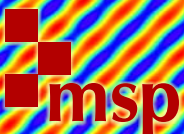


PURE and APPLIED ANALYSIS

PAA

JAN DEREZIŃSKI AND DANIEL SIEMSEN

**AN EVOLUTION EQUATION APPROACH TO
THE KLEIN–GORDON OPERATOR
ON CURVED SPACETIME**



vol. 1 no. 2 2019

AN EVOLUTION EQUATION APPROACH TO THE KLEIN–GORDON OPERATOR ON CURVED SPACETIME

JAN DEREZIŃSKI AND DANIEL SIEMSEN

We develop a theory of the Klein–Gordon equation on curved spacetimes. Our main tool is the method of (nonautonomous) evolution equations on Hilbert spaces. This approach allows us to treat low regularity of the metric, of the electromagnetic potential and of the scalar potential. Our main goal is a construction of various kinds of propagators needed in quantum field theory.

1. Introduction	215
2. Assumptions and setting	222
3. The energy space and the dynamical space	225
4. Instantaneous energy spaces and instantaneous dynamical spaces	226
5. Evolution	228
6. Solutions of the Klein–Gordon equation	232
7. Classical propagators	234
8. Instantaneous nonclassical propagators	238
9. Asymptotic nonclassical propagators	241
Appendix A. Second-order differential operators	243
Appendix B. Concrete assumptions	245
Appendix C. Nonautonomous evolution equations	248
Appendix D. Heinz–Kato inequality	256
Appendix E. Finite speed of propagation	256
Acknowledgements	260
References	260

1. Introduction

We consider the *Klein–Gordon operator on a Lorentzian manifold (M, g) minimally coupled to an electromagnetic potential A and with a scalar potential Y* . In local coordinates it can be written as

$$K := \square_A + Y = |g|^{-1/2} (D_\mu - A_\mu) |g|^{1/2} g^{\mu\nu} (D_\nu - A_\nu) + Y, \quad (1-1)$$

where $|g| = |\det[g_{\mu\nu}]|$ and $D_\mu = -i\partial_\mu$. As in our recent work [Dereziński and Siemssen 2018], we are interested in inverses and bisolutions of the Klein–Gordon operator K .

MSC2010: primary 35L05, 47D06; secondary 58J45, 81Q10, 81T20.

Keywords: Klein–Gordon equation, Klein–Gordon operator, propagator, evolution equation, quantum field theory, quantum field theory in curved spacetimes.

Heuristically, they are defined as follows:

- An operator G is a *bisolution* of K if it satisfies

$$KG = 0 \quad \text{and} \quad GK = 0.$$

- An operator G is an *inverse* of K if it satisfies

$$KG = \mathbb{1} \quad \text{and} \quad GK = \mathbb{1}.$$

To make these statements rigorous, one needs to specify the spaces between which these operators act, making sure that the composition of K and G is well-defined. Often, G can be understood as an operator from $C_c^\infty(M)$ to $C^\infty(M)$.

The Klein–Gordon operator has several distinguished inverses and bisolutions. They are known by many names, e.g., “propagator” or “two-point function”. Inverses are often also called “Green’s functions”.

The most well-known propagators are probably the *forward (retarded) propagator* G^\vee and the *backward (advanced) propagator* G^\wedge . Their difference $G^{\text{PJ}} := G^\vee - G^\wedge$ is sometimes called the *Pauli–Jordan propagator*, which is the name we use. In the literature one can also find other names, such as “commutator function” or “causal propagator”.¹ These three propagators are important in the Cauchy problem of the classical theory. Therefore, we will call them jointly *classical propagators*. It is well known that on globally hyperbolic spacetimes the classical propagators exist and are unique.

In quantum field theory, one needs also other propagators: two inverses, the *Feynman propagator* G^{F} and the *anti-Feynman propagator* $G^{\bar{\text{F}}}$, as well as the *positive- and negative-frequency bisolutions* $G^{(\pm)}$. We will call them jointly *nonclassical propagators*. A positive-frequency bisolution yields the two-point function of a vacuum state — a pure quasifree state whose Gelfand–Naimark–Segal (GNS) representation yields a Hilbert space for the quantum field theory. The integral kernel of the Feynman propagator coincides with the expectation value of time-ordered products of quantum fields. It is used to evaluate Feynman diagrams.

The analysis of the Klein–Gordon equation is especially simple if the spacetime is stationary and the Hamiltonian is positive. On the mathematical side, if in addition the Hamiltonian is bounded away from zero (the “positive-mass case”), we have a natural Hilbert space structure for the Cauchy data. The most obvious choice is the so-called *energy Hilbert space*. It is also natural to consider a whole scale of Hilbert spaces, which includes the energy space. The generator of the dynamics is self-adjoint on all of these spaces. Thus the functional analytic setting for stationary spacetimes in the “positive-mass case” is rather clean and simple. If we assume that the Hamiltonian is only positive, without a positive lower bound, (the “zero-mass case”), then the functional-analytic setup becomes slightly more technically involved, but the general picture remains the same.

¹We try to use as much as possible the terminology from classic textbooks on quantum field theory. For instance, “Pauli–Jordan function” is the name used for G^{PJ} already in [Bogoliubov and Shirkov 1980]. The same authors call G^{F} the “causal Green’s function”, since the choice of G^{F} for the evaluation of Feynman diagrams expresses causality in quantum field theory. Therefore, using the name “causal propagator” for G^{PJ} clashes with the traditional terminology and, we believe, should be discouraged.

On the physical side, on a stationary spacetime with a positive Hamiltonian, it is clear how to define the nonclassical propagators. The positive- and negative-frequency bisolutions, as well as the Feynman and anti-Feynman propagators, are constructed from the spectral projections of the generator of the dynamics. These constructions, at least implicitly, can be found in various works devoted to quantum field theory on curved spacetimes. In a systematic way the static case has been worked out recently in [Dereziński and Siemssen 2018]; see also [Dereziński and Gérard 2013, Chapter 18]. In [Dereziński and Siemssen 2018], we assumed in addition the “positive-mass condition”, and the results of that paper can be easily generalized to stationary spacetimes (using, e.g., the stationary special case of Sections 2.1 and 2.2 as a starting point).

The positivity of the Hamiltonian plays an important role in the construction of nonclassical propagators. This is related to the fact that nonpositive Hamiltonians lead to problems in quantum field theory, which are often collectively called the *Klein paradox*. The original paper by Klein involved fermions and the Dirac equation with a large step potential causing spontaneous *pair creation*. One can easily resolve the fermionic Klein paradox in the second quantized theory. Splitting the Hilbert space into the particle and antiparticle subspaces and applying second quantization makes the quantum Hamiltonian positive definite. The corresponding problem for bosons is much more serious. If the classical Hamiltonian is not positive, it will not become positive by quantization. Besides, in this case there is no positive scalar product preserved by the evolution, as is the case for Dirac fermions. This typically leads to the so-called *superradiance*. In mathematical terms it means that the scattering operator has a norm greater than 1, or it does not exist at all because the norm of the evolution grows all the time.

This paper is devoted to the study of the Klein–Gordon equation on rather general (possibly, nonstationary) spacetimes. We construct both the classical propagators and certain families of nonclassical propagators. Let us first describe the basic steps of our construction of the classical propagators:

- (1) We assume that there is a manifold Σ such that the spacetime M is diffeomorphic to $\mathbb{R} \times \Sigma$. This diffeomorphism provides a global time function t whose level sets Σ_t are assumed to be spacelike. It also defines a flow whose generator ∂_t is assumed to be timelike.
- (2) We rewrite the Klein–Gordon equation as a (nonautonomous) first-order equation for the Cauchy data on Σ_t . Thus the generator of the evolution can be written as a 2×2 matrix.
- (3) We make various assumptions on the metric, electromagnetic and scalar potentials. The assumptions on their regularity are rather weak; however, they are global in spacetime. We assume that the positive-mass condition holds for all times; that is, all instantaneous Hamiltonians have a strictly positive lower bound.
- (4) We apply functional-analytic methods from the theory of *nonautonomous evolution equations*, as developed in [Kato 1970]. Note that, unlike in [Dereziński and Siemssen 2018], in the nonstationary case we do not have a unique distinguished energy space. Instead, we have a whole time-dependent family of instantaneous-energy Hilbert spaces describing the Cauchy data at each time. Under the assumptions we impose, these spaces can be identified with one another. They have a variable scalar product, but a common topology — thus the Cauchy data at each time belong to a single *Hilbertizable space*.
- (5) The Pauli–Jordan propagator essentially coincides with one of the matrix elements of the evolution operator. One can then write down the forward and backward propagators by inserting the Heaviside

function in the appropriate places. Thus if one uses the method of evolution equations, the Pauli–Jordan propagator becomes the central object, whereas in typical approaches found in the literature, e.g., [Bär et al. 2007], the forward and backward propagators are obtained first and then used to define the Pauli–Jordan propagator. We find this (trivial) observation curious.

The assumptions of our paper, in particular their global character and the positive-mass assumptions, are adapted to the needs of nonclassical propagators, which are our main interest. However, if one is interested only in classical propagators, some of these assumptions can be relaxed.

When the Hamiltonian is merely bounded from below, we can reduce the problem to the positive-mass case by a perturbation argument. Then one can construct the evolution, and hence also the classical propagators. We remark about this fact at the end of Section 5.

Another point that can be relaxed are the global assumptions. We know that the propagation of solutions to the Klein–Gordon equation has finite speed — this can be proven independently under a weak assumption on the regularity; see, e.g., Appendix E. Therefore, to construct the evolution, it is sufficient to have local information about our system. We do not discuss this point further in our paper.

As already stated above, our main interest is the nonclassical propagators. Unfortunately, in the nonstationary case it is not obvious how to define them. The most popular view on this subject says that instead of a single positive-frequency bisolution one should consider a whole class of bisolutions locally similar to the Minkowski two-point function, known as *Hadamard states*. There exists considerable literature about them; in particular we would like to mention [Radzikowski 1996; Kay and Wald 1991]. Properties of Hadamard states play a central role in most formulations of perturbation theory and renormalization on curved spacetimes; see, e.g., [Hollands and Wald 2001; 2002]. Moreover, the expectation value of time-ordered fields in every Hadamard state is the integral kernel of an inverse of K and can be viewed as a possible generalization of the usual Feynman propagator to the generic case.

One of possibilities is to use spectral projections of the generator of the evolution at a fixed instance of time, as we describe in Section 8. This allows us to define *instantaneous positive- and negative-frequency bisolutions*, which yield the so-called *instantaneous vacua*. One also has the corresponding *instantaneous Feynman inverses*.

One can criticize these propagators on physical grounds. Not only do they depend on an arbitrary and unphysical choice of a preferred time, but it is folklore knowledge that they are generally not Hadamard states. In a forthcoming article we will show, using methods from our formalism, that an instantaneous positive-frequency bisolution yields a Hadamard state if the Klein–Gordon operator K is infinitesimally stationary at the Cauchy surface where the positive/negative-frequency splitting was performed.

Spacetimes that become asymptotically stationary in the past and the future form a class that in our opinion is especially natural from the point of view of quantum field theory and scattering theory. For such spacetimes one can define positive/negative-frequency bisolutions corresponding to the asymptotic past and future; see Section 9. We can call them *in/out-positive/negative-frequency bisolutions*. One can argue that the corresponding *in-vacuum* yields the representation of incoming states (prepared in the experiment) and the corresponding *out-vacuum* gives the representation of final observables (measured in the experiment). Therefore, the in- and out-states are not only distinguished, they also have a clear and

important physical meaning. If the spacetime becomes stationary sufficiently fast, it can be shown that the states thus defined are Hadamard [Gérard and Wrochna 2017].

As we described above, and is well known, spacetimes with asymptotically stationary past and future possess two pairs of distinguished and physically well-motivated propagators: the in/out-positive/negative-frequency bisolution. It is perhaps less known that a large class of such spacetimes possesses another pair of natural and physically motivated propagators: the so-called *canonical Feynman* and *anti-Feynman propagators (inverses)*. The Feynman propagator appears naturally when we evaluate Feynman diagrams. A study of these propagators will be presented in a forthcoming article, where the formalism and results of the present paper will play an important role.

Let us mention that the canonical Feynman and anti-Feynman propagators are related to the intriguing and poorly understood question about the self-adjointness of the Klein–Gordon operator. It is easy to see that the Klein–Gordon operator is Hermitian; however, the existence of a distinguished self-adjoint extension seems to be difficult to prove and is known only in special cases: in the static case [Dereziński and Siemssen 2018] and (since very recently) for a class of asymptotically Minkowskian spaces [Vasy 2017]. Note that heuristically the canonical Feynman or anti-Feynman propagator is the boundary value at zero from above or below, respectively, of the resolvent of the Klein–Gordon operator. One could also argue that the adjective *canonical* is not needed for both propagators, that they should simply be called *the Feynman* and *anti-Feynman* propagators.

Let us compare our work with the literature. The construction of classical propagators is described in numerous sources. Typically, one shows first the well-posedness of the Cauchy problem. Then the existence of the classical propagators and their properties easily follow; see, e.g., [Kay 1978; Dimock and Kay 1982], and also the more recent works [Drago and Gérard 2017; Gérard and Wrochna 2014; Gérard et al. 2017]. Standard methods include the Hadamard parametrix method [Bär et al. 2007; Friedlander 1975] and energy estimates obtained via the divergence theorem. Another popular method relies on the factorization of the Klein–Gordon operator into the product of first-order scalar operators (see, e.g., the treatment of [Hörmander 1985], which also covers n -th order hyperbolic equations). A brief history of the Cauchy problem for hyperbolic equations with references to various approaches can be found in [Hörmander 1985, Notes to Chapter XXIII].

In our opinion, the method of evolution equations used in this paper provides a natural and powerful approach to analyze the Klein–Gordon equation on curved spacetimes, especially concerning questions relevant to quantum field theory. Therefore, we were greatly surprised that it is difficult to find a treatment of this problem similar to ours in the existing literature. We are only aware of one more publication where the methods of evolution equations have been applied to the problem at hand in the nonstationary case: in [Furlani 1997] the evolution is constructed under quite restrictive assumptions, namely, assuming that Cauchy surfaces are compact and have a decreasing volume along a finite time-interval. The treatment of some papers, such as [Dimock and Kay 1982; Kay 1978; Gérard et al. 2017; Gérard and Wrochna 2014], may also resemble our method. However, in almost all papers that we know, the existence of the evolution is taken for granted, is given by the local theory, and is not constructed within the formalism of evolution equations on some Banach spaces.

The literature devoted to classical propagators on curved spacetimes usually does not use a global functional-analytic setting. As we discussed above, from the point of view of classical propagators, the method of our paper seems to impose unnecessary limitations, because of the global assumptions on the spacetime. However, to define and study nonclassical propagators, some kind of global assumptions are usually indispensable.

Most authors do not consider low-regularity situations; for an exception we refer to [Sanchez and Vickers 2018]. For example, the propagators are typically understood from $C_c^\infty(M)$ to $C^\infty(M)$. As far as we know, the constructions found in the literature require more stringent regularity assumptions than ours.

Throughout our paper we impose rather weak assumptions on the regularity of various objects (the metric, electromagnetic potential and the scalar potential). Nevertheless, we did not write this work with any particular nonregular examples in mind, even though low regularity is present in some interesting physical applications (e.g., boundaries of astrophysical objects, shock waves) and singularities appear generically in solutions of the Einstein equation. Instead, the main reason for the chosen approach is our conviction that weak assumptions play an important theoretical role, because they impose a certain discipline on a mathematical theory, forcing us to find better arguments and a more natural setting for the problem.

We think that our approach is rather natural and direct if one wants to treat the simplest examples of spacetimes (from the point of view of quantum field theory) such as local perturbations of Minkowski spacetime and cosmological spacetimes. However, it is also flexible enough to treat some less obvious examples, such as certain nonglobally hyperbolic spacetimes, including spacetimes with boundaries, provided we impose appropriate boundary conditions. This includes for example compactifications of anti-de Sitter spacetime with appropriate conditions on its timelike boundary (see [Dappiaggi et al. 2018a; 2018b] for a recent discussion of boundary conditions on anti-de Sitter spacetime and spacetimes with a timelike boundary).

Finally, let us remark that Kato's theory of nonautonomous evolution equations has also been successfully applied in the context of quantum field theory for the Dirac equation on curved spacetimes; see, e.g., [Häfner 2009; Nicolas 2002]. The Dirac equation is simpler in this respect than the Klein–Gordon equation. For the Dirac equation there exists a natural Hilbert space. For the (nonstationary) Klein–Gordon equation no such choice exists: one is forced to work with a family of Hilbertizable spaces. Studying the evolution for the Klein–Gordon equation in time-dependent families of Hilbert spaces has also been fruitful in the context of spherical gravitational collapse (i.e., in static Schwarzschild spacetime with time-dependent boundary conditions); see [Bachelot 1999].

1.1. Notation and conventions. Throughout this paper we adopt essentially the same notation and conventions as in [Dereziński and Siemssen 2018] but for the convenience of the reader we repeat the relevant conventions. We also introduce some new notation.

Suppose that T is an operator on a Banach space \mathcal{X} . We denote by $\text{Dom } T$ its domain and by $\text{Ran } T$ its range. For its spectrum we write $\text{sp } T$ and for the resolvent set $\text{rs } T$.

Suppose that T is an operator on a Hilbert space \mathcal{H} with inner product $(\cdot | \cdot)$. If T is positive, i.e., $(u | Tu) \geq 0$, we write $T \geq 0$. If also $\text{Ker } T = \{0\}$, then we write $T > 0$.

A useful function is the so-called “Japanese bracket”, defined as $\langle T \rangle := (1 + |T|^2)^{1/2}$.

A topological vector space \mathcal{X} is called *Hilbertizable* if there exists a scalar product on \mathcal{X} that determines its topology and makes it into a Hilbert space. Clearly, two scalar products determine the topology of \mathcal{X} if and only if they are equivalent.

The p -times continuously differentiable \mathcal{X} -valued functions on a manifold M are denoted by $C^p(M; \mathcal{X})$; if $\mathcal{X} = \mathbb{C}$, we simply write $C^p(M)$. Sets of compactly supported or bounded functions are indicated by a subscript “c” or “b”.

$\text{AC}(\mathbb{R})$ denotes the set of absolutely continuous functions, i.e., functions whose distributional derivative belongs to $L^1_{\text{loc}}(\mathbb{R})$. $\text{AC}^1(\mathbb{R})$ denotes the set of functions whose distributional derivative belongs to $\text{AC}(\mathbb{R})$.

When calculating integrals, we denote by \int' the “Cauchy principal value” at infinity, e.g.,

$$\int'_{i\mathbb{R}} f(t) dt = \lim_{R \rightarrow \infty} \int_{-iR}^{iR} f(t) dt.$$

Observe that we pass to infinity symmetrically in the lower and upper integration limits.

Suppose we fix a positive density γ on M . The space $L^2(M, \gamma)$ of square-integrable functions on M is then defined as the completion of $C_c^\infty(M)$ with respect to the scalar product

$$(u | v)_\gamma := \int_M \bar{u} v \gamma, \quad u, v \in C_c^\infty(M).$$

If g is the metric tensor g on M (of any signature), then we set $|g| := |\det[g_{\mu\nu}]|$. M is then equipped with a canonical density $|g|^{1/2}$. Sometimes it is however convenient to fix a density γ independent of the metric tensor.

Often it is convenient to use the formalism of (complexified) half-densities on M . If γ is a positive density on M , then $\gamma^{1/2}$ is a half-density. The canonical example for a half-density on a pseudo-Riemannian manifold is $|g|^{1/4}$. Since the integral over a density on a manifold is well-defined, half-densities come equipped with a natural L^2 -structure

$$(\tilde{u} | \tilde{v}) = \int_M \bar{\tilde{u}} \tilde{v}, \quad \tilde{u}, \tilde{v} \in C_c^\infty(\Omega^{1/2}M).$$

We denote by $L^2(\Omega^{1/2}M)$ the completion of $C_c^\infty(\Omega^{1/2}M)$ with respect to the corresponding norm. Note that if we fix an everywhere-positive density γ , then

$$L^2(M, \gamma) \ni u \mapsto \tilde{u} := u\gamma^{1/2} \in L^2(\Omega^{1/2}M)$$

is the natural unitary identification of the L^2 -space in the scalar formalism and in the half-density formalism.

The operator $D = -i\partial$ acts naturally on scalars, and $D^\gamma = \gamma^{1/2}D\gamma^{-1/2}$ acts naturally on half-densities.

In our paper we generally prefer to use the half-density formalism rather than the scalar formalism. The Klein–Gordon operator K is presented in (1-1) in the scalar formalism. Transformed to the half-density formalism it is

$$K_{\frac{1}{2}} := |g|^{1/4}K|g|^{-1/4} = |g|^{-1/4}(D_\mu - A_\mu)|g|^{1/2}g^{\mu\nu}(D_\nu - A_\nu)|g|^{-1/4} + Y. \quad (1-2)$$

In what follows we drop the subscript $\frac{1}{2}$ from $K_{1/2}$ and by K we will mean (1-2).

2. Assumptions and setting

2.1. 1 + 3 splitting. We consider smooth manifolds M and Σ such that there exists a (fixed) diffeomorphism $\mathbb{R} \times \Sigma \rightarrow M$. This means that we have a distinguished time function t on M , and the leaves $\Sigma_t = \{t\} \times \Sigma$ provide a foliation of M with a family of diffeomorphisms $\epsilon_t : \Sigma \rightarrow \Sigma_t \subset M$. We define the time vector field

$$\partial_t := \frac{d}{dt} \epsilon_t.$$

Note that $dt \cdot \partial_t = 1$.

We assume that M is equipped with a continuous Lorentzian metric g ; i.e., (M, g) is a *spacetime*. The restriction of g to the tangent space of Σ_t defines a time-dependent family of metrics on Σ , denoted by $g_\Sigma(t) := \epsilon_t^* g$. We make the assumption that all $g_\Sigma(t)$ are Riemannian, or, equivalently, that the covector dt is everywhere timelike. This assumption allows us to define the *lapse function* α :

$$\frac{1}{\alpha^2} := -g^{-1}(dt, dt) > 0.$$

Note that at this moment we do not assume that the vector ∂_t is everywhere timelike, which is equivalent to

$$g_\Sigma(\beta, \beta) < \alpha^2. \quad (2-1)$$

This assumption will be forced on us later on by [Assumption \(1.b\)](#). The part of ∂_t orthogonal to the leaves of the foliation is the *shift vector*

$$\beta := \partial_t + \alpha^2 g^{-1}(dt, \cdot).$$

The inverse metric can now be written as

$$g^{-1} = -\frac{1}{\alpha^2} (\partial_t - \beta) \otimes (\partial_t - \beta) + g_\Sigma^{-1}. \quad (2-2)$$

In coordinates, we have

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -\alpha^2 dt^2 + g_{\Sigma,ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt), \\ g^{\mu\nu} \partial_\mu \partial_\nu &= -\frac{1}{\alpha^2} (\partial_t - \beta^i \partial_i)^2 + g_\Sigma^{ij} \partial_i \partial_j. \end{aligned}$$

The generic notation for a point of M will be (t, \vec{x}) . We often suppress the spatial dependence of objects defined on M ; e.g., we identify $f(t) = f(t, \cdot)$ for some function f on M . Sometimes we also suppress the time-dependence, but it should be kept in mind that the central quantities considered here, the metric g , the electromagnetic potential A and the scalar potential Y , generically are time-dependent. Sometimes we denote derivatives with respect to t (i.e., the action of the vector field ∂_t) by a dot.

2.2. Klein–Gordon operator. The main object of our paper is the Klein–Gordon operator (1-2). Instead of the operator K on $L^2(M)$, it is more convenient to work with the operator

$$\tilde{K} := \alpha K \alpha.$$

With the inverse metric expressed as (2-2), it can be written as

$$\begin{aligned}\tilde{K} &= -\gamma^{-1/2}(D_t - D_i\beta^i + V)\gamma(D_t - \beta^j D_j + V)\gamma^{-1/2} + \gamma^{-1/2}(D_i - A_i)\alpha^2\gamma g_\Sigma^{ij}(D_j - A_j)\gamma^{-1/2} + \alpha^2 Y \\ &= -(D_t + W^*)(D_t + W) + L,\end{aligned}$$

where we introduce

$$\begin{aligned}\gamma &:= \alpha^{-2}|g|^{1/2} = \alpha^{-1}|g_\Sigma|^{1/2}, \\ V &:= -A_0 + A_i\beta^i, \\ W &:= \beta^i D_i + V - \frac{1}{2}\gamma^{-1}(D_t\gamma - \beta^i D_i\gamma), \\ L &:= D_i^{A,\gamma*} \tilde{g}_\Sigma^{ij} D_j^{A,\gamma} + \tilde{Y},\end{aligned}$$

and we use the shorthands

$$\begin{aligned}\tilde{g}_\Sigma^{ij}(t) &:= \alpha(t)^2 g_\Sigma^{ij}(t), \\ \tilde{Y}(t) &:= \alpha(t)^2 Y(t), \\ D^{A,\gamma}(t) &:= \gamma(t)^{1/2}(D - A(t))\gamma(t)^{-1/2}.\end{aligned}$$

Clearly, propagators for \tilde{K} induce corresponding propagators for K .

2.3. First-order formalism. For each $t \in \mathbb{R}$, we (formally) define

$$B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & W(t)^* \end{pmatrix}.$$

Setting $u_1(t) = u(t)$ and $u_2(t) = -(D_t + W(t))u(t)$, we find that

$$(\partial_t + iB(t)) \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = 0$$

if and only if u is a (weak) solution of the Klein–Gordon equation $\tilde{K}u = 0$. Therefore we occasionally call $\partial_t + iB(t)$ the *first-order Klein–Gordon operator*. The half-densities $u_1(t)$ and $u_2(t)$ may be called the *Cauchy data* for u at time t .

2.4. Assumptions local in time.

Assumption 1. We suppose that the following assumptions hold:

- (1.a) For all $t \in \mathbb{R}$, $L(t)$ extends to a positive invertible self-adjoint operator on $L^2(\Omega^{1/2}\Sigma)$ (denoted by the same symbol).
- (1.b) There exists $a \in C(\mathbb{R})$ such that $a(t) < 1$ and $\|W(t)L(t)^{-1/2}\| \leq a(t)$.
- (1.c) There exists a positive $C \in L^1_{\text{loc}}(\mathbb{R})$ such that for all $|t - s| \leq 1$

$$\|L(t)^{-1/2}(L(t) - L(s))L(t)^{-1/2}\| + 2\|(W(t) - W(s))L(t)^{-1/2}\| \leq \left| \int_s^t C(r) dr \right|, \quad (2-3)$$

where we place the absolute value on the right-hand side to account for the arbitrary order of t and s .

(1.d) $t \mapsto \alpha(t)^{\pm 1}$ are norm-continuous on $L(s)^{-1/2}L^2(\Omega^{1/2}\Sigma)$ for any $s \in \mathbb{R}$, and $t \mapsto \dot{\alpha}(t)$ is norm-continuous on $L^2(\Omega^{1/2}\Sigma)$.

A few remarks about these assumptions are in order:

First, [Assumption \(1.a\)](#) can always be realized if $\gamma(t)^{-1}\partial_t\gamma(t)$, $A_i(t) \in L^2_{\text{loc}}(\Sigma)$, $\tilde{g}_{ij}(t) \in L^\infty_{\text{loc}}(\Sigma)$ and $\tilde{Y}(t) \in L^1_{\text{loc}}(\Sigma)$ such that $\tilde{Y}(t)$ is bounded from below by a positive constant. In that case $L(t)$ can be understood as the form

$$(u | L(t) v) = \int_{\Sigma} (\overline{(D_i^{A,\gamma}(t) u)} \tilde{g}_{\Sigma}^{ij}(t) (D_j^{A,\gamma}(t) v) + \bar{u} \tilde{Y}(t) v),$$

on its (natural) maximal form domain $\text{Dom } L(t)^{1/2} \supset C_c^\infty(\Omega^{1/2}\Sigma)$ (but it is not generally clear if $C_c^\infty(\Omega^{1/2}\Sigma)$ is a form core). This form then defines a self-adjoint operator in the usual way. The details of this construction are given in [Appendix A](#); its main aspects can be found in [\[Kato 1966, Theorem VI.2.6\]](#).

Next, [Assumption \(1.b\)](#) means that $\|W(t)L(t)^{-1/2}\| < 1$. Thus the electrostatic potential $V(t)$ together with the variation of the metric expressed by $\gamma(t)^{-1}\dot{\gamma}(t)$ and the shift vector β cannot be too big compared to $L(t)$. This has to be true already on the level of the principal symbols of W and L . Therefore, for each $x = (t, \vec{x}) \in M$ and $p \in T_{\vec{x}}^*\Sigma_t$, we need to have

$$|\beta^k(x) p_k (\tilde{g}_{\Sigma}^{ij}(x) p_i p_j)^{-1/2}| < 1.$$

This is equivalent to

$$\tilde{g}_{\Sigma,ij} \beta^i \beta^j < 1, \tag{2-4}$$

where $\tilde{g}_{\Sigma,ij} = \alpha^{-2} g_{\Sigma,ij}$ is the inverse of \tilde{g}_{Σ}^{ij} , and consequently [\(2-4\)](#) is equivalent to [\(2-1\)](#). Thus [Assumption \(1.b\)](#) implies that ∂_t is timelike. This excludes, e.g., the ergosphere region of Kerr spacetime in stationary coordinates — in such a case one needs to switch to the nonstationary corotating coordinates.

Together, [Assumptions \(1.a\)](#) and [\(1.b\)](#) guarantee that the Hamiltonian is positive and has a positive lower bound (the “positive-mass assumption”). The positivity of the Hamiltonian and its positive lower bound have two aspects. First, they are essentially necessary if we want to construct nonclassical propagators. Second, this assumption helps us to introduce a natural family of Hilbertizable spaces, which are used in the analysis of the evolution. (A similar analysis would be possible with a positive Hamiltonian, but without a positive lower bound; however there would be some additional technical problems).

Nevertheless, as far as the derivation of the evolution and the classical propagators is concerned, [Assumption \(1.a\)](#) can be relaxed. In fact, for the existence of the evolution it is sufficient that there exists a constant $b > 0$ such that these assumptions are satisfied by $L(t) + b$; see also [Corollary 5.5](#). In this case in general we do not have a positive Hamiltonian and our analysis of nonclassical propagators does not apply.

Among other things, [Assumption \(1.c\)](#) guarantees that for any t, s there exists $c(t, s) > 0$ such that

$$L(t) \leq c(t, s)L(s). \tag{2-5}$$

Therefore, for $\delta \in [-1, 1]$ we can define the Hilbertizable spaces

$$\mathcal{K}^\delta := L(t)^{-\delta/2}L^2(\Omega^{1/2}\Sigma),$$

where the Hilbertian structures on the right-hand side are equivalent for different t because of [\(2-5\)](#).

Finally, [Assumption \(1.d\)](#) implies the norm-continuity of $t \mapsto \alpha(t)^{\pm 1}$ on \mathcal{K}^δ for $\delta \in [-1, 1]$. Indeed, by this assumption, $t \mapsto \alpha(t)^{\pm 1}$ are norm-continuous on $\mathcal{K}^{1/2}$, hence by duality also on $\mathcal{K}^{-1/2}$, and then we can interpolate using, e.g., the Heinz–Kato inequality ([Theorem D.1](#)).

While it should be obvious how [Assumptions \(1.a\)](#), [\(1.b\)](#) and [\(1.d\)](#) can be realized in an example, [Assumption \(1.c\)](#) is slightly less obvious. Therefore in [Appendix B](#) we briefly explain how [Assumption \(1.c\)](#) can follow from more concrete assumptions on the metric, the vector potential and the scalar potential.

2.5. Assumptions global in time. While we always require that [Assumption 1](#) holds, the following additional assumptions are only imposed when we derive asymptotic properties of propagators.

Assumption 2. (2.a) $L(t)$ is uniformly bounded away from zero.

(2.b) There exists $a < 1$ such that $\|W(t)L(t)^{-1/2}\| \leq a$ for all t .

(2.c) There exists a positive $C \in L^1(\mathbb{R})$ such that for all $t, s \in \mathbb{R}$

$$\|L(t)^{-1/2}(L(t) - L(s))L(t)^{-1/2}\| + 2\|(W(t) - W(s))L(t)^{-1/2}\| \leq \left| \int_s^t C(r) \, dr \right|,$$

where we place the absolute value on the right-hand side to account for the arbitrary order of t and s .

(2.d) $t \mapsto \alpha(t)^{\pm 1}$ are uniformly bounded on \mathcal{K}^1 and $t \mapsto \dot{\alpha}$ is uniformly bounded on \mathcal{K}^0 .

Note that, by the same argument as for [Assumption \(1.d\)](#), one can show that [Assumption \(2.d\)](#) implies the uniform boundedness of $t \mapsto \alpha(t)^{\pm 1}$ on \mathcal{K}^δ for $\delta \in [-1, 1]$.

3. The energy space and the dynamical space

We will occasionally use the Hilbert space

$$\mathcal{H} := L^2(\Omega^{1/2}\Sigma) \oplus L^2(\Omega^{1/2}\Sigma) = \mathcal{K}^0 \oplus \mathcal{K}^0,$$

with the canonical inner product also denoted by $(\cdot | \cdot)$ and the corresponding norm $\|\cdot\|$.

The Hilbert space \mathcal{H} plays only an auxiliary role in our work. More important are the Hilbertizable spaces \mathcal{H}_λ , $\lambda \in [-1, 1]$, defined as

$$\mathcal{H}_\lambda := \mathcal{K}^{(\lambda+1)/2} \oplus \mathcal{K}^{(\lambda-1)/2}. \quad (3-1)$$

Note that for any t

$$\mathcal{H}_\lambda = (L(t) \oplus L(t))^{-\lambda/4} \mathcal{H}_0, \quad \lambda \in [-1, 1]. \quad (3-2)$$

We will treat the space \mathcal{H}_0 as the central element of the family [\(3-2\)](#), identifying \mathcal{H}_0 with \mathcal{H}_0^* , the *antidual* of \mathcal{H}_0 (the space of bounded antilinear functionals on \mathcal{H}_0). Then we have a natural identification of $\mathcal{H}_{-\lambda}$ with \mathcal{H}_λ^* .

The central role in this work is played by the *energy space*, the *dynamical space* and the antidual of the energy space:

$$\mathcal{H}_{\text{en}} := \mathcal{H}_1 = (L(t)^{-1/2} \oplus \mathbb{1})\mathcal{H} = H_0(t)^{-1/2}\mathcal{H}, \quad (3-3a)$$

$$\mathcal{H}_{\text{dyn}} := \mathcal{H}_0 = (L(t)^{-1/4} \oplus L(t)^{1/4})\mathcal{H}, \quad (3-3b)$$

$$\mathcal{H}_{\text{en}}^* := \mathcal{H}_{-1} = (\mathbb{1} \oplus L(t)^{1/2})\mathcal{H} = (QH_0(t)Q)^{1/2}\mathcal{H}, \quad (3-3c)$$

where we set

$$H_0(t) := L(t) \oplus \mathbb{1} = \begin{pmatrix} L(t) & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

and we also used the *charge form*

$$(u | Qv) := (u_1 | v_2) + (u_2 | v_1), \quad Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

It is evident that the charge form is bounded on \mathcal{H} . More importantly, it is also bounded on \mathcal{H}_{dyn} (but, e.g., not on \mathcal{H}_{en}).

Note that

$$\text{Im}(u | Qv) = \frac{1}{2i}((u | Qv) - (v | Qu))$$

is a symplectic form on \mathcal{H}_{dyn} . Therefore, the formalism based on the charge form is equivalent to the symplectic formalism, commonly used in the literature.

4. Instantaneous energy spaces and instantaneous dynamical spaces

An important role in our paper is played by the instantaneous Hamiltonian, defined formally for each t as

$$H(t) = QB(t) = B(t)^*Q.$$

One can rigorously define $H(t)$ as a form-bounded perturbation of $H_0(t)$:

Proposition 4.1. *The operator*

$$H(t) := \begin{pmatrix} L(t) & W(t)^* \\ W(t) & \mathbb{1} \end{pmatrix}$$

is self-adjoint on \mathcal{H} with the form domain \mathcal{H}_{en} . We have

$$(1 - a(t))H_0(t) \leq H(t) \leq (1 + a(t))H_0(t), \tag{4-1}$$

where $0 \leq a(t) < 1$ was introduced in [Assumption \(1.b\)](#).

Proof. We show only the right-hand side of the inequality [\(4-1\)](#). Set $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Using the Cauchy–Schwarz inequality and [Assumption \(1.b\)](#), we find

$$\begin{aligned} (u | H(t)u) &\leq \|L(t)^{1/2}u_1\|^2 + \|u_2\|^2 + 2\|W(t)u_1\|\|u_2\| \\ &\leq \|L(t)^{1/2}u_1\|^2 + \|u_2\|^2 + 2a(t)\|L(t)^{1/2}u_1\|\|u_2\| \\ &\leq (1 + a(t))(\|L(t)^{1/2}u_1\|^2 + \|u_2\|^2) \\ &= (1 + a(t))(u | H_0(t)u). \end{aligned} \quad \square$$

We define for each time $t \in \mathbb{R}$ the (*instantaneous*) *energy scalar products* given by

$$(u | v)_{\text{en},t} := (u | H(t)v)$$

on \mathcal{H}_{en} . By [\(4-1\)](#) the scalar product $(\cdot | \cdot)_{\text{en},t}$ is compatible with the topology of \mathcal{H}_{en} . We call the resulting Hilbert space the *instantaneous energy space at t* and denote it by $\mathcal{H}_{\text{en},t}$.

Similarly, we can also define the operator $QH(t)^{-1}Q$. We find that its form domain is $\mathcal{H}_{\text{en}}^*$. Indeed,

$$(1 + a(t))^{-1}QH_0(t)^{-1}Q \leq QH(t)^{-1}Q \leq (1 - a(t))^{-1}QH_0(t)^{-1}Q. \quad (4-2)$$

Then we define for each t the scalar product

$$(u | v)_{\text{en}^*, t} := (u | QH(t)^{-1}Qv)$$

and note that it is compatible with the topology of $\mathcal{H}_{\text{en}}^*$; we denote the resulting Hilbert space by $\mathcal{H}_{\text{en}, t}^*$.

The central operator in this work is $B(t)$. In the next section we construct the evolution generated by $B(t)$, solving the first-order Klein–Gordon equation.

Proposition 4.2. *Considered as an operator on $\mathcal{H}_{\text{en}, t}^*$ with domain \mathcal{H}_{en} ,*

$$B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & W(t)^* \end{pmatrix}$$

is self-adjoint and 0 is in its resolvent set.

Proof. For notational simplicity, we drop the time-dependence of $B(t)$ and the other objects.

First note that, by definition, $H_0(t)^{-1/2} = (L(t)^{-1/2} \oplus \mathbb{1})$ maps \mathcal{H} to \mathcal{H}_{en} , and $(QH_0(t)Q)^{-1/2} = (\mathbb{1} \oplus L(t)^{-1/2})$ maps $\mathcal{H}_{\text{en}}^*$ to \mathcal{H} . Now, to check that $B(t)$ is well-defined, we calculate

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{-1/2} \end{pmatrix} \begin{pmatrix} W & \mathbb{1} \\ L & W^* \end{pmatrix} \begin{pmatrix} L^{-1/2} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} WL^{-1/2} & \mathbb{1} \\ \mathbb{1} & L^{-1/2}W^* \end{pmatrix},$$

which is bounded by [Assumption \(1.b\)](#).

Next, we show that $0 \in \text{rs } B$, and consequently also that B is closed. We rewrite B as

$$B = \begin{pmatrix} \mathbb{1} & 0 \\ W^* & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ L - W^*W & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ W & \mathbb{1} \end{pmatrix}$$

and check that B^{-1} is bounded from $\mathcal{H}_{\text{en}}^*$ to \mathcal{H}_{en} :

$$\begin{pmatrix} L^{1/2} & 0 \\ 0 & \mathbb{1} \end{pmatrix} B^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{1/2} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ -L^{-1/2}W^* & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & (\mathbb{1} - L^{-1/2}W^*WL^{-1/2})^{-1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -WL^{-1/2} & \mathbb{1} \end{pmatrix},$$

where the first and last factors on the right-hand side are bounded by [Assumption \(1.b\)](#), and

$$\mathbb{1} - L^{-1/2}W^*WL^{-1/2}$$

is invertible because $\|L^{-1/2}W^*WL^{-1/2}\| < 1$, also by [Assumption \(1.b\)](#).

Finally, we check that B is Hermitian on $\mathcal{H}_{\text{en}}^*$. We calculate

$$(QH_0Q)^{-1}B^{-1} = (BQH_0Q)^{-1} = (QH_0QH_0Q)^{-1} = (QH_0QB^*)^{-1} = B^{*-1}(QH_0Q)^{-1}. \quad \square$$

We can now define for each time $t \in \mathbb{R}$ a whole scale of Hilbert spaces

$$\mathcal{H}_{\lambda, t} := |B(t)|^{-(1+\lambda)/2} \mathcal{H}_{\text{en}, t}^*, \quad \lambda \in \mathbb{R},$$

with scalar products

$$(u | v)_{\lambda,t} := (u | |B(t)|^{1+\lambda} v)_{\text{en}^*,t}, \quad u, v \in \mathcal{H}_{\lambda,t}.$$

Above we performed the polar decomposition with respect to the Hilbert space $\mathcal{H}_{\text{en},t}^*$, where we have

$$|B(t)| = \sqrt{B(t)^2} = \sqrt{QH(t)QH(t)}.$$

It follows from its definition, that $B(t)$ extends/restricts to a self-adjoint operator on each of the spaces $\mathcal{H}_{\lambda,t}$. When $B(t)$ is interpreted as an operator on $\mathcal{H}_{\lambda,t}$, its domain is $\mathcal{H}_{\lambda+2,t}$.

Clearly the scales $\mathcal{H}_{\lambda,t}$ contain $\mathcal{H}_{\text{en},t}^* = \mathcal{H}_{-1,t}$. They also contain the (instantaneous) energy spaces $\mathcal{H}_{\text{en},t} = \mathcal{H}_{1,t}$, because a short calculation shows $H(t) = QH(t)^{-1}Q|B(t)|^2$. Furthermore, we define the (instantaneous) dynamical spaces

$$\mathcal{H}_{\text{dyn},t} := \mathcal{H}_{0,t},$$

which are treated as the central spaces in these scales. Note that $\mathcal{H}_{\text{dyn},t}$ is the form domain of $B(t)$. We identify $\mathcal{H}_{0,t}^*$ with $\mathcal{H}_{0,t}$, and hence $\mathcal{H}_{\lambda,t}^*$ is identified with $\mathcal{H}_{-\lambda,t}$. Thus we obtain the rigged Hilbert space setting

$$\mathcal{H}_{\text{en},t} \subset \mathcal{H}_{\text{dyn},t} \subset \mathcal{H}_{\text{en},t}^*.$$

Proposition 4.3. *In the sense of Hilbertizable spaces, we have*

$$\mathcal{H}_{\lambda,t} = \mathcal{H}_{\lambda}, \quad \lambda \in [-1, 1], \tag{4-3}$$

thus justifying our notation. In particular,

$$\mathcal{H}_{\text{en},t} = \mathcal{H}_{\text{en}}, \quad \mathcal{H}_{\text{dyn},t} = \mathcal{H}_{\text{dyn}}, \quad \mathcal{H}_{\text{en},t}^* = \mathcal{H}_{\text{en}}^*.$$

Proof. It follows from (4-1) and (4-2) that $\mathcal{H}_{\text{en},t} = \mathcal{H}_{\text{en}}$ and $\mathcal{H}_{\text{en},t}^* = \mathcal{H}_{\text{en}}^*$. Since both $L(t)^{1/2} \oplus L(t)^{1/2}$ and $|B|$ can be understood as invertible bounded operators from \mathcal{H}_{en} to $\mathcal{H}_{\text{en}}^*$, there exists $c > 1$ such that

$$c^{-1} \|(L(t) \oplus L(t))^{1/2} u\|_{\text{en}^*} \leq \| |B(t)| u \|_{\text{en}^*} \leq c \|(L(t) \oplus L(t))^{1/2} u\|_{\text{en}^*}.$$

By interpolation (e.g., using the Heinz–Kato inequality, [Theorem D.1](#)),

$$c^{-\delta} \|(L(t) \oplus L(t))^{\delta/2} u\|_{\text{en}^*} \leq \| |B(t)|^\delta u \|_{\text{en}^*} \leq c^\delta \|(L(t) \oplus L(t))^{\delta/2} u\|_{\text{en}^*}$$

for $\delta \in [0, 1]$. It follows that the norms for \mathcal{H}_{λ} and $\mathcal{H}_{\lambda,t}$ with $\lambda \in [-1, 1]$ are equivalent and thus (4-3) follows. □

Note that for $|\lambda| > 1$ the spaces $\mathcal{H}_{\lambda,t}$ may depend on t and do not have to coincide with \mathcal{H}_{λ} .

5. Evolution

In the last section we laid the foundations for an application of the theory of nonautonomous evolution equations to the situation at hand, i.e., the first-order Klein–Gordon equation

$$\partial_t u(t) + iB(t)u(t) = 0.$$

Autonomous evolution equations (viz., with a time-independent generator) possess a well-understood theory in terms of the theory of strongly continuous semigroups and groups. The theory for nonautonomous evolution equations is significantly more complicated and subtle. In [Appendix C](#) we discuss the relevant results based on [\[Kato 1970\]](#).

Here we apply [Theorem C.10](#) to the operator $B(t)$ on the spaces

$$\mathcal{X}_t = \mathcal{H}_{\text{en},t}^* \quad \text{and} \quad \mathcal{Y}_t = \mathcal{H}_{\text{en},t}. \quad (5-1)$$

For this purpose, we need to check whether the conditions (a)–(c) of [Theorem C.10](#) hold. The self-adjointness condition (c) is clearly true; see [Section 3](#). The next proposition implies that condition (b), a continuity condition on the norms of the Hilbert spaces $\mathcal{H}_{\text{en},t}$ and $\mathcal{H}_{\text{en},t}^*$, holds:

Proposition 5.1. *Let $C \in L^1_{\text{loc}}(\mathbb{R})$ as in [Assumption \(1.c\)](#), $a(t) \in C(\mathbb{R})$ as in [Assumption \(1.b\)](#) and $|t - s| \leq 1$ with $t \geq s$. Set*

$$c_{s,t} := \sup_{\tau \in [s,t]} (1 - a(\tau))^{-1}.$$

Then, for $\lambda \in [-1, 1]$,

$$\|u\|_{\lambda,t} \exp\left(-c_{s,t} \int_s^t C(\tau) \, d\tau\right) \leq \|u\|_{\lambda,s} \leq \|u\|_{\lambda,t} \exp\left(c_{s,t} \int_s^t C(\tau) \, d\tau\right). \quad (5-2)$$

Proof. First we show (5-2) for $\lambda = 1$, i.e., for the energy space.

By [Assumption \(1.c\)](#), we have

$$\begin{aligned} & \| (L(t))^{-1/2} \oplus \mathbb{1} (H(s) - H(t)) (L(t))^{-1/2} \oplus \mathbb{1} \| \\ & \leq \| L(t)^{-1/2} (L(s) - L(t)) L(t)^{-1/2} \| + 2 \| (W(s) - W(t)) L(t)^{-1/2} \| \\ & \leq \int_s^t C(\tau) \, d\tau. \end{aligned} \quad (5-3)$$

Equation (4-1) then implies

$$\| H(t)^{-1/2} (L(t) \oplus \mathbb{1}) H(t)^{-1/2} \| \leq c_{s,t}. \quad (5-4)$$

Putting together (5-3) and (5-4), we obtain

$$\| H(t)^{-1/2} (H(s) - H(t)) H(t)^{-1/2} \| \leq c_{s,t} \int_s^t C(\tau) \, d\tau.$$

Consequently we have

$$\| \|u\|_{\text{en},s}^2 - \|u\|_{\text{en},t}^2 \| \leq \|u\|_{\text{en},t}^2 \left(c_{s,t} \int_s^t C(\tau) \, d\tau \right).$$

Therefore

$$\|u\|_{\text{en},s}^2 \leq \|u\|_{\text{en},t}^2 \left(1 + c_{s,t} \int_s^t C(\tau) \, d\tau \right) \leq \|u\|_{\text{en},t}^2 \exp\left(c_{s,t} \int_s^t C(\tau) \, d\tau\right)$$

and, exchanging the roles of t and s , we can similarly derive

$$\|u\|_{\text{en},s}^2 \geq \|u\|_{\text{en},t}^2 \exp\left(-c_{s,t} \int_s^t C(\tau) \, d\tau\right),$$

so that the inequality (5-2) for $\lambda = 1$ follows.

For $\lambda = -1$ the inequality follows by duality. Using interpolation, we can then extend the inequality to the remaining values of λ . \square

To show that condition (a) of [Theorem C.10](#) holds, we only need to show the norm-continuity of $t \mapsto B(t)$; the remaining statements are obvious.

Proposition 5.2. *With $C \in L^1_{\text{loc}}(\mathbb{R})$ as in [Assumption \(1.c\)](#), $c_{s,t}$ as in (4-1) and $|t - s| \leq 1$,*

$$\|(B(s) - B(t))u\|_{\text{en}^*,t} \leq \|u\|_{\text{en},t} \left| c_{s,t} \int_s^t C(\tau) \, d\tau \right|,$$

where we place the absolute value on the right-hand side because $t \geq s$ or $t \leq s$. In particular, $t \mapsto B(t)$ is norm-continuous as an operator from $\mathcal{H}_{\text{en},t}$ to $\mathcal{H}_{\text{en},t}^*$.

Proof. We reduce the problem to the inequalities

$$\begin{aligned} \|(\mathbb{1} \oplus L(t)^{-1/2})(B(s) - B(t))(L(t)^{-1/2} \oplus \mathbb{1})\| &= \|\mathcal{Q}(L(t)^{-1/2} \oplus \mathbb{1})\mathcal{Q}(B(s) - B(t))(L(t)^{-1/2} \oplus \mathbb{1})\| \\ &\leq \|(L(t)^{-1/2} \oplus \mathbb{1})(H(s) - H(t))(L(t)^{-1/2} \oplus \mathbb{1})\| \\ &\leq \left| \int_s^t C(\tau) \, d\tau \right| \end{aligned}$$

and proceed much as in the proof of [Proposition 5.1](#). Since the integral is continuous, the required norm-continuity follows. \square

It follows that we can globally define an evolution for $B(t)$:

Theorem 5.3. *There exists a unique, strongly continuous family of bounded operators $\{U(t, s)\}_{s,t \in \mathbb{R}}$ on $\mathcal{H}_{\text{en}}^*$, the evolution generated by $B(t)$, with the following properties:*

(i) *For all $r, s, t \in \mathbb{R}$, we have the identities*

$$U(t, t) = \mathbb{1}, \quad U(t, r)U(r, s) = U(t, s). \quad (5-5)$$

(ii) *For $\lambda \in [-1, 1]$, $U(t, s)\mathcal{H}_\lambda \subset \mathcal{H}_\lambda$, $(t, s) \mapsto U(t, s)$ is strongly \mathcal{H}_λ -continuous and satisfies the bounds*

$$\|U(t, r)\|_{\lambda,s} \leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau\right), \quad (5-6a)$$

$$\|U(r, t)\|_{\lambda,s} \leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau\right), \quad (5-6b)$$

with C, c as in [Proposition 5.1](#) and $r \leq s \leq t$, where $|t - r| \leq 1$.

(iii) *For all $u \in \mathcal{H}_{\text{en}}$, $U(t, s)u$ is continuously differentiable in $s, t \in \mathbb{R}$ with respect to the strong topology of $\mathcal{H}_{\text{en}}^*$ and it satisfies*

$$i\partial_t U(t, s)u = B(t)U(t, s)u, \quad (5-7a)$$

$$-i\partial_s U(t, s)u = U(t, s)B(s)u. \quad (5-7b)$$

Proof. Propositions 5.1 and 5.2 as well as the results of Section 3 show that Theorem C.10 can be applied to our operator $B(t)$, understood as an operator from \mathcal{H}_{en} to $\mathcal{H}_{\text{en}}^*$ (or, equivalently, as a form on \mathcal{H}_{dyn} with form domain \mathcal{H}_{en}). We thus obtain for every sufficiently small compact interval $I \subset \mathbb{R}$ an evolution $U(t, s)$ with the properties (i)–(iv) of Theorem C.10. In particular, we have for $r, t \in I$ and $r \leq s \leq t$

$$\begin{aligned} \|U(t, r)\|_{\text{en},s} &\leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau\right), \\ \|U(t, r)\|_{\text{en}^*,s} &\leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau\right). \end{aligned}$$

The same bounds also hold for $\|U(r, t)\|_{\text{en},s}$ and $\|U(r, t)\|_{\text{en}^*,s}$. By interpolation we find (5-6).

We cover \mathbb{R} by compact intervals. Using the identity (5-5), we thereby define the evolution $U(t, s)$ on the whole real axis by gluing. For finite s, t , it has the properties (i)–(iv) of Theorem C.10. \square

Equation (5-6) states that $U(t, s)$ is bounded for finite t, s . To obtain stronger results later, we can choose more stringent assumptions:

Corollary 5.4. *If Assumption (2.c) holds, and we set $C_1(t) := 2(1-a)^{-1}C(t)$, then*

$$\|U(t, s)\|_{\lambda,r} \leq \exp\left(\int_{\mathbb{R}} C_1(\tau) \, d\tau\right)$$

for all $r, s, t \in \mathbb{R}$ and any $\lambda \in [-1, 1]$.

In Assumption (1.a) we supposed that $L(t)$ is positive and invertible. Actually, the main results of this section remain true if $L(t)$ is only bounded from below:

Corollary 5.5. *Instead of Assumption 1, suppose that there exists a constant $b > 0$ such that Assumption 1 holds for $L(t) + b$. Then Theorem 5.3 holds with respect to the scale of Hilbert spaces and constants obtained from $L(t) + b$, and with the bounds (5-6) replaced by*

$$\begin{aligned} \|U(t, r)\|_{\lambda,s} &\leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau + (t-r)b\|(L(s) + b)^{-1/2}\|\right), \\ \|U(r, t)\|_{\lambda,s} &\leq \exp\left(2c_{r,t} \int_r^t C(\tau) \, d\tau + (t-r)b\|(L(s) + b)^{-1/2}\|\right). \end{aligned}$$

Proof. Replacing $L(t)$ in $B(t)$ by $L(t) + b$, we obtain a new operator

$$B_b(t) = B(t) + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

We also replace $L(t)$ by $L(t) + b$ in all definitions concerning the Hilbert and Hilbertizable spaces that we need. According to Theorem 5.3, $B_b(t)$ has an evolution $U_b(t, s)$ with the properties stated in the theorem. Note that $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ is a bounded operator on $\mathcal{H}_{\text{en}}^*$. Indeed, this follows from the boundedness of

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L^{-1/2}b & 0 \end{pmatrix}$$

on \mathcal{H} . Since $B(t)$ is a bounded perturbation of $B_b(t)$, we can apply Theorem C.11 to find the evolution for $B(t)$. \square

Remark 5.6. Our choice of spaces (5-1) to prove Theorem 5.3 is natural, especially given our low-regularity setup. Under more restrictive assumptions on the smoothness and boundedness of coefficients of the Klein–Gordon operator K , other spaces in the scale $\mathcal{H}_{\lambda,t}$, $\lambda \in \mathbb{R}$, could be used. This would lead to improved regularity results of the type $U(t,s)\mathcal{H}_\lambda \subset \mathcal{H}_\lambda$ and continuous differentiability of $U(t,s)\mathcal{H}_\lambda$ in $\mathcal{H}_{\lambda-2}$.

Remark 5.7. In the stationary case, i.e., if B does not depend on time, there exist distinguished Hilbert spaces \mathcal{H}_λ and the evolution family $U(t,s)$ simplifies to a unitary group $U(t-s) = U(t,s)$ on \mathcal{H}_λ .

6. Solutions of the Klein–Gordon equation

Solutions of the Klein–Gordon equation are closely related to solutions of the first-order Klein–Gordon equation.

Let us introduce the projection onto the second component:

$$\pi_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := u_2.$$

We also define embeddings

$$\iota_2 u := \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \rho u := \begin{pmatrix} u \\ -(D_t + W)u \end{pmatrix}.$$

A formal calculation then shows that²

$$\tilde{K} = -i\pi_2(\partial_t + iB)\rho \quad \text{and} \quad K = -i\alpha^{-1}\pi_2(\partial_t + iB)\rho\alpha^{-1}. \tag{6-1}$$

Therefore, if $Ku = f$ or, equivalently, $\tilde{K}\tilde{u} = \tilde{f}$ with $\tilde{u} = \alpha^{-1}u$, $\tilde{f} = \alpha f$, then

$$-i(\partial_t + iB)\rho\tilde{u} = \iota_2\tilde{f}.$$

The projection π_2 and the embeddings ρ , ι_2 , which relate solutions of the Klein–Gordon equation and the first-order Klein–Gordon equation, can be understood between various spaces. It follows from the definition of \mathcal{H}_λ in (3-1) that, for $\lambda \in [-1, 1]$,

$$\pi_2 : \mathcal{H}_\lambda \rightarrow \mathcal{K}^{(\lambda-1)/2}, \tag{6-2a}$$

$$\pi_2 Q : \mathcal{H}_\lambda \rightarrow \mathcal{K}^{(\lambda+1)/2}, \tag{6-2b}$$

$$\iota_2 : \mathcal{K}^{(\lambda-1)/2} \rightarrow \mathcal{H}_\lambda. \tag{6-2c}$$

These projections and embeddings already allow us to easily prove an existence and uniqueness result regarding solutions of the Klein–Gordon equation with Cauchy data in the energy space:

Theorem 6.1. *Let $s \in \mathbb{R}$, $\begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \in \mathcal{H}_{\text{en}}$ and $f \in L^1_{\text{loc}}(\mathbb{R}; \mathcal{K}^0)$. Set*

$$\begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \end{pmatrix} = \alpha(s)^{-1} \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \quad \text{and} \quad \tilde{f} = \alpha f.$$

²Note that there is a sign error in the corresponding equation in [Dereziński and Siemssen 2018] which also affects the definition of the associated propagators.

Then $u = \alpha \tilde{u}$, with

$$\tilde{u}(t) = \pi_2 Q U(t, s) \begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \end{pmatrix} + i \int_s^t \pi_2 Q U(t, r) \iota_2 \tilde{f}(r) dr,$$

is the unique solution of $Ku = f$ such that

$$u \in C(\mathbb{R}; \mathcal{K}^1) \cap C^1(\mathbb{R}; \mathcal{K}^0) \quad \text{and} \quad \rho \tilde{u}(s) = \begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \end{pmatrix}. \quad (6-3)$$

Proof. We have the following special cases of (6-2):

$$\iota_2 : \mathcal{K}^0 \rightarrow \mathcal{H}_{\text{en}}, \quad (6-4a)$$

$$\pi_2 Q : \mathcal{H}_{\text{en}} \rightarrow \mathcal{K}^1, \quad (6-4b)$$

$$\pi_2 Q : \mathcal{H}_{\text{en}}^* \rightarrow \mathcal{K}^0. \quad (6-4c)$$

By (6-4a), (6-4b) and Assumption (1.d), u belongs to $C(\mathbb{R}; \mathcal{K}^1)$. By (6-4a), (6-4c) and Assumption (1.d), $\partial_t u$ belongs to $C(\mathbb{R}; \mathcal{K}^0)$. Hence the first part of (6-3) is true. The second part of (6-3) is obvious.

Set

$$\begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} = U(t, s) \begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \end{pmatrix} + i \int_s^t U(t, r) \iota_2 \tilde{f}(r) dr. \quad (6-5)$$

Differentiating (6-5) we obtain

$$i \partial_t \begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} = B(t) \begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} - \iota_2 \tilde{f}(t). \quad (6-6)$$

Clearly, $\tilde{u}(t) = \tilde{u}_1(t)$. The first component of (6-6) yields $\tilde{u}_2(t) = -(D_t + W(t))\tilde{u}_1(t)$. Hence

$$\rho \tilde{u}(t) = \begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix}. \quad (6-7)$$

The second component of (6-6) and then insertion of (6-7) yield

$$\tilde{f}(t) = -i \pi_2 (\partial_t + iB) \begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} = -i \pi_2 (\partial_t + iB) \rho \tilde{u}(t) = \tilde{K} \tilde{u}(t),$$

whence we have shown that \tilde{u} solves $\tilde{K} \tilde{u} = \tilde{f}$ and thus $Ku = f$.

Uniqueness of the solution follows from the uniqueness of the evolution $U(t, s)$, and the linearity of K, ρ by the standard argument: if u, u' satisfy

$$Ku = Ku' = f \quad \text{and} \quad \rho \tilde{u}(s) = \rho \tilde{u}'(s) = \begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \end{pmatrix},$$

where $\tilde{u}' = \alpha^{-1} u'$, then $K(u - u') = 0$, $\rho(\tilde{u} - \tilde{u}')(s) = 0$ and thus $u = u'$. \square

It is well known that solutions of the Klein–Gordon equation propagate slower than the speed of light. The method of evolution equations together with the freedom of the choice of the time-variable provide a rather obvious heuristic argument for the propagation at a finite speed. However, when one tries to

convert this argument into a rigorous proof, technical problems appear which make such a proof difficult to formulate.

In the literature the finiteness of the speed of propagation is usually shown for the Klein–Gordon equation with smooth coefficients. In [Appendix E](#), in particular in [Theorem E.1](#), we show that solutions of the Klein–Gordon propagate at a finite speed also in a low-regularity setup typical for our paper.

7. Classical propagators

Having constructed the evolution for $B(t)$ in [Section 5](#), it is not difficult to find the classical propagators for the first-order Klein–Gordon operator $\partial_t + iB$. To wit, the *Pauli–Jordan propagator* E^{PJ} and the *forward/backward propagator* $E^{\vee/\wedge}$ are given by the integral kernels

$$E^{\text{PJ}}(t, s) := U(t, s), \tag{7-1a}$$

$$E^{\vee}(t, s) := \theta(t - s) U(t, s), \tag{7-1b}$$

$$E^{\wedge}(t, s) := -\theta(s - t) U(t, s), \tag{7-1c}$$

where θ denotes the Heaviside step function, via

$$(E^{\bullet} f)(t) = \int_{\mathbb{R}} E^{\bullet}(t, s) f(s) \, ds. \tag{7-2}$$

Theorem 7.1. *Let $\lambda \in [-1, 1]$:*

(i) *The classical propagators E^{PJ} and $E^{\vee/\wedge}$ are well-defined between the spaces*

$$E^{\bullet} : L^1_c(\mathbb{R}; \mathcal{H}_\lambda) \rightarrow C(\mathbb{R}; \mathcal{H}_\lambda),$$

$$E^{\bullet} : L^1_c(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow C^1(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(ii) *The forward and backward propagators $E^{\vee/\wedge}$ are well-defined between the spaces*

$$E^{\vee/\wedge} : L^1_{\text{loc}}(I; \mathcal{H}_\lambda) \rightarrow C(I; \mathcal{H}_\lambda),$$

$$E^{\vee/\wedge} : L^1_{\text{loc}}(I; \mathcal{H}_{\text{en}}) \rightarrow C^1(I; \mathcal{H}_{\text{en}}^*),$$

where $I = [a, +\infty[$ or $]-\infty, a]$, respectively, for some $a \in \mathbb{R}$.

(iii) *If [Assumption 2](#) is satisfied, the classical propagators E^{PJ} and $E^{\vee/\wedge}$ are bounded between the spaces*

$$E^{\bullet} : L^1(\mathbb{R}; \mathcal{H}_\lambda) \rightarrow C_b(\mathbb{R}; \mathcal{H}_\lambda),$$

$$E^{\bullet} : L^1(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow C^1_b(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(iv) E^{PJ} *is a bisolution of $\partial_t + iB$:*

$$(\partial_t + iB)E^{\text{PJ}} f = 0, \quad f \in L^1_c(\mathbb{R}; \mathcal{H}_{\text{en}}), \tag{7-3}$$

$$E^{\text{PJ}}(\partial_t + iB) f = 0, \quad f \in L^1_c(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap \text{AC}_c(\mathbb{R}; \mathcal{H}_{\text{en}}^*). \tag{7-4}$$

(v) $E^{\vee/\wedge}$ are the unique inverses of $\partial_t + iB$ such that

$$(\partial_t + iB)E^{\vee/\wedge} f = f, \quad f \in L^1_{\text{loc}}(I, \mathcal{H}_{\text{en}}), \quad (7-5)$$

$$E^{\vee/\wedge}(\partial_t + iB)f = f, \quad f \in L^1_{\text{loc}}(I; \mathcal{H}_{\text{en}}) \cap \text{AC}(I, \mathcal{H}_{\text{en}}^*), \quad (7-6)$$

with $I = [a, +\infty[$ or $]-\infty, a]$, respectively, for some $a \in \mathbb{R}$.

(vi) The relation $E^{\text{PJ}} = E^{\vee} - E^{\wedge}$ holds.

Proof. Parts (i)–(iii) follow from the properties of the evolution $U(t, s)$ (see [Theorem 5.3](#) and [Corollary 5.4](#)) and the definition of the kernels (7-1).

Consider next (iv) and (v). We first need to check that the products contained in these properties are well-defined. Indeed, by (i), the maps

$$E^{\bullet} : L^1_{\text{c}}(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow C(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap C^1(\mathbb{R}; \mathcal{H}_{\text{en}}^*), \quad (7-7a)$$

$$(\partial_t + iB) : C(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap C^1(\mathbb{R}; \mathcal{H}_{\text{en}}^*) \rightarrow C(\mathbb{R}; \mathcal{H}_{\text{en}}^*), \quad (7-7b)$$

are well-defined, which shows that (7-3) and (7-5) make sense. Similarly, by (i), we have

$$(\partial_t + iB) : L^1_{\text{c}}(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap \text{AC}_{\text{c}}(\mathbb{R}; \mathcal{H}_{\text{en}}^*) \rightarrow L^1_{\text{c}}(\mathbb{R}; \mathcal{H}_{\text{en}}^*), \quad (7-8a)$$

$$E^{\bullet} : L^1_{\text{c}}(\mathbb{R}; \mathcal{H}_{\text{en}}^*) \rightarrow C(\mathbb{R}; \mathcal{H}_{\text{en}}^*), \quad (7-8b)$$

and hence the products in (7-4) and (7-6) make sense. Then we show (7-3)–(7-6) using (7-2) and (5-7). For (7-4) and (7-6) we also need to apply an integration by parts. \square

We can also state an L^2 version of [Theorem 7.1\(iii\)](#):

Theorem 7.2. *Let $s > \frac{1}{2}$ and $\lambda \in [-1, 1]$. If [Assumption 2](#) is satisfied, the classical propagators E^{PJ} and $E^{\vee/\wedge}$ are bounded between the spaces*

$$E^{\bullet} : \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{H}_{\lambda}) \rightarrow \langle t \rangle^s L^2(\mathbb{R}; \mathcal{H}_{\lambda}),$$

$$E^{\bullet} : \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow \langle t \rangle^s \langle \partial_t \rangle^{-1} L^2(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

Proof. We use the embeddings

$$\langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{X}) \subset L^1(\mathbb{R}; \mathcal{X}) \quad \text{and} \quad \langle t \rangle^s L^2(\mathbb{R}; \mathcal{X}) \supset C_{\text{b}}(\mathbb{R}; \mathcal{X})$$

for any Banach space \mathcal{X} and $s > \frac{1}{2}$. \square

The classical propagators for the first-order Klein–Gordon operator can also be understood between various spaces other than those considered in [Theorems 7.1](#) and [7.2](#), but our choices are quite natural. At the same time, this setup leads to an almost straightforward derivation of the propagators for the Klein–Gordon operator K .

Since $\partial_t + iB$ and K are related via (6-1), also the propagators of these operators are closely related. At least formally, it can be shown that if E^{\bullet} is a propagator for $\partial_t + iB$, then $i\pi_2 Q E^{\bullet} \iota_2$ is a propagator for \tilde{K} , and hence

$$G^{\bullet} = i\alpha\pi_2 Q E^{\bullet} \iota_2 \alpha \quad (7-9)$$

is a propagator for the Klein–Gordon operator K . As we shall see now, this is indeed true if the domain of G^\bullet is carefully chosen:

Theorem 7.3. *Let $\delta \in [0, 1]$:*

(i) *The classical propagators G^{PJ} and $G^{\vee/\wedge}$ are well-defined between the spaces*

$$G^\bullet : L_c^1(\mathbb{R}; \mathcal{K}^{-\delta}) \rightarrow C(\mathbb{R}; \mathcal{K}^{1-\delta}), \quad (7-10)$$

$$G^\bullet : L_c^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C^1(\mathbb{R}; \mathcal{K}^0). \quad (7-11)$$

(ii) *The forward and backward propagators $G^{\vee/\wedge}$ are well-defined between the spaces*

$$G^{\vee/\wedge} : L_{\text{loc}}^1(I; \mathcal{K}^{-\delta}) \rightarrow C(I; \mathcal{K}^{1-\delta}),$$

$$G^{\vee/\wedge} : L_{\text{loc}}^1(I; \mathcal{K}^0) \rightarrow C^1(I; \mathcal{K}^0),$$

where $I = [a, +\infty[$ or $]-\infty, a]$, respectively, for some $a \in \mathbb{R}$.

(iii) *If Assumption 2 is satisfied, the classical propagators G^{PJ} and $G^{\vee/\wedge}$ are bounded between the spaces*

$$G^\bullet : L^1(\mathbb{R}; \mathcal{K}^{-\delta}) \rightarrow C_b(\mathbb{R}; \mathcal{K}^{1-\delta}),$$

$$G^\bullet : L^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C_b^1(\mathbb{R}; \mathcal{K}^0).$$

(iv) G^{PJ} *is a bisolution of K :*

$$K G^{\text{PJ}} f = 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^0), \quad (7-12)$$

$$G^{\text{PJ}} K f = 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^1) \cap \text{AC}_c(\mathbb{R}; \mathcal{K}^0) \cap \text{AC}_c^1(\mathbb{R}; \mathcal{K}^{-1}). \quad (7-13)$$

(v) $G^{\vee/\wedge}$ *are the unique inverses of K such that*

$$K G^{\vee/\wedge} f = f, \quad f \in L_{\text{loc}}^1(I; \mathcal{K}^0), \quad (7-14)$$

$$G^{\vee/\wedge} K f = f, \quad f \in L_{\text{loc}}^1(I; \mathcal{K}^1) \cap \text{AC}(\mathbb{R}; \mathcal{K}^0) \cap \text{AC}^1(I; \mathcal{K}^{-1}), \quad (7-15)$$

with $I = [a, +\infty[$ or $]-\infty, a]$, respectively, for some $a \in \mathbb{R}$.

(vi) *The relation $G^{\text{PJ}} = G^\vee - G^\wedge$ holds.*

Proof. These results are a direct consequence of [Theorem 7.1](#). In (i)–(iii) we used (6-2) and [Assumption \(1.d\)](#).

Let us check that the products in (iv) and (v) are well-defined. From the definition of ρ we can read off that

$$\rho : C(\mathbb{R}; \mathcal{K}^1) \cap C^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C(\mathbb{R}; \mathcal{H}_{\text{en}}),$$

$$\rho : L_c^1(\mathbb{R}; \mathcal{K}^1) \cap \text{AC}_c(\mathbb{R}; \mathcal{K}^0) \rightarrow L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}),$$

$$\rho : \text{AC}_c(\mathbb{R}; \mathcal{K}^0) \cap \text{AC}_c^1(\mathbb{R}; \mathcal{K}^{-1}) \rightarrow \text{AC}_c(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

Then, by (i) and also using (7-7), we have

$$G^\bullet : L_c^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C(\mathbb{R}; \mathcal{K}^1) \cap C^1(\mathbb{R}; \mathcal{K}^0),$$

$$K : C(\mathbb{R}; \mathcal{K}^1) \cap C^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C^{-1}(\mathbb{R}; \mathcal{K}^0) \cap C(\mathbb{R}; \mathcal{K}^{-1}),$$

where $C^{-1}(\mathbb{R})$ denotes the space of distributional derivatives of continuous functions. This shows that (7-12) and (7-14) make sense. Similarly, by (i) and (7-8), we have

$$K : L_c^1(\mathbb{R}; \mathcal{K}^1) \cap AC_c(\mathbb{R}; \mathcal{K}^0) \cap AC_c^1(\mathbb{R}; \mathcal{K}^{-1}) \rightarrow L_c^1(\mathbb{R}; \mathcal{K}^{-1}),$$

$$G^\bullet : L_c^1(\mathbb{R}; \mathcal{K}^{-1}) \rightarrow C(\mathbb{R}; \mathcal{K}^0),$$

and hence the products in (7-13) and (7-15) make sense. \square

Here is an L^2 version of [Theorem 7.3\(iii\)](#):

Theorem 7.4. *Let $s > \frac{1}{2}$. If [Assumption 2](#) is satisfied, the classical propagators G^{PJ} and $G^{\vee/\wedge}$ are bounded between the spaces*

$$G^\bullet : \langle t \rangle^{-s} L^2(\Omega^{1/2} M) \rightarrow \langle t \rangle^s L(t)^{-1/2} L^2(\Omega^{1/2} M), \quad (7-16a)$$

$$G^\bullet : \langle t \rangle^{-s} L^2(\Omega^{1/2} M) \rightarrow \langle t \rangle^s \langle \partial_t \rangle^{-1} L^2(\Omega^{1/2} M). \quad (7-16b)$$

Proof. By (7-10), for $\delta \in [0, 1]$ we have

$$G^\bullet : \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{K}^{-\delta}) \rightarrow \langle t \rangle^s L^2(\mathbb{R}; \mathcal{K}^{1-\delta}). \quad (7-17)$$

Setting $\delta = 0$ we obtain

$$G^\bullet : \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{K}^0) \rightarrow \langle t \rangle^s L(t)^{-1/2} L^2(\mathbb{R}; \mathcal{K}^0). \quad (7-18)$$

But $L^2(\mathbb{R}; \mathcal{K}^0) = L^2(\mathbb{R}; L^2(\Omega^{1/2} \Sigma))$ and $L^2(\Omega^{1/2} M)$ can naturally be identified, which proves (7-16a).

It follows from (7-11) that

$$G^\bullet : \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{K}^0) \rightarrow \langle t \rangle^s \langle \partial_t \rangle^{-1} L^2(\mathbb{R}; \mathcal{K}^0).$$

This yields (7-16b). \square

Observe that in other approaches, e.g., [[Bär et al. 2007](#)], the retarded and advanced propagators are the central objects and the Pauli–Jordan propagator is defined as their difference. Here, instead, the Pauli–Jordan propagator follows immediately from the evolution $U(t, s)$ and should be seen as the central object, while the retarded and advanced propagators are derived objects.

Using the Pauli–Jordan propagator G^{PJ} , we can associate to every sufficiently regular compactly supported function a solution of the homogeneous Klein–Gordon equation. In fact, as the following proposition shows, also the converse is true.

Proposition 7.5. *Suppose that $u \in L_{\text{loc}}^1(\mathbb{R}; \mathcal{K}^1) \cap AC(\mathbb{R}; \mathcal{K}) \cap AC^1(\mathbb{R}; \mathcal{K}^{-1})$ satisfies $Ku = 0$. Then there exists a (nonunique) $f \in L_c^1(\mathbb{R}; \mathcal{K}^{-1})$ such that $u = G^{\text{PJ}} f$.*

Proof. Choose any $r, s \in \mathbb{R}$, $r < s$, and $\chi \in C^\infty(M)$ such that $\chi(t) = 0$ for $t < r$, $0 \leq \chi(t) \leq 1$ for $r \leq t \leq s$ and $\chi(t) = 1$ for $t > s$. Clearly,

$$0 = Ku = K\chi u - K(\chi - 1)u$$

and thus $\text{supp}(K\chi u) \subset [r, s] \times \Sigma$. Besides, $K\chi u \in L_c^1(\mathbb{R}; \mathcal{K}^{-1})$. Therefore, we can act with G^{PJ} on $K\chi u$, obtaining

$$G^{\text{PJ}} K\chi u = G^\vee K\chi u - G^\wedge K(\chi - 1)u = u.$$

That is, $f = K\chi u$ is the desired function. \square

Our construction of the classical propagators starts from the propagators for the first-order Klein–Gordon operator; i.e., given E^\bullet , we derive G^\bullet using (7-9). If, instead, G^\bullet is provided, then E^\bullet can be derived:

(i) If G^\bullet is an inverse of K then

$$E^\bullet = -i \begin{pmatrix} -\alpha^{-1}G^\bullet\alpha^{-1}(D_t+W^*) & \alpha^{-1}G^\bullet\alpha^{-1} \\ \mathbb{1}+(D_t+W)\alpha^{-1}G^\bullet\alpha^{-1}(D_t+W^*) & -(D_t+W)\alpha^{-1}G^\bullet\alpha^{-1} \end{pmatrix}$$

is (formally) an inverse of $(\partial_t + iB)$.

(ii) If G^\bullet is a bisolution of K then

$$E^\bullet = -i \begin{pmatrix} -\alpha^{-1}G^\bullet\alpha^{-1}(D_t+W^*) & \alpha^{-1}G^\bullet\alpha^{-1} \\ (D_t+W)\alpha^{-1}G^\bullet\alpha^{-1}(D_t+W^*) & -(D_t+W)\alpha^{-1}G^\bullet\alpha^{-1} \end{pmatrix}$$

is (formally) a bisolution of $(\partial_t + iB)$.

Note the subtle difference in the formulas for inverses and bisolutions. No such difference appears in (7-9), which yields G^\bullet given E^\bullet .

8. Instantaneous nonclassical propagators

Consider an arbitrary reference time τ . According to Proposition 4.2, $B(\tau)$ is a self-adjoint operator on $\mathcal{H}_{\text{en},\tau}^*$. Therefore we can use the functional calculus to define the projections onto the positive and negative parts of the spectrum of $B(\tau)$:

$$\Pi_\tau^{(\pm)} := \mathbb{1}_{[0,\infty[}(\pm B(\tau)). \quad (8-1)$$

Zero is in the resolvent set of $B(\tau)$, and therefore the projections in (8-1) are complementary.

Proposition 8.1. $\Pi_\tau^{(\pm)}$ restrict to complementary projections on \mathcal{H}_λ for $\lambda \in [-1, 1]$, and have the following properties:

- (i) $\Pi_\tau^{(\pm)} B(\tau) = B(\tau) \Pi_\tau^{(\pm)}$.
- (ii) $\Pi_\tau^{(+)} - \Pi_\tau^{(-)} = \text{sgn } B(\tau)$.
- (iii) $\text{sp}(\pm \Pi_\tau^{(\pm)} B(\tau)) \subset]0, \infty[$.
- (iv) $\Pi_\tau^{(\pm)}$ are self-adjoint with respect to $\mathcal{H}_{\lambda,\tau}$.

Moreover, the projections $\Pi_\tau^{(\pm)}$ split $\mathcal{H}_{\lambda,\tau}$ into subspaces of positive and negative charge (with respect to the charge form Q):

$$\mathbf{Proposition 8.2.} \quad \pm(u \mid Q \Pi_\tau^{(\pm)} u) = \pm(\Pi_\tau^{(\pm)} u \mid Qu) = \pm(\Pi_\tau^{(\pm)} u \mid Q \Pi_\tau^{(\pm)} u) \geq 0 \quad (8-2)$$

for all $u \in \mathcal{H}_\lambda$ with $\lambda \in [-1, 1]$.

Proof. The proof is the same as that of [Dereziński and Siemssen 2018, Proposition 6.3]. □

The projections $\Pi_\tau^{(\pm)}$ can be used to define *instantaneous positive/negative-frequency bisolutions* $E_\tau^{(\pm)}$, given by their integral kernels as

$$E_\tau^{(\pm)}(t, s) := \pm U(t, \tau) \Pi_\tau^{(\pm)} U(\tau, s). \quad (8-3)$$

Using step functions, we then define the kernels of the *instantaneous Feynman and anti-Feynman inverses* of $\partial_t + \mathbf{i}B$:

$$E_\tau^{\text{F}}(t, s) := \theta(t - s) E_\tau^{(+)}(t, s) + \theta(s - t) E_\tau^{(-)}(t, s),$$

$$E_\tau^{\bar{\text{F}}}(t, s) := -\theta(t - s) E_\tau^{(-)}(t, s) - \theta(s - t) E_\tau^{(+)}(t, s).$$

It is easy to see that these kernels can also be expressed using the retarded and advanced propagators:

$$E_\tau^{\text{F}}(t, s) = E^\wedge(t, s) + E_\tau^{(+)}(t, s) = E^\vee(t, s) + E_\tau^{(-)}(t, s), \quad (8-4a)$$

$$E_\tau^{\bar{\text{F}}}(t, s) = E^\vee(t, s) - E_\tau^{(+)}(t, s) = E^\wedge(t, s) - E_\tau^{(-)}(t, s). \quad (8-4b)$$

As before, these kernels define the corresponding propagators via (7-2):

Theorem 8.3. *Let $\lambda \in [-1, 1]$:*

(i) *The instantaneous nonclassical propagators $E_\tau^{(\pm)}$ and $E_\tau^{\text{F}/\bar{\text{F}}}$ are well-defined between the spaces*

$$E_\tau^\bullet : L_c^1(\mathbb{R}; \mathcal{H}_\lambda) \rightarrow C(\mathbb{R}; \mathcal{H}_\lambda),$$

$$E_\tau^\bullet : L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow C^1(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(ii) *If Assumption 2 is satisfied, $E_\tau^{(\pm)}$ and $E_\tau^{\text{F}/\bar{\text{F}}}$ are bounded between the spaces*

$$E_\tau^\bullet : L^1(\mathbb{R}; \mathcal{H}_\lambda) \rightarrow C_b(\mathbb{R}; \mathcal{H}_\lambda),$$

$$E_\tau^\bullet : L^1(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow C_b^1(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(iii) *$E_\tau^{(\pm)}$ are bisolutions of $\partial_t + \mathbf{i}B$:*

$$(\partial_t + \mathbf{i}B) E_\tau^{(\pm)} f = 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}),$$

$$E_\tau^{(\pm)} (\partial_t + \mathbf{i}B) f = 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap \text{AC}_c(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(iv) *$E_\tau^{\text{F}/\bar{\text{F}}}$ are inverses of $\partial_t + \mathbf{i}B$:*

$$(\partial_t + \mathbf{i}B) E_\tau^{\text{F}/\bar{\text{F}}} f = f, \quad f \in L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}),$$

$$E_\tau^{\text{F}/\bar{\text{F}}} (\partial_t + \mathbf{i}B) f = f, \quad f \in L_c^1(\mathbb{R}; \mathcal{H}_{\text{en}}) \cap \text{AC}_c(\mathbb{R}; \mathcal{H}_{\text{en}}^*).$$

(v) *The instantaneous nonclassical propagators satisfy the relations*

$$E_\tau^{\text{F}} = E^\wedge + E_\tau^{(+)} = E^\vee + E_\tau^{(-)}, \quad E_\tau^{\text{F}} + E_\tau^{\bar{\text{F}}} = E^\vee + E^\wedge, \quad E_\tau^{(+)} - E_\tau^{(-)} = E^{\text{PJ}},$$

$$E_\tau^{\bar{\text{F}}} = E^\vee - E_\tau^{(+)} = E^\wedge - E_\tau^{(-)}, \quad E_\tau^{\text{F}} - E_\tau^{\bar{\text{F}}} = E_\tau^{(+)} + E_\tau^{(-)}.$$

Proof. The various properties of the nonclassical propagators can be shown along the same lines as in Theorem 7.1 so we will omit the proofs. Property (v) in particular follows from (8-4) and its linear combinations. \square

As for the classical propagators, we can also find an L^2 version of [Theorem 8.3\(ii\)](#):

Theorem 8.4. *Let $s > \frac{1}{2}$ and $\lambda \in [-1, 1]$. If [Assumption 2](#) is satisfied, the instantaneous nonclassical propagators $E_\tau^{(\pm)}$ and $E_\tau^{F/\bar{F}}$ are bounded between the spaces*

$$\begin{aligned} E_\tau^\bullet &: \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{H}_\lambda) \rightarrow \langle t \rangle^s L^2(\mathbb{R}; \mathcal{H}_\lambda), \\ E_\tau^\bullet &: \langle t \rangle^{-s} L^2(\mathbb{R}; \mathcal{H}_{\text{en}}) \rightarrow \langle t \rangle^s \langle \partial_t \rangle^{-1} L^2(\mathbb{R}; \mathcal{H}_{\text{en}}^*). \end{aligned}$$

Similarly to (7-9), we define the instantaneous nonclassical propagators $G_\tau^{(\pm)}$ and $G_\tau^{F/\bar{F}}$ for the Klein-Gordon operator K by

$$G_\tau^{(\pm)} := \alpha \pi_2 Q E_\tau^{(\pm)} \iota_2 \alpha, \quad G_\tau^{F/\bar{F}} := i \alpha \pi_2 Q E_\tau^{F/\bar{F}} \iota_2 \alpha.$$

Note the absence of the complex unit in the definition of $G_\tau^{(\pm)}$ so that $G_\tau^{(\pm)}$ define positive forms; see property (vi) below.

Analogously to [Theorem 7.3](#), we find:

Theorem 8.5. *Let $\delta \in [-1, 1]$:*

(i) *The instantaneous nonclassical propagators $G_\tau^{(\pm)}$ and $G_\tau^{F/\bar{F}}$ are well-defined between the spaces*

$$\begin{aligned} G_\tau^\bullet &: L_c^1(\mathbb{R}; \mathcal{K}^{-\delta}) \rightarrow C(\mathbb{R}; \mathcal{K}^{1-\delta}), \\ G_\tau^\bullet &: L_c^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C^1(\mathbb{R}; \mathcal{K}^0). \end{aligned}$$

(ii) *If [Assumption 2](#) is satisfied, $G_\tau^{(\pm)}$ and $G_\tau^{F/\bar{F}}$ are bounded between the spaces*

$$\begin{aligned} G_\tau^\bullet &: L^1(\mathbb{R}; \mathcal{K}^{-\delta}) \rightarrow C_b(\mathbb{R}; \mathcal{K}^{1-\delta}), \\ G_\tau^\bullet &: L^1(\mathbb{R}; \mathcal{K}^0) \rightarrow C_b^1(\mathbb{R}; \mathcal{K}^0). \end{aligned}$$

(iii) *$G_\tau^{(\pm)}$ are bisolutions of K :*

$$\begin{aligned} K G_\tau^{(\pm)} f &= 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^0), \\ G_\tau^{(\pm)} K f &= 0, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^1) \cap \text{AC}_c(\mathbb{R}; \mathcal{K}^0) \cap \text{AC}_c^1(\mathbb{R}; \mathcal{K}^{-1}). \end{aligned}$$

(iv) *$G_\tau^{F/\bar{F}}$ are inverses of K :*

$$\begin{aligned} K G_\tau^{F/\bar{F}} f &= f, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^0), \\ G_\tau^{F/\bar{F}} K f &= f, \quad f \in L_c^1(\mathbb{R}; \mathcal{K}^1) \cap \text{AC}_c(\mathbb{R}; \mathcal{K}^0) \cap \text{AC}_c^1(\mathbb{R}; \mathcal{K}^{-1}). \end{aligned}$$

(v) *The instantaneous nonclassical propagators satisfy the relations:*

$$\begin{aligned} G_\tau^F &= G^\wedge + i G_\tau^{(+)} = G^\vee + i G_\tau^{(-)}, & G_\tau^F + G_\tau^{\bar{F}} &= G^\vee + G^\wedge, & G_\tau^{(+)} - G_\tau^{(-)} &= -i G^{\text{PJ}}, \\ G_\tau^{\bar{F}} &= G^\vee - i G_\tau^{(+)} = G^\wedge - i G_\tau^{(-)}, & G_\tau^F - G_\tau^{\bar{F}} &= i G_\tau^{(+)} + i G_\tau^{(-)}. \end{aligned}$$

(vi) *The instantaneous positive/negative-frequency bisolutions are positive:*

$$(f | G_\tau^{(\pm)} f) = \int_M \bar{f} G_\tau^{(\pm)} f \geq 0$$

for $f \in L_c^1(\mathbb{R}; \mathcal{K}^0)$.

Proof. We only show (vi); the remaining properties follow from corresponding properties of E_τ^\bullet in Theorem 8.3 and can be shown as in Theorem 7.3. For (vi), we note that

$$\begin{aligned} (f | G_\tau^{(\pm)} f) &= \iint (\iota_2 \tilde{f}(t) | Q E_\tau^{(\pm)}(t, s) \iota_2 \tilde{f}(s)) \, ds \, dt \\ &= (\tilde{u}(\tau) | Q \Pi_\tau^{(\pm)} \tilde{u}(\tau)) \geq 0 \end{aligned}$$

by Proposition 8.2, where we set

$$\tilde{f} = \alpha f \quad \text{and} \quad \tilde{u}(\tau) = \int U(\tau, t) \tilde{f}(t) \, dt \in \mathcal{H}_{\text{en}}. \quad \square$$

The L^2 version of Theorem 8.5(ii) is:

Theorem 8.6. *Let $s > \frac{1}{2}$. If Assumption 2 is satisfied, the instantaneous nonclassical propagators $G_\tau^{(\pm)}$ and $G_\tau^{\text{F}/\text{F}}$ are bounded between the spaces*

$$\begin{aligned} G_\tau^\bullet &: \langle t \rangle^{-s} L^2(\Omega^{1/2} M) \rightarrow \langle t \rangle^s L(t)^{-1/2} L^2(\Omega^{1/2} M), \\ G_\tau^\bullet &: \langle t \rangle^{-s} L^2(\Omega^{1/2} M) \rightarrow \langle t \rangle^s \langle \partial_t \rangle^{-1} L^2(\Omega^{1/2} M). \end{aligned}$$

In the static case, the nonclassical propagators defined above do not depend on τ . They are the natural propagators to consider in that situation; see also our earlier work [Dereziński and Siemssen 2018].

In the nonstatic case, however, the instantaneous nonclassical propagators just defined have deficiencies from the physical point of view; see, e.g., [Fulling 1979]. First of all, their definition hinges on the arbitrary choice of a fixed instance of time and, even more seriously, on the choice of a time function. Secondly, instantaneous positive-frequency bisolutions usually do not satisfy the microlocal spectrum condition of [Radzikowski 1996] (in other words, they do not define Hadamard states).

Nevertheless, the situation improves if the Klein–Gordon operator is infinitesimally static at the time when the positive/negative-frequency splitting is performed. In a forthcoming article we will show (using methods of evolution equations) that the corresponding instantaneous positive-frequency bisolutions, which we define in the following section, satisfy then the microlocal spectrum condition of [Radzikowski 1996].

9. Asymptotic nonclassical propagators

Throughout this section we assume that Assumption 2 is satisfied. It follows, in particular, that $B(t)$ converges to $B(\pm\infty)$ as $t \rightarrow \pm\infty$ in norm as an operator from \mathcal{H}_{en} to $\mathcal{H}_{\text{en}}^*$. We define the *out/in-positive/negative-frequency projections*

$$\begin{aligned} \Pi_+^{(\pm)} &:= \mathbb{1}_{[0, \infty[}(\pm B(+\infty)), \\ \Pi_-^{(\pm)} &:= \mathbb{1}_{[0, \infty[}(\pm B(-\infty)). \end{aligned}$$

Theorem 9.1. *The strong limits*

$$\Pi_+^{(\pm)}(t) := \text{s-lim}_{\tau \rightarrow +\infty} U(t, \tau) \Pi_+^{(\pm)} U(\tau, t), \quad (9-1a)$$

$$\Pi_-^{(\pm)}(t) := \text{s-lim}_{\tau \rightarrow -\infty} U(t, \tau) \Pi_-^{(\pm)} U(\tau, t) \quad (9-1b)$$

exist as bounded operators on \mathcal{H}_λ with $\lambda \in [-1, 1]$. They satisfy the obvious analogs of Propositions 8.1 and 8.2. Additionally,

$$U(s, t)\Pi_+^{(\pm)}(t)U(t, s) = \Pi_+^{(\pm)}(s), \tag{9-2}$$

$$U(s, t)\Pi_-^{(\pm)}(t)U(t, s) = \Pi_-^{(\pm)}(s). \tag{9-3}$$

Proof. We only prove the theorem for (9-1a) because the proof for (9-1b) is the same. We have

$$U(t, r)\Pi_+^{(\pm)}U(r, t) = U(t, r)e^{i(t-r)B(+\infty)}\Pi_+^{(\pm)}e^{i(r-t)B(+\infty)}U(r, t).$$

We analyze separately the limit $r \rightarrow +\infty$ of the operators left and right of the projection. Since both operators are bounded on $\mathcal{H}_{\lambda, \tau}$, $\lambda \in [-1, 1]$, uniformly in t, r for arbitrary $\tau \in \mathbb{R}$, it is sufficient to show the convergence on \mathcal{H}_{en} with respect to the norm on $\mathcal{H}_{\text{en}, \tau}^*$.

We may assume that $r > t$. For $u \in \mathcal{H}_{\text{en}}$ we have

$$\begin{aligned} U(t, r)e^{i(t-r)B(+\infty)}u &= u + \int_t^r \partial_s(U(t, s)e^{i(t-s)B(+\infty)})u \, ds \\ &= u - i \int_t^r U(t, s)(B(s) - B(+\infty))e^{i(t-s)B(+\infty)}u \, ds, \end{aligned}$$

by the fundamental theorem of calculus and Theorem 5.3(iii). Taking the norm of this expression in $\mathcal{H}_{\text{en}, \tau}^*$, we find

$$\|U(t, r)e^{i(t-r)B(+\infty)}u - u\|_{\text{en}^*, \tau} \leq C\|u\|_{\text{en}, \tau} \int_t^r \|(\mathbb{1} \oplus L(\tau)^{-1/2})(B(s) - B(+\infty))(L(\tau)^{-1/2} \oplus \mathbb{1})\| \, ds,$$

since $U(t, s)$ is uniformly bounded on $\mathcal{H}_{\text{en}, \tau}^*$.

It follows from the proof of Proposition 5.2 that

$$\|(\mathbb{1} \oplus L(\tau)^{-1/2})(B(s) - B(+\infty))(L(\tau)^{-1/2} \oplus \mathbb{1})\|$$

is uniformly bounded. Therefore,

$$\|U(t, r)e^{i(t-r)B(+\infty)}u - u\|_{\text{en}^*, \tau} \rightarrow 0$$

as $t, r \rightarrow +\infty$ and the desired convergence follows.

The proof for $U(t, r)e^{i(t-r)B(+\infty)}$ is essentially the same. The main difference is that we use the uniform boundedness of $U(t, s)$ on $\mathcal{H}_{\text{en}, \tau}$. □

We also define

$$E_+^{(\pm)}(t, s) := \pm U(t, \tau)\Pi_+^{(\pm)}(\tau)U(\tau, s), \tag{9-4}$$

$$E_-^{(\pm)}(t, s) := \pm U(t, \tau)\Pi_-^{(\pm)}(\tau)U(\tau, s). \tag{9-5}$$

Clearly, the definitions above do not depend on τ .

The kernels $E_\pm^{(\pm)}(t, s)$ yield the *positive/negative-frequency bisolutions at future and past infinity*. They are often called *out* and *in*, or jointly *asymptotic*. Moreover, we may use them together with the advanced

and retarded propagators to define corresponding *asymptotic Feynman* and *anti-Feynman propagators*:

$$\begin{aligned} E_{\pm}^{\text{F}} &= E^{\wedge} + E_{\pm}^{(+)} = E^{\vee} + E_{\pm}^{(-)}, \\ E_{\pm}^{\overline{\text{F}}} &= E^{\vee} - E_{\pm}^{(+)} = E^{\wedge} - E_{\pm}^{(-)}. \end{aligned}$$

As before, the propagators E_{\pm}^{\bullet} for $\partial_t + iB$ induce the corresponding propagators G_{\pm}^{\bullet} for K . Obviously, the asymptotic nonclassical propagators defined here have analogs to Theorems 8.3 and 8.5; we only have to replace occurrences of τ with \pm .

The asymptotic propagators defined above have various advantages over the instantaneous ones of the previous section. For one, they do not depend on an arbitrarily chosen instant of time. Under rather broad assumptions one can show that they even do not depend on the choice of the time function, but only on the spacetime itself. Finally, as recently discussed in [Gérard and Wrochna 2017], if the spacetime becomes asymptotically static sufficiently fast, they satisfy the microlocal spectrum condition of [Radzikowski 1996].

Appendix A: Second-order differential operators

Consider a manifold Σ . Every second-order Hermitian differential operator on $L^2(\Omega^{1/2}\Sigma)$ can locally be written as

$$L = D_i g^{ij}(x) D_j - A^i(x) D_i - D_i A^i(x) + Y_0(x), \quad (\text{A-1})$$

where $g^{ij} = g^{ji}$, Y_0 and A^i are real-valued.

L can be often rewritten in the form

$$L = (D_i - A_i) g^{ij} (D_j - A_j) + Y_1. \quad (\text{A-2})$$

This is possible in particular if g^{ij} is everywhere nondegenerate, viz., g determines a (pseudo-)Riemannian structure on M . Then (A-2) holds with

$$A_i := g_{ij} A^j, \quad Y_1 := Y_0 - A^i g_{ij} A^j,$$

where g_{ij} denotes the inverse of g^{ij} .

Let γ be an everywhere nonzero function. Then the operator L can be rewritten as

$$L = \gamma^{-1/2} (D_i - A_i) \gamma g^{ij} (D_j - A_j) \gamma^{-1/2} + Y_{\gamma}, \quad (\text{A-3})$$

where

$$Y_{\gamma} := Y - \frac{1}{2} (D_i g^{ij} \gamma^{-1} (D_j \gamma)) - \frac{1}{4} g^{ij} \gamma^{-2} (D_i \gamma) (D_j \gamma).$$

In particular, if we set $\gamma := |g|^{1/2}$, where $|g| := |\det[g_{ij}]|$ is the canonical density induced by the metric, and $Y := Y_{|g|^{1/2}}$, then (A-3) yields the geometric form of the operator L :

$$L = |g|^{-1/4} (D_i - A_i) |g|^{1/2} g^{ij} (D_j - A_j) |g|^{-1/4} + Y. \quad (\text{A-4})$$

If g is a metric tensor, A a 1-form, and Y a scalar, then the right-hand side of (A-4) transforms covariantly and L is well-defined as a differential operator acting on half-densities. We can rewrite (A-4) using the

Levi-Civita derivative ∇ for g as

$$L = g^{ij} (i\nabla_i + A_i)(i\nabla_j + A_j) + Y. \tag{A-5}$$

Note that in (A-5) the right ∇ acts on half-densities and the left ∇ acts on half-densitized covectors.

If the metric is Riemannian, the differential part of the operator (A-4) can be called a (*magnetic*) *Laplace–Beltrami operator*, and the full operator can be called a (*magnetic*) *Schrödinger operator*. If the metric is Lorentzian, the differential part of the operator (A-4) can be called an (*electromagnetic*) *d’Alembertian*, and the full operator can be called an (*electromagnetic*) *Klein–Gordon operator*.

It is however sometimes convenient to consider a density γ independent of the metric tensor g , i.e., to work with (A-3) instead of (A-4). Using the derivative

$$D^{A,\gamma} := \gamma^{1/2}(D - A)\gamma^{-1/2}, \tag{A-6}$$

L can be written as a quadratic form on half-densities:

$$(u | Lv) = \int_{\Sigma} ((\overline{D_i^{A,\gamma} u}) g^{ij} (D_j^{A,\gamma} v) + \bar{u} Y_{\gamma} v). \tag{A-7}$$

Assumption 3. In the remaining part of this appendix we assume that g is a Riemannian metric. We also assume that $\gamma^{-1} \partial_i \gamma$, $A_i \in L^2_{\text{loc}}(\Sigma)$, $g^{ij} \in L^{\infty}_{\text{loc}}(\Sigma)$ and $Y_{\gamma} \in L^1_{\text{loc}}(\Sigma)$ such that $Y_{\gamma} \geq C$ for some $C \in \mathbb{R}$.

We will see that under the above assumption L can be understood as a self-adjoint operator on $L^2(\Omega^{1/2}\Sigma)$ in at least two natural ways. First we reinterpret (A-7) by introducing the form

$$l_{\text{mx}}[u, v] = \int_{\Sigma} ((\overline{D_i^{A,\gamma} u}) g^{ij} (D_j^{A,\gamma} v) + \bar{u} Y_{\gamma} v) \tag{A-8}$$

on its maximal form domain

$$\text{dom } l_{\text{mx}} = \{u \in L^2(\Omega^{1/2}\Sigma) \mid D^{A,\gamma} u \in L^2(\Omega^{1/2}T^*\Sigma, g), Y_{\gamma}^{1/2} u \in L^2(\Omega^{1/2}\Sigma)\}.$$

Here we denote by $L^2(\Omega^{1/2}T^*\Sigma, g)$ the completion of $C^{\infty}_c(\Omega^{1/2}T^*\Sigma)$ with respect to the norm given by

$$u \mapsto \left(\int_{\Sigma} \bar{u}_i g^{ij} u_j \right)^{1/2}.$$

We remark that $C^{\infty}_c(\Omega^{1/2}\Sigma) \subset \text{dom } l_{\text{mx}}$.

The following is a standard proof and has been adapted from [Leinfelder and Simader 1981, Lemma 1].

Lemma A.1. *The form l_{mx} is closed and Hermitian. It defines a unique self-adjoint operator L_{mx} on*

$$\text{Dom } L_{\text{mx}} = \{v \in \text{dom } l_{\text{mx}} \mid |l_{\text{mx}}[u, v]| \leq C_v \|u\| \text{ for all } u \in L^2(\Omega^{1/2}\Sigma)\}$$

satisfying

$$(u | L_{\text{mx}} v) = l_{\text{mx}}[u, v]$$

for $u \in \text{dom } l_{\text{mx}}$ and $v \in \text{Dom } L_{\text{mx}}$. Moreover, $\text{dom } l_{\text{mx}} = \text{Dom } L_{\text{mx}}^{1/2}$.

Proof. Suppose that $\{u_n\} \subset \text{dom } l_{\text{mx}}$ is a Cauchy sequence with respect to the norm

$$\text{dom } l_{\text{mx}} \ni u \mapsto (l_{\text{mx}}[u, u] + (1 - C)\|u\|^2)^{1/2}.$$

Then there exist $u, v \in L^2(\Omega^{1/2}\Sigma)$ and $w \in L^2(\Omega^{1/2}T^*\Sigma, g)$ such that

$$u_n \rightarrow u, \quad Y_\gamma^{1/2}u_n \rightarrow v \quad \text{in } L^2(\Omega^{1/2}\Sigma)$$

and

$$D^{A,\gamma}u_n \rightarrow w \quad \text{in } L^2(\Omega^{1/2}T^*\Sigma, g).$$

Moreover, $Y_\gamma^{1/2}u_n \rightarrow Y_\gamma^{1/2}u$ and $D^{A,\gamma}u_n \rightarrow D^{A,\gamma}u$ weakly, and thus $v = Y_\gamma^{1/2}u$ and $w = D^{A,\gamma}u$ because v, w must coincide with the weak limits. It follows that l_{mx} is a closed form (and manifestly Hermitian). Therefore, by the first representation theorem [Kato 1966, Theorem VI.2.6], l_{mx} defines a unique self-adjoint operator with the stated properties. \square

An alternative to l_{mx} is the form l_{mn} given by the completion of the form (A-8) on $C_c^\infty(\Omega^{1/2}\Sigma)$, and the corresponding operator L_{mn} . l_{mn} may have a strictly smaller domain than l_{mx} because of boundary effects. If $l_{\text{mn}} = l_{\text{mx}}$, then $C_c^\infty(\Omega^{1/2}\Sigma)$ is a core of l_{mx} . Note that for $\Sigma = \mathbb{R}^3$ with the Euclidean metric this is known to be true; see, e.g., [Leinfelder and Simader 1981].

Certainly the setting considered in this appendix is not the most general possible. For example, the assumption that Y is bounded from below can certainly be relaxed.

Appendix B: Concrete assumptions

The objective of this appendix is to elucidate how Assumption (1.c) may be realized in practice. Recall that $(\Sigma, \tilde{g}_\Sigma(t))$ is a family of Riemannian manifolds, $\gamma(t) > 0$ are densities on Σ , $A(t)$ are real-valued 1-forms and $\tilde{Y}(t)$ are real-valued scalar potentials. For simplicity, we write \tilde{g} for \tilde{g}_Σ . As in Assumption 3 in Appendix A, we assume that $\gamma^{-1}(t)\partial_i\gamma(t)$, $A_i(t) \in L_{\text{loc}}^2(\Sigma)$, $\tilde{g}^{ij} \in L_{\text{loc}}^\infty(\Sigma)$, and $\tilde{Y} \in L_{\text{loc}}^1(\Sigma)$ is bounded from below.

Let us recall the definition of the operators $W(t)$ and $L(t)$ on $L^2(\Omega^{1/2}\Sigma)$:

$$\begin{aligned} W(t) &:= \beta(t)^i D_i + V(t) - \frac{1}{2}\gamma(t)^{-1}(D_t\gamma(t) - \beta(t)^i D_i\gamma(t)), \\ (u | L(t) v) &:= \int_\Sigma \left(\overline{(D_i^{A,\gamma}(t)u)} \tilde{g}^{ij}(t) (D_j^{A,\gamma}(t)v) + \bar{u} \tilde{Y}(t)v \right), \end{aligned} \quad (\text{B-1})$$

where $L(t)$ is interpreted, say, as the maximal operator given by (B-1), as in Appendix A. Assumption (1.c) now says that there exists a positive $C \in L_{\text{loc}}^1(\mathbb{R})$ such that for all $|t - s| \leq 1$

$$\|L(t)^{-1/2}(L(t) - L(s))L(t)^{-1/2}\| + 2\|(W(t) - W(s))L(t)^{-1/2}\| \leq \left| \int_s^t C(r) dr \right| \quad (\text{B-2})$$

for some $C \in L_{\text{loc}}^1(\mathbb{R})$.

We also introduce the family of norms

$$\|X\|_t = \left(\int_\Sigma \tilde{g}^{ij}(t) \bar{X}_i X_j \right)^{1/2}$$

for half-densitized 1-forms X on Σ .

Proposition B.1. *Suppose that there are positive $C_Y, C_g, C_W \in L^1_{\text{loc}}(\mathbb{R})$, $C_A, C_\gamma \in L^2_{\text{loc}}(\mathbb{R})$ such that for all $|t - s| \leq 1$*

$$\begin{aligned} \|L(t)^{-1/2} \partial_s \tilde{Y}(s) L(t)^{-1/2}\| &\leq C_Y(s), \\ \|\partial_s W(s) L(t)^{-1/2}\| &\leq C_W(s), \\ \|\partial_s A(s) L(t)^{-1/2}\|_t &\leq C_A(s), \\ \|\partial_s \gamma(s)^{-1} d\gamma(s) L(t)^{-1/2}\|_t &\leq C_\gamma(s), \\ |\partial_s \tilde{g}^{ij}(s) X_i X_j| &\leq C_g(s) \tilde{g}^{ij}(t) X_i X_j, \quad X \in C(T^*\Sigma). \end{aligned}$$

Then (B-2) holds and thus Assumption (1.c) is true.

Proof. To avoid notational clutter within this proof, we simply write D_i for $D_i^{A,\gamma}$. Clearly, the assumptions of the proposition imply

$$\|L(t)^{-1/2} (\tilde{Y}(t) - \tilde{Y}(s)) L(t)^{-1/2}\| \leq \left| \int_s^t C_Y(r) dr \right|, \quad (\text{B-3a})$$

$$\|(W(t) - W(s)) L(t)^{-1/2}\| \leq \left| \int_s^t C_W(r) dr \right|, \quad (\text{B-3b})$$

$$\|(A(t) - A(s)) L(t)^{-1/2}\|_t \leq \left| \int_s^t C_A(r) dr \right|, \quad (\text{B-3c})$$

$$\|(\gamma(t)^{-1} d\gamma(t) - \gamma(s)^{-1} d\gamma(s)) L(t)^{-1/2}\|_t \leq \left| \int_s^t C_\gamma(r) dr \right|, \quad (\text{B-3d})$$

$$|\tilde{g}^{ij}(t) X_i X_j - \tilde{g}^{ij}(s) X_i X_j| \leq \left| \int_s^t C_g(r) dr \right| \tilde{g}^{ij}(t) X_i X_j. \quad (\text{B-3e})$$

We compute

$$\begin{aligned} &(u | (L(t) - L(s))u) \\ &= \int_\Sigma \tilde{g}^{ij}(t) ((\overline{D_i(t)u})(D_j(t)u - D_j(s)u) + (\overline{D_i(t)u - D_i(s)u})(D_j(t)u) \\ &\quad - (\overline{D_i(t)u - D_i(s)u})(D_j(t)u - D_j(s)u)) \\ &+ \int_\Sigma (\tilde{g}^{ij}(t) - \tilde{g}^{ij}(s)) ((\overline{D_i(t)u})(D_j(t)u) - (\overline{D_i(t)u})(D_j(t)u - D_j(s)u) \\ &\quad - (\overline{D_i(t)u - D_i(s)u})(D_j(t)u) + (\overline{D_i(t)u - D_i(s)u})(D_j(t)u - D_j(s)u)) \\ &+ \int_\Sigma (\tilde{Y}(t) - \tilde{Y}(s)) |u|^2, \end{aligned}$$

where

$$D_i(t) - D_i(s) = -A_i(t) + A_i(s) + \frac{i}{2} \gamma(t)^{-1} \partial_i \gamma(t) - \frac{i}{2} \gamma(s)^{-1} \partial_i \gamma(s).$$

Estimating each term separately using (B-3), we find

$$|(u | (L(t) - L(s))u)| \leq \tilde{C}(t, s) (u | L(t)u),$$

where

$$\tilde{C}(t, s) = 2 \left| \int_s^t C_D(r) dr \right| + \left| \int_s^t C_D(r) dr \right|^2 + \left| \int_s^t C_g(r) dr \right| \left(1 + \left| \int_s^t C_D(r) dr \right| \right)^2 + \left| \int_s^t C_Y(r) dr \right|,$$

with $C_D = C_A + C_\gamma/2$. After two applications of

$$\left| \int_s^t C_D(r) \, dr \right|^2 \leq |t - s| \left| \int_s^t C_D(r)^2 \, dr \right| \leq \left| \int_s^t C_D(r)^2 \, dr \right|,$$

which is a simple consequence of the Cauchy–Schwarz inequality, we obtain

$$\tilde{C}(t, s) \leq \left| \int_s^t (c(t)(2C_D + C_D^2) + C_\gamma + C_g) \, dr \right|,$$

where $c(t) := 1 + \int_{t-1}^{t+1} C_g(r) \, dr$. Thus [Assumption \(1.c\)](#) is true with $C(t) = \tilde{C}(t) + C_W(t)$. \square

The inequalities [\(B-3\)](#) in the last proposition were stated with respect to $L(t)$. For a more convenient criterion, fix a (time-independent) Riemannian metric g_0 on Σ and set $\gamma_0 := |g_0|^{1/2}$. Consider the operator L_0 defined by the form

$$(u \mid L_0 v) := \int_{\Sigma} ((\overline{D_i^{\gamma_0} u}) g_0^{ij}(t) (D_j^{\gamma_0} v) + \bar{u} v).$$

Proposition B.2. *Assume that there exists a positive $C_g \in C(\mathbb{R})$ such that*

$$\tilde{g}^{ij}(t) X_i X_j \geq C_g(t) g_0^{ij} X_i X_j. \quad (\text{B-4})$$

Further, suppose that there exist $\varepsilon_0 \in C(\mathbb{R})$, $\varepsilon_0(t) \in]0, 1[$, and a positive $C_0 \in C(\mathbb{R})$ such that

$$\varepsilon_0(t) \gamma_0^2 \gamma(t)^{-2} (\partial_i \gamma_0^{-1} \gamma(t)) \tilde{g}^{ij}(t) (\partial_j \gamma_0^{-1} \gamma(t)) + \tilde{Y}(t) \geq C_0(t). \quad (\text{B-5})$$

Then there exists a positive $C \in C(\mathbb{R})$ such that L_0 satisfies the inequality

$$\|L(t)^{1/2} u\| \geq C(t) \|L_0^{1/2} u\|, \quad u \in \text{Dom } L(t)^{1/2}. \quad (\text{B-6})$$

Proof. Let $\varepsilon(t) := (1 - 4\varepsilon_0(t))^{-1}$, so that $\varepsilon_0(t) = \frac{1}{4}(1 - \varepsilon(t)^{-1})$. Then

$$\begin{aligned} & (u \mid L(t)u) \\ & \geq \int_{\Sigma} (-(D_i^{\gamma}(t)|u|) \tilde{g}^{ij}(t) (D_j^{\gamma}(t)|u|) + \tilde{Y}(t) |u|^2) \\ & \geq \int_{\Sigma} (\varepsilon(t) - 1) (D_i^{\gamma_0}|u|) \tilde{g}^{ij}(t) (D_j^{\gamma_0}|u|) + \int_{\Sigma} (\varepsilon_0(t) \gamma_0^2 \gamma(t)^{-2} (\partial_i \gamma_0^{-1} \gamma(t)) \tilde{g}^{ij}(t) (\partial_j \gamma_0^{-1} \gamma(t)) + \tilde{Y}(t)) |u|^2 \\ & \geq \min(C_g(t)(1 - \varepsilon(t)), C_0(t)) (|u| \mid L_0 |u|). \end{aligned}$$

In the first step we used the diamagnetic inequality

$$|(\partial_x - iV(x))f(x)| \geq |\partial_x |f(x)||$$

almost everywhere for real V and f such that $(\partial_x - iV)f$ exists almost everywhere. In the second step we used the Cauchy–Schwarz inequality. \square

We can apply the preceding proposition to restate [Proposition B.1](#) using L_0 instead of $L(t)$. For this purpose we introduce another norm on half-densitized 1-forms:

$$\|X\| = \left(\int_{\Sigma} g_0^{ij} \bar{X}_i X_j \right)^{1/2}.$$

Proposition B.3. *In addition to (B-4) and (B-5) we suppose that for some $C_g \in C(\mathbb{R})$*

$$\tilde{g}^{ij}(t) X_i X_j \leq C_g(t) g_0^{ij} X_i X_j, \quad X \in C(T^*\Sigma).$$

Moreover, we assume that there are positive $C_{Y,0}, C_{g,0}, C_{W,0} \in L^1_{\text{loc}}(\mathbb{R}), C_{A,0}, C_{\gamma,0} \in L^2_{\text{loc}}(\mathbb{R})$ such that for all $t \in \mathbb{R}$

$$\begin{aligned} \|L_0^{-1/2} |\partial_t \tilde{Y}(t)| L_0^{-1/2}\| &\leq C_{Y,0}(t), \\ \|\partial_t W(t) L_0^{-1/2}\| &\leq C_{W,0}(t), \\ \|\partial_t A(t) L_0^{-1/2}\| &\leq C_{A,0}(t), \\ \|\partial_t \gamma(t)^{-1} d\gamma(t) L_0^{-1/2}\| &\leq C_{\gamma,0}(t), \\ |\partial_t \tilde{g}^{ij}(t) X_i X_j| &\leq C_{g,0}(t) g_0^{ij} X_i X_j, \quad X \in C(T^*\Sigma). \end{aligned}$$

Then Assumption (1.c) is true.

Appendix C: Nonautonomous evolution equations

To make this paper more self-contained, we explain in this appendix relevant aspects of the theory of linear evolution equations. We are more general than strictly necessary for the purposes of this paper, but in anticipation of our upcoming work this generality could be useful. The results stated in this appendix can be found in similar form in [Kato 1970] and in the monographs [Pazy 1983; Tanabe 1997]. We also wish to refer to the appendix of [Bach and Bru 2016], which uses slightly different assumptions that essentially coincide with ours for the Hilbertian case. Finally, we would like to mention [Schmid and Griesemer 2017] which also discusses the theory of nonautonomous evolution equations on uniformly convex Banach spaces.

Let \mathcal{X} be a Banach space. We recall that a linear operator A on \mathcal{X} is the generator of a strongly continuous (one-parameter) semigroup $[0, \infty[\ni t \mapsto e^{tA}$ if and only if A is densely defined, closed and there exist constants $M \geq 1, \beta \in \mathbb{R}$ such that its resolvent satisfies

$$\|(A - \lambda)^{-n}\| \leq M(\lambda - \beta)^{-n}, \quad \lambda > \beta, \quad n = 1, 2, \dots \tag{C-1}$$

Then we have $\|e^{tA}\| \leq M e^{\beta t}$ and say that e^{tA} is a semigroup of type (M, β) . If both A and $-A$ generate strongly continuous semigroups, they generate a strongly continuous (one-parameter) group $\mathbb{R} \ni t \mapsto e^{tA}$.

If

$$\|(A - \lambda)^{-1}\| \leq (\lambda - \beta)^{-1}, \quad \lambda > \beta, \tag{C-2}$$

then (C-1) is true with $M = 1$. Thus $\|e^{tA}\| \leq M e^{\beta t}$, so that $e^{t(A-\beta)}$ is a semigroup of contractions.

Let \mathcal{Y} be another Banach space which is densely and continuously embedded in \mathcal{X} .

Definition C.1. By the *part of A on \mathcal{Y}* we mean the operator \tilde{A} , which is the restriction of A to the domain

$$\text{Dom}(\tilde{A}) := \{y \in \text{Dom}(A) \cap \mathcal{Y} \mid Ay \in \mathcal{Y}\}.$$

Definition C.2. \mathcal{Y} is called *A -admissible* if the semigroup $e^{tA}, t \in [0, \infty[$, leaves \mathcal{Y} invariant and its restriction to \mathcal{Y} is a strongly continuous semigroup on \mathcal{Y} .

In the following we consider a family $\{A(t)\}_{t \in [0, T]}$ of generators of a strongly continuous semigroup. We chose the interval $[0, T]$ for convenience and definiteness; the generalization to other intervals is straightforward.

Definition C.3. The family $\{A(t)\}_{t \in [0, T]}$ is called *stable* with stability constants $M \geq 1$, $\beta \in \mathbb{R}$, if

$$\left\| \prod_{j=1}^k (A(t_j) - \lambda)^{-1} \right\| \leq M(\lambda - \beta)^{-k}, \quad \lambda > \beta,$$

for all finite sequences $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k = 1, 2, \dots$. Here and below such products are time-ordered (viz., factors with a larger t_j are to the left of factors with a smaller t_j).

Proposition C.4. If $\{A(t)\}_{t \in [0, T]}$ is stable with stability constants M, β , then

$$\left\| \prod_{j=1}^k e^{\mu_j A(t_j)} \right\| \leq M e^{\beta(\mu_1 + \dots + \mu_k)}, \quad \mu_j \geq 0.$$

Proof. The proof is straightforward; see, e.g., [Tanabe 1997, Proposition 7.3]. □

The following simple generalization of [Kato 1970, Proposition 3.4] gives a criterion for the stability using an assumption of the form (C-2) for a time-dependent norm:

Proposition C.5. For each $t \in [0, T]$, let $\|\cdot\|_t$ be an equivalent norm on \mathcal{X} and $C \in L^1[0, T]$ positive such that

$$\|u\|_s \leq \|u\|_t \exp \left| \int_s^t C(r) dr \right|, \quad u \in \mathcal{X}, \quad s, t \in [0, T]. \tag{C-3}$$

If $\{A(t)\}$ satisfies

$$\|(A(t) - \lambda)^{-1}\|_t \leq (\lambda - \beta)^{-1}, \quad \lambda > \beta, \tag{C-4}$$

for all $t \in [0, T]$, then for any $s \in [0, T]$

$$\left\| \prod_{j=1}^k (A(t_j) - \lambda)^{-1} \right\|_s \leq (\lambda - \beta)^{-k} \exp \left(\int_0^T 2C(r) dr \right), \quad t_1 \leq s \leq t_k,$$

for every finite sequence $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$.

Proof. Repeated application of (C-3) and (C-4) yields

$$\begin{aligned} \left\| \prod_{j=1}^k (A(t_j) - \lambda)^{-1} u \right\|_{t_k} &\leq (\lambda - \beta)^{-1} \left\| \prod_{j=1}^{k-1} (A(t_j) - \lambda)^{-1} u \right\|_{t_k} \\ &\leq (\lambda - \beta)^{-1} \exp \left(\int_{t_{k-1}}^{t_k} C(r) dr \right) \left\| \prod_{j=1}^{k-1} (A(t_j) - \lambda)^{-1} u \right\|_{t_{k-1}} \\ &\vdots \\ &\leq (\lambda - \beta)^{-k} \exp \left(\int_{t_1}^{t_k} C(r) dr \right) \|u\|_{t_1}. \end{aligned}$$

Applying (C-3) twice more (for s and t_k , as well as s and t_1), we obtain the desired result. □

Let us start with a rather general theorem on the construction of evolution operators; see also [Kato 1970, Theorem 4.1; Tanabe 1997, Theorem 7.1]. Note that the properties of the evolution operator described in this theorem are rather modest.

Theorem C.6. *Assume that:*

- (a) $\{A(t)\}_{t \in [0, T]}$ is stable with constants M, β .
- (b) \mathcal{Y} is $A(t)$ -admissible for each t , and the part $\tilde{A}(t)$ of $A(t)$ in \mathcal{Y} is stable with constants $\tilde{M}, \tilde{\beta}$.
- (c) $\mathcal{Y} \subset \text{Dom } A(t)$ so that $A(t) \in B(\mathcal{Y}, \mathcal{X})$ for each t , and $t \mapsto A(t)$ is norm-continuous in the norm of $B(\mathcal{Y}, \mathcal{X})$.

Then there exists a unique family of bounded operators $\{U(t, s)\}_{0 \leq s \leq t \leq T}$, on \mathcal{X} , called the evolution (operator) generated by $A(t)$, with the following properties:

- (i) For all $0 \leq r \leq s \leq t \leq T$, we have the identities

$$U(t, t) = \mathbb{1}, \quad U(t, s)U(s, r) = U(t, r).$$

- (ii) $(t, s) \mapsto U(t, s)$ is strongly \mathcal{X} -continuous and $\|U(t, s)\|_{\mathcal{X}} \leq Me^{\beta(t-s)}$.
- (iii) For all $y \in \mathcal{Y}$ and $0 \leq s \leq t \leq T$,

$$\partial_t^+ U(t, s)y|_{t=s} = A(s)y, \tag{C-5a}$$

$$-\partial_s U(t, s)y = U(t, s)A(s)y, \tag{C-5b}$$

where the right derivative ∂_t^+ and the derivative ∂_s (right derivative if $s = 0$ and left derivative if $s = t$) are in the strong topology of \mathcal{X} .

Proof. We approximate $A(t)$ by step functions: Set

$$A_n(t) = A(T \lfloor tn/T \rfloor / n),$$

where $\lfloor \cdot \rfloor$ denotes the floor function, viz., rounding to the integral part. Since $t \mapsto A(t)$ is norm-continuous in the norm of $B(\mathcal{Y}, \mathcal{X})$, we have

$$\|A_n(t) - A(t)\|_{B(\mathcal{Y}, \mathcal{X})} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{C-6}$$

uniformly in t . It follows immediately that also $A_n(t)$ and $\tilde{A}_n(t)$ are stable with constants M, β and $\tilde{M}, \tilde{\beta}$, respectively.

Corresponding to $A_n(t)$ we construct approximating evolution operators $U_n(t, s)$ by setting

$$U_n(t, s) = e^{(t-s)A_n(s)}$$

if s, t belong to the closure of an interval where A_n is constant, and by imposing the relation

$$U_n(t, s) = U_n(t, r)U_n(r, s)$$

to determine $U_n(t, s)$ for other values of s, t . Clearly, $U_n(t, t) = \mathbb{1}$ and $(t, s) \mapsto U_n(t, s)$ is strongly \mathcal{X} -continuous. We also have

$$\|U_n(t, s)\|_{\mathcal{X}} \leq Me^{\beta(t-s)}, \quad \|U_n(t, s)\|_{\mathcal{Y}} \leq \tilde{M}e^{\tilde{\beta}(t-s)} \tag{C-7}$$

by Proposition C.4, and $U_n(t, s)\mathcal{Y} \subset \mathcal{Y}$ because \mathcal{Y} is $A(t)$ -admissible. Furthermore, because $\mathcal{Y} \subset \text{Dom } A(t)$ we have for $y \in \mathcal{Y}$

$$\begin{aligned} \partial_t U_n(t, s)y &= A_n(t)U_n(t, s)y, \\ \partial_s U_n(t, s)y &= -U_n(t, s)A_n(s)y, \end{aligned}$$

for any t or s , respectively, that is not on the boundary of an interval where A_n is constant.

Next we show that $U_n(t, s)$ converges to $U(t, s)$ strongly in \mathcal{X} uniformly in s, t : By the fundamental theorem of calculus, we have

$$U_n(t, r)y - U_m(t, r)y = \int_r^t U_n(t, s)(A_n(s) - A_m(s))U_m(s, r)y \, ds.$$

Applying (C-7), we thus obtain

$$\|U_n(t, r)y - U_m(t, r)y\|_{\mathcal{X}} \leq M\tilde{M}e^{\gamma(t-r)}\|y\|_{\mathcal{Y}} \int_r^t \|A_n(s) - A_m(s)\|_{B(\mathcal{Y}, \mathcal{X})} \, ds,$$

where $\gamma = \max(\beta, \tilde{\beta})$. Therefore it follows from (C-6) that $U_n(t, s)y$ converges in the strong topology of \mathcal{X} uniformly in s, t . Since \mathcal{Y} is dense in \mathcal{X} and $U_n(t, s)$ is uniformly bounded in n , $U_n(t, s)$ converges strongly in \mathcal{X} and we set

$$U(t, s) = \text{s-lim}_{n \rightarrow \infty} U_n(t, s).$$

It is immediate that the properties (i) and (ii) follow from the corresponding properties for $U_n(t, s)$.

Finally, we show uniqueness and (iii): If $\{V(t, s)\}_{0 \leq s \leq t \leq T}$ satisfies (i)–(iii) for a stable family of operators $\{A'(t)\}_{t \in [0, T]}$ with the same stability constants, then we apply the fundamental theorem of calculus to find

$$U_n(t, s)y - V(t, s)y = \int_s^t U_n(t, r)(A_n(r) - A'(r))V(r, s)y \, dr,$$

and therefore

$$\|U_n(t, s)y - V(t, s)y\|_{\mathcal{X}} \leq M\tilde{M}e^{\gamma(t-s)}\|y\|_{\mathcal{Y}} \int_s^t \|A_n(r) - A'(r)\|_{B(\mathcal{Y}, \mathcal{X})} \, dr. \tag{C-8}$$

If we set $A'(t) = A(t)$ and let $n \rightarrow \infty$, we thus find that $U(t, s)y = V(t, s)y$ and by density $U(t, s) = V(t, s)$ on the whole of \mathcal{X} . We conclude that $U(t, s)$ is unique.

Now, in (C-8), we set $A'(t) = A(\tau) = \text{const}$ for $\tau \in [0, T]$, divide by $t - s$ and let $n \rightarrow \infty$ to obtain

$$(t - s)^{-1}\|U(t, s)y - e^{(t-s)A(\tau)}y\|_{\mathcal{X}} \leq (t - s)^{-1}M\tilde{M}e^{\gamma(t-s)}\|y\|_{\mathcal{Y}} \int_s^t \|A_n(r) - A(\tau)\|_{B(\mathcal{Y}, \mathcal{X})} \, dr.$$

On the one hand, for $\tau = s$, we find (C-5a) in the limit $t \rightarrow s$. On the other hand, setting $\tau = t$ and letting $t \rightarrow s$, we find

$$\partial_s^- U(t, s)y|_{s=t} = -A(t)y. \tag{C-9}$$

To find (C-5b), we check the right and left derivative separately. Applying (C-5a) and (C-9), we obtain

$$\begin{aligned} \partial_s^+ U(t, s)y &= \text{s-lim}_{h \searrow 0} h^{-1}(U(t, s+h)y - U(t, s)y) \\ &= U(t, s+h) \text{s-lim}_{h \searrow 0} h^{-1}(y - U(s+h, s)y) = -U(t, s)A(s)y, \end{aligned} \tag{C-10a}$$

$$\begin{aligned} \partial_s^- U(t, s)y &= s\text{-}\lim_{h \searrow 0} h^{-1}(U(t, s)y - U(t, s-h)y) \\ &= U(t, s) s\text{-}\lim_{h \searrow 0} h^{-1}(y - U(s, s-h)y) = -U(t, s)A(s)y. \end{aligned} \quad (\text{C-10b})$$

Therefore we have completed the proof also for (iii). \square

We say that a Banach space \mathcal{Y} possesses a predual if there exists a Banach space \mathcal{Y}_* such that \mathcal{Y} is the dual of \mathcal{Y}_* . Having fixed a predual \mathcal{Y}_* , we can equip \mathcal{Y} with the so-called weak* topology, which is generated by the seminorms $y \mapsto |\xi(y)|$, where $\xi \in \mathcal{Y}_*$. Note in particular that every reflexive Banach space possesses a unique predual (namely, its dual). For reflexive Banach spaces the weak* convergence clearly coincides with the weak convergence.

For Banach spaces possessing a predual one can slightly improve the previous theorem; see also [Kato 1970, Theorem 5.1].

Theorem C.7. *In addition to the assumptions of Theorem C.6, assume that:*

(d) \mathcal{Y} possesses a predual.

Then, in addition to (i)–(iii), the evolution $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ has the following properties:

(iv) $U(t, s)\mathcal{Y} \subset \mathcal{Y}$, $(t, s) \mapsto U(t, s)$ is weakly* continuous and

$$\|U(t, r)\|_{\mathcal{Y}} \leq \tilde{M}e^{\tilde{\beta}(t-s)}, \quad 0 \leq r \leq s \leq t \leq T. \quad (\text{C-11})$$

Proof. Note that for fixed $s, t \in [0, T]$ and $y \in \mathcal{Y}$, $U_n(t, s)y$ is a uniformly bounded sequence in \mathcal{Y} , and thus, by the Banach–Alaoglu theorem, it contains a weakly* convergent subsequence. Moreover, by our previous results, $U_n(t, s)y \rightarrow U(t, s)y$ in \mathcal{X} . But $U(t, s)y$ must be equal to the weak* limit, and thus lie in \mathcal{Y} ; i.e., $U(t, s)\mathcal{Y} \subset \mathcal{Y}$. The inequality then follows from (C-7).

Now, let $(t_j)_j, (s_j)_j$ be sequences with $t_j \rightarrow t$, $s_j \rightarrow s$ and $y \in \mathcal{Y}$. Recall that $U(t_j, s_j)y \rightarrow U(t, s)y$ in \mathcal{X} because $(t, s) \mapsto U(t, s)$ is \mathcal{X} -strongly continuous. By the Banach–Alaoglu theorem, since $U(t_j, s_j)$ is uniformly bounded, $U(t_j, s_j)y$ contains a weakly* convergent subsequence. The weak* limit of $U(t_j, s_j)y$ is thus $U(t, s)y$ and must lie in \mathcal{Y} . In other words, $U(t, s)$ is weakly* continuous on \mathcal{Y} . \square

We recall that a normed space is called *uniformly convex* if for every $\varepsilon > 0$ and unit vectors $\|x\| = \|y\| = 1$ there exists $\delta > 0$ such that

$$\|x - y\| \geq \varepsilon \quad \implies \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

We note that all uniformly convex Banach spaces are reflexive. Additionally, on uniformly convex Banach spaces, $\|x_n\| \rightarrow \|x\|$ and the weak convergence $x_n \rightharpoonup x$ implies the strong convergence. All Hilbert spaces are uniformly convex.

If we assume that the Banach space \mathcal{Y} is uniformly convex, stronger results about the evolution can be derived. They are described in the following theorem, which is a part of [Kato 1970, Theorem 5.2]:

Theorem C.8. *In addition to the assumptions of Theorem C.6, assume that:*

(d') \mathcal{Y} is uniformly convex.

(e) For every t there exist on \mathcal{Y} an equivalent norm $\|\cdot\|_{\mathcal{Y},t}$ as well as a positive $C \in L^1[0, T]$ such that

$$\|y\|_{\mathcal{Y},s} \leq \|y\|_{\mathcal{Y},t} \exp \left| \int_s^t C(r) dr \right|, \quad s, t \in [0, T]. \quad (\text{C-12})$$

Additionally, there exists $\tilde{\beta} \in \mathbb{R}$ such that

$$\|(\tilde{A}(t) - \lambda)^{-1}\|_{\mathcal{Y},t} \leq (\lambda - \tilde{\beta})^{-1}, \quad \lambda > \tilde{\beta},$$

for all $t \in [0, T]$.

Then, in addition to (i)–(iii), the evolution $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ has the following property, which is an improved version of (iv):

(iv') $U(t, s)$ preserves \mathcal{Y} , is \mathcal{Y} -strongly continuous in s for fixed t and \mathcal{Y} -strongly right-continuous in t for fixed s , and

$$\|U(t, r)\|_{\mathcal{Y},s} \leq \exp \left(\int_r^t (\tilde{\beta} + 2C(\tau)) d\tau \right), \quad 0 \leq r \leq s \leq t \leq T. \quad (\text{C-13})$$

Proof. Since \mathcal{Y} is uniformly convex, it is also reflexive, and thus Theorem C.7(iv) holds. Then we use Propositions C.4 and C.5 to find (C-13).

Let us prove the strong continuity. By (iv), for any $y \in \mathcal{Y}$ we have $w\text{-}\lim_{t,r \rightarrow s} U(t, r)y \rightarrow y$. Using this, and then the bound (C-13), we obtain

$$\begin{aligned} \|y\| &\leq \liminf_{r,t \rightarrow s} \|U(t, r)y\|_{\mathcal{Y},s} \leq \limsup_{r,t \rightarrow s} \|U(t, r)y\|_{\mathcal{Y},s} \\ &\leq \limsup_{r,t \rightarrow s} \exp \left(\int_r^t (\tilde{\beta} + 2C(\tau)) d\tau \right) \|y\| = \|y\|. \end{aligned}$$

Hence, $\lim_{r,t \rightarrow s} \|U(t, r)y\|_{\mathcal{Y},s} = \|y\|$. But \mathcal{Y} is uniformly convex, so this implies

$$\lim_{r,t \rightarrow s} U(t, r)y = y.$$

Let $0 \leq s \leq s' \leq t \leq T$ and $y \in \mathcal{Y}$. Then

$$\|U(t, s')y - U(t, s)y\|_{\mathcal{Y}} \leq \|U(t, s')\|_{\mathcal{Y}} \|y - U(s', s)y\|_{\mathcal{Y}} \rightarrow 0$$

as $s' \rightarrow s$ or $s \rightarrow s'$. Similarly, for $0 \leq s \leq t \leq t' \leq T$ we find

$$\|U(t', s)y - U(t, s)y\|_{\mathcal{Y}} \leq \|(U(t', t) - \mathbb{1})U(t, s)y\|_{\mathcal{Y}} \rightarrow 0$$

as $t' \rightarrow t$. □

In the previous theorem we still had to distinguish between the t - and s -properties of $U(t, s)$. If the reversed operator $-A(T - t)$ also satisfies the assumptions of the theorems above, this distinction can be dropped; see also [Kato 1970, Remark 5.3]:

Theorem C.9. *Suppose that both $\{A(t)\}_{t \in [0, T]}$ and the reversed family $\{-A(T - t)\}_{t \in [0, T]}$ satisfy the assumptions of Theorems C.6 and C.8. Then the unique family of bounded operators $\{U(t, s)\}_{s, t \in \mathbb{R}}$ described in the previous theorems satisfies the following improved versions of (i), (iii), and (iv'):*

(i') For all $r, s, t \in [0, T]$, we have the identities

$$U(t, t) = \mathbb{1}, \quad U(t, s)U(s, r) = U(t, r).$$

(iii') For all $y \in \mathcal{Y}$ and $s, t \in [0, T]$,

$$\partial_t U(t, s)y = A(t)U(t, s)y, \tag{C-14a}$$

$$-\partial_s U(t, s)y = U(t, s)A(s)y, \tag{C-14b}$$

where the derivatives (right/left derivatives at the boundaries of $[0, T]$) are in the strong topology of \mathcal{X} .

(iv'') $(t, s) \mapsto U(t, s)$ preserves \mathcal{Y} , is \mathcal{Y} -strongly continuous and satisfies (C-13).

Proof. Denote the evolution for $\{A(t)\}_{t \in [0, T]}$ by $U(t, s)$ and the evolution for $\{-A(T-t)\}_{t \in [0, T]}$ by $V(t, s)$. For $0 \leq s \leq t \leq T$, we define

$$U(s, t) = V(T-s, T-t).$$

From the approximations $U_n(t, s)$ and $V_n(t, s)$, it is easy to see that

$$U(t, s)U(s, t) = \mathbb{1}$$

for $s, t \in \mathbb{R}$. This proves (i').

It is clear that

$$\partial_t U(t, s)y|_{t=s} = A(s)y,$$

$$-\partial_s U(t, s)y = U(t, s)A(s)y$$

for $s, t \in [0, T]$. Then we can proceed as in (C-10) to find also

$$\partial_t U(t, s)y = A(t)U(t, s)y.$$

Finally, the strong continuity of $U(t, s)$ follows from (iv') applied to both $U(t, s)$ and $V(t, s)$, which implies, in particular, that $U(t, s)$ is strongly right- and left-continuous in t for fixed s . □

Theorem C.9 implies the following; see also [Yajima 2011, Theorem 3.2]:

Theorem C.10. *Let \mathcal{X} and \mathcal{Y} be Hilbert spaces such that \mathcal{Y} is densely and continuously embedded in \mathcal{X} . Let $I \subset \mathbb{R}$ be a compact interval, and $\{A(t)\}_{t \in I}$ a family of densely defined, closed operators on \mathcal{X} . Suppose that the following is satisfied:*

- (a) $\mathcal{Y} \subset \text{Dom } A(t)$ so that $A(t) \in B(\mathcal{Y}, \mathcal{X})$ and $t \mapsto A(t)$ is norm-continuous in the norm of $B(\mathcal{Y}, \mathcal{X})$.
- (b) For every $t \in I$, there exist on \mathcal{X} and \mathcal{Y} Hilbert structures $(\cdot | \cdot)_{\mathcal{X}, t}$ and $(\cdot | \cdot)_{\mathcal{Y}, t}$, which are equivalent to the original ones and for a positive $C \in L^1(I)$ and all $s, t \in I$

$$\|x\|_{\mathcal{X}, s} \leq \|x\|_{\mathcal{X}, t} \exp \left| \int_s^t C(r) \, dr \right|,$$

$$\|y\|_{\mathcal{Y}, s} \leq \|y\|_{\mathcal{Y}, t} \exp \left| \int_s^t C(r) \, dr \right|.$$

Denote the corresponding Hilbert spaces by \mathcal{X}_t and \mathcal{Y}_t .

(c) $A(t)$ is self-adjoint with respect to \mathcal{X}_t and the part $\tilde{A}(t)$ of $A(t)$ in \mathcal{Y}_t is self-adjoint in \mathcal{Y}_t .

Then there exists a unique family of bounded operators $\{U(t, s)\}_{s,t \in I}$, in \mathcal{X} , called the **evolution (operator)** generated by $A(t)$, with the following properties:

(i) For all $r, s, t \in I$, we have the identities

$$U(t, t) = \mathbb{1}, \quad U(t, s)U(s, r) = U(t, r).$$

(ii) $U(t, s)$ is \mathcal{X} -strongly continuous and

$$\|U(t, s)\|_{\mathcal{X},s} \leq \exp \left| \int_s^t 2C(r) \, dr \right|, \quad s, t \in I.$$

(iii) For all $y \in \mathcal{Y}$ and $s, t \in I$,

$$\begin{aligned} i\partial_t U(t, s)y &= A(t)U(t, s)y, \\ -i\partial_s U(t, s)y &= U(t, s)A(s)y, \end{aligned}$$

where the derivatives (right/left derivatives at the boundaries of I) are in the strong topology of \mathcal{X} .

(iv) $U(t, s)\mathcal{Y} \subset \mathcal{Y}$, $U(t, s)$ is \mathcal{Y} -strongly continuous and

$$\|U(t, s)\|_{\mathcal{Y},s} \leq \exp \left| \int_s^t 2C(r) \, dr \right|, \quad s, t \in I.$$

The following perturbation theorem is essentially [Kato 1966, Theorem 4.5]. We leave the proof as an exercise to the reader.

Theorem C.11. *Suppose that $\{A(t)\}_{t \in [0, T]}$ satisfies the assumptions of Theorem C.6. Let $\{B(t)\}_{t \in [0, T]}$ be a family of bounded operators in \mathcal{X} such that $t \mapsto B(t)$ is strongly continuous with respect to \mathcal{X} and $K = \sup_t \|B(t)\|_{\mathcal{X}}$. Then there exists a unique evolution $V(t, s)$ for $\{A(t) + B(t)\}_{t \in [0, T]}$ satisfying the properties (i)–(iii), but with the estimate*

$$\|V(t, s)\| \leq M e^{(\beta + KM)(t-s)}.$$

Suppose that $\{A(t)\}_{t \in [0, T]}$ also satisfies the stronger assumptions of Theorem C.7, C.8 or C.9, and $\{B(t)\}_{t \in [0, T]}$ preserves \mathcal{Y} and its part $\{\tilde{B}(t)\}_{t \in [0, T]}$ in \mathcal{Y} is bounded in \mathcal{Y} with $\tilde{K} = \sup_t \|B(t)\|_{\mathcal{Y}}$. Then the evolution $V(t, s)$ satisfies the corresponding stronger properties, where the estimate (C-11) needs to be multiplied by $e^{\tilde{K}\tilde{M}(t-s)}$, and the estimate (C-13) by $e^{\tilde{K}(t-s)}$.

The evolution $V(t, s)$ in the theorem above is given symbolically by

$$V = U + U * B * U + U * B * U * B * U + \dots,$$

where $* B *$ denotes a Volterra-type convolution with “density” $B(t)$. For example,

$$(U * B * U)(t, r) = \int_r^t U(t, s)B(s)U(s, r) \, ds.$$

Appendix D: Heinz–Kato inequality

We recall the Heinz–Kato inequality [Heinz 1951; Kato 1961], which is an elementary but very useful result for the interpolation of operators:

Theorem D.1. *Suppose that A, B are positive operators on Hilbert spaces \mathcal{X}, \mathcal{Y} , respectively. If T is a bounded operator from \mathcal{X} to \mathcal{Y} such that $T(\text{Dom } A) \subset \text{Dom } B$ and*

$$\|Tx\| \leq C_0\|x\|, \quad \|BTx\| \leq C_1\|Ax\|,$$

for $x \in \text{Dom } A$, then

$$\|B^\lambda Tx\| \leq C_0^\lambda C_1^{1-\lambda} \|A^\lambda x\|, \quad \lambda \in [0, 1]. \tag{D-1}$$

Appendix E: Finite speed of propagation

In this appendix we prove the finite speed of propagation for solutions of the Klein–Gordon equation with coefficients of low regularity.

In this section we prefer to work with the Klein–Gordon equation in the scalar formalism, given by (1-1), which can be locally written as

$$Ku := -g^{\mu\nu}(\nabla_\mu - iA_\mu)(\nabla_\nu - iA_\nu)u + Yu, \tag{E-1}$$

with pseudo-Riemannian metric g and the corresponding Levi-Civita derivative ∇ , vector potential A , and scalar potential Y . Our standing assumptions in this appendix are as follows:

Assumption 4. $M = \mathbb{R} \times \Sigma$ is equipped with a continuous Lorentzian metric $g = -\alpha^2 dt^2 + g_\Sigma$, where $\alpha > 0$ and g_Σ are continuous, and g_Σ restricts to a family of Riemannian metrics on Σ . (Recall that every globally hyperbolic spacetime can be brought into this form [Bernal and Sánchez 2005].) We assume that $A_\mu(t) \in L^\infty_{\text{loc}}(\Sigma)$ for all t , and $A_\mu, \dot{A}_\mu, Y \in L^\infty_{\text{loc}}(M)$. Moreover, in every compact neighborhood $U \subset M$ there is $C_g > 0$ such that

$$|\dot{g}^{\mu\nu} X_\mu X_\nu| \leq C_g |g^{\mu\nu} X_\mu X_\nu|$$

almost everywhere in U for all covectors X .

Under these assumption we will show the following theorem on the finite speed of propagation:

Theorem E.1. *If $u \in C^1(\mathbb{R}; L^2_{\text{loc}}(\Sigma))$ with $\partial_t u \in C(\mathbb{R}; L^2_{\text{loc}}(\Sigma))$ and $Ku \in L^2_{\text{loc}}(M)$, then*

$$\text{supp } u \subset J(\text{supp } Ku \cup \{t\} \times (\text{supp } u(t) \cup \text{supp } \dot{u}(t)))$$

for any $t \in \mathbb{R}$. That is, u is supported in the causal shadow of the union of Ku and of the support of its Cauchy data on $\{t\} \times \Sigma$.

Equation (E-1) can be obtained via the Euler–Lagrange equations from the *Lagrangian density*

$$\mathcal{L}[u] := -|g|^{1/2}((\partial_\mu + iA_\mu)\bar{u})g^{\mu\nu}((\partial_\nu - iA_\nu)u) + Y|u|^2.$$

To the Lagrangian density \mathcal{L} we can associate the *momentum flux density*

$$\mathcal{P}^\mu[u] := -\delta_0^\mu \mathcal{L}[u] + \frac{\partial \mathcal{L}[u]}{\partial(\partial_\mu \bar{u})} \partial_t \bar{u} + \frac{\partial \mathcal{L}[u]}{\partial(\partial_\mu u)} \partial_t u.$$

If the action for \mathcal{L} is invariant under infinitesimal time-translations, Noether's theorem says that the momentum flux is conserved. If the action is not time-translation invariant, \mathcal{P} is in general not conserved, but it is still a useful quantity.

The energy density $\mathcal{E} = \mathcal{P}^0$ obtained from \mathcal{L} is not necessarily positive. Therefore, for technical reasons it will be convenient to replace \mathcal{L} with the modified Lagrangian density

$$\tilde{\mathcal{L}}[u] := -|g|^{1/2} \left((\partial_\mu + iA_\mu) \bar{u} \right) g^{\mu\nu} \left((\partial_\nu - iA_\nu) u \right) - (1 + \alpha^{-2} A_0^2) |u|^2,$$

denoting the corresponding momentum flux density by $\tilde{\mathcal{P}}$. Using the special form of the metric, we find the energy density

$$\tilde{\mathcal{E}}[u] := \tilde{\mathcal{P}}^0[u] = |g|^{1/2} \left(\alpha^{-2} |\dot{u}|^2 + ((\partial_i + iA_i) \bar{u}) g_\Sigma^{ij} ((\partial_j - iA_j) u) + |u|^2 \right)$$

and the spatial momentum flux density

$$\tilde{\mathcal{P}}^i[u] = \mathcal{P}^i[u] = -|g|^{1/2} \left(\dot{u} g_\Sigma^{ij} ((\partial_j - iA_j) u) + \dot{u} g_\Sigma^{ij} ((\partial_j + iA_j) \bar{u}) \right).$$

Below we will integrate $\partial_\mu \tilde{\mathcal{P}}^\mu$ over a region which is delimited by two constant-time surfaces and the backward lightcone of a point as described in [Figure 1](#). To rewrite this integral as an integral over the boundary of said region via Stokes' theorem, it is useful to assume that $\partial J_g^\pm(\Omega)$ is a Lipschitz topological hypersurface; see [\[Beem et al. 1996, Theorem 3.9\]](#). Here we denote by $J_g^\pm(\Omega)$ the causal future (+) or causal past (−) of Ω , i.e., the set of points which can be reached from Ω by future-directed or, respectively, past-directed causal curves with respect to the metric g . Moreover, we write $J_g(\Omega) = J_g^+(\Omega) \cup J_g^-(\Omega)$.

If g is not smooth (or at least C^2), it is not guaranteed that $\partial J_g^\pm(\Omega)$ is a Lipschitz topological hypersurface. However, we can approximate g by smooth metrics:

If a Lorentzian metric \hat{g} has strictly larger lightcones than g , i.e., each nonvanishing g -causal vector X^μ ($g_{\mu\nu} X^\mu X^\nu \leq 0$) is \hat{g} -timelike ($\hat{g}_{\mu\nu} X^\mu X^\nu < 0$), then we write

$$\hat{g} \succ g.$$

As shown in [\[Chruściel and Grant 2012, Proposition 1.2\]](#), there always exists a *smooth* Lorentzian metric \hat{g} with strictly larger lightcones which approximates g arbitrarily well.

Proposition E.2. *Let $\hat{g} \succ g$ be smooth and consider the situation depicted in [Figure 1](#). Then there exists $C > 0$ such that*

$$e^{C(s-t)} \int_{K_t} \tilde{\mathcal{E}}[u](t) \leq \int_{K_s} \tilde{\mathcal{E}}[u](s) + \int_{\Omega} |g|^{1/2} |Ku|^2 \tag{E-2}$$

for all $u \in C^1(\mathbb{R}; L_{\text{loc}}^2(\Sigma))$ with $\partial_i u \in C(\mathbb{R}; L_{\text{loc}}^2(\Sigma))$ and $Ku \in L_{\text{loc}}^2(M)$.

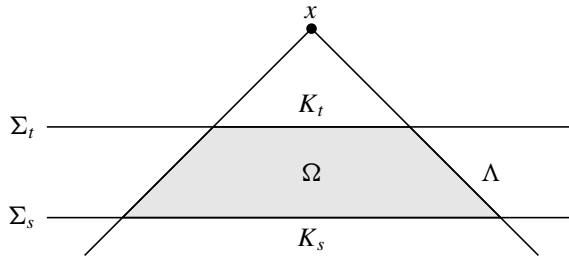


Figure 1. The truncated cone given by the backward lightcone $J_{\tilde{g}}^-(x)$ of a point, and two constant-time surfaces $\Sigma_t = \{t\} \times \Sigma$ and Σ_s (with $t > s$). We write $K_t = J_{\tilde{g}}^-(x) \cap (\{t\} \times \Sigma)$ and K_s for the caps, and $\Lambda = \partial J_{\tilde{g}}^-(x) \cap ([s, t] \times \Sigma)$ for the mantle of the truncated cone $\Omega = J_{\tilde{g}}^-(x) \cap ([s, t] \times \Sigma)$.

Proof. We derive

$$\begin{aligned} \partial_\mu \tilde{\mathcal{P}}^\mu[u] &= -\partial_t \tilde{\mathcal{L}}[u] + \left(\partial_\mu \frac{\partial \tilde{\mathcal{L}}[u]}{\partial(\partial_\mu \bar{u})} \right) \dot{\bar{u}} + \frac{\partial \tilde{\mathcal{L}}}{\partial(\partial_\mu \bar{u})} \partial_\mu \partial_t \bar{u} + \left(\partial_\mu \frac{\partial \tilde{\mathcal{L}}}{\partial(\partial_\mu u)} \right) \dot{u} + \frac{\partial \tilde{\mathcal{L}}[u]}{\partial(\partial_\mu u)} \partial_\mu \partial_t u \\ &= -\partial_t \tilde{\mathcal{L}}[u] + \left(|g|^{1/2} \tilde{K}u + \frac{\partial \tilde{\mathcal{L}}[u]}{\partial \bar{u}} \right) \dot{\bar{u}} + \frac{\partial \tilde{\mathcal{L}}[u]}{\partial(\partial_\mu \bar{u})} \partial_t \partial_\mu \bar{u} + \left(|g|^{1/2} \tilde{K}\bar{u} + \frac{\partial \tilde{\mathcal{L}}[u]}{\partial u} \right) \dot{u} + \frac{\partial \tilde{\mathcal{L}}[u]}{\partial(\partial_\mu u)} \partial_t \partial_\mu u \\ &= -2|g|^{1/2} \operatorname{Re}(\dot{\bar{u}} \tilde{K}u) - \frac{\partial \tilde{\mathcal{L}}[u]}{\partial g^{\mu\nu}} \dot{g}^{\mu\nu} - \frac{\partial \tilde{\mathcal{L}}[u]}{\partial A_\mu} \dot{A}_\mu - \frac{\partial \tilde{\mathcal{L}}[u]}{\partial |g|} \partial_t |g| \\ &= |g|^{1/2} (2 \operatorname{Re}(\dot{\bar{u}} \tilde{K}u) + ((\partial_\mu + iA_\mu)\bar{u}) \dot{g}^{\mu\nu} ((\partial_\nu - iA_\nu)u) - 2\alpha^{-3} \dot{\alpha} A_0^2 |u|^2 \\ &\quad - 2 \operatorname{Im}(\bar{u} \dot{A}_\mu g^{\mu\nu} (\partial_\nu - iA_\nu)u) + 2\alpha^{-2} A_0 \dot{A}_0 |u|^2 - \frac{1}{2} |g|^{-1} (\partial_t |g|) \tilde{\mathcal{L}}[u]). \end{aligned}$$

where, in the second step, we used the Euler–Lagrange equations with

$$\tilde{K} = K - Y + 1 + \alpha^{-2} A_0^2$$

being the Klein–Gordon operator associated to $\tilde{\mathcal{L}}$. Estimating each term separately using our assumptions and the Cauchy–Schwarz inequality yields

$$\partial_\mu \tilde{\mathcal{P}}^\mu[u] \leq |g|^{1/2} (|Ku|^2 + C_1 \alpha^{-2} |\dot{u}|^2 + C_2 ((\partial_i + iA_i)\bar{u}) g_\Sigma^{ij} ((\partial_j + iA_j)u) + C_3 |u|^2)$$

for $C_1, C_2, C_3 > 0$ which do not depend on u . Therefore we find

$$\int_\Omega \partial_\mu \tilde{\mathcal{P}}^\mu[u] \leq \int_\Omega (|g|^{1/2} |Ku|^2 + C \tilde{\mathcal{E}}[u]) \tag{E-3}$$

for some constant $C > 0$.

By Stokes’ theorem,

$$\int_\Omega \partial_\mu \tilde{\mathcal{P}}^\mu[u] = \int_{\partial\Omega} n_\mu \tilde{\mathcal{P}}^\mu[u] = \int_{K_t} \tilde{\mathcal{E}}[u](t) - \int_{K_s} \tilde{\mathcal{E}}[u](s) + \int_\Lambda n_\mu \tilde{\mathcal{P}}^\mu[u], \tag{E-4}$$

where n is the outward-directed normal field to $\partial\Omega$. For any future-directed causal covector field ξ (i.e., $g^{\mu\nu}\xi_\mu\xi_\nu \leq 0$ and $\xi_0 \geq 0$) with $|\vec{\xi}| = (g_\Sigma^{ij}\xi_i\xi_j)^{1/2}$,

$$\begin{aligned}\xi_\mu \tilde{\mathcal{P}}^\mu[u] &= \xi_0 \tilde{\mathcal{E}}[u] - 2|g|^{1/2} \operatorname{Re}(\xi_i \dot{u} g_\Sigma^{ij} (\partial_j - iA_j)u) \\ &\geq \xi_0 \tilde{\mathcal{E}}[u] - |g|^{1/2} \alpha |\vec{\xi}| (\alpha^{-2} |\dot{u}|^2 + ((\partial_i + iA_i)\bar{u}) g_\Sigma^{ij} ((\partial_j - iA_j)u)) \\ &\geq (\xi_0 - \alpha |\vec{\xi}|) \tilde{\mathcal{E}}[u] \geq 0\end{aligned}$$

almost everywhere. Consequently, we can estimate the last term in (E-4) as $\int_\Lambda n_\mu \tilde{\mathcal{P}}^\mu \geq 0$.

Combining (E-3) and (E-4), we obtain

$$\int_{K_t} \tilde{\mathcal{E}}[u](t) - \int_{K_s} \tilde{\mathcal{E}}[u](s) \leq \int_s^t \left(\int_{K_r} (|g|^{1/2} |Ku(r)|^2 + C \tilde{\mathcal{E}}[u](r)) \right) dr,$$

and thus (E-2) by Grönwall's inequality. \square

Now, using the proposition above, we can show the finite speed of propagation:

Theorem E.3. *If $u \in C^1(\mathbb{R}; L_{\text{loc}}^2(\Sigma))$ with $\partial_i u \in C(\mathbb{R}; L_{\text{loc}}^2(\Sigma))$ and $Ku \in L_{\text{loc}}^2(M)$, then*

$$\begin{aligned}\operatorname{supp} u \cap M_\pm &\subset J_g^\pm((\operatorname{supp} Ku \cap M_\pm) \cup \{t\} \times (\operatorname{supp} u(t) \cup \operatorname{supp} \dot{u}(t))), \\ \operatorname{supp} u &\subset J_g(\operatorname{supp} Ku \cup \{t\} \times (\operatorname{supp} u(t) \cup \operatorname{supp} \dot{u}(t)))\end{aligned}\tag{E-5}$$

for any $t \in \mathbb{R}$, where $M_+ = [t, +\infty[\times \Sigma$ and $M_- =]-\infty, t] \times \Sigma$.

Proof. Note that, as a subset of Σ , we have $\operatorname{supp} \tilde{\mathcal{E}}[u](t) = \operatorname{supp} u(t) \cup \operatorname{supp} \dot{u}(t)$. We show that $u(x) = 0$ for any

$$x \in M \setminus J_{\hat{g}}^+((\operatorname{supp} Ku \cap M_+) \cup \{t\} \times \operatorname{supp} \tilde{\mathcal{E}}[u](t))$$

by an application of Proposition E.2 for all smooth $\hat{g} \succ g$. For any such x , $J_{\hat{g}}^-(x)$ does not intersect $(\operatorname{supp} Ku \cap M_+) \cup \{t\} \times \operatorname{supp} \tilde{\mathcal{E}}[u](t)$. Proposition E.2 now shows that u vanishes in $J_{\hat{g}}^-(x) \cap M_+$ and thus also at x .

We have thus shown that

$$\operatorname{supp} u \cap M_\pm \subset J_g^\pm((\operatorname{supp} Ku \cap M_\pm) \cup \{t\} \times (\operatorname{supp} u(t) \cup \operatorname{supp} \dot{u}(t)))$$

for all smooth $\hat{g} \succ g$. It follows that (E-5) holds, because a vector is g -causal if and only if it is \hat{g} -timelike for all smooth $\hat{g} \succ g$ by [Chruściel and Grant 2012, Proposition 1.5] and therefore

$$J_g^\pm(\Omega) = \bigcap_{\hat{g} \succ g} J_{\hat{g}}^\pm(\Omega), \quad \Omega \subset M.$$

The embedding for J^- follows by time reversal and the remaining embedding by the union of the embeddings for J^+ and J^- . \square

Acknowledgements

We would like to thank Kenji Yajima for useful discussions. The work of Siemssen was supported by a grant of the Polish National Science Center (NCN) based on decision no. DEC-2015/16/S/ST1/00473. The work of Dereziński was supported by the National Science Center under the grant UMO-2014/15/B/ST1/00126.

References

- [Bach and Bru 2016] V. Bach and J.-B. Bru, *Diagonalizing quadratic bosonic operators by non-autonomous flow equations*, Mem. Amer. Math. Soc. **1138**, Amer. Math. Soc., Providence, RI, 2016. [MR](#) [Zbl](#)
- [Bachelot 1999] A. Bachelot, “The Hawking effect”, *Ann. Inst. H. Poincaré Phys. Théor.* **70**:1 (1999), 41–99. [MR](#) [Zbl](#)
- [Bär et al. 2007] C. Bär, N. Ginoux, and F. Pfäffle, *Wave equations on Lorentzian manifolds and quantization*, European Mathematical Society, Zürich, 2007. [MR](#) [Zbl](#)
- [Beem et al. 1996] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian geometry*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics **202**, Marcel Dekker, New York, 1996. [MR](#) [Zbl](#)
- [Bernal and Sánchez 2005] A. N. Bernal and M. Sánchez, “Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes”, *Comm. Math. Phys.* **257**:1 (2005), 43–50. [MR](#) [Zbl](#)
- [Bogoliubov and Shirkov 1980] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, John Wiley & Sons, New York, 1980. [MR](#)
- [Chruściel and Grant 2012] P. T. Chruściel and J. D. E. Grant, “On Lorentzian causality with continuous metrics”, *Classical Quantum Gravity* **29**:14 (2012), art. id. 145001. [MR](#) [Zbl](#)
- [Dappiaggi et al. 2018a] C. Dappiaggi, N. Drago, and H. Ferreira, “Fundamental solutions for the wave operator on static Lorentzian manifolds with timelike boundary”, preprint, 2018. [arXiv](#)
- [Dappiaggi et al. 2018b] C. Dappiaggi, H. Ferreira, and A. Marta, “Ground states of a Klein–Gordon field with Robin boundary conditions in global anti-de Sitter spacetime”, preprint, 2018. [arXiv](#)
- [Dereziński and Gérard 2013] J. Dereziński and C. Gérard, *Mathematics of quantization and quantum fields*, Cambridge University Press, 2013. [MR](#) [Zbl](#)
- [Dereziński and Siemssen 2018] J. Dereziński and D. Siemssen, “Feynman propagators on static spacetimes”, *Rev. Math. Phys.* **30**:3 (2018), art. id. 1850006. [MR](#) [Zbl](#)
- [Dimock and Kay 1982] J. Dimock and B. S. Kay, “Classical wave operators and asymptotic quantum field operators on curved space-times”, *Ann. Inst. H. Poincaré Sect. A (N.S.)* **37**:2 (1982), 93–114. [MR](#) [Zbl](#)
- [Drago and Gérard 2017] N. Drago and C. Gérard, “On the adiabatic limit of Hadamard states”, *Lett. Math. Phys.* **107**:8 (2017), 1409–1438. [MR](#) [Zbl](#)
- [Friedlander 1975] F. G. Friedlander, *The wave equation on a curved space-time*, Cambridge Monographs on Mathematical Physics **2**, Cambridge University Press, 1975. [MR](#) [Zbl](#)
- [Fulling 1979] S. A. Fulling, “Remarks on positive frequency and Hamiltonians in expanding universes”, *Gen. Relativity Gravitation* **10**:10 (1979), 807–824. [MR](#) [Zbl](#)
- [Furlani 1997] E. P. Furlani, “Evolution operators and scattering theory for massive scalar fields on curved spacetimes”, *J. Phys. A* **30**:17 (1997), 6065–6079. [MR](#) [Zbl](#)
- [Gérard and Wrochna 2014] C. Gérard and M. Wrochna, “Construction of Hadamard states by pseudo-differential calculus”, *Comm. Math. Phys.* **325**:2 (2014), 713–755. [MR](#) [Zbl](#)
- [Gérard and Wrochna 2017] C. Gérard and M. Wrochna, “Hadamard property of the *in* and *out* states for Klein–Gordon fields on asymptotically static spacetimes”, *Ann. Henri Poincaré* **18**:8 (2017), 2715–2756. [MR](#) [Zbl](#)
- [Gérard et al. 2017] C. Gérard, O. Oulghazi, and M. Wrochna, “Hadamard states for the Klein–Gordon equation on Lorentzian manifolds of bounded geometry”, *Comm. Math. Phys.* **352**:2 (2017), 519–583. [MR](#) [Zbl](#)
- [Häfner 2009] D. Häfner, *Creation of fermions by rotating charged black holes*, Mém. Soc. Math. Fr. (N.S.) **117**, Soc. Math. France, Paris, 2009. [MR](#) [Zbl](#)

- [Heinz 1951] E. Heinz, “Beiträge zur Störungstheorie der Spektralzerlegung”, *Math. Ann.* **123** (1951), 415–438. [MR](#) [Zbl](#)
- [Hollands and Wald 2001] S. Hollands and R. M. Wald, “Local Wick polynomials and time ordered products of quantum fields in curved spacetime”, *Comm. Math. Phys.* **223**:2 (2001), 289–326. [MR](#) [Zbl](#)
- [Hollands and Wald 2002] S. Hollands and R. M. Wald, “Existence of local covariant time ordered products of quantum field in curved spacetime”, *Comm. Math. Phys.* **231**:2 (2002), 309–345. [MR](#) [Zbl](#)
- [Hörmander 1985] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren der mathematischen Wissenschaften **274**, Springer, 1985. [MR](#) [Zbl](#)
- [Kato 1961] T. Kato, “A generalization of the Heinz inequality”, *Proc. Japan Acad.* **37** (1961), 305–308. [MR](#) [Zbl](#)
- [Kato 1966] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften **132**, Springer, 1966. [MR](#) [Zbl](#)
- [Kato 1970] T. Kato, “Linear evolution equations of “hyperbolic” type”, *J. Fac. Sci. Univ. Tokyo Sect. I* **17** (1970), 241–258. [MR](#) [Zbl](#)
- [Kay 1978] B. S. Kay, “Linear spin-zero quantum fields in external gravitational and scalar fields, I: A one particle structure for the stationary case”, *Comm. Math. Phys.* **62**:1 (1978), 55–70. [MR](#)
- [Kay and Wald 1991] B. S. Kay and R. M. Wald, “Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon”, *Phys. Rep.* **207**:2 (1991), 49–136. [MR](#) [Zbl](#)
- [Leinfelder and Simader 1981] H. Leinfelder and C. G. Simader, “Schrödinger operators with singular magnetic vector potentials”, *Math. Z.* **176**:1 (1981), 1–19. [MR](#) [Zbl](#)
- [Nicolas 2002] J.-P. Nicolas, *Dirac fields on asymptotically flat space-times*, Dissertationes Math. **408**, Instytut Matematyczny Polskiej Akademii Nauk, Warsaw, 2002. [MR](#) [Zbl](#)
- [Pazy 1983] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer, 1983. [MR](#) [Zbl](#)
- [Radzikowski 1996] M. J. Radzikowski, “Micro-local approach to the Hadamard condition in quantum field theory on curved space-time”, *Comm. Math. Phys.* **179**:3 (1996), 529–553. [MR](#) [Zbl](#)
- [Sanchez and Vickers 2018] Y. S. Sanchez and J. Vickers, “Green operators for low regularity spacetimes”, *J. Phys. Conf. Ser.* **968** (2018), art. id. 012011.
- [Schmid and Griesemer 2017] J. Schmid and M. Griesemer, “Well-posedness of non-autonomous linear evolution equations in uniformly convex spaces”, *Math. Nachr.* **290**:2-3 (2017), 435–441. [MR](#) [Zbl](#)
- [Tanabe 1997] H. Tanabe, *Functional analytic methods for partial differential equations*, Monographs and Textbooks in Pure and Applied Mathematics **204**, Marcel Dekker, New York, 1997. [MR](#) [Zbl](#)
- [Vasy 2017] A. Vasy, “Essential self-adjointness of the wave operator and the limiting absorption principle on Lorentzian scattering spaces”, preprint, 2017. [arXiv](#)
- [Yajima 2011] K. Yajima, “Schrödinger equations with time-dependent unbounded singular potentials”, *Rev. Math. Phys.* **23**:8 (2011), 823–838. [MR](#) [Zbl](#)

Received 16 Oct 2018. Revised 8 Feb 2019. Accepted 6 Mar 2019.

JAN DEREZIŃSKI: jan.derezinski@fuw.edu.pl

Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Warsaw, Poland

DANIEL SIEMSEN: siemssen@uni-wuppertal.de

Department of Mathematics and Informatics, University of Wuppertal, Wuppertal, Germany

EDITORS-IN-CHIEF

- Charles L. Epstein University of Pennsylvania
cle@math.upenn.edu
- Maciej Zworski University of California at Berkeley
zworski@math.berkeley.edu

EDITORIAL BOARD

- Sir John M. Ball University of Oxford
ball@maths.ox.ac.uk
- Michael P. Brenner Harvard University
brenner@seas.harvard.edu
- Charles Fefferman Princeton University
cf@math.princeton.edu
- Susan Friedlander University of Southern California
susanfri@usc.edu
- Anna Gilbert University of Michigan
annacg@umich.edu
- Leslie F. Greengard Courant Institute, New York University, and
Flatiron Institute, Simons Foundation
greengard@cims.nyu.edu
- Yan Guo Brown University
yan_guo@brown.edu
- Claude Le Bris CERMICS - ENPC
lebris@cermics.enpc.fr
- Robert J. McCann University of Toronto
mccann@math.toronto.edu
- Michael O'Neil Courant Institute, New York University
oneil@cims.nyu.edu
- Jill Pipher Brown University
jill_pipher@brown.edu
- Johannes Sjöstrand Université de Dijon
johannes.sjostrand@u-bourgogne.fr
- Vladimir Šverák University of Minnesota
sverak@math.umn.edu
- Daniel Tataru University of California at Berkeley
tataru@berkeley.edu
- Michael I. Weinstein Columbia University
miw2103@columbia.edu
- Jon Wilkening University of California at Berkeley
wilken@math.berkeley.edu
- Enrique Zuazua DeustoTech-Bilbao, and
Universidad Autónoma de Madrid
enrique.zuazua@deusto.es

PRODUCTION

- Silvio Levy (Scientific Editor)
production@msp.org

Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

See inside back cover or msp.org/paa for submission instructions.

The subscription price for 2019 is US \$495/year for the electronic version, and \$555/year (+\$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Pure and Applied Analysis (ISSN 2578-5885 electronic, 2578-5893 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

PAA peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing
<http://msp.org/>

© 2019 Mathematical Sciences Publishers

PURE and APPLIED ANALYSIS

vol. 1 no. 2 2019

- About small eigenvalues of the Witten Laplacian 149
LAURENT MICHEL
- Semiclassical resolvent estimates for short-range L^∞ potentials 207
GEORGI VODEV
- An evolution equation approach to the Klein–Gordon operator on curved spacetime 215
JAN DEREZIŃSKI and DANIEL SIEMSEN
- The interior of dynamical extremal black holes in spherical symmetry 263
DEJAN GAJIC and JONATHAN LUK