L^2(\mathbb{R}^n)$ ,  $0 < \varepsilon < 1$ ,  $s > \frac{1}{2}$  and  $E > 0$  is a fixed energy level independent of  $h$ . Such bounds are known in various situations. For example, for long-range real-valued  $C^1$  potentials it is proved in [Datchev 2014] when  $n \geq 3$  and in [Shapiro 2019] when  $n = 2$  that

$$g_s^\pm(h, \varepsilon) \leq Ch^{-1}, \tag{1-2}$$

with some constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . Previously, the bound (1-2) was proved for smooth potentials in [Burq 2002] and an analog of (1-2) for Hölder potentials was proved in [Vodev 2014b]. A high-frequency analog of (1-2) on more complex Riemannian manifolds was also proved in [Burq 1998; Cardoso and Vodev 2002]. In all these papers the regularity of the potential (and of the perturbation in general) plays an essential role. Without any regularity, the problem of bounding  $g_s^\pm$  from above by an explicit function of  $h$  gets quite tough. Nevertheless, it was recently shown in [Shapiro 2018] that for real-valued compactly supported  $L^\infty$  potentials one has the bound

$$g_s^\pm(h, \varepsilon) \leq Ch^{-4/3} \log(h^{-1}), \tag{1-3}$$

with some constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . The bound (1-3) was also proved in [Klopp and Vogel 2019], still for real-valued compactly supported  $L^\infty$  potentials but with the weight  $\langle x \rangle^{-s}$  replaced by a cut-off function. When  $n = 1$  it was shown in [Dyatlov and Zworski 2019] that we have the better bound (1-2) instead of (1-3). When  $n \geq 2$ , however, the bound (1-3) seems hard to improve without extra conditions on the potential. The problem of showing that the bound (1-3) is optimal is largely open. In contrast, it is well known that the bound (1-2) cannot be improved in general; e.g., see [Datchev et al. 2015].

In this note we show that the bound (1-3) still holds for noncompactly supported  $L^\infty$  potentials when  $n \geq 3$ . Our main result is the following.

**Theorem 1.1.** *Under the condition (1-1), there exists  $h_0 > 0$  such that for all  $0 < h \leq h_0$  the bound (1-3) holds true.*

**Remark.** It is easy to see from the proof, see the inequality (4-2), that the bound (1-3) holds also for a complex-valued potential  $V$  satisfying (1-1), provided that its imaginary part satisfies the condition

$$\mp \operatorname{Im} V(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

To prove this theorem we adapt the Carleman estimates proved in [Shapiro 2018] simplifying some key arguments as, for example, the construction of the phase function  $\varphi$ . This is made possible by defining the key function  $F$  in Section 3 differently, without involving the second derivative  $\varphi''$ . The consequence is that we do not need to seek  $\varphi'$  as a solution to a differential equation as done in [Shapiro 2018], but it suffices to define it explicitly. Note also that similar (but simpler) Carleman estimates were used in [Vodev 2014a] to prove high-frequency resolvent estimates for the magnetic Schrödinger operator with large  $L^\infty$  magnetic potentials.

## 2. Construction of the phase and weight functions

We will first construct the weight function. We begin by introducing the continuous function

$$\mu(r) = \begin{cases} (r+1)^2 - 1 & \text{for } 0 \leq r \leq a, \\ (a+1)^2 - 1 + (a+1)^{-2s+1} - (r+1)^{-2s+1} & \text{for } r \geq a, \end{cases}$$

where

$$\frac{1}{2} < s < \frac{1}{2}(\delta - 2) \tag{2-1}$$

and  $a = h^{-m}$  with some parameter  $m > 0$  to be fixed in the proof of Lemma 2.3 below depending only on  $\delta$  and  $s$ . Clearly, the first derivative (in sense of distributions) of  $\mu$  satisfies

$$\mu'(r) = \begin{cases} 2(r+1) & \text{for } 0 \leq r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

The main properties of the functions  $\mu$  and  $\mu'$  are given in the following.

**Lemma 2.1.** *For all  $r > 0, r \neq a$ , we have the inequalities*

$$2r^{-1}\mu(r) - \mu'(r) \geq 0, \tag{2-2}$$

$$\mu'(r) \geq C_1(r+1)^{-2s}, \tag{2-3}$$

$$\frac{\mu(r)^2}{\mu'(r)} \leq C_2 a^4 (r+1)^{2s} \tag{2-4}$$

with some constants  $C_1, C_2 > 0$ .

*Proof.* For  $r < a$  the left-hand side of (2-2) is equal to 2, while for  $r > a$  it is bounded from below by

$$2r^{-1}(a^2 + 2a - s) > 2a^2 r^{-1} > 0,$$

provided  $a$  is taken large enough. Furthermore, we clearly have (2-3) for  $r < a$  with  $C_1 = 2$ , while for  $r > a$  it holds with  $C_1 = 2s - 1$ . Therefore, (2-3) holds with  $C_1 = \min\{2, 2s - 1\}$ . The bound (2-4) follows with  $C_2 = 2C_1^{-1}$  from (2-3) and the observation that  $\mu(r)^2 \leq (a+1)^4 \leq 2a^4$  for all  $r$ .  $\square$

We now turn to the construction of the phase function  $\varphi \in C^1([0, +\infty))$  such that  $\varphi(0) = 0$  and  $\varphi(r) > 0$  for  $r > 0$ . We define the first derivative of  $\varphi$  by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-1} - \tau(a+1)^{-1} & \text{for } 0 \leq r \leq a, \\ 0 & \text{for } r \geq a, \end{cases}$$

where

$$\tau = \tau_0 h^{-1/3}, \tag{2-5}$$

with some parameter  $\tau_0 \gg 1$  independent of  $h$  to be fixed in Lemma 2.3 below. Clearly, the first derivative of  $\varphi'$  satisfies

$$\varphi''(r) = \begin{cases} -\tau(r+1)^{-2} & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases}$$

**Lemma 2.2.** *For all  $r \geq 0$  we have the bound*

$$h^{-1}\varphi(r) \lesssim h^{-4/3} \log \frac{1}{h}. \tag{2-6}$$

*Proof.* We have

$$\max \varphi = \int_0^a \varphi'(r) dr \leq \tau \int_0^a (r+1)^{-1} dr = \tau \log(a+1),$$

which clearly implies (2-6) in view of the choice of  $\tau$  and  $a$ .  $\square$

For  $r \neq a$ , set

$$A(r) = (\mu\varphi'^2)'(r),$$

$$B(r) = \frac{(\mu(r)(h^{-1}(r+1)^{-\delta} + |\varphi''(r)|))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)}.$$

The following lemma will play a crucial role in the proof of the Carleman estimates in the next section.

**Lemma 2.3.** *Given any  $C > 0$  independent of the variable  $r$  and the parameters  $h$ ,  $\tau$  and  $a$ , there exist  $\tau_0 = \tau_0(C) > 0$  and  $h_0 = h_0(C) > 0$  so that for  $\tau$  satisfying (2-5) and for all  $0 < h \leq h_0$  we have the inequality*

$$A(r) - CB(r) \geq -\frac{1}{2}E\mu'(r) \quad (2-7)$$

for all  $r > 0$ ,  $r \neq a$ .

*Proof.* For  $r < a$  we have

$$\begin{aligned} A(r) &= -(\varphi'^2)'(r) + \tau^2 \partial_r (1 - (r+1)(a+1)^{-1})^2 \\ &= -2\varphi'(r)\varphi''(r) - 2\tau^2(a+1)^{-1}(1 - (r+1)(a+1)^{-1}) \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - 2\tau^2(a+1)^{-1} \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - \tau^2 a^{-1}\mu'(r) \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - \mathcal{O}(h^{m-1})\mu'(r), \end{aligned}$$

where we have used that  $\mu'(r) \geq 2$ . Taking  $m > 2$  we get

$$A(r) \geq 2\tau(r+1)^{-2}\varphi'(r) - \mathcal{O}(h)\mu'(r) \quad (2-8)$$

for all  $r < a$ . We will now bound the function  $B$  from above. Let first  $0 < r \leq \frac{1}{2}a$ . Since in this case we have

$$\varphi'(r) \geq \frac{1}{3}\tau(r+1)^{-1},$$

we obtain

$$\begin{aligned} B(r) &\lesssim \frac{\mu(r)(h^{-2}(r+1)^{-2\delta} + \varphi''(r)^2)}{h^{-1}\varphi'(r)} \\ &\lesssim (\tau h)^{-1} \frac{\mu(r)(r+1)^{2-2\delta}}{\varphi'(r)^2} \tau(r+1)^{-2}\varphi'(r) + h \frac{\mu(r)\varphi''(r)^2}{\mu'(r)\varphi'(r)} \mu'(r) \\ &\lesssim \tau^{-3}h^{-1}(r+1)^{6-2\delta} \tau(r+1)^{-2}\varphi'(r) + \tau h \mu'(r) \\ &\lesssim \tau_0^{-3} \tau(r+1)^{-2}\varphi'(r) + \tau_0 h^{2/3} \mu'(r), \end{aligned}$$

where we have used that  $\delta > 3$ . This bound, together with (2-8), clearly implies (2-7), provided  $\tau_0^{-1}$  and  $h$  are taken small enough depending on  $C$ .

Let now  $\frac{1}{2}a < r < a$ . Then we have the bound

$$\begin{aligned} B(r) &\leq \left( \frac{\mu(r)}{\mu'(r)} \right)^2 (h^{-1}(r+1)^{-\delta} + |\varphi''(r)|)^2 \mu'(r) \\ &\lesssim (h^{-2}(r+1)^{2-2\delta} + \tau^2(r+1)^{-2})\mu'(r) \\ &\lesssim (h^{-2}a^{2-2\delta} + \tau^2 a^{-2})\mu'(r) \\ &\lesssim (h^{2m(\delta-1)-2} + h^{2m-2/3})\mu'(r) \lesssim h\mu'(r), \end{aligned}$$

provided  $m$  is taken large enough. Again, this bound, together with (2-8), implies (2-7).

It remains to consider the case  $r > a$ . Using that  $\mu = \mathcal{O}(a^2)$ , together with (2-3), and taking into account that  $s$  satisfies (2-1), we get

$$\begin{aligned} B(r) &= \frac{(\mu(r)(h^{-1}(r+1)^{-\delta}))^2}{\mu'(r)} \\ &\lesssim h^{-2}a^4(r+1)^{4s-2\delta}\mu'(r) \lesssim h^{-2}a^{4+4s-2\delta}\mu'(r) \\ &\lesssim h^{2m(\delta-2-2s)-2}\mu'(r) \lesssim h\mu'(r), \end{aligned}$$

provided that  $m$  is taken large enough. Since in this case  $A(r) = 0$ , the above bound clearly implies (2-7).  $\square$

### 3. Carleman estimates

Our goal in this section is to prove the following:

**Theorem 3.1.** *Suppose (1-1) holds and let  $s$  satisfy (2-1). Then, for all functions  $f \in H^2(\mathbb{R}^n)$  such that  $\langle x \rangle^s (P(h) - E \pm i\varepsilon)f \in L^2$  and for all  $0 < h \ll 1$ ,  $0 < \varepsilon \leq ha^{-2}$ , we have the estimate*

$$\|\langle x \rangle^{-s} e^{\varphi/h} f\|_{L^2} \leq Ca^2 h^{-1} \|\langle x \rangle^s e^{\varphi/h} (P(h) - E \pm i\varepsilon)f\|_{L^2} + Ca\tau(\varepsilon/h)^{1/2} \|e^{\varphi/h} f\|_{L^2}, \tag{3-1}$$

with a constant  $C > 0$  independent of  $h, \varepsilon$  and  $f$ .

*Proof.* We pass to the polar coordinates  $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ ,  $r = |x|$ ,  $w = x/|x|$ , and recall that  $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, r^{n-1} dr dw)$ . In what follows we denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and the scalar product in  $L^2(\mathbb{S}^{n-1})$ . We will make use of the identity

$$r^{(n-1)/2} \Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}, \tag{3-2}$$

where  $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(n-1)(n-3)$  and  $\Delta_w$  denotes the negative Laplace–Beltrami operator on  $\mathbb{S}^{n-1}$ . Set  $u = r^{(n-1)/2} e^{\varphi/h} f$  and

$$\begin{aligned} \mathcal{P}^\pm(h) &= r^{(n-1)/2} (P(h) - E \pm i\varepsilon) r^{-(n-1)/2}, \\ \mathcal{P}_\varphi^\pm(h) &= e^{\varphi/h} \mathcal{P}^\pm(h) e^{-\varphi/h}. \end{aligned}$$

Using (3-2) we can write the operator  $\mathcal{P}^\pm(h)$  in the coordinates  $(r, w)$  as

$$\mathcal{P}^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon + V,$$

where we have put  $\mathcal{D}_r = -ih\partial_r$  and  $\Lambda_w = -h^2\tilde{\Delta}_w$ . Since the function  $\varphi$  depends only on the variable  $r$ , this implies

$$\mathcal{P}_\varphi^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon - \varphi'^2 + h\varphi'' + 2i\varphi'\mathcal{D}_r + V.$$

For  $r > 0$ ,  $r \neq a$ , introduce the function

$$F(r) = -\langle (r^{-2}\Lambda_w - E - \varphi'(r)^2)u(r, \cdot), u(r, \cdot) \rangle + \|\mathcal{D}_r u(r, \cdot)\|^2$$

and observe that its first derivative is given by

$$F'(r) = \frac{2}{r} \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + ((\varphi')^2)' \|u(r, \cdot)\|^2 - 2h^{-1} \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ \pm 2\varepsilon h^{-1} \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + 4h^{-1} \varphi' \|\mathcal{D}_r u(r, \cdot)\|^2 + 2h^{-1} \operatorname{Im} \langle (V + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle.$$

Thus, if  $\mu$  is the function defined in the previous section, we obtain the identity

$$\mu' F + \mu F' = (2r^{-1} \mu - \mu') \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + (E\mu' + (\mu(\varphi')^2)') \|u(r, \cdot)\|^2 \\ - 2h^{-1} \mu \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \pm 2\varepsilon h^{-1} \mu \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 + 2h^{-1} \mu \operatorname{Im} \langle (V + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle.$$

Using that  $\Lambda_w \geq 0$ , together with (2-2), we get the inequality

$$\mu' F + \mu F' \geq (E\mu' + (\mu(\varphi')^2)') \|u(r, \cdot)\|^2 + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \frac{1}{3} \mu' \|\mathcal{D}_r u(r, \cdot)\|^2 - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \\ - 3h^{-2} \mu^2 (\mu' + 4h^{-1} \varphi' \mu)^{-1} \|(V + h\varphi'') u(r, \cdot)\|^2 - \frac{1}{3} (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ \geq (E\mu' + (\mu(\varphi')^2)' - C\mu^2 (\mu' + h^{-1} \varphi' \mu)^{-1} (h^{-1} (r+1)^{-\delta} + |\varphi''|)^2) \|u(r, \cdot)\|^2 \\ - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2),$$

with some constant  $C > 0$ . Now we use Lemma 2.3 to conclude that

$$\mu' F + \mu F' \geq \frac{1}{2} E \mu' \|u(r, \cdot)\|^2 - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2).$$

We now integrate this inequality with respect to  $r$  and use that, since  $\mu(0) = 0$ , we have

$$\int_0^\infty (\mu' F + \mu F') dr = 0.$$

Thus we obtain the estimate

$$\frac{1}{2} E \int_0^\infty \mu' \|u(r, \cdot)\|^2 dr \\ \leq 3h^{-2} \int_0^\infty \frac{\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 dr + \varepsilon h^{-1} \int_0^\infty \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr. \quad (3-3)$$

Using that  $\mu = \mathcal{O}(a^2)$  together with (2-3) and (2-4) we get from (3-3)

$$\int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr \\ \leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 dr + C \varepsilon h^{-1} a^2 \int_0^\infty (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr, \quad (3-4)$$

with some constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . On the other hand, we have the identity

$$\operatorname{Re} \int_0^\infty \langle 2i\varphi' \mathcal{D}_r u(r, \cdot), u(r, \cdot) \rangle dr = \int_0^\infty h\varphi'' \|u(r, \cdot)\|^2 dr$$

and hence

$$\begin{aligned} \operatorname{Re} \int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r, \cdot), u(r, \cdot) \rangle dr &= \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr + \int_0^\infty \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle dr \\ &\quad - \int_0^\infty (E + \varphi'^2) \|u(r, \cdot)\|^2 dr + \int_0^\infty \langle V u(r, \cdot), u(r, \cdot) \rangle dr. \end{aligned}$$

This implies

$$\begin{aligned} \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr &\leq \mathcal{O}(\tau^2) \int_0^\infty \|u(r, \cdot)\|^2 dr \\ &\quad + \gamma \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr + \gamma^{-1} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \end{aligned} \quad (3-5)$$

for every  $\gamma > 0$ . We take now  $\gamma$  small enough, independent of  $h$ , and recall that  $\varepsilon h^{-1} a^2 \leq 1$ . Thus, combining the estimates (3-4) and (3-5), we get

$$\begin{aligned} \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr \\ \leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr + C \varepsilon h^{-1} a^2 \tau^2 \int_0^\infty \|u(r, \cdot)\|^2 dr, \end{aligned} \quad (3-6)$$

with a new constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . It is an easy observation now that the estimate (3-6) implies (3-1).  $\square$

#### 4. Resolvent estimates

In this section we will derive the bound (1-3) from Theorem 3.1. Indeed, it follows from the estimate (3-1) and Lemma 2.2 that for  $0 < h \ll 1$ ,  $0 < \varepsilon \leq h a^{-2}$  and  $s$  satisfying (2-1) we have

$$\|\langle x \rangle^{-s} f\|_{L^2} \leq M \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2} + M \varepsilon^{1/2} \|f\|_{L^2}, \quad (4-1)$$

where

$$M = \exp(C h^{-4/3} \log(h^{-1})),$$

with a constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . On the other hand, since the operator  $P(h)$  is symmetric, we have

$$\begin{aligned} \varepsilon \|f\|_{L^2}^2 &= \pm \operatorname{Im} \langle (P(h) - E \pm i\varepsilon) f, f \rangle_{L^2} \\ &\leq (2M)^{-2} \|\langle x \rangle^{-s} f\|_{L^2}^2 + (2M)^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}^2. \end{aligned} \quad (4-2)$$

We rewrite (4-2) in the form

$$M \varepsilon^{1/2} \|f\|_{L^2} \leq \frac{1}{2} \|\langle x \rangle^{-s} f\|_{L^2} + 2M^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}. \quad (4-3)$$



We now combine (4-1) and (4-3) to get

$$\|\langle x \rangle^{-s} f\|_{L^2} \leq 4M^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}. \quad (4-4)$$

It follows from (4-4) that the resolvent estimate

$$\|\langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2} \leq 4M^2 \quad (4-5)$$

holds for all  $0 < h \ll 1$ ,  $0 < \varepsilon \leq ha^{-2}$  and  $s$  satisfying (2-1). On the other hand, for  $\varepsilon \geq ha^{-2}$  the estimate (4-5) holds in a trivial way. Indeed, in this case, since the operator  $P(h)$  is symmetric, the norm of the resolvent is bounded above by  $\varepsilon^{-1} = \mathcal{O}(h^{-2m-1})$ . Finally, observe that if (4-5) holds for  $s$  satisfying (2-1), it holds for all  $s > \frac{1}{2}$ .

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