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MULTIDIMENSIONAL NONLINEAR GEOMETRIC OPTICS FOR TRANSPORT OPERATORS WITH APPLICATIONS TO STABLE SHOCK FORMATION

JARED SPECK

In $n \geq 1$ spatial dimensions, we study the Cauchy problem for a genuinely nonlinear quasilinear transport equation coupled to a quasilinear symmetric hyperbolic subsystem of a rather general type. For an open set (relative to a suitable Sobolev topology) of regular initial data that are close to the data of a simple plane wave, we give a sharp, constructive proof of shock formation in which the transport variable remains bounded but its first-order Cartesian coordinate partial derivatives blow up in finite time. Moreover, we prove that, at least at the low derivative levels, the singularity does not propagate into the symmetric hyperbolic variables: they and their first-order Cartesian coordinate partial derivatives remain bounded, even though they interact with the transport variable all the way up to its singularity. The formation of the singularity is tied to the finite-time degeneration, relative to the Cartesian coordinates, of a system of geometric coordinates adapted to the characteristics of the transport operator. Two crucial features of the proof are that relative to the geometric coordinates, all solution variables remain smooth, and that the finite-time degeneration coincides with the intersection of the transport characteristics. Compared to prior shock formation results in more than one spatial dimension, in which the blowup occurred in solutions to quasilinear wave equations, the main new features of the present work are: (i) we develop a theory of nonlinear geometric optics for transport operators, which is compatible with the coupling and which allows us to implement a quasilinear geometric vector field method, even though the regularity properties of the corresponding eikonal function are less favorable compared to the wave equation case and (ii) we allow for a full quasilinear coupling; i.e., the principal coefficients in all equations are allowed to depend on all solution variables.

MSC2010: primary 35L67; secondary 35L45.
Keywords: blowup, characteristics, eikonal equation, eikonal function, simple wave, singularity, stable blowup, vector field method, wave breaking.

1. Introduction

The study of quasilinear hyperbolic PDE systems is one of the most classical pursuits in mathematics, and it is also among the most active. Such systems are of intense theoretical interest, in no small part due to the fact that their study lies at the core of the revered field of nonlinear hyperbolic conservation laws (more generally "balance laws"); we refer readers to [Dafermos 2010] for a detailed discussion of the history of nonlinear hyperbolic balance laws as well as a comprehensive introduction to the main results of the field and the main techniques behind their proofs, with an emphasis on the case of one spatial dimension. The subject of quasilinear hyperbolic systems is of physical interest as well since they are used to model a vast range of physical phenomena.

A fundamental issue surrounding the study of the initial value problem for such PDEs is that solutions can develop singularities in finite time, starting from regular initial data. In one spatial dimension, the theory is in a rather advanced state, and in many cases, the known well-posedness results are able to accommodate the formation of shock singularities as well as their subsequent interactions; see the aforementioned work of Dafermos. The advanced status of the one-space-dimensional theory is highly indebted to the availability of estimates in the space of functions of bounded variation (BV). In contrast, Rauch [1986] showed that for quasilinear hyperbolic systems in more than one spatial dimension, well-posedness in BV class *generally does not hold*. For this reason, energy estimates in L^2-based Sobolev spaces play an essential role in multiple spatial dimensions, and as a consequence, even the question of whether or not there is stable singularity formation (starting from regular initial data) can be exceptionally challenging. That is, in proving a constructive shock formation result in more than one spatial dimension, one cannot avoid the exacting task of deriving energy estimates that hold up to the singularity; below we will elaborate on this difficulty.

In view of the remarks above, it is not surprising that the earliest blowup results for quasilinear hyperbolic PDEs in more than one spatial dimension without symmetry assumptions were not constructive, but were instead based on proofs by contradiction, with influential contributions coming from, for example, John [1981] for a class of wave equations and Sideris [1984; 1985] for a class of hyperbolic systems in the former work and for the compressible Euler equations in the latter. The main idea of the proofs was to show that for smooth solutions with suitable initial data, certain spatially averaged quantities satisfy ordinary differential inequalities that force them to blow up, contradicting the assumption of smoothness.

Although the blowup results mentioned in the previous paragraph are compelling, their chief drawback is that they provide no information about the nature of the singularity, other than an upper bound on the solution's classical lifespan. In particular, such results are not useful if one aims to extract sharp information about the blowup mechanism and blowup time, or if one aims to uniquely continue the solution past the singularity in a weak sense. In contrast, many state-of-the-art blowup results for hyperbolic PDEs yield a detailed description of the singularity formation, even in the challenging setting of more than one spatial dimension. This is especially true for results on the formation of shocks starting from smooth initial conditions, a topic that has enjoyed remarkable progress in the last decade, as we describe in Section 1G. Our main results are in this vein, our motivation being to advance the rigorous mathematical theory of the formation of shocks. We recall that a shock singularity[1] is such that some derivative of the solution blows up in finite time, while the solution itself remains bounded. Shock singularities are of interest in part due to their rather mild nature, which leaves open the hope that one might be able to extend the solution uniquely past the shock, in a weak sense, under suitable selection criteria. In the case of the relativistic Euler equations and the compressible Euler equations in multiple spatial dimensions, this hope has been realized in the form of Christodoulou's recent breakthrough resolution [2019] of the *restricted shock development* problem without symmetry assumptions; see Section 1G2 for further discussion.

We now provide a very rough statement of our results; see Theorem 1.5 on page 453 for a more detailed summary and Theorem 10.1 on page 509 for the complete statements.

Theorem 1.1 (stable shock formation (very rough version)). *In an* ***arbitrary number*** *of spatial dimensions, there are many quasilinear hyperbolic PDE systems, comprising a transport equation satisfying an appropriate genuinely nonlinear-type assumption coupled to a symmetric hyperbolic subsystem, such that the following occurs: there exists an open set of initial data* ***without symmetry assumptions*** *such that the transport variable remains bounded but its first derivatives blow up in finite time. More precisely, the derivatives of the transport variable in directions* ***tangent*** *to the transport characteristics remain bounded, while its derivative with respect to any unit-length* ***transversal*** *vector field blows up. Moreover, the singularity does not propagate into the symmetric hyperbolic variables; they remain bounded, as do their first derivatives in* ***all*** *directions.*[2]

Remark 1.2 (rescaling the transversal derivative so as to "cancel" the blowup). We note already that a key part of the proof is showing the derivative of the transport variable in the transversal direction $\breve{X}$ also remains bounded. This does not contradict Theorem 1.1 for the following reason: the vector field $\breve{X}$ is constructed so that its Cartesian components go to 0 as the shock forms, in a manner that exactly compensates for the blowup of an "order-unity-length" transversal derivative of the transport variable. Roughly, the situation can be described as follows, where Ψ is the transport variable and the remaining quantities will be rigorously defined later in the article: $|X\Psi|$ blows up,[3] $|\breve{X}\Psi|$ remains bounded, $\breve{X} = \mu X$, and the weight μ vanishes for the first time at the shock; one could say that $|X\Psi|$ blows up like C/μ as $\mu \downarrow 0$, where C is the size of $|\breve{X}\Psi|$ at the shock; see Section 1F4 for a more in-depth discussion of this point.

Remark 1.3 (the heart of the proof and the kind of initial data under study). The heart of the proof of Theorem 1.1 is to control the singular terms and to show that the shock actually happens, i.e., that chaotic interactions do not prevent the shock from forming or cause a more severe kind of singularity. In an effort to focus on only the singularity formation, we have chosen to study the simplest nontrivial set of initial data to which our methods apply: perturbations of the data corresponding to simple plane symmetric waves (see Section 1D for further discussion), where we assume plentiful initial Sobolev regularity.

[1] The formation of a shock is sometimes referred to as "wave breaking".

[2] Our proof allows for the possibility that the second-order Cartesian coordinate partial derivatives of the symmetric hyperbolic variables might blow up at the locations of the transport variable singularities.

[3] Here and throughout, if Z is a vector field and f is a scalar function, then $Zf := Z^{\alpha}\,\partial_{\alpha} f$ is the derivative of f in the direction Z.

The corresponding solutions do not experience dispersion, so there are no time or radial weights in our estimates. We will describe the initial data in more detail in Section 1F3.

Remark 1.4 (extensions to other kinds of hyperbolic subsystems). From our proof, one can infer that the assumption of symmetric hyperbolicity for the subsystem from Theorem 1.1 is in itself not important; we therefore anticipate that similar shock formation results should hold for systems comprising quasilinear transport equations coupled to many other types of hyperbolic subsystems, such as wave equations or regularly hyperbolic, in the sense of [Christodoulou 2000], subsystems.

1A. ***Paper outline.*** • In the remainder of Section 1, we give a more detailed description of our main results, summarize the main ideas behind the proofs, place our work in context by discussing prior works on shock formation, and summarize some of our notation.

• In Section 2, we precisely define the class of systems to which our main results apply.

• In Section 3, we construct the majority of the geometric objects that play a role in our analysis. We also derive evolution equations for some of the geometric quantities.

• In Section 4, we derive energy identities.

• In Section 5, we state the number of derivatives that we use to close our estimates, state our size assumptions on the data, and state bootstrap assumptions that are useful for deriving estimates.

• In Section 6, we derive pointwise estimates for solutions to the evolution equations and their derivatives, up to top order.

• In Section 7, we derive some properties of the change of variables map from geometric to Cartesian coordinates.

• In Section 8, which is the main section of the paper, we derive a priori estimates for all of the quantities under study.

• In Section 9, we provide some continuation criteria that, in the last section, we use to show that the solution survives up to the shock.

• In Section 10, we state and prove the main theorem.

1B. ***The role of nonlinear geometric optics in proving Theorem 1.1.*** In prior constructive stable shock formation results in more than one spatial dimension (which we describe in Section 1G2), the blowup occurred in the derivatives of a solution to a quasilinear wave equation. In the present work, the blowup occurs in the derivatives of the solution to the transport equation. The difference is significant in that to obtain a sharp picture of shock formation, one must rely on a geometric version of the vector field method that is precisely tailored to the family of characteristics whose intersection is tied to the blowup. The key point is that the basic regularity properties of the characteristics and the geometric vector fields (which seem essential for the proofs) that are adapted to them are different in the wave equation and transport equation cases. In fact, in the transport equation case, *the Cartesian components of the geometric vector fields are one degree less differentiable compared to the wave equation case*; see (1F.1) for the set of geometric vector fields that we use in the present article. Although one might anticipate that the reduced regularity will lead to new complications, in the present paper, we are able to handle the loss of regularity using a strategy that, in fact, leads to simplifications compared to the wave equation case: roughly by treating the first-order *Cartesian* coordinate partial derivatives of the symmetric hyperbolic variables as new unknowns, we are able to *allow the loss of regularity* in the geometric vector fields. The fact that we can allow the loss is ultimately tied to the fact that in the present article, the variable that forms a shock is a solution to a first-order equation (in contrast to the case of wave equations). We emphasize that our approach is considerably different from, and in some ways simpler than, approaches that have been taken in proving shock formation in solutions to quasilinear wave equations, a context in which the known proofs fundamentally rely on avoiding[4] the loss of regularity; while the special structure of wave equations indeed allows one to avoid the loss of regularity in the eikonal function and the corresponding geometric vector fields, the known approaches to avoiding the loss introduce enormous technical complications into the analysis. We will discuss these fundamental points in more detail in Sections 1E and 1F5.

Although the blowup mechanism for solutions to the transport equations under study is broadly similar to the Riccati-type mechanism that drives singularity formation in the simple one-space-dimensional example of Burgers' equation[5] (see Section 1D for related discussion), the proof of our main theorem is much more complicated, owing in part to the aforementioned difficulty of having to derive energy estimates in multiple spatial dimensions. The overall strategy of our proof is to construct a system of geometric coordinates adapted to the transport characteristics, *relative to which the solution remains smooth*, in part because the geometric coordinates "hide"[6] the Riccati-type term mentioned above. In more than one spatial dimension, the philosophy of constructing geometric coordinates to regularize the problem of shock formation seems to have originated in Alinhac's work [1999a; 1999b; 2001] on quasilinear wave equations; see Section 1G2 for further discussion. As will become abundantly clear, our construction of the geometric coordinates and other related quantities is tied to the following fundamental ingredient in our approach: our development of a theory of *nonlinear geometric optics for quasilinear transport equations*, tied to an eikonal function, that is compatible with full quasilinear coupling to the symmetric hyperbolic subsystem. We use nonlinear geometric optics to construct geometric vector field differential operators (see (1F.1)) adapted to the characteristics as well as to detect the singularity formation. By "compatible", we mean, especially, from the perspective of regularity considerations. Indeed, in any situation in which one uses nonlinear geometric optics to study a quasilinear hyperbolic PDE system, *one must ensure that the regularity of the corresponding eikonal function is consistent with that of the solution*. By "full quasilinear coupling", we mean that in the systems that we study, the *principal coefficients in all equations are allowed to depend on all solution variables*.

[4] Actually, as we describe in Section 1G2, Alinhac's approach handles the loss of regularity through a Nash–Moser iteration scheme. However, Alinhac's Nash–Moser approach suffers from some technical limitations that seem to obstruct one's ability to track the behavior of the solution up to the boundary of the maximal development. In turn, this poses an obstacle to even properly setting up the shock development problem; see Section 1G2 for further discussion.

[5] The Riccati term appears after one spatial differentiation of Burgers' equation.

[6] In one spatial dimension, this is sometimes referred to as "straightening out the characteristics" via a change of coordinates.

Upon introducing nonlinear geometric optics into the problem, we encounter the following key difficulty, which we alluded to above:

> Some of the geometric vector fields that we construct (see (1F.1)) have Cartesian components that are one degree less differentiable than the transport variable, as we explain in Section 1F5.

On the one hand, due to the full quasilinear coupling, it seems that we must use the geometric vector fields when commuting the symmetric hyperbolic subsystem to obtain higher-order estimates; this allows us to avoid generating uncontrollable commutator error terms involving "bad derivatives" (i.e., in directions transversal to the transport characteristics) of the shock-forming transport variable. On the other hand, the loss of regularity of the Cartesian components of the geometric vector fields leads, at the top-order derivative level, to commutator error terms in the symmetric hyperbolic subsystem that are uncontrollable in that they have insufficient regularity. To overcome this difficulty, we employ the following strategy:

> We never commute the symmetric hyperbolic subsystem a top-order number of times with a pure string of geometric vector fields; instead, we first commute the symmetric hyperbolic subsystem with a single Cartesian coordinate partial derivative, and then follow up the Cartesian derivative with commutations by the geometric vector fields.

The strategy above allows us to avoid the loss of a derivative, but it generates commutator error terms depending on a single Cartesian coordinate partial derivative, which are dangerous because they are transversal to the transport characteristics. Indeed, the first-order Cartesian coordinate partial derivatives of the transport variable blow up at the shock. Fortunately, by using a weight[7] adapted to the characteristics, we are able to control such error terms featuring a single Cartesian differentiation, all the way up to the singularity.

We close this subsection by providing some remarks on prior implementations of nonlinear geometric optics in the study of the maximal development[8] of initial data for quasilinear hyperbolic PDEs without symmetry assumptions. The approach was pioneered by Christodoulou and Klainerman [1993] in their celebrated proof of the stability of Minkowski spacetime as a solution to the Einstein vacuum equations.[9] Since perturbative global existence results for hyperbolic PDEs typically feature estimates with "room to spare", in many cases, it is possible to close the proofs by relying on a version of *approximate* nonlinear geometric optics, which features approximate eikonal functions whose level sets approximate the characteristics. The advantage of using approximate eikonal functions is that is that their regularity theory is typically very simple. For example, such an approach was taken by Lindblad and Rodnianski [2010] in their proof of the stability of the Minkowski spacetime relative to wave coordinates. Their proof was less precise than Christodoulou and Klainerman's but significantly shorter since, unlike Christodoulou and Klainerman, Lindblad and Rodnianski relied on approximate eikonal functions whose level sets were

[7] The weight is the quantity μ from Remark 1.2, and we describe it in detail below.

[8] The maximal development is, roughly, the largest possible classical solution that is uniquely determined by the data. Readers can consult [Sbierski 2016; Wong 2013] for further discussion.

[9] Roughly, [Christodoulou and Klainerman 1993] contains a small-data global existence result for Einstein's equations.

standard Minkowski light cones; in particular, Lindblad and Rodnianski were able to close their proof using C^∞ vector fields tied to the background Minkowskian geometry.

The use of eikonal functions for proving shock formation for quasilinear wave equations in more than one spatial dimension without symmetry assumptions was pioneered by Alinhac [1999a; 1999b; 2001], and his approach was later remarkably sharpened/extended by Christodoulou [2007]. In contrast to global existence problems, in proofs of shock formation without symmetry assumptions, the *use of an eikonal function adapted to the true characteristics* (*as opposed to approximate ones*) *seems essential*, since the results yield that the singularity formation exactly coincides with the intersection of the characteristics. One can also draw an analogy between works on shock formation and works on low regularity well-posedness for quasilinear wave equations, such as [Klainerman and Rodnianski 2003; 2005; Smith and Tataru 2005; Klainerman et al. 2015], where the known proofs fundamentally rely on eikonal functions whose levels sets are true characteristics.

1C. ***A more precise statement of the main results.*** For the systems under study, we assume that the number of spatial dimensions is $n \geq 1$, where n is arbitrary. For convenience, we study the dynamics of solutions in spacetimes of the form $\mathbb{R} \times \Sigma$, where

$$\Sigma = \mathbb{R} \times \mathbb{T}^{n-1} \tag{1C.1}$$

is the spatial manifold and $\mathbb{T}^{n-1}$ is the standard $(n-1)$-dimensional torus (i.e., $[0, 1)^{n-1}$ with the endpoints identified and equipped with the usual smooth orientation). The factor $\mathbb{T}^{n-1}$ in (1C.1) will correspond to perturbations away from plane symmetry. Our assumption on the topology of Σ is for technical convenience only; since our results are localized in spacetime, one could derive similar stable blowup results for arbitrary spatial topology.[10] Throughout, $\{x^\alpha\}_{\alpha=0,\ldots,n}$ are a fixed set of Cartesian spacetime coordinates on $\mathbb{R}\times\Sigma$, where $t := x^0 \in \mathbb{R}$ is the time coordinate, $\{x^i\}_{i=1,\ldots,n}$ are the spatial coordinates on Σ, $x^1 \in \mathbb{R}$ is the "noncompact space coordinate", and $\{x^i\}_{i=2,\ldots,n}$ are standard locally defined coordinates on $\mathbb{T}^{n-1}$ such that $(\partial_2, \ldots, \partial_n)$ is a positively oriented frame. We denote the Cartesian coordinate partial derivative vector fields by $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$, and we sometimes use the alternate notation $\partial_t := \partial_0$. Note that the vector fields $\{\partial_\alpha\}_{\alpha=0,\ldots,n}$ can be globally defined so as to form a smooth frame, even though the $\{x^i\}_{i=2,\ldots,n}$ are only locally defined. For mathematical convenience, in our main results, we consider *nearly plane symmetric solutions*, where by our conventions, exact plane symmetric solutions depend only on t and x^1. We now roughly summarize our main results; see Theorem 10.1 for precise statements.

Theorem 1.5 (stable shock formation (rough version)).
Assumptions. *Consider the following coupled system*[11] *with initial data posed on the constant-time hypersurface* $\Sigma_0 := \{0\} \times \mathbb{R} \times \mathbb{T}^{n-1} \simeq \mathbb{R} \times \mathbb{T}^{n-1}$:

$$L^\alpha(\Psi, v)\, \partial_\alpha \Psi = 0, \tag{1C.2a}$$

$$A^\alpha(\Psi, v)\, \partial_\alpha v = 0, \tag{1C.2b}$$

[10] However, assumptions on the data that lead to shock formation generally must be adapted to the spatial topology.

[11] Throughout we use Einstein's summation convention. Greek lowercase "spacetime" indices vary over $0, 1, \ldots, n$, while Latin lowercase "spatial" indices vary over $1, 2, \ldots, n$.

where Ψ is a scalar function, $v = (v^1, \dots, v^M)$ is an array (M is arbitrary), and the A^α are symmetric $M \times M$ matrices. Assume that $L^1(\Psi, v)$ satisfies a genuinely nonlinear-type condition tied to its dependence on Ψ (specifically condition (2B.1)) and that for small Ψ and v, the constant-time hypersurfaces Σ_t and the $\mathcal{P}_u$ are ***spacelike***[12] *for the subsystem (1C.2b). Here and throughout, the $\mathcal{P}_u$ are L-characteristics, which are the family of (solution-dependent) hypersurfaces equal to the level sets of the eikonal function u, that is, the solution to the eikonal equation (see footnote 3 regarding the notation) $Lu = 0$ with the initial condition $u|_{\Sigma_0} = 1 - x^1$.*

To close the proof, we make the following assumptions on the data, which we propagate all the way up to the singularity:

> *Along Σ_0, the array v, all of its derivatives, and the $\mathcal{P}_u$-tangential derivatives of Ψ are small* ***relative***[13] *to quantities constructed out of a first-order $\mathcal{P}_u$-transversal derivative of Ψ (see Section 5D for the precise smallness assumptions, which involve geometric derivatives). Moreover, along $\mathcal{P}_0$, all derivatives of v up to top order are relatively small.*

Conclusions. There exists an open set (relative to a suitable Sobolev topology) of data that are close to the data of a simple plane wave (where a simple plane wave is such that $\Psi = \Psi(t, x^1)$ and $v \equiv 0$), given along the unity-thickness subset Σ_0^1 of Σ_0 and a finite portion of $\mathcal{P}_0$, such that the solution behaves as follows:

> $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$ *blows up in finite time, while $|\Psi|$, $\{|v^J|\}_{1\le J\le M}$, and $\{|\partial_\alpha v^J|\}_{0\le\alpha\le n, 1\le J\le M}$ remain uniformly bounded.*

The blowup coincides with the intersection of the $\mathcal{P}_u$, which in turn is precisely characterized by the vanishing of the inverse foliation density $\mu := 1/\partial_t u$ of the $\mathcal{P}_u$, which satisfies $\mu|_{\Sigma_0^1} \approx 1$; see Figure 1 for a picture in which a shock is about to form (in the region up top, where μ is small). Moreover, one can complete (t, u) to form a geometric coordinate system $(t, u, \vartheta^2, \dots, \vartheta^n)$ on spacetime with the following key property, central to the proof:

> *No singularity occurs in Ψ, $\{v^J\}_{1\le J\le M}$, $\{\partial_\alpha v^J\}_{0\le\alpha\le n, 1\le J\le M}$, or their derivatives with respect to the geometric coordinates*[14] *up to top order.*

Put differently, the problem of shock formation can be transformed into an equivalent problem in which one proves nondegenerate estimates relative to the geometric coordinates and, at the same time, proves that the geometric coordinates degenerate in a precise fashion with respect to the Cartesian coordinates as $\mu \downarrow 0$.

Remark 1.6 (nontrivial interactions all the way up to the singularity). We emphasize that in Theorem 1.5, v can be nonzero at the singularity in $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$. This means, in particular, that the problem cannot be reduced to the study of blowup for the much easier case of a decoupled scalar transport equation.

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[12] This means that $A^\alpha \omega_\alpha$ is positive definite, where the one-form ω is conormal to the surface and satisfies $\omega_0 > 0$.

[13] We also assume an absolute smallness condition on $\|\Psi\|_{L^\infty(\Sigma_0)}$.

[14] In practice, we will derive estimates for the derivatives of the solution with respect to the vector fields depicted in Figure 1.

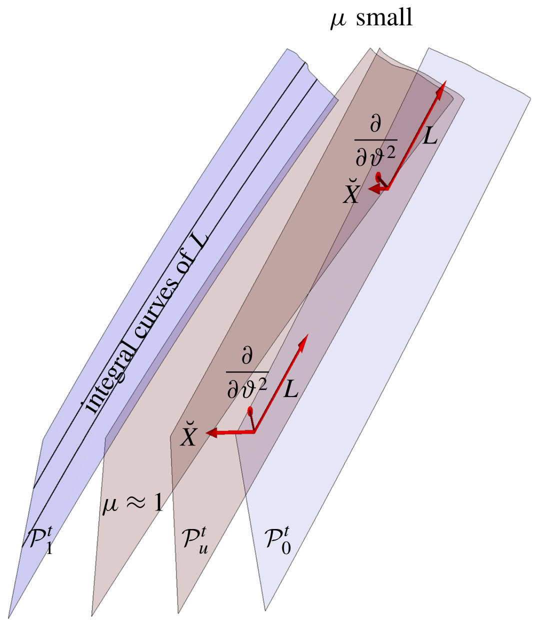

Figure 1. The dynamics until close to the time of the shock when $n = 2$.

Remark 1.7 (extensions to allow for semilinear terms). We expect that the results of Theorem 1.5 could be extended to allow for the presence of arbitrary smooth semilinear terms on the right-hand sides of (1C.2a)–(1C.2b) that are functions of (Ψ, v). The extension would be straightforward to derive for semilinear terms that vanish when $v = 0$ (for example, terms of type $v \cdot \Psi$). The reason is that our main results imply that such semilinear terms remain small, in suitable norms, up to the shock. In fact, such semilinear terms completely vanish for the exact simple waves whose perturbations we treat in Theorem 1.5; see Section 1D for further discussion of simple waves. Consequently, a set of initial data similar to the one from Theorem 1.5 would also lead to the formation of a shock in the presence of such semilinear terms. In contrast, for semilinear terms that do not vanish when $v = 0$ (for example, terms of type Ψ^2), the analysis would be more difficult and the assumptions on the data might have to be changed to produce shock-forming solutions. In particular, such semilinear terms can, at least for data with Ψ large, radically alter the behavior of some solutions. This can be seen in the simple model problem of the inhomogeneous Burgers-type equation $\partial_t \Psi + \Psi\, \partial_x \Psi = \Psi^2$. This equation admits the family of ODE-type blowup solutions $\Psi_{(\mathrm{ODE});T}(t) := (T - t)^{-1}$, whose singularity is much more severe than the shocks that typically form when the semilinear term Ψ^2 is absent.

Remark 1.8 (description of a portion of the maximal development). We expect that the approach we take in proving our main theorem is precise enough that it can be extended to yield sharp information about the behavior of the solution up the boundary of the maximal development, as Christodoulou [2007, Chapter 15] did in his related work (which we describe in more detail in Section 1G2). For brevity, we do not pursue this issue in the present article. However, in the detailed version of our main results (i.e., Theorem 10.1), we set the stage for the possible future study of the maximal development by proving a "one-parameter family of results", indexed by $U_0 \in (0, 1]$; one would need to vary U_0 to study the maximal development. Here and throughout, U_0 corresponds to an initial data region $\Sigma_0^{U_0}$ of thickness U_0; see Figure 2 on page 472 and Section 1F2 for further discussion. For $U_0 = 1$, which is implicitly assumed in Theorem 1.5, a shock forms in the maximal development of the data given along[15] $\Sigma_0^{U_0} \cup \mathcal{P}_0$. However, for small U_0, a shock does not necessarily form in the maximal development of the data given along $\Sigma_0^{U_0} \cup \mathcal{P}_0$ within the amount of time that we attempt to control the solution.

1D. ***Further discussion on simple plane symmetric waves.*** Theorem 1.5 shows, roughly, that the well-known stable blowup of $\partial_x \Psi$ in solutions to the one-space-dimensional Burgers' equation

$$\partial_t \Psi + \Psi\, \partial_x \Psi = 0 \tag{1D.1}$$

is stable under a full quasilinear coupling of (1D.1) to other hyperbolic subsystems, under perturbations of the coefficients in the transport equation, and under increasing the number of spatial dimensions. We now further explain what we mean by this. A special case of Theorem 1.5 occurs when $v \equiv 0$ and Ψ depends only on t and x^1 (plane symmetry). In this simplified context, the blowup of $\max_{\alpha=0,1} |\partial_\alpha \Psi|$ for solutions to (1C.2a) can be proved using a simple argument based on the method of characteristics, similar to the argument that is typically used to prove blowup in the case of Burgers' equation. Solutions with $v \equiv 0$ are sometimes referred to as *simple waves* since they can be described by a single nonzero scalar component. From this perspective, we see that Theorem 1.5 yields the stability of simple plane wave shock formation for the transport variable in solutions to the system (1C.2a)–(1C.2b).

1E. ***The main new ideas behind the proof.*** The proof of Theorem 1.5 is based in part on ideas used in earlier works on shock formation in more than one spatial dimension. We review these works in Section 1G. Here we summarize the two most novel aspects behind the proof of Theorem 1.5.

• (nonlinear geometric optics for transport equations) As in all prior shock formation results in more than one spatial dimension, our proof relies on nonlinear geometric optics, that is, the eikonal function u. The use of an eikonal function is essentially the method of characteristics implemented in more than one spatial dimension. All of the prior works were such that the blowup occurred in a solution to a quasilinear wave equation and thus the theory of nonlinear geometric optics was adapted to the corresponding "wave characteristics". In this article, we advance the theory of nonlinear geometric optics for transport equations. Although the theory is simpler in some ways, compared to the case of wave equations, it is, as our prior discussion has suggested, also more degenerate in the following sense: *the regularity theory for the eikonal function u is less favorable in that u is one degree less differentiable in some directions compared to the case of wave equations*. We therefore must close the proof of Theorem 1.5 under this decreased differentiability. We defer further discussion of this point until Section 1F5. Here, we will simply further motivate our use of nonlinear geometric optics in proving shock formation.

First, we note that in more than one spatial dimension, it does not seem possible to close the proof using only the Cartesian coordinates; indeed, Theorem 1.5 shows that the blowup of Ψ precisely corresponds to

[15] Actually, we only need to specify the data along the subset $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{A}_*^{-1}}$ of $\Sigma_0^{U_0} \cup \mathcal{P}_0$; see Section 1F2 for discussion of this subset and the data-dependent parameter $\mathring{A}_*$.

the vanishing of the inverse foliation density μ of the characteristics, which is equivalent to the blowup of $\partial_t u$. Hence, it is difficult to imagine how a sharp, constructive proof of stable blowup would work without referencing an eikonal function. In view of these considerations, we construct a geometric coordinate system $(t, u, \vartheta^2, \dots, \vartheta^n)$ adapted to the transport operator vector field L and prove that Ψ, v, $V_\alpha := \partial_\alpha v$, and their geometric coordinate partial derivatives remain regular all the way up to the singularity in $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$. The blowup of $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$ occurs because the change of variables map between geometric and Cartesian coordinates *degenerates*, which is in turn tied to the vanishing of μ; the Jacobian determinant of this map is in fact proportional to μ; see Lemma 3.25. The coordinate t is the standard Cartesian time function. The geometric coordinate function u is the eikonal function described in Theorem 1.5. The initial condition $u|_{\Sigma_0} = 1 - x^1$ is adapted to the approximate plane symmetry of the initial data. We similarly construct the "geometric torus coordinates" $\{\vartheta^j\}_{j=2,\dots,n}$ by solving $L\vartheta^j = 0$ with the initial condition $\vartheta^j|_{\Sigma_0} = x^j$. The main challenge is to derive regular estimates relative to the geometric coordinates for all quantities, including the solution variables and geometric quantities constructed out of the geometric coordinates.

- (full quasilinear coupling) Because we are able to close the proof with decreased regularity for u (compared to the case of wave equations), we are able to handle full quasilinear coupling between all solution variables. This is an interesting advancement over prior works, where the principal coefficients in the evolution equation for the shock-forming variable were allowed to depend only on the shock-forming variable itself and on other solution variables that satisfy a wave equation with the *same principal part* as the shock-forming variable; i.e., in (1C.2a), we allow $L^\alpha = L^\alpha(\Psi, v)$, where the principal part of the evolution equation (1C.2b) for v is *distinct* (by assumption) from L.

1F. ***A more detailed overview of the proof.*** In this subsection, we provide an overview of the proof of our main results. Our analysis is based in part on some key ideas originating in earlier works, which we review in Section 1G. Our discussion in this subsection is, at times, somewhat loose; our rigorous analysis begins in Section 2.

1F1. *Setup and geometric constructions.* In Sections 2–3, we construct the geometric coordinate system $(t, u, \vartheta^2, \dots, \vartheta^n)$ described in Section 1B, which is central for all that follows. We also construct many related geometric objects, including the inverse foliation density μ (see Definition 3.5 for the precise definition) of the characteristics $\mathcal{P}_u$ of the eikonal function u, i.e., of the level sets of u. As we mentioned earlier, our overall strategy is to show that the solution remains regular with respect to the geometric coordinates, all the way up to the top derivative level, to show that μ vanishes in finite time, and to show that the vanishing of μ is exactly tied to the blowup of $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$. It turns out that when deriving estimates, it is important to replace the geometric coordinate partial derivative vector field $\frac{\partial}{\partial u}$ with a Σ_t-tangent vector field that we denote by $\breve{X}$, which is similar to $\frac{\partial}{\partial u}$ but generally not parallel to it; see Figure 1 on page 455 for a picture of $\breve{X}$. In the context of the present paper, the main advantage of $\breve{X}$ is that it enjoys the following key property: the vector field $X = \mu^{-1}\breve{X}$ has Cartesian components that remain uniformly bounded, all the way up to the shock. Put differently, we have $\breve{X} = \mu X$, where we will show that X is a vector field of order-unity Euclidean length (and thus the Euclidean length of $\breve{X}$ is $\mathcal{O}(\mu)$). We further explain the significance of this in Section 1F4, when we outline the proof that the shock forms. In total, when deriving estimates for the derivatives of quantities, we differentiate them with respect to elements of the vector field frame

$$\mathscr{Z} := \{L, \breve{X}, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}, \tag{1F.1}$$

which spans the tangent space of spacetime at each point with $\mu > 0$. Here, ${}^{(i)}\Theta := \frac{\partial}{\partial \vartheta^i}$, L is the vector field from (1C.2a) and, by construction, we have $L = \frac{\partial}{\partial t}$ (see (3C.5)). The vector fields L and ${}^{(i)}\Theta$ are tangent to the $\mathcal{P}_u$, while $\breve{X}$ is transversal and normalized by $\breve{X} u = 1$ (see (3C.6)); see Figure 1 on page 455 for a picture of the frame. Note that since $\breve{X}$ is of length $\mathcal{O}(\mu)$, the uniform boundedness of $|\breve{X}\Psi|$ is consistent with the formation of a singularity in $|X\Psi|$ and thus in the Cartesian coordinate partial derivatives of Ψ when $\mu \downarrow 0$; see Section 1F4 for further discussion of this point.

We now highlight a crucial ingredient in our proof, alluded to earlier: we treat the Cartesian coordinate partial derivatives of v^J as independent unknowns V_α^J, defined by

$$V_\alpha^J := \partial_\alpha v^J. \tag{1F.2}$$

As we stressed already in Section 1B, our reliance on V_α^J *allows us to avoid commuting* (1C.2b) *up to top order with elements of* $\mathscr{Z}$, which allows us to avoid certain top-order commutator terms that would result in the loss of a derivative. Moreover, as we noted in Theorem 1.5, a key aspect of our framework is our proof that the quantities V_α remain bounded up to the singularity in $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$. To achieve this, we will control V_α by studying its evolution equation subsystem $A^\beta \partial_\beta V_\alpha = -(\partial_\alpha A^\beta) V_\beta$, whose inhomogeneous terms are controllable under the scope of our approach.

1F2. *A more precise description of the spacetime regions under study.* For convenience, we study only the future portion of the solution that is completely determined by the data lying in the subset $\Sigma_0^{U_0} \subset \Sigma_0$ of thickness U_0 and on a portion of the transport characteristic $\mathcal{P}_0$, where $0 < U_0 \leq 1$ is a parameter, fixed until Theorem 10.1; see Figure 2 on page 472. We will study spacetime regions such that $0 \leq u \leq U_0$, where u is the eikonal function described above. We have introduced the parameter U_0 because one would need to allow U_0 to vary in order to study the behavior of the solution up the boundary of the maximal development, as we mentioned in Remark 1.8.

In our analysis, we will use a bootstrap argument in which we only consider times t with $0 \leq t < 2\mathring{A}_*^{-1}$, where $\mathring{A}_* > 0$ is a data-dependent parameter described in Section 1F3 (see also Definition 5.1). Our main theorem shows that if $U_0 = 1$, then a shock forms at a time equal to a small perturbation of $\mathring{A}_*^{-1}$; see Section 1F4 for an outline of the proof. For this reason, in proving our main results, we only take into account the portion of the data lying in $\Sigma_0^{U_0}$ and in the subset $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$ of the characteristic $\mathcal{P}_0$; from domain-of-dependence considerations, one can infer that only this portion can influence the solution in the regions under study.

Remark 1.9. For the remainder of Section 1F, we will suppress further discussion of U_0 by setting $U_0 = 1$.

1F3. *Data-size assumptions, bootstrap assumptions, and pointwise estimates.* In Section 5, we state our assumptions on the data and formulate bootstrap assumptions that are useful for deriving estimates. Our assumptions on the data involve the parameters $\mathring{\alpha} > 0$, $\mathring{\epsilon} \geq 0$, $\mathring{A} > 0$, and $\mathring{A}_* > 0$, where, for our proofs to close, $\mathring{\alpha}$ must be chosen to be small in an absolute sense and $\mathring{\epsilon}$ must be chosen to be small in a *relative sense* compared to $\mathring{A}^{-1}$ and $\mathring{A}_*$ (see Section 5D for a precise description of the required smallness). The following remarks capture the main ideas behind the data-size parameters:

(1) $\mathring{\alpha} = \|\Psi\|_{L^\infty(\Sigma_0^1)}$ is the size of Ψ.

(2) $\mathring{\epsilon}$ is the size, in appropriate norms, of the derivatives of Ψ up to top order in which *at least one* $\mathcal{P}_u$*-tangential differentiation occurs*, and of v, V and *all of their derivatives* up to top order with respect to the elements of the vector field frame $\mathscr{Z}$ defined in (1F.1). We emphasize that we will study perturbations of plane symmetric shock-forming solutions such that $\mathring{\epsilon} = 0$. That is, the case $\mathring{\epsilon} = 0$ corresponds to a plane symmetric simple wave in which $v \equiv 0$. We state the total number of derivatives that we use to close the estimates in Sections 5A and 5B2. We also highlight that to close our proof, we never need to differentiate any quantity with more than one copy of the $\mathcal{P}_u$-transversal vector field $\breve{X}$. This approach is possible in part because of the following crucial fact, proved in Lemma 3.22: commuting the elements of the frame $\mathscr{Z}$ with each other yields a vector field belonging to span$\{{}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$.

(3) $\mathring{A} = \|\breve{X}\Psi\|_{L^\infty(\Sigma_0^1)}$ is the size of the $\mathcal{P}_u$-transversal derivative of Ψ.

(4) $\mathring{A}_* = \sup_{\Sigma_0^1}[\mathcal{G}\breve{X}\Psi]_-$ is a modified measure of the size of the $\mathcal{P}_u$-transversal derivative of Ψ, where $\mathcal{G} \neq 0$ is a coefficient determined by the nonlinearities and $[f]_- := |\min\{f, 0\}|$.

(5) When $t = 0$, other geometric quantities that we use in studying solutions obey similar size estimates, where any differentiation of a quantity with respect to a $\mathcal{P}_u$-tangential vector field leads to $\mathcal{O}(\mathring{\epsilon})$-smallness; see Lemma 5.5. A crucial exception occurs for $L\mu$, which initially is of relatively large size $\mathcal{O}(\mathring{A})$ in view of its evolution equation $L\mu \sim \breve{X}\Psi + \cdots$ (see (3G.1a) for the precise evolution equation).

(6) The relative smallness of $\mathring{\epsilon}$ corresponds to initial data that are close to that of a simple plane symmetric wave, as we described in Section 1D.

One of the main steps in our analysis is to propagate the size assumptions above all the way up to the shock. To this end, on a region of the form $(t, u, \vartheta) \in [0, T_{(\text{Boot})}) \times [0, U_0] \times \mathbb{T}^{n-1}$, we make L^∞-type bootstrap assumptions that capture the expectation that the size assumptions stated above hold. In particular, *the bootstrap assumptions capture our expectation that no singularity will form in any quantity relative to the geometric coordinates*. Moreover, since $V_\alpha^J = \partial_\alpha v^J$, the bootstrap assumptions for the smallness[16] of V *capture our expectation that the Cartesian coordinate partial derivatives of* v *should remain bounded*; indeed, this is a key aspect of our proof that we use to control various error terms depending on V. As we mentioned earlier, a crucial point is that we have set the problem up so that the shock forms at time $T_{(\text{Lifespan})} < 2\mathring{A}_*^{-1}$. Therefore, we make the assumption

$$0 < T_{(\text{Boot})} < 2\mathring{A}_*^{-1}, \tag{1F.3}$$

which leaves us with ample margin of error to show that a shock forms. In particular, in view of (1F.3), we can bound factors of t, $\exp(t)$, etc. by a constant $C > 0$ depending on $\mathring{A}_*^{-1}$, and the estimates will

[16]We note that the bootstrap assumptions refer to a parameter $\varepsilon > 0$ that, in our main theorem, we will show is controlled by $\mathring{\epsilon}$; for brevity, we will avoid further discussion of ε until Section 5C2.

close as long as $\mathring{\epsilon}$ is sufficiently small; see Section 1H for further discussion on our conventions regarding the dependence of constants C.

In Section 6, with the help of the bootstrap assumptions and data-size assumptions described above, we commute all evolution equations, including (1C.2a)–(1C.2b) and evolution equations for μ and related geometric quantities, with elements of the $\mathscr{Z}$ up to top order and derive pointwise estimates for the error terms. Actually, due to the special structures of the equations relative to the geometric coordinates, *we never need to commute the evolution equations satisfied by v, V, or μ with the transversal vector field $\breve{X}$*. Moreover, for the other geometric quantities, we need to commute their evolution equations *at most once* with $\breve{X}$. We clarify, however, that we commute all equations many times with the elements of the $\mathcal{P}_u$-tangential subset $\mathscr{P} := \{L, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$.

1F4. *Sketch of the formation of the shock.* Let us assume that the bootstrap assumptions and pointwise estimates described in Section 1F3 hold for a sufficiently long amount of time. We will sketch how they can be used to give a simple proof of shock formation, that is, that $\mu \downarrow 0$ and $\partial\Psi$ blows up. The main estimates in this regard are provided by Lemma 6.8; here we sketch them. First, using (3G.1a), the bootstrap assumptions, and the pointwise estimates, we deduce the following evolution equation for the inverse foliation density: $L\mu(t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](t, u, \vartheta) + \cdots$, where the "blowup coefficient" $\mathcal{G} \neq 0$ was described in Section 1F3 and "$\cdots$" denotes small error terms, which we ignore here. Next, we note the following pointwise estimate, which falls under the scope of the discussion in Section 1F3: $L(\mathcal{G}\breve{X}\Psi) = \cdots$ (smallness is gained since L is a $\mathcal{P}_u$-tangential differentiation). Recalling that $L = \frac{\partial}{\partial t}$, we use the fundamental theorem of calculus and the smallness of $L(\mathcal{G}\breve{X}\Psi)$ to deduce $[\mathcal{G}\breve{X}\Psi](t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \cdots$. Inserting this estimate into the one above for $L\mu$, we find that $L\mu(t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \cdots$. From the fundamental theorem of calculus and the initial condition $\mu(0, u, \vartheta) = 1 + \cdots$, we obtain $\mu(t, u, \vartheta) = 1 + t[\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \cdots$. From this estimate and the definition of $\mathring{A}_*$, we obtain $\min_{(u,\vartheta)\in[0,1]\times\mathbb{T}^{n-1}} \mu(t, u, \vartheta) = 1 - t\mathring{A}_* + \cdots$. Hence, μ vanishes for the first time at $T_{(\text{Lifespan})} = \mathring{A}_*^{-1} + \cdots$, as desired. Moreover, the reasoning used above can easily be extended to show that $|\breve{X}\Psi|(t, u, \vartheta) \gtrsim 1$ at any point (t, u, ϑ) such that $\mu(t, u, \vartheta) < \frac{1}{4}$. Recalling that $\breve{X} = \mu X$, where X has order-unity Euclidean length, we see that the following holds:

$$|X\Psi| \text{ must blow up like } C/\mu \text{ as } \mu \downarrow 0.$$

This argument shows, in particular, that the vanishing of μ *exactly coincides with the blowup of* $\max_{\alpha=0,\dots,n} |\partial_\alpha\Psi|$.

1F5. *Considerations of regularity.* This subsubsection is an interlude in which we highlight some issues tied to considerations of regularity. Our discussion will distinguish the problem of shock formation for transport equations from the (by now) well-understood case of quasilinear wave equations, which we further describe in Section 1G2. To illustrate the issues, we will highlight some features of our analysis, with a focus on derivative counts. In Lemma 3.21, we derive the following evolution equation for the Cartesian components of ${}^{(i)}\Theta$: $L{}^{(i)}\Theta^j = {}^{(i)}\Theta L^j$, where ${}^{(i)}\Theta = \frac{\partial}{\partial\vartheta^i}$. Recalling that $L = \frac{\partial}{\partial t}$, that $V_\alpha^J = \partial_\alpha v^J$, and that L^j is a smooth function of (Ψ, v), we infer, from standard energy estimates for transport equations, that ${}^{(i)}\Theta^j$ should have the same degree of Sobolev differentiability as $\partial\Psi$ and V. In particular, we expect that ${}^{(i)}\Theta^j$ should be *one degree less differentiable* than Ψ. For similar reasons, μ, V, and some other geometric quantities that play a role in our analysis are also one degree less differentiable than Ψ. The following point is crucial for our approach:

> We are able to close the energy estimates for Ψ up to top order even though, upon commuting Ψ's transport equation, we generate error terms that depend on the "less differentiable" quantities.

That is, in controlling Ψ, we must carefully ensure that all error terms feature an allowable amount of regularity. Moreover, the same care must be taken throughout the paper, by which we mean that we must ensure that we can close the estimates for all quantities using a consistent number of derivatives. In particular, we stress that it is precisely due to considerations of the regularity of the Cartesian components of ${}^{(i)}\Theta$ and $\breve{X}$ that we have introduced the quantities $V_\alpha^J = \partial_\alpha v^J$, as we explained in Section 1F1.

In the case of quasilinear wave equations with principal part $(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi$, the derivative counts are different. For example, the inverse foliation density μ enjoys the *same* Sobolev regularity as the wave equation solution variable Ψ in directions tangent to the characteristics, a gain of one tangential derivative compared to the present work. Moreover, for quasilinear wave equations, a similar gain in tangential differentiability also holds for some other key geometric objects, which we will not describe here. The gain is available because certain special combinations of quantities constructed out of the eikonal function and the wave equation solution variable satisfy an unexpectedly good evolution equation, with source terms that have better-than-expected regularity; see Section 1G2 or the survey article [Holzegel et al. 2016] for further discussion. Moreover, this gain seems *essential* for closing some of the top-order energy estimates in the wave equation case, the reason being that one must commute the geometric vector fields through the *second-order* wave operator, which eats up the gain. As we explain in Section 1G2, one pays a steep price in gaining back the derivative: the resulting energy estimates allow for possible energy blowup at the high geometric derivative levels (a potential phenomenon that is related to, but distinct from, the formation of a shock), a difficulty which we do not encounter in the present work.

We close this subsubsection by again highlighting that we are able to prove shock formation for systems with full quasilinear coupling (in the sense explained in the second paragraph of Section 1B) precisely because we are able to close our estimates using geometric quantities that are one degree less differentiable than Ψ, and that the viability of allowing the loss of differentiability leads to simplifications in the proof compared to the case of quasilinear wave equations. In contrast, in the case of quasilinear wave equations, due to the apparent necessity of avoiding a loss of differentiability in various geometric quantities, it does not seem possible to prove shock formation for general systems of quasilinear wave equations with multiple propagation speeds; the special combinations of quantities mentioned in the previous paragraph, which are needed to close the geometric energy estimates in the case of quasilinear wave equations, seem to be unstable under a full quasilinear coupling of multiple speed wave systems. Here is one representative manifestation of this issue: the problem of multispace-dimensional shock formation for covariant wave equation systems (see footnote 24 on page 467 regarding the notation) of the form

$$\square_{g_1(\Psi_1,\Psi_2)}\Psi_1 = 0, \tag{1F.4a}$$

$$\square_{g_2(\Psi_1,\Psi_2)}\Psi_2 = 0, \tag{1F.4b}$$

where g_1 and g_2 are Lorentzian metrics,[17] is open whenever $g_1 \neq g_2$, even though shock formation for systems with $g_1 = g_2$ and for scalar equations $\square_{g(\Psi)}\Psi = 0$ is well-understood [Speck 2016]. We note, however, that stable shock formation has been understood for some wave equation systems such that the quasilinear part of the shock-forming variable's wave equation has a decoupled structure. Specifically, in [Speck 2018], in two spatial dimensions, we proved a stable shock formation result for the variable Ψ_1 for systems in the unknowns (Ψ_1, Ψ_2) of the form

$$\square_{g_1(\Psi_1)}\Psi_1 = \mathcal{N}_1(\Psi_1, \partial\Psi_1, \Psi_2, \partial\Psi_2), \tag{1F.5a}$$

$$(g_2^{-1})^{\alpha\beta}(\Psi_1, \Psi_2, \partial\Psi_2)\,\partial_\alpha\partial_\beta\Psi_2 = \mathcal{N}_2(\Psi_1, \partial\Psi_1, \Psi_2, \partial\Psi_2), \tag{1F.5b}$$

under appropriate assumptions on the semilinear terms $\mathcal{N}_1$ and $\mathcal{N}_2$ as well as the assumption that the wave propagation speed corresponding to g_1 is faster than the wave propagation speed corresponding to g_2, i.e., that Ψ_1 is the "fastest wave variable"; see Section 1G2 for further discussion of this result. We clarify that a key structural feature, exploited in [Speck 2018], is that in (1F.5a), the metric g_1 corresponding to the shock-forming variable Ψ_1 depends only on Ψ_1; this is tantamount to the assumption of partial decoupling of the most difficult quasilinear terms.

1F6. *Energy estimates.* In Section 8, we derive the main technical estimates of the article: energy estimates up to top order for Ψ, v, V, μ, and related geometric quantities. Energy estimates are an essential ingredient in the basic regularity theory of quasilinear hyperbolic systems in multiple spatial dimensions, and in this article, they are also important because they yield improvements of our bootstrap assumptions described in Section 1F3. We now describe the energies, which we construct in Section 4. To control the transport variable Ψ, we construct geometric energies along Σ_t. To control the symmetric hyperbolic variables v and V, we construct μ-weighted energies along Σ_t as well as *non-μ-weighted* energies along the transport characteristics $\mathcal{P}_u$. With Σ_t^u defined to be the subset of Σ_t in which the eikonal function takes on values in between 0 and u and $\mathcal{P}_u^t$ defined to be the subset of $\mathcal{P}_u$ corresponding to times between 0 and t, our energies $\mathbb{E}^{(\mathrm{Shock})}[P\Psi](t,u), \dots$, and our characteristic fluxes $\mathbb{E}^{(\mathrm{Regular})}[V](t,u), \dots$ satisfy, with $P \in \mathscr{P} = \{L, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$ (see Section 4 for the details)

$$\mathbb{E}^{(\mathrm{Shock})}[P\Psi](t,u) := \int_{\Sigma_t^u} (P\Psi)^2\, d\vartheta\, du', \tag{1F.6a}$$

$$\mathbb{E}^{(\mathrm{Regular})}[v](t,u) \approx \int_{\Sigma_t^u} \mu|v|^2\, d\vartheta\, du', \qquad \mathbb{F}^{(\mathrm{Regular})}[v](t,u) \approx \int_{\mathcal{P}_u^t} |v|^2\, d\vartheta\, dt', \tag{1F.6b}$$

$$\mathbb{E}^{(\mathrm{Regular})}[V](t,u) \approx \int_{\Sigma_t^u} \mu|V|^2\, d\vartheta\, du', \quad \mathbb{F}^{(\mathrm{Regular})}[V](t,u) \approx \int_{\mathcal{P}_u^t} |V|^2\, d\vartheta\, dt'. \tag{1F.6c}$$

In our analysis, we of course must also control various higher-order energies, but here we ignore this issue. The degenerate μ weights featured in $\mathbb{E}^{(\mathrm{Regular})}[v]$ and $\mathbb{E}^{(\mathrm{Regular})}[V]$ arise from expressing the standard energy for symmetric hyperbolic systems in terms of the geometric coordinates; roughly, the weight μ appears because Σ_t is transversal to the $\mathcal{P}_u$ and because dx^1 is "well-approximated by" $\mu\, du'$.

[17] That is, for $i = 1, 2$, the matrix of Cartesian components of g_i has signature $(-, +, \dots, +)$.

For controlling certain error integrals that arise in the energy identities, *it is crucial that the characteristic fluxes* $\mathbb{F}^{(\text{Regular})}[v]$ *and* $\mathbb{F}^{(\text{Regular})}[V]$ *do not feature any degenerate* μ *weight*. These characteristic fluxes are positive definite only because our structural assumptions on the equations ensure that the propagation speed of v and V is strictly slower than that of Ψ (see (2C.1) for the precise assumptions). Readers can consult Lemma 4.2 and its proof to better understand the role of these assumptions.

We now outline the derivation of the energy estimates; see Section 8 for precise statements and proofs. Let us define[18] the controlling quantity $\mathbb{W}(t,u)$ to be the sum of the energies and characteristic fluxes in (1F.6a)–(1F.6c) and their analogs up to the top derivative level (corresponding to differentiations with respect to the geometric vector fields). The initial data that we study in our main theorem satisfy (by assumption) $\mathbb{W}(0,1) \lesssim \mathring{\epsilon}^2$ and $\mathbb{W}(2\mathring{A}_*^{-1},0) \lesssim \mathring{\epsilon}^2$, where $\mathring{\epsilon}$ is the small parameter described in Section 1F3. We again stress that $\mathbb{W}(t,u) \equiv 0$ for simple plane waves.

Next, we note that energy identities, based on applying the divergence theorem on the geometric coordinate region $[0,t]\times[0,u]\times\mathbb{T}^{n-1}$, together with the pointwise estimates for error terms mentioned in Section 1F3, lead to the inequality

$$\mathbb{W}(t,u) \leq C\mathring{\epsilon}^2 + C\int_{t'=0}^{t}\int_{u'=0}^{u}\int_{\mathbb{T}^{n-1}}\{|P\Psi|^2+|v|^2+|V|^2\}(t',u',\vartheta)\,d\vartheta\,du'\,dt' + \cdots, \tag{1F.7}$$

where the terms "$\cdots$" depend on the geometric derivatives of Ψ, v, and V up to top order and the derivatives of various geometric quantities up to top order; the terms "$\cdots$" can be bounded using arguments similar to the ones we sketch below, so we will not discuss them further here. In view of the definition of $\mathbb{W}$, we deduce the following inequality from (1F.7):

$$\mathbb{W}(t,u) \leq C\mathring{\epsilon}^2 + C\int_{t'=0}^{t}\mathbb{W}(t',u)\,dt' + C\int_{u'=0}^{u}\mathbb{W}(t,u')\,du' + \cdots. \tag{1F.8}$$

Then from (1F.8) and Gronwall's inequality with respect to t and u, we conclude, ignoring the terms "$\cdots$" and taking into account (1F.3), that the following a priori estimate holds for $(t,u)\in[0,T_{(\text{Boot})})\times[0,U_0]$ (see Proposition 8.6 for the details):

$$\mathbb{W}(t,u) \lesssim \mathring{\epsilon}^2\exp(C\mathring{A}_*^{-1}) \lesssim \mathring{\epsilon}^2. \tag{1F.9}$$

The estimate (1F.9) represents the realization of our hope that the solution remains regular relative to the geometric coordinates, up to the top derivative level.

We now stress the following key point: the characteristic fluxes $\mathbb{F}^{(\text{Regular})}[v]$ and $\mathbb{F}^{(\text{Regular})}[V]$ *are needed to control the terms* $|v|^2+|V|^2$ *on the right-hand side of* (1F.7); without the characteristic fluxes, instead of the term $C\int_{u'=0}^{u}\mathbb{W}(t,u')\,du'$ on the right-hand side of (1F.8), we would instead have the term $C\int_{t'=0}^{t}\mathbb{W}(t',u)/(\min_{\Sigma_{t'}^u}\mu)\,dt'$, whose denominator vanishes as the shock forms. Such a term would have led to a priori estimates allowing for the possibility that at all derivative levels, the geometric energies blow up as the shock forms. This in turn would have been inconsistent with the bootstrap assumptions

[18] Our definition of $\mathbb{W}(t,u)$ given here is schematic. See Definition 8.1 for the precise definition of the controlling quantity, which we denote by $\mathbb{Q}(t,u)$.

described in Section 1F3 and would have obstructed our approach of showing that the solution remains regular relative to the geometric coordinates.

1F7. *Combining the estimates.* Once we have obtained the a priori energy estimates, we can derive improvements of our L^∞-type bootstrap assumptions via Sobolev embedding (see Corollary 8.8). These steps, together with the estimates from Section 1F4 showing that μ vanishes in finite time, are the main steps in the proof of the main theorem. We need a few additional technical results to complete the proof, including some results guaranteeing that the geometric and Cartesian coordinates are diffeomorphic up to the shock (see Section 7) and some fairly standard continuation criteria (see Section 9), which in total ensure that the solution survives up to the shock. We combine all of these results in Section 10, where we prove the main theorem.

1G. ***Connections to prior work.*** Many aspects of the approach outlined in Section 1F have their genesis in earlier works, which we now describe.

1G1. *Results in one spatial dimension.* In one spatial dimension and in symmetry classes whose PDEs are effectively one-dimensional, there are many results, by now considered classical, that use the method of characteristics to exhibit the formation of shocks in initially smooth solutions to various quasilinear hyperbolic systems. Important examples include Riemann's work [1860] (in which he developed the method of Riemann invariants), Lax's proof [1964] of stable blowup for 2×2 genuinely nonlinear systems via the method of Riemann invariants, Lax's blowup results [1972; 1973] for scalar conservation laws, John's extension [1974] of Lax's work to systems in one spatial dimension with more than two unknowns (which required the development of new ideas since the method of Riemann invariants does not apply), and the recent work of Christodoulou and Perez [2016], in which they significantly sharpened [John 1974]. The main obstacle to extending the results mentioned above to more than one spatial dimension is that one must complement the method of characteristics with an ingredient that, due to the singularity formation, is often accompanied by enormous technical complications: energy estimates that are adapted to and that hold up to the singularity. We further explain these technical complications in the next subsubsection.

1G2. *Results in more than one spatial dimension.* The first breakthrough results on shock formation in more than one spatial dimension without symmetry assumptions were proved by Alinhac [1999a; 1999b; 2001] for small-data solutions to scalar quasilinear wave equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\,\partial_\alpha\,\partial_\beta\Phi = 0 \tag{1G.1}$$

that fail to satisfy the null condition. Here, $g(\partial\Phi)$ is a Lorentzian metric equal to the Minkowski metric plus an error term of size $\mathcal{O}(\partial\Phi)$. As we do in this paper, Alinhac constructed a set of geometric coordinates tied to an eikonal function u, which in the context of his problems was a solution the fully nonlinear eikonal equation

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\,\partial_\alpha u\,\partial_\beta u = 0. \tag{1G.2}$$

Much like in our work here, the level sets of u are characteristic hypersurfaces for (1G.1). They are also known, in the context of Lorentzian geometry, as *null hypersurfaces*, in view of their intimate connection to the g-null[19] vector field $-(g^{-1})^{\alpha\beta}\,\partial_\beta u$. In his works, Alinhac identified a set of small compactly supported initial data satisfying a nondegeneracy condition such that $\max_{\alpha,\beta=0,\dots,n}|\partial_\alpha\partial_\beta\Phi|$ blows up in finite time due to the intersection of the characteristics, while $|\Phi|$ and $\max_{\alpha=0,\dots,n}|\partial_\alpha\Phi|$ remain bounded. Moreover, relative to the geometric coordinates, Φ and $\{\partial_\alpha\Phi\}_{\alpha=0,\dots,n}$ remain smooth, except possibly at the very high derivative levels (we will elaborate upon this just below).

In proving his results, Alinhac faced three serious difficulties. We will focus only on the case of three spatial dimensions, though Alinhac obtained similar results in two spatial dimensions. The first difficulty is that for small data, solutions to (1G.1) experience a long period of dispersive decay, which seems to work against the formation of a shock and which necessitated the application of Klainerman's commuting vector field method [1985; 1986] in which the vector fields have time and radial weights. We stress that such dispersive behavior is not exhibited by the solutions that we study in this article and hence our vector fields do not feature time or radial weights. Alinhac showed that after an era[20] of dispersive decay, the nonlinearity in (1G.1) takes over and drives the formation of the shock.

The second main difficulty faced by Alinhac is that to follow the solution up the singularity, it seems necessary to commute the equations with geometric vector fields constructed out of the eikonal function, and these vector fields seem to lead to the loss of a derivative when commuted through the wave operator. Specifically, the geometric vector fields Z have Cartesian components that depend on ∂u, and hence commuting them through the wave equation (1G.2) leads to an equation of the schematic form $(g^{-1})^{\alpha\beta}(\partial\Phi)\,\partial_\alpha\partial_\beta(Z\Phi)=\partial^2 Z\cdot\partial\Phi+\cdots$. The difficulty is that standard wave equation energy estimates suggest, due to the source term $\partial^2 Z$, that Φ enjoys only the same Sobolev regularity as $Z\sim\partial u$, whereas standard energy estimates for the eikonal equation (1G.2) only allow one to prove that ∂u enjoys the same Sobolev regularity as $\partial^2\Phi$; this *suggests*, misleadingly, that the approach of using vector fields constructed out of an eikonal function will lead to the loss of a derivative. To overcome this difficulty, Alinhac obtained the nonlinear solution, up to the shock, as the limit of iterates that solve singular linearized problems, and he used a rather technical Nash–Moser iteration scheme featuring a free boundary in order to recover the loss of a derivative. For technical reasons, his reliance on the Nash–Moser iteration allowed him to follow "most" small-data solutions to the constant-time hypersurface of first blowup, and not further. More precisely, his approach only allowed him to treat "nondegenerate" data such that the first singularity is *isolated* in the constant-time hypersurface of first blowup. We again emphasize that in our work here, we encounter a similar difficulty concerning the regularity of the geometric vector fields, but since our PDE systems are first-order, we are able to overcome it in a different way, in fact by *allowing* for reduced regularity in the geometric vector fields; see Sections 1B and 1F5.

The third and most challenging difficulty encountered by Alinhac is the following: when proving energy estimates relative to the geometric coordinates, it seems necessary to rely on energies that feature degenerate weights that vanish as the shock forms; the weights are direct analogs of the inverse foliation density μ from Theorem 1.5. These weights make it difficult to control certain error terms in the energy

[19] That is, if $\hat{L}^\alpha := -(g^{-1})^{\alpha\beta}\,\partial_\beta u$, then by (1G.2), we have $g(\hat{L},\hat{L})=0$.

[20] Roughly the era of dispersive decay lasts for a time interval of length $\exp(c/\epsilon)$, where ϵ is the size of the data in a weighted Sobolev norm.

identities, which in turn leads to a priori estimates allowing for the following possibility: as the shock forms, the high-order energies might blow up like a positive power of[21] $1/\mu$. We stress that the possible high-order energy blowup encountered by Alinhac occurs relative to the geometric coordinates and is related to — but *distinct* from — the formation of the shock singularity (in which $\max_{\alpha,\beta=0,\dots,n} |\partial_\alpha \partial_\beta \Phi|$ blows up). To close the proof, Alinhac had to show that the possible high-order geometric energy blowup does not propagate down too far to the lower geometric derivative levels, i.e., that the solution remains fairly smooth relative to the geometric coordinates. This "descent scheme" costs many derivatives, and for this reason, the data must belong to a Sobolev space of rather high order for the estimates to close. We stress that although the energies that we use in the present paper also contain the same degenerate μ weights, we encounter different kinds of error terms in our energy estimates, tied in part to the fact that our systems are first-order and tied in part to our strategy of estimating the quantity V_α^J defined by (1F.2). For this reason, our a priori energy estimates relative to the geometric coordinates are regular in that *even the top-order geometric energies remain uniformly bounded up to the shock.*

In the remarkable work [Christodoulou 2007], Alinhac's shock formation results are significantly sharpened for the quasilinear wave equations of irrotational (i.e., vorticity-free) relativistic fluid mechanics in three spatial dimensions, which form a subclass of wave equations of type (1G.1). These wave equations arise from formulating the relativistic Euler equations in terms of a fluid potential Φ, which is possible when the vorticity vanishes. The equations studied by Christodoulou enjoy special features that he exploited in his proofs, such as having an Euler–Lagrange formulation with a Lagrangian that is invariant under the Poincaré group. The main results proved by Christodoulou are as follows: (i) there is an open (relative to a Sobolev space of high, nonexplicit order) set of small[22] data such that the only possible singularities that can form in the solution are shocks driven by the intersection of the characteristics; (ii) there is an open subset of the data from (i), not restricted by nondegeneracy assumptions of the type imposed by Alinhac, such that a shock does in fact form in finite time; and (iii) for those solutions that form shocks, Christodoulou gave a complete description of the maximal classical development of the data near the singularity, which intersects the future of the constant-time hypersurface of first blowup. His sharp description of the maximal development seems necessary for even properly setting up the *shock development problem*. This is the problem of uniquely locally continuing the solution past the singularity to the Euler equations in a *weak sense*, a setting in which one must also construct the "shock hypersurface", across which the solution jumps (the solution is smooth on either side of the shock hypersurface). The shock development problem in relativistic fluid mechanics was solved in spherical symmetry in [Christodoulou and Lisibach 2016] and, in yet another breakthrough work [Christodoulou 2019], for the nonrelativistic compressible Euler equations and the relativistic Euler equations without symmetry assumptions in a restricted case (known as the restricted shock development problem) such that the jump in entropy across the shock hypersurface was ignored. The work [Christodoulou 2019] is

[21] In the context of wave equations, μ is often defined as follows: $\mu = -1/((g^{-1})^{\alpha\beta}\, \partial_\alpha u\, \partial_\beta t)$, where t is the Cartesian time function.

[22] In the context of [Christodoulou 2007], "small" means a small perturbation of the nontrivial constant-state fluid solutions, which take the form $\Phi = kt$, where $k > 0$ is a constant.

the first of its type in more than one spatial dimension. We remark that in one spatial dimension, there are general results of this type. For example, for the existence of (weak — but unique under suitable admissibility criteria) solutions to strictly hyperbolic systems in one spatial dimension with *small total variation* (a context that allows for the presence of and interaction of "small" shock waves), we refer readers to the aforementioned work [Dafermos 2010, Chapter XV].

Compared to Alinhac's approach, the main technical improvement afforded by Christodoulou's approach [2007] to proving shock formation is that it avoids the loss of a derivative through a sharper, more direct method; instead of using Alinhac's Nash–Moser scheme, Christodoulou found special combinations of geometric quantities that satisfy good evolution equations, and he combined them with elliptic estimates on codimension-two spacelike hypersurfaces.[23] This approach to avoiding the loss of a derivative in wave equation eikonal functions originated in the aforementioned proof [Christodoulou and Klainerman 1993] of the stability of Minkowski spacetime, and it was extended by Klainerman and Rodnianski [2003] to the case of general scalar quasilinear wave equations in their study of low-regularity well-posedness for wave equations of the form $-\partial_t^2\Psi + g^{ab}(\Psi)\,\partial_a\partial_b\Psi = 0$. In total, Christodoulou's approach allowed him to control the solution up to the shock using a traditional "forwards" approach, without the free boundary found in Alinhac's iteration scheme. However, as in Alinhac's work, Christodoulou's energy estimates allowed for the possibility that the high-order energies might blow up as the shock forms. Therefore, like Alinhac, Christodoulou had to give a separate, technical argument to show that any high-order energy singularity does not propagate down too far to the lower geometric derivative levels.

In [Speck 2016], we extended Christodoulou's sharp shock formation results to the case of general quasilinear wave equations of type (1G.1) in three spatial dimensions that fail to satisfy the null condition, to the case of covariant wave equations of the type[24] $\square_{g(\Psi)}\Psi = 0$ that fail to satisfy the null condition, and to inhomogeneous versions of these wave equations featuring "admissible" semilinear terms. Similar results were proved in [Christodoulou and Miao 2014] for a subset of these equations, namely those wave equations arising from nonrelativistic compressible fluid mechanics with vanishing vorticity. All of the results mentioned so far in this subsubsection are explained in detail in the survey article [Holzegel et al. 2016].

In the wake of the results above, there have been significant further advancements, which we now describe. In [Speck et al. 2016], we extended the shock formation results of [Speck 2016] to a new, physically relevant regime of initial conditions for wave equations in two spatial dimensions such that the solutions are close to simple outgoing plane symmetric waves, much like the setup of the present article. For the initial conditions studied in [Speck et al. 2016], the solutions do not experience dispersive decay. Hence, we used a new analytic framework to control the solution up to the shock, based on "close-to-simple-plane-wave"-type smallness assumptions on the data that are similar in spirit to the assumptions that we make on the data in the present article. The results of [Speck et al. 2016] can be viewed as an extension, to the case of quasilinear wave equations without symmetry assumptions, of the aforementioned blowup results of [Lax 1964] for 2×2 genuinely nonlinear systems, and as an extension of well-known blowup results for first-order quasilinear scalar conservation laws in an arbitrary number

[23]These codimension-two surfaces are analogs of the $(n-1)$-dimensional tori $\mathcal{T}_{t,u}$ from Definition 3.2.

[24]$\square_g$ is the covariant wave operator of g. Relative to arbitrary coordinates, $\square_g\Psi = (1/\sqrt{|\det g|})\,\partial_\alpha(\sqrt{|\det g|}(g^{-1})^{\alpha\beta}\,\partial_\beta\Psi)$.

of spatial dimensions; see, for example, [Dafermos 2010, Section 6.1] for a discussion of finite-time shock formation for scalar equations on $\mathbb{R}^{1+n}$ of the form $\partial_t \Phi + \sum_{a=1}^{n} \partial_a [G(\Phi)] = 0$ under appropriate assumptions on the nonlinearity G and the initial data. For special classes of wave equations in three spatial dimensions with cubic nonlinearities, Miao and Yu [2017] proved similar shock formation results for a set of large initial data featuring a single scaling parameter, similar to the short pulse ansatz exploited in the breakthrough work [Christodoulou 2009] on the formation of trapped surfaces in solutions to the Einstein vacuum equations. For the same wave equations studied in [Miao and Yu 2017], Miao [2018] recently made a related-but-distinct ansatz on the initial data and proved the existence of an open set of solutions that exist classically on the time interval $(-\infty, T_{(\mathrm{Shock})})$ but blow up at time $T_{(\mathrm{Shock})} \approx -1$.

All of the works mentioned above concern systems whose characteristics have a simple structure: they correspond to a single wave operator. We now describe some recent shock formation results in which the systems have more complicated principal parts, leading to multiple speeds of propagation and distinct families of characteristics. The first result of this type without symmetry assumptions was our joint work [Luk and Speck 2018] with J. Luk, which concerned the compressible Euler equations in two spatial dimensions under an arbitrary[25] barotropic[26] equation of state. Specifically, in [Luk and Speck 2018], we extended the shock formation results of [Christodoulou and Miao 2014] for the compressible Euler equations to allow for the presence of small amounts of vorticity at the location of the singularity. The vorticity satisfies a transport equation and, as it turns out, remains Lipschitz with respect to the Cartesian coordinates, all the way up to the shock. More precisely, the shock occurs in the "sound wave part" of the system rather than in the vorticity, and, as in all prior works, the shock is driven by the intersection of a family of characteristic hypersurfaces corresponding to a Lorentzian metric (known as the *acoustical metric* in the context of fluid mechanics). In particular, [Luk and Speck 2018] yielded the first proof of stable shock formation without symmetry assumptions in solutions to a hyperbolic system featuring multiple speeds, where all solution variables were allowed to interact up to the singularity.

The results proved in [Luk and Speck 2018] were based on a new wave-transport-div-curl formulation of the compressible Euler equations under a barotropic equation of state, which we derived in [Luk and Speck 2016]. The new formulation exhibits remarkable null structures and regularity properties, tied in part to the availability of elliptic estimates for the vorticity in three spatial dimensions (vorticity stretching does not occur in two spatial dimensions, and in its absence, one does not need elliptic estimates to control the vorticity). In a forthcoming work, we will extend the shock formation results of [Luk and Speck 2018] to the much more difficult case of three spatial dimensions, where to control the vorticity up to top order in a manner compatible with the wave part of the system, one must rely on the elliptic estimates, which allow one to show that the vorticity is exactly as differentiable as the velocity with respect to geometric vector fields adapted to the sound wave characteristics. In [Speck 2017], we extended the results of [Luk and Speck 2016] to allow for an arbitrary equation of state in which the pressure

[25]There is one exceptional equation of state, known as that of the Chaplygin gas, to which the results of [Luk and Speck 2018] do not apply. In one spatial dimension, the resulting PDE system is *totally linearly degenerate*, and it is widely believed that shocks do not form in (initially smooth) solutions to such systems.

[26]A barotropic equation of state is such that the pressure is a function of the density.

depends on the density and entropy. The formulation of the equations in [Speck 2017] exhibits further remarkable properties that, in our forthcoming work, we will use to prove a stable shock formation result in three spatial dimensions in which the vorticity and entropy are allowed to be nonzero at the singularity. In the work [Speck 2018] (which we mentioned at the end of Section 1F5), in two spatial dimensions, we proved the first stable shock formation result for systems of quasilinear wave equations featuring *multiple wave speeds* of propagation; i.e., the systems featured more than one distinct quasilinear wave operator. The main result yielded an open set of data such that the "fastest" wave forms a shock in finite time, while the remaining solution variables remain regular up to the singularity in the fast wave, much like in Theorem 1.5. The initial conditions were perturbations of simple plane waves, similar to the setup for the case of the scalar wave equations studied in [Speck et al. 2016] and similar to the setup of the present article. The main new difficulty that we faced in [Speck 2018] is that the geometric vector fields adapted to the shock-forming fast wave, which seem to be an essential ingredient for following the fast wave all the way to its singularity, exhibit very poor commutation properties with the slow wave operator. Indeed, commuting the geometric vector fields all the way through the slow wave operator produces error terms that are uncontrollable, both from the point of view of regularity and from the point of view of the strength of the singular commutator terms that this generates. To overcome this difficulty, we relied on a first-order reformulation of the slow wave equation which, though somewhat limiting in the precision it affords, allows us to avoid commuting all the way through the slow wave operator and hence to avoid the uncontrollable error terms.

1H. ***Notation, index conventions, and conventions for "constants".*** We now summarize some of our notation. Some of the concepts referred to here are defined later in the article. Throughout, $\{x^\alpha\}_{\alpha=0,1,\dots,n}$ denote the standard Cartesian coordinates on spacetime $\mathbb{R}\times\Sigma$, where $x^0\in\mathbb{R}$ is the time variable and $(x^1,x^2,\dots,x^n)\in\Sigma=\mathbb{R}\times\mathbb{T}^{n-1}$ are the space variables. We denote the corresponding Cartesian partial derivative vector fields by $\partial_\alpha =: \frac{\partial}{\partial x^\alpha}$ (the $\frac{\partial}{\partial x^\alpha}$ are globally defined and smooth even though $\{x^i\}_{i=2}^n$ are only locally defined) and we often use the alternate notation $t := x^0$ and $\partial_t := \partial_0$.

- Lowercase Greek spacetime indices α, β, etc. correspond to the Cartesian spacetime coordinates and vary over $0,1,\dots,n$. Lowercase Latin spatial indices a, b, etc. correspond to the Cartesian spatial coordinates and vary over $1,2,\dots,n$. An exception to the latter rule occurs for the geometric torus coordinate vector fields ${}^{(i)}\Theta$ from (3A.5), in which the labeling index i varies over $2,\dots,n$. Uppercase Latin indices such as J correspond to the components v^J of the array of symmetric hyperbolic variables and typically vary from 1 to M.

- We use Einstein's summation convention in that repeated indices are summed over their respective ranges.

- Unless otherwise indicated, all quantities in our estimates that are not explicitly under an integral are viewed as functions of the geometric coordinates (t,u,ϑ) of Definition 3.4. Unless otherwise indicated, quantities under integrals have the functional dependence established below in Definition 3.26.

- If Q_1 and Q_2 are two operators, then $[Q_1,Q_2]=Q_1Q_2-Q_2Q_1$ denotes their commutator.

- $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$.
- $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.
- $A = \mathcal{O}(B)$ means that $|A| \lesssim |B|$.
- Constants such as C and c are free to vary from line to line. *These constants, as well as implicit constants, are allowed to depend in an increasing, continuous fashion on the data-size parameters* $\mathring{A}$ *and* $\mathring{A}_*^{-1}$ *from Section 5B. However, the constants can be chosen to be independent of the parameters* $\mathring{\alpha}$, $\mathring{\epsilon}$, *and* ε *whenever the following conditions hold:* (i) $\mathring{\epsilon}$ *and* ε *are sufficiently small relative to* 1, *relative to* $\mathring{A}^{-1}$, *and relative to* $\mathring{A}_*$, *and* (ii) $\mathring{\alpha}$ *is sufficiently small relative to* 1, in the sense described in Section 5D.
- Constants $C_{\blacklozenge}$ are also allowed to vary from line to line, but unlike C and c, the $C_{\blacklozenge}$ are universal in that, as long as $\mathring{\alpha}$, $\mathring{\epsilon}$, and ε are sufficiently small relative to 1, they do not depend on $\mathring{\alpha}$, ε, $\mathring{\epsilon}$, $\mathring{A}$, or $\mathring{A}_*^{-1}$.
- $A = \mathcal{O}_{\blacklozenge}(B)$ means that $|A| \leq C_{\blacklozenge}|B|$, with $C_{\blacklozenge}$ as above.
- $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ respectively denote the standard floor and ceiling functions.

2. Rigorous setup of the problem and fundamental definitions

In this section, we state the equations that we will study and state our basic assumptions on the nonlinearities.

2A. ***Statement of the equations.*** Our main results concern systems in $1+n$ spacetime dimensions and $1+M$ unknowns of the form

$$L\Psi = 0, \tag{2A.1a}$$

$$A^\alpha \, \partial_\alpha v = 0, \tag{2A.1b}$$

where, in our main theorem, the scalar function Ψ forms a shock, $M \geq 1$ is an integer,[27]

$$v := (v^J)_{J=1,\dots,M} \tag{2A.2}$$

denotes the "symmetric hyperbolic variables" (whose first-order Cartesian coordinate partial derivatives will remain bounded up to the singularity in $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$), L is a vector field whose Cartesian components are given smooth functions of Ψ and v, that is, $L^\alpha = L^\alpha(\Psi, v)$, and the A^α are *symmetric* $M \times M$ matrices whose components $A_I^{\alpha;J} = A_J^{\alpha;I}$ are given smooth functions of Ψ and v. Note that (2A.1b) is equivalent to the M scalar equations $A_J^{\alpha;I} \, \partial_\alpha v^J = 0$, where $1 \leq I \leq M$, and we sum over the repeated occurrences of α and J. For convenience, we assume the normalization conditions

$$L^0 \equiv 1, \tag{2A.3a}$$

$$L^1|_{(\Psi,v)=(0,0)} = 1. \tag{2A.3b}$$

More generally, if we were to assume that $(L^0|_{(\Psi,v)=(0,0)}, L^1|_{(\Psi,v)=(0,0)}) \neq (0,0)$, then we could achieve (2A.3a)–(2A.3b) by performing a linear change of coordinates in the (t, x^1)-plane and then dividing (2A.1a) by a scalar.

[27] Our results also apply in the case $M = 0$, though we omit discussion of this simpler case.

As we stressed in the introduction, an essential aspect of our analysis is that we treat the Cartesian coordinate partial derivatives of v^J as independent quantities. For this reason, we define

$$V_\alpha^J := \partial_\alpha v^J, \quad V_\alpha := (V_\alpha^J)_{1 \le J \le M}, \quad V := (V_\alpha^J)_{0 \le \alpha \le n, 1 \le J \le M}. \tag{2A.4}$$

As a straightforward consequence of (2A.1b) and definition (2A.4), we obtain the following evolution equation for V_α:

$$A^\beta \partial_\beta V_\alpha = -(\partial_\alpha A^\beta) V_\beta. \tag{2A.5}$$

2B. ***The genuinely nonlinear-type assumption.*** Recall that we can view $L^1 = L^1(\Psi, v)$. To ensure that shocks can form in nearly plane symmetric solutions, we assume that

$$\left.\frac{\partial L^1}{\partial \Psi}\right|_{(\Psi,v)=(0,0)} \neq 0. \tag{2B.1}$$

By continuity, it follows from (2B.1) that $\frac{\partial L^1}{\partial \Psi} \neq 0$ whenever $|\Psi| + |v|$ is sufficiently small.

2C. ***Assumptions on the speed of propagation for the symmetric hyperbolic subsystem.*** In this subsection, we state our assumptions on the speed of propagation for the symmetric hyperbolic subsystem (2A.1b). Specifically, we assume that the matrices

$$A^0|_{(\Psi,v)=(0,0)} \text{ and } A^0|_{(\Psi,v)=(0,0)} - A^1|_{(\Psi,v)=(0,0)} \text{ are positive definite.} \tag{2C.1}$$

We now explain the significance of (2C.1). The positivity of $A^0|_{(\Psi,v)=(0,0)}$ ensures that for solution values near the "background state" $(\Psi, v) = (0, 0)$, the hypersurfaces Σ_t are spacelike for (2A.1b), that is, for the evolution equation satisfied by the non-shock-forming variable v. By (2A.3a), the Σ_t are also spacelike for (2A.1a); i.e., L is transversal to Σ_t. The positivity of $A^0|_{(\Psi,v)=(0,0)} - A^1|_{(\Psi,v)=(0,0)}$ will ensure that for solution values near the background state, hypersurfaces close to the flat planes $\{t - x^1 = \text{const.}\}$ are spacelike for (2A.1b). This assumption is significant because for the solutions that we will study, we will construct (in Section 3A) a family $\{\mathcal{P}_u\}_{u\in[0,1]}$ of hypersurfaces that are characteristic for (2A.1a) (that is, for the operator L) and that are close to the flat planes $\{t - x^1 = \text{const.}\}$. Put differently, the $\mathcal{P}_u$ will be characteristic for the evolution equation for Ψ but spacelike for the evolution equation for v, which essentially means that for solution values near the background state, Ψ propagates at a strictly faster speed than v (and also strictly faster than V, since the principal coefficients in the evolution equations for v and V_α are the same).

3. Geometric constructions

In this section, we define/construct most of the geometric objects that we use to analyze solutions. We defer the construction of our L^2-type energies until Section 4.

3A. ***The eikonal function and the geometric coordinates.*** In this subsection, we construct the geometric coordinates that we use to follow the solution all the way to the shock. The most important of these is the eikonal function.

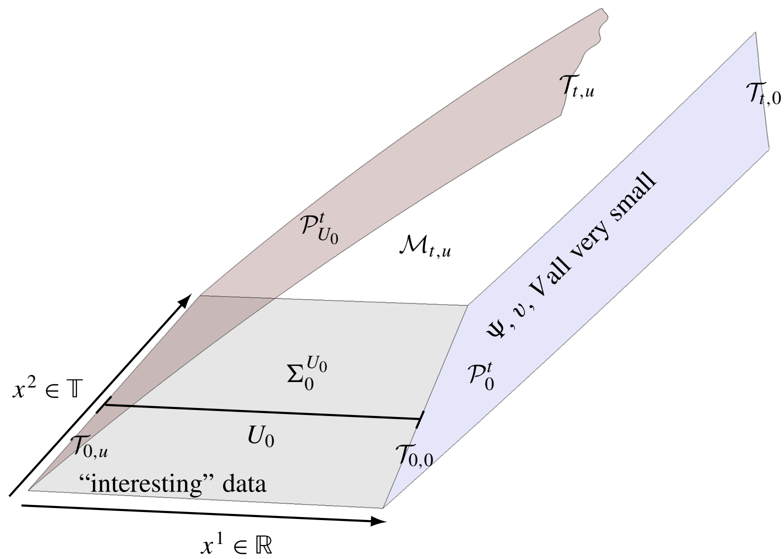

Figure 2. The spacetime region under study in the case $n = 2$.

Definition 3.1. The eikonal function is the solution u to the following transport initial value problem, where L is the transport operator vector field from (2A.1a):

$$Lu = 0, \quad u|_{\Sigma_0} = 1 - x^1. \tag{3A.1}$$

For reasons described in Remark 1.8 and Section 1F2, we now fix a real parameter U_0 satisfying

$$0 < U_0 \leq 1. \tag{3A.2}$$

We will restrict out attention to spacetime regions with $0 \leq u \leq U_0$.

Our analysis will take place on the following subsets of spacetime, which are tied to the eikonal function; see Figure 2 for a picture of the setup.

Definition 3.2. We define the following subsets of spacetime, where $x := (x^1, x^2, \ldots, x^n)$ denotes a point in $\mathbb{R} \times \mathbb{T}^{n-1}$ and (t, x) denotes a point in $\mathbb{R} \times \mathbb{R} \times \mathbb{T}^{n-1}$:

$$\Sigma_{t'} := \{(t, x) \mid t = t'\}, \tag{3A.3a}$$

$$\Sigma_{t'}^{u'} := \{(t, x) \mid t = t', \ 0 \leq u(t, x) \leq u'\}, \tag{3A.3b}$$

$$\mathcal{P}_{u'} := \{(t, x) \mid u(t, x) = u'\}, \tag{3A.3c}$$

$$\mathcal{P}_{u'}^{t'} := \{(t, x) \mid 0 \leq t \leq t', \ u(t, x) = u'\}, \tag{3A.3d}$$

$$\mathcal{T}_{t',u'} := \mathcal{P}_{u'}^{t'} \cap \Sigma_{t'}^{u'} = \{(t, x) \mid t = t', \ u(t, x) = u'\}, \tag{3A.3e}$$

$$\mathcal{M}_{t',u'} := \bigcup_{u \in [0,u']} \mathcal{P}_u^{t'} \cap \{(t, x) \mid 0 \leq t < t'\}. \tag{3A.3f}$$

We refer to the Σ_t and Σ_t^u as "constant time slices", the $\mathcal{P}_u^t$ as "characteristics", and the $\mathcal{T}_{t,u}$ as "tori". Note that $\mathcal{M}_{t,u}$ is "open-at-the-top" by construction.

To complete the geometric coordinate system, we now construct local coordinates on the tori $\mathcal{T}_{t,u}$.

Definition 3.3. We define the local geometric torus coordinates $(\vartheta^2, \dots, \vartheta^n)$ to be the solutions to the following initial value problems, where L is the transport operator vector field from (2A.1a):

$$L\vartheta^i = 0, \quad \vartheta^i|_{\Sigma_0} = x^i, \quad (i = 2, 3, \dots, n). \tag{3A.4}$$

Note that we can view $(\vartheta^2, \dots, \vartheta^n)$ as locally defined coordinates on $\mathcal{T}_{t,u} \simeq \mathbb{T}^{n-1}$.

Definition 3.4. We refer to $(t, u, \vartheta^2, \dots, \vartheta^n)$ as the geometric coordinates, and we set $\vartheta := (\vartheta^2, \dots, \vartheta^n)$. We denote the corresponding partial derivative vector fields by

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial u}, \quad {}^{(i)}\Theta := \frac{\partial}{\partial \vartheta^i}, \quad (i = 2, \dots, n). \tag{3A.5}$$

Note that the ${}^{(i)}\Theta$ are $\mathcal{T}_{t,u}$-tangent by construction. Moreover, we note even though the coordinate functions ϑ^i are only locally defined on $\mathcal{T}_{t,u}$, the vector fields $\{{}^{(i)}\Theta\}_{i=2,\dots,n}$ can be defined so as to form a smooth (relative to the geometric coordinates) global positively oriented frame on $\mathcal{T}_{t,u}$.

3B. ***The inverse foliation density.*** We now define $\mu > 0$, the inverse foliation density of the characteristics $\mathcal{P}_u$. When μ goes to 0, the characteristics intersect and, as our main theorem shows, $\max_{\alpha=0,\dots,n} |\partial_\alpha \Psi|$ blows up. That is, $\mu \downarrow 0$ *signifies the formation of a shock singularity.*

Definition 3.5 (inverse foliation density). We define $\mu > 0$ as follows:

$$\mu := \frac{1}{\partial_t u}. \tag{3B.1}$$

We observe that from (2A.3a)–(2A.3b) and (3A.1), it follows that when $|\Psi| + |v|$ is sufficiently small (as will be the case in our main theorem), we have

$$\mu|_{\Sigma_0} = 1 + \mathcal{O}_{\blacklozenge}(|\Psi|) + \mathcal{O}_{\blacklozenge}(|v|). \tag{3B.2}$$

In particular, if Ψ and v are initially small, then μ is initially close to 1.

3C. ***Vector fields and one-forms adapted to the characteristics and the blowup coefficient.*** In this subsection, we construct various vector fields and one-forms that are adapted to the characteristics $\mathcal{P}_u$. We also derive some of their basic properties. We also define the blowup coefficient $\mathcal{G}$ which, when nonzero, signifies the genuinely nonlinear nature of the transport equation (2A.1a).

Definition 3.6 (the eikonal function gradient one-forms). We define λ and ξ to be the one-forms with the following Cartesian components $(0 \le \alpha \le n,\ 1 \le j \le n)$:

$$\lambda_\alpha := \mu\, \partial_\alpha u, \tag{3C.1a}$$

$$\xi_0 := 0, \quad \xi_j := \mu\, \partial_j u. \tag{3C.1b}$$

Remark 3.7. From (3B.1) and (3C.1a), we deduce that

$$\lambda_0 = 1. \tag{3C.2}$$

The following definition captures the strength of the coefficient of the main term that drives the shock formation (as is evidenced by the estimates (6C.8a)–(6C.8b)). The definition is adapted to the x^1-direction since, in our main theorem, we study solutions with approximate plane symmetry (where by plane symmetric solutions, we mean ones that depend only on t and x^1).

Definition 3.8 (the blowup coefficient). Viewing $L^1 = L^1(\Psi, v)$, we define the blowup coefficient $\mathcal{G}$ as

$$\mathcal{G} := \frac{\partial L^1}{\partial \Psi}\xi_1. \tag{3C.3}$$

Remark 3.9 ($\mathcal{G} \neq 0$). The solutions that we will study will be such that ξ_1 is a small perturbation of -1; see definition (3D.3d) and the estimate (6C.7a). Hence, from (2B.1), it follows that $\mathcal{G} \neq 0$ for $|\Psi| + |v|$ sufficiently small (as will be the case for the solutions under study).

In the next definition, we define a pair of $\mathcal{P}_u$-transversal vector fields that we use to study the solution.

Definition 3.10 ($\mathcal{P}_u$-transversal vector fields). We define the Cartesian components of the Σ_t-tangent vector fields X and $\breve{X}$ as follows ($1 \leq j \leq n$):

$$X^j := -L^j, \tag{3C.4a}$$

$$\breve{X}^j := \mu X^j = -\mu L^j. \tag{3C.4b}$$

We now derive some basic properties of L and $\breve{X}$.

Lemma 3.11 (basic properties of L and $\breve{X}$). *Relative to the geometric coordinates, we have*

$$L = \frac{\partial}{\partial t}. \tag{3C.5}$$

Moreover, the following identity holds:

$$\breve{X}u = 1. \tag{3C.6}$$

Finally, there exists a $\mathcal{T}_{t,u}$-tangent vector field Ξ such that

$$\breve{X} = \frac{\partial}{\partial u} - \Xi. \tag{3C.7}$$

Proof. To prove (3C.5), we note that $Lu = L\vartheta^j = 0$ by construction. Also taking into account (2A.3a), we conclude (3C.5).

To prove (3C.6), we first use the eikonal equation (3A.1) and the assumption (2A.3a) to deduce the identity $\partial_t u = -L^a\, \partial_a u$. Multiplying this identity by μ and appealing to definition (3B.1), we deduce that $1 = -\mu L^a\, \partial_a u$, which, in view of definition (3C.4b), yields (3C.6). The existence of a $\mathcal{T}_{t,u}$-tangent vector field such that (3C.7) holds then follows as a simple consequence of (3C.6) and the identity $\breve{X}t = 0$ (that is, the fact that $\breve{X}$ is Σ_t-tangent). □

Lemma 3.12 (basic identities for the eikonal function gradient one-forms). *The following identities hold*:

$$L^\alpha \lambda_\alpha = 0, \quad L^a \xi_a = -1, \tag{3C.8a}$$

$$X^\alpha \lambda_\alpha = 1, \quad X^a \xi_a = 1. \tag{3C.8b}$$

Moreover, if Y is any $\mathcal{T}_{t,u}$-tangent vector field, then

$$Y^\alpha \lambda_\alpha = 0, \quad Y^a \xi_a = 0. \tag{3C.8c}$$

Proof. The identities in (3C.8a) are a straightforward consequence of (3A.1), definitions (3C.1a)–(3C.1b), (2A.3a), and (3C.2). The identities in (3C.8b) follow from (3C.4b), (3C.6), definitions (3C.1a)–(3C.1b), and the fact that $X^0 = 0$. To obtain (3C.8c), we first note that for $\mathcal{T}_{t,u}$-tangent vector fields Y, we have $Y \in \text{span}\{{}^{(i)}\Theta\}_{i=2,\ldots,n}$ and thus $Yu := Y^\alpha \partial_\alpha u = 0$. The identities in (3C.8c) follow from this fact, definitions (3C.1a)–(3C.1b), and the fact that $Y^0 = 0$. □

To obtain estimates for the solution's derivatives, we will commute the equations with the vector fields belonging to the following sets.

Definition 3.13. We define the following sets of commutation vector fields:

$$\mathscr{Z} := \{L, \breve{X}, {}^{(2)}\Theta, {}^{(3)}\Theta, \ldots, {}^{(n)}\Theta\}, \tag{3C.9a}$$

$$\mathscr{P} := \{L, {}^{(2)}\Theta, {}^{(3)}\Theta, \ldots, {}^{(n)}\Theta\}. \tag{3C.9b}$$

Remark 3.14. Note that $\mathscr{P}$ consists of precisely the $\mathcal{P}_u$-tangent elements of $\mathscr{Z}$.

3D. ***Perturbed parts of various scalar functions.*** In this subsection, we define the perturbed parts of various scalar functions that we have constructed. The perturbed quantities, which are decorated with the subscript or superscript "Small", vanish for the background solution $(\Psi, v) = (0, 0)$.

Definition 3.15 (the perturbed parts of various scalar functions). Let L be the vector field from (2A.1a), let $\{{}^{(i)}\Theta\}_{i=2,\ldots,n}$ be the geometric torus vector fields from (3A.5), and let ξ be the one-form defined in (3C.1b). We define the following "background" quantities, which are constants ($j = 1, \ldots, n$):

$$\tilde{L}^j := L^j|_{(\Psi,v)=(0,0)}, \tag{3D.1a}$$

$$\widetilde{X}^j := X^j|_{(\Psi,v)=(0,0)} = -L^j|_{(\Psi,v)=(0,0)}. \tag{3D.1b}$$

In (3D.1a)–(3D.1b), we are viewing L^j and X^j to be functions of (Ψ, v) (this is possible for X^j by (3C.4a)). Note that by (2A.3b) and (3C.4a), we have

$$\tilde{L}^1 = 1, \quad \widetilde{X}^1 = -1. \tag{3D.2}$$

We also define the perturbed quantities

$$L^j_{(\text{Small})} := L^j - \tilde{L}^j, \tag{3D.3a}$$

$$X^j_{(\text{Small})} := X^j - \widetilde{X}^j = -L^j_{(\text{Small})}, \tag{3D.3b}$$

$${}^{(i)}\Theta^j_{(\text{Small})} := {}^{(i)}\Theta^j - \delta^{ij}, \tag{3D.3c}$$

$$\xi_j^{(\text{Small})} := \xi_j + \delta_j^1, \tag{3D.3d}$$

where the second equality in (3D.3b) follows from (3C.4a) and δ^{ij} and δ_j^1 are standard Kronecker deltas.

3E. ***Arrays of unknowns and schematic notation.*** We use the following arrays for convenient shorthand notation.

Definition 3.16 (shorthand notation for various solution variables). We define the following arrays γ and $\underline{\gamma}$ of scalar functions:

$$\gamma := (\Psi, v^J, V_\alpha^J, \xi_i^{(\text{Small})}, {}^{(j)}\Theta^k_{(\text{Small})})_{0\leq\alpha\leq n,\, 1\leq i,\, k\leq n,\, 2\leq j\leq n,\, 1\leq J\leq M}, \tag{3E.1a}$$

$$\underline{\gamma} := (\mu, \Psi, v^J, V_\alpha^J, \xi_i^{(\text{Small})}, {}^{(j)}\Theta^k_{(\text{Small})})_{0\leq\alpha\leq n,\, 1\leq i,\, k\leq n,\, 2\leq j\leq n,\, 1\leq J\leq M}. \tag{3E.1b}$$

Remark 3.17 (schematic functional dependence). In the remainder of the article, we use the notation $\mathrm{f}(s_1, s_2, \ldots, s_m)$ to schematically depict an expression that depends smoothly on the scalar functions $s_1, s_2, \ldots, s_m$. Note that in general, $\mathrm{f}(0) \neq 0$.

Remark 3.18 (the meaning of the symbol P). Throughout, P schematically denotes a differential operator that is tangent to the characteristics $\mathcal{P}_u$, typically L or ${}^{(i)}\Theta$. We use such notation when the precise details of P are not important.

3F. ***Cartesian partial derivatives in terms of geometric vector fields.*** In the next lemma, we expand the vector fields $\{\partial_\alpha\}_{\alpha=0,\ldots,n}$ in terms of the geometric commutation vector fields.

Lemma 3.19 (Cartesian partial derivatives in terms of geometric vector fields). *There exist smooth scalar functions* $\mathrm{f}_{ij}(\gamma)$ *such that the Cartesian vector fields* ∂_α *can be expanded as follows in terms of the elements of the set* $\mathscr{X}$ *defined in* (3C.9a) *whenever* $|\gamma|$ *is sufficiently small, where* ξ_j *is defined in* (3C.1b):

$$\partial_t = L + X, \tag{3F.1a}$$

$$\partial_j = \xi_j X + \sum_{i=2}^{n} \mathrm{f}_{ij}(\gamma)\,{}^{(i)}\Theta \quad (1 \leq j \leq n). \tag{3F.1b}$$

Proof. Equation (3F.1a) follows from (2A.3a), (3C.4a), and the fact that $X^0 = 0$.

To prove (3F.1b), we first note that for any fixed j with $1 \leq j \leq n$, since ∂_j is Σ_t-tangent and since $\{X, {}^{(2)}\Theta, \ldots, {}^{(n)}\Theta\}$ spans the tangent space of Σ_t, there exist unique (j-dependent) scalars $\alpha_1, \ldots, \alpha_n$ such that $\partial_j = \alpha_1 X + \sum_{i=2}^n \alpha_i\,{}^{(i)}\Theta$. Using both sides of this expansion to differentiate the eikonal function u and using (3C.4b) and (3C.6), we obtain the identity $\partial_j u = \alpha_1 \mu^{-1}$. In view of definition (3C.1b), we conclude that $\alpha_1 = \xi_j$, as is stated on the right-hand side of (3F.1b). Next, for $1 \leq j, k \leq n$, we allow both sides of the expansion to differentiate the Cartesian coordinate x^k to obtain the identity $\delta_j^k = \alpha_1 X^k + \sum_{i=2}^n \alpha_i\,{}^{(i)}\Theta^k$. For fixed j, we can view this as an identity whose left-hand side is the n-dimensional vector with components $(\delta_j^1, \ldots, \delta_j^n)^\top$ and whose right-hand side is equal to the product of a matrix $M_{n\times n}$ and the n-dimensional vector $(\alpha_1, \ldots, \alpha_n)^\top$, where $\top$ denotes transpose. From Definition 3.15, we see that

$$M_{n\times n} = \left(\begin{array}{c|c} -1 & \mathbf{0}_{1\times(n-1)} \\ \hline *_{(n-1)\times 1} & \mathbb{I}_{(n-1)\times(n-1)} \end{array}\right) + M_{n\times n}^{(\text{Small})},$$

where the entries of $*_{(n-1)\times 1}$ are of the schematic form $\mathrm{f}(\gamma)$ and the entries of $M_{n\times n}^{(\mathrm{Small})}$ are of the schematic form $\gamma\mathrm{f}(\gamma)$ (and thus are small when $|\gamma|$ is small). Hence, when $|\gamma|$ is small, we can invert $M_{n\times n}$ to conclude that the α_i are smooth functions of γ, which completes the proof of (3F.1b). □

3G. ***Evolution equations for the Cartesian components of various geometric quantities.*** In this subsection, we derive transport equations for the Cartesian components of various geometric quantities that are adapted to the characteristics $\mathcal{P}_u$. Later, we will use these transport equations to derive estimates for these quantities.

Lemma 3.20 (transport equations for μ, ξ_j, and $\xi_j^{(\mathrm{Small})}$). *The scalar functions μ, ξ_j, and $\xi_j^{(\mathrm{Small})}$, which are defined respectively in* (3B.1), (3C.1b), *and* (3D.3d), *satisfy the following transport equations, where the scalar functions $\mathrm{f}_{ij}(\gamma)$ are as in Lemma 3.19* ($i=2,\dots,n,\ j=1,\dots,n$):

$$L\mu = (\breve{X}L^a)\xi_a + \mu(LL^a)\xi_a, \tag{3G.1a}$$

$$L\xi_j = L\xi_j^{(\mathrm{Small})} = (LL^a)\xi_a\xi_j - \sum_{i=2}^{n}\mathrm{f}_{ij}(\gamma)({}^{(i)}\Theta L^a)\xi_a. \tag{3G.1b}$$

Moreover, there exist functions that are smooth whenever $|\gamma|$ is sufficiently small and that are schematically denoted by f *such that the following initial conditions hold along Σ_0*:

$$\mu|_{\Sigma_0} = 1 + (\Psi, v)\cdot\mathrm{f}(\Psi, v), \tag{3G.2a}$$

$$\xi_j|_{\Sigma_0} = \{-1 + (\Psi, v)\cdot\mathrm{f}(\Psi, v)\}\delta_j^1, \tag{3G.2b}$$

$$\xi_j^{(\mathrm{Small})}|_{\Sigma_0} = (\Psi, v)\cdot\mathrm{f}(\Psi, v)\delta_j^1. \tag{3G.2c}$$

Proof. Differentiating the eikonal equation (3A.1) with ∂_α and using (2A.3a), we obtain

$$L\,\partial_\alpha u = -(\partial_\alpha L^a)\,\partial_a u. \tag{3G.3}$$

Setting $\alpha = 0$ in (3G.3) and appealing to definition (3B.1), we deduce

$$L\mu = \mu(\partial_t L^a)(\mu\,\partial_a u). \tag{3G.4}$$

From (3G.4), (3F.1a), (3C.4b), and definition (3C.1b), we conclude (3G.1a).

Next, we set $\alpha = j$ in (3G.3), multiply the equation by μ, and use definition (3C.1b) and (3G.4) to compute that

$$\begin{aligned} L(\mu\,\partial_j u) &= -(\partial_j L^a)(\mu\,\partial_a u) + (\partial_t L^a)(\mu\,\partial_a u)(\mu\,\partial_j u) \\ &= -(\partial_j L^a)\xi_a + (\partial_t L^a)\xi_a\xi_j. \end{aligned} \tag{3G.5}$$

From (3G.5) and (3F.1a)–(3F.1b), we conclude (3G.1b).

To prove (3G.2a), we use (2A.3a)–(2A.3b), (3A.1), definition (3B.1), (3D.2), and (3D.3a) to obtain $(1/\mu)|_{\Sigma_0} = \partial_t u|_{\Sigma_0} = -L^a\,\partial_a u|_{\Sigma_0} = L^1|_{\Sigma_0} = 1 + (\Psi, v)\cdot\mathrm{f}(\Psi, v)$, from which (3G.2a) easily follows (when $|\Psi|$ and $|v|$ are small). To prove (3G.2b), we use definition (3C.1b) and the argument above to deduce that $\xi_j|_{\Sigma_0} = -(\mu\delta_j^1)|_{\Sigma_0} = \{-1 + (\Psi, v)\cdot\mathrm{f}(\Psi, v)\}\delta_j^1$, as desired. Equation (3G.2c) then follows from (3G.2b) and definition (3D.3d). □

In the next lemma, we derive transport equations for the Cartesian components of the geometric torus coordinate partial derivative vector fields.

Lemma 3.21 (transport equations for the Cartesian components of ${}^{(i)}\Theta$). *The Cartesian components* ${}^{(i)}\Theta^j$ *of the* $\mathcal{T}_{t,u}$*-tangent vector fields from* (3A.5) *and their perturbed parts* ${}^{(i)}\Theta^j_{(\mathrm{Small})}$ *defined in* (3D.3c) *are solutions to the following transport equation initial value problem*:

$$L{}^{(i)}\Theta^j = {}^{(i)}\Theta L^j, \quad {}^{(i)}\Theta^j|_{\Sigma_0} = \delta^{ij}, \tag{3G.6a}$$

$$L{}^{(i)}\Theta^j_{(\mathrm{Small})} = {}^{(i)}\Theta L^j, \quad {}^{(i)}\Theta^j|_{\Sigma_0} = 0, \tag{3G.6b}$$

where δ^{ij} *is the standard Kronecker delta.*

Proof. L and ${}^{(i)}\Theta$ are geometric coordinate partial derivative vector fields and they therefore commute: $[L, {}^{(i)}\Theta] = 0$. Relative to Cartesian coordinates, the vanishing commutator can be expressed as $L{}^{(i)}\Theta^j = {}^{(i)}\Theta L^j$, which is the desired evolution equation in (3G.6a). Next, we observe that along Σ_0, ${}^{(i)}\Theta = \partial_i$ by construction. Hence, ${}^{(i)}\Theta^j|_{\Sigma_0} = {}^{(i)}\Theta|_{\Sigma_0} x^j = \partial_i x^j = \delta^{ij}$, which yields the initial condition (3G.6a). Equation (3G.6b) then follows from definition (3D.3c) and (3G.6a). □

3H. *Vector field commutator properties.* In this subsection, we derive some basic properties of various vector field commutators.

Lemma 3.22. *The following vector fields are* $\mathcal{T}_{t,u}$*-tangent* ($i = 2, \dots, n$):

$$[L, \breve{X}], \quad [L, {}^{(i)}\Theta], \quad [\breve{X}, {}^{(i)}\Theta], \quad (i = 2, \dots, n). \tag{3H.1}$$

Moreover, there exist smooth functions, denoted by subscripted versions of f, *such that the following identities hold whenever* $|\gamma|$ *is sufficiently small* (*see Remark 3.18 regarding the notation*) ($i, i_1, i_2 = 2, \dots, n$):

$$[L, {}^{(i)}\Theta] = [{}^{(i_1)}\Theta, {}^{(i_2)}\Theta] = 0, \tag{3H.2a}$$

$$[L, \breve{X}] = \sum_{i=2}^{n} \mathrm{f}_i(\underline{\gamma}, L\Psi, \breve{X}\Psi){}^{(i)}\Theta, \tag{3H.2b}$$

$$[\breve{X}, {}^{(i)}\Theta] = \sum_{j=2}^{n} \mathrm{f}_{ij}(\underline{\gamma}, \breve{X}\gamma, P\Psi, P\mu){}^{(j)}\Theta. \tag{3H.2c}$$

Proof. Since (3C.5) implies that L is a geometric coordinate partial derivative vector field and since, by definition, the same is true of ${}^{(i)}\Theta$, we conclude (3H.2a).

To prove (3H.2b), we first use (3C.5), (3C.6), and the fact that $\breve{X}$ is Σ_t-tangent to deduce that $[L, \breve{X}]t = [L, \breve{X}]u = 0$. Hence, $[L, \breve{X}]$ is $\mathcal{T}_{t,u}$-tangent. Therefore, there exist unique scalars α_i such that the following identity holds for $j = 1, 2, \dots, n$: $[L, \breve{X}]^j = \sum_{i=2}^{n} \alpha_i {}^{(i)}\Theta^j$. Next, we use the fact that $L^a = \mathrm{f}(\Psi, v)$, (3C.4a)–(3C.4b), and the evolution equation (3G.1a) to deduce the schematic identity $[L, \breve{X}]^j = L(\mu X^j) - \breve{X}L^j = \mathrm{f}(\underline{\gamma}, L^\alpha V_\alpha, \breve{X}^a V_a, L\Psi, \breve{X}\Psi) = \mathrm{f}(\underline{\gamma}, L\Psi, \breve{X}\Psi)$. Next, considering the index range $2 \le j \le n$, we view the identity $[L, \breve{X}]^j = \sum_{i=2}^{n} \alpha_i {}^{(i)}\Theta^j$ as an identity whose left-hand side is the $(n-1)$-dimensional vector with Cartesian components equal to $([L, \breve{X}]^2, \dots, [L, \breve{X}]^n)^\top$ and whose right-hand side is the product of the $(n-1)\times(n-1)$ matrix $M_{(n-1)\times(n-1)} := ({}^{(i)}\Theta^j)_{i,j=2,\dots,n}$ and the $(n-1)$-dimensional vector $(\alpha_2,\dots,\alpha_n)^\top$, where $\top$ denotes transpose. From definition (3D.3c), we see that $M_{(n-1)\times(n-1)}$ is equal to the identity matrix plus an error matrix whose components are of the schematic form $\gamma\mathrm{f}(\gamma)$. In particular, $M_{(n-1)\times(n-1)}$ is invertible whenever $|\gamma|$ is sufficiently small. Hence, $(\alpha_2,\dots,\alpha_n)^\top$ is the product of a matrix, whose components are of the form $\mathrm{f}(\gamma)$ and the vector $([L,\breve{X}]^2,\dots,[L,\breve{X}]^n)^\top$, whose components are of the form $\mathrm{f}(\underline{\gamma}, L\Psi, \breve{X}\Psi)$. This completes the proof of (3H.2b). The identity (3H.2c) can be proved in a similar fashion and we omit the details. ☐

Corollary 3.23 (evolution equation for Ξ^j). *There exist functions that are smooth whenever $|\gamma|$ is sufficiently small and that are schematically denoted by indexed versions of* f *such that the Cartesian components Ξ^j $(j=1,\dots,n)$ of the $\mathcal{T}_{t,u}$-tangent vector field Ξ from* (3C.7) *satisfy the evolution equation*

$$L\Xi^j = \sum_{i=2}^{n} \Xi^a \mathrm{f}_{ia}(\gamma)\,{}^{(i)}\Theta L^j - \sum_{i=2}^{n} \mathrm{f}_i(\underline{\gamma}, L\Psi, \breve{X}\Psi)\,{}^{(i)}\Theta^j \tag{3H.3}$$

and the initial condition

$$\Xi^j|_{\Sigma_0} = \mathrm{f}^j(\Psi, v), \tag{3H.4}$$

where the f_{ia} *on the right-hand side of* (3H.3) *are as in* (3F.1b), *and the second sum on the right-hand side of* (3H.3) *is precisely the sum on the right-hand side of* (3H.2b).

Proof. From (3C.5) and (3C.7), we deduce that $[L,\Xi]^j = -[L,\breve{X}]^j$. Considering the Cartesian components of both sides of this equation and using (3H.2b), we obtain $L\Xi^j = \Xi^a\,\partial_a L^j - \sum_{i=2}^n \mathrm{f}_i(\underline{\gamma}, L\Psi, \breve{X}\Psi)\,{}^{(i)}\Theta^j$. Finally, we use (3F.1b) to substitute for ∂_a in the expression $\Xi^a\,\partial_a L^j$, and we use (3C.8c) to deduce that the component $\Xi^a \xi_a X L^j$ vanishes. In total, this yields (3H.3).

To prove (3H.4), we use (3C.7) to deduce that $\Xi^j = \Xi x^j = \frac{\partial}{\partial u}x^j - \breve{X}^j$. In view of the way in which the geometric coordinates were constructed, along Σ_0, we have $\frac{\partial}{\partial u} = -\partial_1$. Moreover, in view of (3C.4a)–(3C.4b) and (3G.2a), we deduce that $\breve{X}^j|_{\Sigma_0} = \breve{X}x^j|_{\Sigma_0} = (\mu X^j)|_{\Sigma_0} = \mu|_{\Sigma_0}\mathrm{f}(\Psi, v) = \mathrm{f}(\Psi, v)$, where f depends on j. Combining the calculations above, we conclude (3H.4). ☐

3I. *The change of variables map.* In this subsection, we define the change of variables map from geometric to Cartesian coordinates and derive some of its basic properties.

Definition 3.24. We define $\Upsilon : \mathbb{R}\times\mathbb{R}\times\mathbb{T}^{n-1} \to \mathbb{R}\times\mathbb{R}\times\mathbb{T}^{n-1}$ to be the change of variables map from geometric to Cartesian coordinates; i.e., $\Upsilon^\alpha(t, u, \vartheta^2, \dots, \vartheta^n) = x^\alpha$.

Lemma 3.25 (basic properties of the change of variables map). *The following identities hold*, *where L is the vector field from* (2A.1a), *the ${}^{(i)}\Theta$ are the vector fields from* (3A.5), *$\breve{X}$ is the vector field from* (3C.4b), *and Ξ is the vector field from* (3C.7):

$$\frac{\partial\Upsilon}{\partial(t,u,\vartheta^2,\dots,\vartheta^n)} := \frac{\partial(x^0,x^1,x^2,\dots,x^n)}{\partial(t,u,\vartheta^2,\dots,\vartheta^n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ L^1 & \mu X^1 + \Xi^1 & {}^{(2)}\Theta^1 & {}^{(3)}\Theta^1 & \dots & {}^{(n)}\Theta^1 \\ L^2 & \mu X^2 + \Xi^2 & {}^{(2)}\Theta^2 & {}^{(3)}\Theta^2 & \dots & {}^{(n)}\Theta^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L^n & \mu X^n + \Xi^n & {}^{(2)}\Theta^n & {}^{(3)}\Theta^n & \dots & {}^{(n)}\Theta^n \end{pmatrix}. \tag{3I.1}$$

Moreover, there exists a smooth function of γ vanishing at $\gamma = 0$, schematically denoted by $\gamma \mathrm{f}(\gamma)$, such that

$$\det \frac{\partial(x^0, x^1, x^2, \dots, x^n)}{\partial(t, u, \vartheta^2, \dots, \vartheta^n)} = \frac{\partial(x^1, x^2, \dots, x^n)}{\partial(u, \vartheta^2, \dots, \vartheta^n)} = -\mu\{1 + \gamma \mathrm{f}(\gamma)\}. \tag{3I.2}$$

Similarly, the following identity holds:

$$\det \frac{\partial(x^2, \dots, x^n)}{\partial(\vartheta^2, \dots, \vartheta^n)} = 1 + \gamma \mathrm{f}(\gamma). \tag{3I.3}$$

Proof. The first column of (3I.1) is a simple consequence of (3C.5) and the fact that $Lx^\alpha = L^\alpha$. The second column of (3C.4b) follows similarly from the fact that $\breve{X}$ is Σ_t-tangent (i.e., $\breve{X}t = 0$), (3C.4b), and (3C.7). The remaining $n-1$ columns of (3C.4b) follow similarly from the fact that the vector fields ${}^{(i)}\Theta$ are Σ_t-tangent.

The first equality in (3I.2) is a simple consequence of (3I.1). To derive the second equality in (3I.2), we first note that since $\Xi \in \mathrm{span}\{{}^{(i)}\Theta\}_{i=2,\dots,n}$, we can delete Ξ from the matrix on the right-hand side of (3I.1) without changing its determinant. It follows that

$$\text{left-hand side of (3I.2)} = \mu \det \begin{pmatrix} X^1 & {}^{(2)}\Theta^1 & {}^{(3)}\Theta^1 & \dots & {}^{(n)}\Theta^1 \\ X^2 & {}^{(2)}\Theta^2 & {}^{(3)}\Theta^2 & \dots & {}^{(n)}\Theta^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X^n & {}^{(2)}\Theta^n & {}^{(3)}\Theta^n & \dots & {}^{(n)}\Theta^n \end{pmatrix}.$$

In view of Definition 3.15 and definition (3E.1a), we see that the previous expression is equal to μ times the determinant of $M_{n\times n} + M_{n\times n}^{(\mathrm{Small})}$, where $M_{n\times n}$ and $M_{n\times n}^{(\mathrm{Small})}$ are the matrices from the proof of Lemma 3.19. Using arguments similar to the ones given in the proof of Lemma 3.19, we conclude the identity (3I.2). The identity (3I.3) can be proved via a similar argument, and we omit the details. □

3J. ***Integration forms and integrals.*** In this subsection, we define quantities connected to the two kinds of integration that we use in our analysis: integration with respect to the geometric coordinates and integration with respect to the Cartesian coordinates. In Remark 3.29, we clarify why both kinds of integration play a role in our analysis and why geometric integration is the most important for our analysis. In Lemma 3.30, we quantify the relationship between the two kinds of integration.

3J1. *Geometric integration.*

Definition 3.26 (geometric forms and related integrals). Relative to the geometric coordinates of Definition 3.4, we define the following forms:[28]

$$\begin{aligned} d\vartheta &:= d\vartheta^2 \cdots d\vartheta^n, & d\underline{\varpi} &:= d\vartheta\, du', \\ d\overline{\varpi} &:= d\vartheta\, dt', & d\varpi &:= d\vartheta\, du'\, dt'. \end{aligned} \tag{3J.1}$$

[28]Throughout the paper, we blur the distinction between the (nonnegative) integration measure $d\vartheta$ and the corresponding form $d\vartheta^2 \wedge \cdots \wedge d\vartheta^n$, and similarly for the other quantities appearing in (3J.1). The precise meaning will be clear from context.

If f is a scalar function, then we define

$$\int_{\mathcal{T}_{t,u}} f\, d\vartheta := \int_{\vartheta \in \mathbb{T}^{n-1}} f(t,u,\vartheta)\, d\vartheta, \tag{3J.2a}$$

$$\int_{\Sigma_t^u} f\, d\underline{\varpi} := \int_{u'=0}^{u} \int_{\vartheta \in \mathbb{T}^{n-1}} f(t,u',\vartheta)\, d\vartheta\, du', \tag{3J.2b}$$

$$\int_{\mathcal{P}_u^t} f\, d\overline{\varpi} := \int_{t'=0}^{t} \int_{\vartheta \in \mathbb{T}^{n-1}} f(t',u,\vartheta)\, d\vartheta\, dt', \tag{3J.2c}$$

$$\int_{\mathcal{M}_{t,u}} f\, d\varpi := \int_{t'=0}^{t} \int_{u'=0}^{u} \int_{\vartheta \in \mathbb{T}^{n-1}} f(t',u',\vartheta)\, d\vartheta\, du'\, dt'. \tag{3J.2d}$$

3J2. *Cartesian integration.*

Definition 3.27 (the one-form H). Let λ be the one-form from Definition 3.6. We define H to be the one-form with the following Cartesian components:

$$H_\nu := \frac{1}{(\delta^{\alpha\beta}\lambda_\alpha\lambda_\beta)^{1/2}}\lambda_\nu, \tag{3J.3}$$

where $\delta^{\alpha\beta}$ is the standard inverse Euclidean metric on $\mathbb{R}\times\Sigma$ (that is, $\delta^{\alpha\beta} = \mathrm{diag}(1,1,\dots,1)$ relative to the Cartesian coordinates). Note that H is the Euclidean-unit-length conormal to $\mathcal{P}_u$.

Definition 3.28 (Cartesian coordinate volume and area forms and related integrals). We define

$$d\mathcal{M} := dx^1 dx^2 \cdots dx^n dt, \quad d\Sigma := dx^1 dx^2 \cdots dx^n, \quad d\mathcal{P} \tag{3J.4}$$

to be, respectively, the standard volume form on $\mathcal{M}_{t,u}$ induced by the Euclidean metric[29] on $\mathbb{R}\times\Sigma$, the standard area form induced on Σ_t^u by the Euclidean metric on $\mathbb{R}\times\Sigma$, and the standard area form induced on $\mathcal{P}_u^t$ by the Euclidean metric on $\mathbb{R}\times\Sigma$.

We define the integrals of functions f with respect to the forms above in analogy with the way that we defined the integrals (3J.2a)–(3J.2d). For example,

$$\int_{\Sigma_t^U} f\, d\Sigma := \int_{\{(x^1,\dots,x^n) \mid 0 \le u(t,x^1,\dots,x^n) \le U\}} f(t,x^1,\dots,x^n)\, dx^1 \cdots dx^n,$$

where $u(t,x^1,\dots,x^n)$ is the eikonal function.

Remark 3.29 (the role of the Cartesian forms). We never *estimate* integrals involving the Cartesian forms; before deriving estimates, we will always use Lemma 3.30 below in order to replace the Cartesian forms with the geometric ones of Definition 3.26; we use the Cartesian forms only when deriving energy *identities* relative to the Cartesian coordinates, in which the Cartesian forms naturally appear.

[29] By definition, the Euclidean metric has the components $\mathrm{diag}(1,1,\dots,1)$ relative to the standard Cartesian coordinates $(t,x^1,x^2,\dots,x^n)$ on $\mathbb{R}\times\Sigma$.

3J3. *Comparison between the Cartesian integration measures and the geometric integration measures.* In the next lemma, we quantify the relationship between the Cartesian integration measures and the geometric integration measures.

Lemma 3.30. *There exist scalar functions, schematically denoted by* $\mathrm{f}(\gamma)$, *that are smooth for* $|\gamma|$ *sufficiently small and such that the following relationship holds between the geometric integration measures corresponding to Definition 3.26 and the Cartesian integration measures corresponding to Definition 3.28, where all of the measures are nonnegative* (*see footnote 28*):

$$d\mathcal{M} = \mu\{1+\gamma\mathrm{f}(\gamma)\}\,d\varpi, \quad d\Sigma = \mu\{1+\gamma\mathrm{f}(\gamma)\}\,d\underline{\varpi}, \quad d\mathcal{P} = \{\sqrt{2}+\gamma\mathrm{f}(\gamma)\}\,d\overline{\varpi}. \tag{3J.5}$$

Proof. We prove only the identity $d\mathcal{P} = \{\sqrt{2}+\gamma\mathrm{f}(\gamma)\}\,d\overline{\varpi}$ since the other two identities in (3J.5) are a straightforward consequence of Lemma 3.25 (in particular, the Jacobian determinant[30] expressions in (3I.2)). Throughout this proof, we view $d\overline{\varpi}$ (see (3J.1)) as the n-form $dt \wedge d\vartheta^2 \wedge \cdots \wedge d\vartheta^n$ on $\mathcal{P}_u$, where $dt \wedge d\vartheta^2 = dt \otimes d\vartheta^2 - d\vartheta^2 \otimes dt$, etc. Similarly, we view $d\mathcal{P}$ as the n-form induced on $\mathcal{P}_u$ by the standard Euclidean metric on $\mathbb{R}\times\Sigma$. Then relative to Cartesian coordinates, we have $d\mathcal{P} = (dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n)\cdot W$, where W is the future-directed Euclidean normal to $\mathcal{P}_u$ and $(dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n)\cdot W$ denotes contraction of W against the first slot of $dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n$. Note that $W^\alpha = \delta^{\alpha\beta}H_\beta$, where H_α is defined in (3J.3) and $\delta^{\alpha\beta} = \mathrm{diag}(1, 1, \ldots, 1)$ is the standard inverse Euclidean metric on $\mathbb{R}\times\Sigma$. Since $d\overline{\varpi}$ and $d\mathcal{P}$ are proportional and since $(dt \wedge d\vartheta^2 \wedge \cdots \wedge d\vartheta^n)\cdot(L \otimes {}^{(2)}\Theta \otimes \cdots \otimes {}^{(n)}\Theta) = 1$, it suffices to show that $\{\sqrt{2}+\gamma\mathrm{f}(\gamma)\} = (dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n)\cdot(W \otimes L \otimes {}^{(2)}\Theta \otimes \cdots \otimes {}^{(n)}\Theta)$. To proceed, we note that $(dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n)\cdot(W \otimes L \otimes {}^{(2)}\Theta \otimes \cdots \otimes {}^{(n)}\Theta)$ is equal to the determinant of the $(1+n)\times(1+n)$ matrix

$$N := \begin{pmatrix} W^0 & L^0 & 0 & \cdots & 0 \\ W^1 & L^1 & {}^{(2)}\Theta^1 & \cdots & {}^{(n)}\Theta^1 \\ \vdots & \vdots & \vdots & & \vdots \\ W^n & L^n & {}^{(2)}\Theta^n & \cdots & {}^{(n)}\Theta^n \end{pmatrix}.$$

From (2A.3a)–(2A.3b), Definition 3.6, (3C.2), Definition 3.15, definition (3E.1a), definition (3J.3), and the relation $W^\alpha = \delta^{\alpha\beta}H_\beta$, it follows that

$$N = \left(\begin{array}{cc|c} \frac{\sqrt{2}}{2} & 1 & \\ & & \mathbf{0}_{2\times(n-1)} \\ -\frac{\sqrt{2}}{2} & 1 & \\ \hline \multicolumn{2}{c|}{*_{(n-1)\times 2}} & \mathbb{I}_{(n-1)\times(n-1)} \end{array}\right) + N^{(\mathrm{Small})},$$

where the entries of the submatrix $*_{(n-1)\times 2}$ are of the schematic form $\mathrm{f}(\gamma)$, $\mathbb{I}_{(n-1)\times(n-1)}$ is the identity matrix, and $N^{(\mathrm{Small})}$ is a matrix whose entries are all of the schematic form $\gamma\mathrm{f}(\gamma)$, where f is smooth. From these facts and the basic properties of the determinant, we conclude that $\det N = \sqrt{2}+\gamma\mathrm{f}(\gamma)$, which is the desired identity. □

[30]Note that the minus sign in (3I.2) does not appear in (3J.5) since we are viewing (3J.5) as a relationship between integration measures.

3K. *Notation for repeated differentiation.* In this subsection, we define some notation that we use when performing repeated differentiation.

Definition 3.31. Recall that the commutation vector field sets $\mathscr{Z}$ and $\mathscr{P}$ are defined in Definition 3.13. We label the $n+1$ vector fields in $\mathscr{Z}$ as follows: $Z_{(1)} = L$, $Z_{(2)} = {}^{(2)}\Theta$, $Z_{(3)} = {}^{(3)}\Theta$, $\ldots$, $Z_{(n)} = {}^{(n)}\Theta$, $Z_{(n+1)} = \breve{X}$. Note that $\mathscr{P} = \{Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}\}$. We define the following vector field operators:

- If $\vec{I} = (\iota_1, \iota_2, \ldots, \iota_N)$ is a multi-index of order $|\vec{I}| := N$ with $\iota_1, \iota_2, \ldots, \iota_N \in \{1, 2, \ldots, n+1\}$, then $\mathscr{Z}^{\vec{I}} := Z_{(\iota_1)} Z_{(\iota_2)} \cdots Z_{(\iota_N)}$ denotes the corresponding N-th order differential operator. We write $\mathscr{Z}^N$ rather than $\mathscr{Z}^{\vec{I}}$ when we are not concerned with the structure of $\vec{I}$, and we sometimes omit the superscript when $N = 1$.
- If $\vec{I} = (\iota_1, \iota_2, \ldots, \iota_N)$, then $\vec{I}_1 + \vec{I}_2 = \vec{I}$ means that $\vec{I}_1 = (\iota_{k_1}, \iota_{k_2}, \ldots, \iota_{k_m})$ and $\vec{I}_2 = (\iota_{k_{m+1}}, \iota_{k_{m+2}}, \ldots, \iota_{k_N})$, where $1 \le m \le N$ and $k_1, k_2, \ldots, k_N$ is a permutation of $1, 2, \ldots, N$.
- Sums such as $\vec{I}_1 + \vec{I}_2 + \cdots + \vec{I}_K = \vec{I}$ have an analogous meaning.
- $\mathcal{P}_u$-tangent operators such as $\mathscr{P}^{\vec{I}}$ are defined analogously, except in this case we have $\iota_1, \iota_2, \ldots, \iota_N \in \{1, 2, \ldots, n\}$. We write $\mathscr{P}^N$ rather than $\mathscr{P}^{\vec{I}}$ when we are not concerned with the structure of $\vec{I}$, and we sometimes omit the superscript when $N = 1$.

3L. *Notation involving multi-indices.* In defining our main L^2-controlling quantity (see Definition 8.1), we will refer to the following set of multi-indices.

Definition 3.32 (a set of $\mathscr{Z}$-multi-indices). We define $\mathcal{I}_*^{[1,N];1}$ to be the set of $\mathscr{Z}$ multi-indices $\vec{I}$ (in the sense of Definition 3.31) such that (i) $1 \le |\vec{I}| \le N$, (ii) $\mathscr{Z}^{\vec{I}}$ contains *at least one factor* belonging to $\mathscr{P} = \{L, {}^{(2)}\Theta, {}^{(3)}\Theta, \ldots, {}^{(n)}\Theta\}$, and (iii) $\mathscr{Z}^{\vec{I}}$ contains no more than one factor of $\breve{X}$.

3M. *Norms.* In this subsection, we define the norms that we use in studying the solution.

Definition 3.33 (pointwise norms). We define the following pointwise norms for arrays $v = (v^J)_{1 \le J \le M}$ and $V = (V_\alpha^J)_{0 \le \alpha \le n, 1 \le J \le M}$:

$$|v| := \sum_{J=1}^{M} |v^J|, \quad |V_\alpha| := \sum_{J=1}^{M} |V_\alpha^J|, \quad |V| := \sum_{J=1}^{M} \sum_{\alpha=0}^{n} |V_\alpha^J|. \tag{3M.1}$$

We will use the following L^2 and L^∞ norms in our analysis.

Definition 3.34 (L^2 and L^∞ norms). In terms of the geometric forms of Definition 3.26, we define the following norms for scalar or array-valued functions w:

$$\|w\|_{L^2(\mathcal{T}_{t,u})}^2 := \int_{\mathcal{T}_{t,u}} |w|^2 \, d\vartheta, \quad \|w\|_{L^2(\Sigma_t^u)}^2 := \int_{\Sigma_t^u} |w|^2 \, d\underline{\varpi}, \quad \|w\|_{L^2(\mathcal{P}_u^t)}^2 := \int_{\mathcal{P}_u^t} |w|^2 \, d\overline{\varpi}, \tag{3M.2a}$$

$$\begin{aligned} \|w\|_{L^\infty(\mathcal{T}_{t,u})} &:= \operatorname*{ess\,sup}_{\vartheta \in \mathbb{T}^{n-1}} |w|(t, u, \vartheta), \\ \|w\|_{L^\infty(\Sigma_t^u)} &:= \operatorname*{ess\,sup}_{(u',\vartheta) \in [0,u] \times \mathbb{T}^{n-1}} |w|(t, u', \vartheta), \\ \|w\|_{L^\infty(\mathcal{P}_u^t)} &:= \operatorname*{ess\,sup}_{(t',\vartheta) \in [0,t] \times \mathbb{T}^{n-1}} |w|(t', u, \vartheta). \end{aligned} \tag{3M.2b}$$

3N. ***Strings of commutation vector fields and vector field seminorms.*** We will use the following shorthand notation to capture the relevant structure of our vector field differential operators and to schematically depict estimates.

Definition 3.35. • $\mathscr{Z}^{N;1} f$ denotes an arbitrary string of N commutation vector fields in $\mathscr{Z}$ (see (3C.9a)) applied to f, where the string contains *at most* one factor of the $\mathcal{P}_u^t$-transversal vector field $\breve{X}$. We sometimes write Zf instead of $\mathscr{Z}^{1;1} f$.

• $\mathscr{P}^N f$ denotes an arbitrary string of N commutation vector fields in $\mathscr{P}$ (see (3C.9b)) applied to f. Consistent with Remark 3.18, we sometimes write Pf instead of $\mathscr{P}^1 f$.

• For $N \geq 1$, $\mathscr{Z}_*^{N;1} f$ denotes an arbitrary string of N commutation vector fields in $\mathscr{Z}$ applied to f, where the string contains *at least* one $\mathcal{P}_u$-tangent factor and *at most* one factor of $\breve{X}$. We also set $\mathscr{Z}_*^{0;0} f := f$.

• For $N \geq 1$, $\mathscr{P}_*^N f$ denotes an arbitrary string of N commutation vector fields in $\mathscr{P}$ applied to f, where the string contains *at least one factor* belonging to the geometric torus coordinate partial derivative vector field set $\{{}^{(2)}\Theta, {}^{(3)}\Theta, \dots, {}^{(n)}\Theta\}$ or *at least two factors* of L.

Remark 3.36 (another way to think about operators $\mathscr{P}_*^N$). For exact simple plane wave solutions, if $N \geq 1$ and f is *any* of the quantities that we must estimate, then we have $\mathscr{P}_*^N f \equiv 0$.

We also define seminorms constructed out of sums of the strings of vector fields above:

• $|\mathscr{Z}^{N;1} f|$ simply denotes the magnitude of one of the $\mathscr{Z}^{N;1} f$ as defined above (there is no summation).

• $|\mathscr{Z}^{\leq N;1} f|$ is the *sum* over all terms of the form $|\mathscr{Z}^{N';1} f|$ with $N' \leq N$ and $\mathscr{Z}^{N';1} f$ as defined above. We sometimes write $|\mathscr{Z}^{\leq 1} f|$ instead of $|\mathscr{Z}^{\leq 1;1} f|$.

• $|\mathscr{Z}^{[1,N];1} f|$ is the sum over all terms of the form $|\mathscr{Z}^{N';1} f|$ with $1 \leq N' \leq N$ and $\mathscr{Z}^{N';1} f$ as defined above.

• Sums such as $|\mathscr{P}^{\leq N} f|$, $|\mathscr{P}_*^{[1,N]} f|$, etc. are defined analogously.

• Seminorms such as $\|\mathscr{Z}_*^{[1,N];1} f\|_{L^\infty(\Sigma_t^u)}$ and $\|\mathscr{P}_*^{[1,N]} f\|_{L^\infty(\Sigma_t^u)}$ (see (3M.2)) are defined analogously.

Remark 3.37. In our forthcoming estimates, terms that do not make sense are assumed to be absent. For example in the case $N = 1$, all terms on the right-hand side of (6B.3) are absent except for the term $|\mathscr{P}^{\leq N-1} V|$.

Remark 3.38 (remarks on the symbol "$*$"). Some operators in Definition 3.35 are decorated with a $*$. These operators involve $\mathcal{P}_u$-tangent differentiations that often lead to a gain in smallness in the estimates. More precisely, the operators $\mathscr{P}_*^N$ always lead to a gain in smallness, while the operators $\mathscr{Z}_*^{N;1}$ lead to a gain in smallness except perhaps when they are applied to μ (because $L\mu$ is not generally small).

4. Energy identities

In this section, we define the building block energies and characteristic fluxes that we use to control the solution in L^2 and derive their basic coerciveness properties. We then derive energy identities involving the building blocks. Later in the article, in Definition 8.1, we will use the building blocks to define the main L^2-controlling quantity.

4A. ***Energies and characteristic flux definitions.***

Definition 4.1 (energies and characteristics fluxes). In terms of the geometric forms of Definition 3.26, we define the energy $\mathbb{E}^{(\text{Shock})}[\,\cdot\,]$, which is a functional of scalar-valued functions f, as

$$\mathbb{E}^{(\text{Shock})}[f](t,u) := \int_{\Sigma_t^u} f^2 \, d\underline{\varpi}. \tag{4A.1}$$

In terms of the Cartesian forms of Definition 3.28 and the Euclidean-unit-length one-form H_α defined in (3J.3), we define the energy $\mathbb{E}^{(\text{Regular})}[\,\cdot\,]$ and characteristic flux $\mathbb{F}^{(\text{Regular})}[\,\cdot\,]$, which are functionals of $\mathbb{R}^M$-valued functions w, as

$$\mathbb{E}^{(\text{Regular})}[w](t,u) := \int_{\Sigma_t^u} \delta_{JK} A_I^{0;J}(\Psi, v) w^I w^K \, d\Sigma, \tag{4A.2a}$$

$$\mathbb{F}^{(\text{Regular})}[w](t,u) := \int_{\mathcal{P}_u^t} \delta_{JK} A_I^{\alpha;J}(\Psi, v) H_\alpha w^I w^K \, d\mathcal{P}, \tag{4A.2b}$$

where δ_{JK} is the standard Kronecker delta.

Lemma 4.2 (coerciveness of the energies and characteristic fluxes for the symmetric hyperbolic variables). *If $|\gamma|$ is sufficiently small, then the energy and the characteristic flux from Definition 4.1 enjoy the following coerciveness*:

$$\mathbb{E}^{(\text{Regular})}[w](t,u) \approx \int_{\Sigma_t^u} \mu \delta_{JK} w^J w^K \, d\underline{\varpi}, \tag{4A.3a}$$

$$\mathbb{F}^{(\text{Regular})}[w](t,u) \approx \int_{\mathcal{P}_u^t} \delta_{JK} w^J w^K \, d\overline{\varpi}, \tag{4A.3b}$$

where δ_{JK} is the standard Kronecker delta.

Proof. From the arguments given in the proof of Lemma 3.30, it follows that the one-form H_α defined in (3J.3) can be decomposed as $H_\alpha = \delta_\alpha^0 - \delta_\alpha^1 + H_\alpha^{(\text{Small})}$, where $H_\alpha^{(\text{Small})} = \gamma \mathrm{f}(\gamma)$. Hence, from (2C.1), it follows that when $|\gamma|$ is sufficiently small, we have $\delta_{JK} A_I^{0;J} w^I w^K \approx \delta_{JK} w^J w^K$ and $\delta_{JK} A_I^{\alpha;J} H_\alpha w^I w^K \approx \delta_{JK} w^J w^K$. Appealing to definitions (4A.2a)–(4A.2b) and using the integration measure relationships stated in (3J.5), we conclude (4A.3a)–(4A.3b). □

4B. ***Energy-characteristic flux identities.*** The integral identities in the following proposition form the starting point for our L^2 analysis of solutions. A crucial point is that the left-hand side of (4B.4) features the characteristic flux $\mathbb{F}^{(\text{Regular})}[\,\cdot\,](t,u)$, which by (4A.3b) can be used to control v and V on the characteristic hypersurfaces $\mathcal{P}_u^t$ *without any degenerate μ weight.*

Proposition 4.3 (energy-characteristic flux identities). *Let $L = L^\alpha(\Psi, v)\,\partial_\alpha$ be the vector field from* (2A.1a) *and let f be a solution to the inhomogeneous transport equation*

$$Lf = \mathfrak{F}. \tag{4B.1}$$

Then the following integral identity holds for the energy defined in (4A.1):

$$\mathbb{E}^{(\text{Shock})}[f](t,u) = \mathbb{E}^{(\text{Shock})}[f](0,u) + 2\int_{\mathcal{M}_{t,u}} f \mathfrak{F} \, d\varpi. \tag{4B.2}$$

Moreover, let $A^{\alpha;I}_J(\Psi, v)$ *be the components of the symmetric matrices from* (2A.1b) *and let* w *be a solution to the* (*linear-in-*w) *inhomogeneous symmetric hyperbolic system*

$$\mu A^{\alpha;I}_J \, \partial_\alpha w^J = \mathfrak{F}^I. \tag{4B.3}$$

Then there exist smooth functions, *schematically denoted by* f, *such that the following integral identity holds for the energy and characteristic flux defined in* (4A.2a)–(4A.2b):

$$\begin{aligned}\mathbb{E}^{(\mathrm{Regular})}[w](t,u) + \mathbb{F}^{(\mathrm{Regular})}[w](t,u) = \mathbb{E}^{(\mathrm{Regular})}[w](0,u) + \mathbb{F}^{(\mathrm{Regular})}[w](t,0) \\ + 2\int_{\mathcal{M}_{t,u}} \{1+\gamma \mathrm{f}(\gamma)\}\, \delta_{JK} \mathfrak{F}^J w^K \, d\varpi \\ + \int_{\mathcal{M}_{t,u}} \mathrm{f}_{JK}(\underline{\gamma}, \breve{X}\Psi, P\Psi) w^J w^k \, d\varpi,\end{aligned} \tag{4B.4}$$

where δ_{JK} *is the standard Kronecker delta.*

Proof. The identity (4B.2) is a simple consequence of (4B.1) since $L = \frac{\partial}{\partial t}$ relative to the geometric coordinates (t, u, ϑ).

To prove (4B.4), we define the following vector field relative to the Cartesian coordinates: $\mathscr{J}^\alpha := \delta_{JK} A^{\alpha;J}_I w^I w^K$. Using (4B.3) and the symmetry assumption $A^{\alpha;I}_J = A^{\alpha;J}_I$, we derive (relative to the Cartesian coordinates) the following divergence identity: $\mu\, \partial_\alpha \mathscr{J}^\alpha = 2\delta_{JK}\mathfrak{F}^J w^K + \delta_{JK}(\mu\, \partial_\alpha A^{\alpha;J}_I) w^I w^K$. We now apply the divergence theorem to the vector field $\mathscr{J}$ on the region $\mathcal{M}_{t,u}$, where we use the Cartesian coordinates, the Euclidean metric $\delta^{\alpha\beta} := \mathrm{diag}(1,1,\dots,1)$ on $\mathbb{R}\times\Sigma$, and the Cartesian forms of Definition 3.28 in all computations. Also using that the future-directed Euclidean conormal to Σ_t has Cartesian components δ^0_α and that the future-directed Euclidean conormal to $\mathcal{P}^t_u$ has Cartesian components H_α (see Definition 3.27), we deduce

$$\begin{aligned}&\int_{\Sigma^u_t} \delta_{JK} A^{0;J}_I w^I w^K \, d\Sigma + \int_{\mathcal{P}^t_u} \delta_{JK} A^{\alpha;J}_I H_\alpha w^I w^K \, d\mathcal{P} \\ &\quad = \int_{\Sigma^u_0} \delta_{JK} A^{0;J}_I w^I w^K \, d\Sigma + \int_{\mathcal{P}^t_0} \delta_{JK} A^{\alpha;J}_I H_\alpha w^I w^K \, d\mathcal{P} \\ &\qquad + \int_{\mathcal{M}_{t,u}} \{2\delta_{JK}\mathfrak{F}^J w^K + \delta_{JK}(\mu\,\partial_\alpha A^{\alpha;J}_I) w^I w^K\} \frac{d\mathcal{M}}{\mu}.\end{aligned} \tag{4B.5}$$

Next, using Lemma 3.19 and definition (3E.1b), we can express the integrand $\delta_{JK}(\mu\,\partial_\alpha A^{\alpha;J}_I) w^I w^K$ on the right-hand side of (4B.5) in the following schematic form: $\mathrm{f}_{JK}(\underline{\gamma}, \breve{X}\Psi, P\Psi) w^J w^k$. Also using Lemma 3.30 to express the integration measure $d\mathcal{M}/\mu$ on the right-hand side of (4B.5) as $\{1+\gamma \mathrm{f}(\gamma)\}\, d\varpi$ and appealing to definitions (4A.2a)–(4A.2b), we arrive at the desired identity (4B.4). □

5. The number of derivatives, data-size assumptions, bootstrap assumptions, smallness assumptions, and running assumptions

In this section, we state the number of derivatives that we use to close the forthcoming estimates, state our assumptions on the size of the data, formulate bootstrap assumptions that we use to derive estimates, and describe our smallness assumptions. In Section 5E, we explain why there exist data that satisfy the assumptions.

5A. ***The number of derivatives.*** Throughout the rest of the paper, N_{Top} and N_{Mid} denote two fixed positive integers satisfying the following relations, where n is the number of spatial dimensions:

$$N_{\text{Top}} \geq n+5, \quad N_{\text{Mid}} := \left\lceil \tfrac{1}{2} N_{\text{Top}} \right\rceil + 1. \tag{5A.1}$$

The solutions that we will study are such that, roughly, the order $\leq N_{\text{Top}}$ derivatives of Ψ (with respect to suitable strings of geometric vector fields) are uniformly bounded in the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ and the order $\leq N_{\text{Mid}}$ derivatives of Ψ are uniformly bounded in the norm $\|\cdot\|_{L^\infty(\Sigma_t^u)}$. The remaining quantities that we must estimate satisfy similar bounds but, in some cases, they are one degree less differentiable. The definitions in (5A.1) are convenient in the sense that they will lead to the following: when we derive L^2 estimates for error term products in the commuted equations, all factors in the product except at most one will be uniformly bounded in the norm $\|\cdot\|_{L^\infty(\Sigma_t^u)}$.

5B. ***Data-size assumptions.*** In this subsection, we state our assumptions on the size of the data.

5B1. *The data-size parameter that controls the time of shock formation.* We start with the definition of a data-size parameter $\mathring{A}_*$, which is tied to the time of first shock formation. More precisely, our main theorem shows that $\max_{\alpha=0,\ldots,n} |\partial_\alpha \Psi|$ blows up at a time approximately equal to $\mathring{A}_*^{-1}$.

Definition 5.1 (the crucial quantity that controls the time of shock formation). We define $\mathring{A}_*$ as

$$\mathring{A}_* := \sup_{\Sigma_0^1} [\mathcal{G} \breve{X} \Psi]_-, \tag{5B.1}$$

where $\mathcal{G} \neq 0$ (see Remark 3.9) is the blowup coefficient from Definition 3.8 and $[f]_- := |\min\{f, 0\}|$.

Remark 5.2 (functional dependence of $\mathcal{G}$ along Σ_0). Note that by (3G.2b) and the fact that $L^1 = L^1(\Psi, v)$, we can view $\mathcal{G}$, along Σ_0, as a function of $\Psi|_{\Sigma_0}$ and $v|_{\Sigma_0}$.

Remark 5.3 (positivity of $\mathring{A}_*$). Our main theorem relies on the assumption that $\mathring{A}_* > 0$. Thus, we will make this assumption for the rest of the article.

5B2. *Data-size assumptions.* For technical convenience, we assume that the solution is C^∞ with respect to the Cartesian coordinates along the "data hypersurfaces" $\Sigma_0^{U_0}$ and $\mathcal{P}_u^{2\mathring{A}_*^{-1}}$. However, to close our estimates, we only need to make assumptions on various Sobolev and Lebesgue norms of the data, where the norms are defined in terms of the geometric coordinates and the commutation vector fields $\mathscr{Z}$ defined in (3C.9a). In this subsubsection, we state the norm assumptions, which involve three parameters, denoted by $\mathring{\alpha}$, $\mathring{\epsilon}$, and $\mathring{A}$, that complement the parameter $\mathring{A}_*$. We note that $\mathring{A}$ does not need to be small, and that the same is true for the parameter $\mathring{A}_*$ from Definition 5.1. We will describe our smallness assumptions on $\mathring{\alpha}$ and $\mathring{\epsilon}$ in Section 5D.

We assume that the data satisfy the following size assumptions (see Section 3N regarding the vector field differential operator notation).

<u>L^2 assumptions along Σ_0^1.</u>

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}];1}\Psi\|_{L^2(\Sigma_0^1)},\ \|\mathscr{Z}^{\leq N_{\mathrm{Top}}-1;1}v\|_{L^2(\Sigma_0^1)},\ \|\mathscr{Z}^{\leq N_{\mathrm{Top}}-1;1}V\|_{L^2(\Sigma_0^1)} \leq \mathring{\epsilon}. \tag{5B.2}$$

<u>L^∞ assumptions along Σ_0^1.</u>

$$\|\Psi\|_{L^\infty(\Sigma_0^1)} \leq \mathring{\alpha}, \tag{5B.3a}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Mid}}];1}\Psi\|_{L^\infty(\Sigma_0^1)},\ \|\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1}v\|_{L^\infty(\Sigma_0^1)},\ \|\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1}V\|_{L^\infty(\Sigma_0^1)} \leq \mathring{\epsilon}, \tag{5B.3b}$$

$$\|\breve{X}\Psi\|_{L^\infty(\Sigma_0^1)} \leq \mathring{A}. \tag{5B.3c}$$

<u>Assumptions along $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$.</u>

$$\|\mathscr{Z}^{\leq N_{\mathrm{Top}}-1;1}v\|_{L^2(\mathcal{P}_0^{2\mathring{A}_*^{-1}})},\ \|\mathscr{Z}^{\leq N_{\mathrm{Top}}-1;1}V\|_{L^2(\mathcal{P}_0^{2\mathring{A}_*^{-1}})} \leq \mathring{\epsilon}. \tag{5B.4}$$

<u>Assumptions along $\mathcal{T}_{0,u}$.</u> We assume that for $u \in [0, 1]$, we have

$$\|\mathscr{P}^{\leq N_{\mathrm{Top}}-2}v\|_{L^2(\mathcal{T}_{0,u})},\ \|\mathscr{P}^{\leq N_{\mathrm{Top}}-2}V\|_{L^2(\mathcal{T}_{0,u})} \leq \mathring{\epsilon}. \tag{5B.5}$$

Remark 5.4. Roughly, we will study solutions that are perturbations of nontrivial solutions with $\mathring{\epsilon} = 0$. Note that $\mathring{\epsilon} = 0$ corresponds to a simple plane symmetric wave, as we described in Section 1D. Note also that $\mathring{\alpha}$, $\mathring{A}_*$, and $\mathring{A}$ are generally nonzero for simple plane symmetric waves.

5B3. *Estimates for the initial data of the remaining geometric quantities.* To close our proof, we will have to estimate the scalar functions μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta_{(\mathrm{Small})}^j$, and Ξ^j featured in the array (3E.1b) and definition (3C.7). In this subsubsection, as a preliminary step, we estimate the size of their data along Σ_0^1.

Lemma 5.5 (estimates for the data of μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta_{(\mathrm{Small})}^j$, and Ξ^j). *Under the data-size assumptions of Section 5B2, there exists a constant $C > 0$ depending on the parameter $\mathring{A}$ from* (5B.3c) *and a constant $C_\blacklozenge > 0$ that **does not depend on** $\mathring{A}$ such that the following estimates hold for the scalar functions μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta_{(\mathrm{Small})}^j$ and Ξ^j defined in Definitions 3.5 and 3.15 and* (3C.7), *whenever $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small* (*see Section 3N regarding the vector field notation*):

$$\|\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu\|_{L^2(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.6a}$$

$$\|\mu-1\|_{L^\infty(\Sigma_0^1)} \leq C_\blacklozenge(\mathring{\alpha}+\mathring{\epsilon}), \tag{5B.6b}$$

$$\|L\mu\|_{L^\infty(\Sigma_0^1)} \leq C, \tag{5B.6c}$$

$$\|\mathscr{P}_*^{[1,N_{\mathrm{Mid}}-1]}\mu\|_{L^\infty(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.6d}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^2(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.7a}$$

$$\|\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_0^1)} \leq C_\blacklozenge(\mathring{\alpha}+\mathring{\epsilon})\delta_j^1, \tag{5B.7b}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Mid}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.7c}$$

$$\|\breve{X}\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_0^1)} \leq C, \tag{5B.7d}$$

$$\|\mathscr{Z}^{\leq N_{\mathrm{Top}}-1;1\,(i)}\Theta^j_{(\mathrm{Small})}\|_{L^2(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.8a}$$

$$\|\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1\,(i)}\Theta^j_{(\mathrm{Small})}\|_{L^\infty(\Sigma_0^1)} \leq C\mathring{\epsilon}, \tag{5B.8b}$$

$$\|\mathscr{P}^{\leq N_{\mathrm{Top}}-1}\,\Xi^j\|_{L^2(\Sigma_0^1)} \leq C, \tag{5B.9a}$$

$$\|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\,\Xi^j\|_{L^\infty(\Sigma_0^1)} \leq C. \tag{5B.9b}$$

Remark 5.6 (the "nonsmall" quantities). Note that the only estimates not featuring the smallness parameters $\mathring{\alpha}$ or $\mathring{\epsilon}$ are (5B.6c), (5B.7d), (5B.9a), and (5B.9b).

Proof sketch. We only sketch the proof since it is standard but has a tedious component that is similar to other analysis that we carry out later: commutator estimates of the type proved in Lemma 6.2, based on the vector field commutator identities (3H.2a)–(3H.2c).

To proceed, we use Lemmas 3.20 and 3.21, Corollary 3.23, and the fact that L^α and X^α are smooth functions of (Ψ, v) (the latter by (3C.4a)) to deduce the following schematic relationships, which hold along Σ_0 (where f is smooth):

$$(\mu-1)|_{\Sigma_0} = (\Psi, v)\cdot \mathrm{f}(\Psi, v), \tag{5B.10}$$

$$\xi_j^{(\mathrm{Small})}|_{\Sigma_0} = (\Psi, v)\cdot \mathrm{f}(\Psi, v)\delta_j^1, \tag{5B.11}$$

$$^{(i)}\Theta^j_{(\mathrm{Small})}|_{\Sigma_0} = 0, \tag{5B.12}$$

$$\Xi^j|_{\Sigma_0} = \mathrm{f}(\Psi, v), \tag{5B.13}$$

as well as the following evolution equations, also written in schematic form (where $P \in \mathscr{P}$):

$$L\mu = \mathrm{f}(\gamma)\breve{X}\Psi + \mu\mathrm{f}(\gamma)L\Psi + \mu\mathrm{f}(\gamma)V, \tag{5B.14}$$

$$L\xi_j^{(\mathrm{Small})} = \mathrm{f}(\gamma)P\Psi + \mathrm{f}(\gamma)V, \tag{5B.15}$$

$$L^{(i)}\Theta^j_{(\mathrm{Small})} = \mathrm{f}(\gamma)P\Psi + \mathrm{f}(\gamma)V, \tag{5B.16}$$

$$L\Xi^j = (\Xi^1, \dots, \Xi^n)\cdot\mathrm{f}(\gamma, P\Psi) + \mathrm{f}(\underline{\gamma}, L\Psi, \breve{X}\Psi). \tag{5B.17}$$

By repeatedly differentiating (5B.14)–(5B.17) with the elements of $\mathscr{Z}$ and using the commutator identities (3H.2a)–(3H.2c), we can algebraically express all quantities that we need to estimate in terms of the derivatives of μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta^j_{(\mathrm{Small})}$, and Ξ^j with respect to the (Σ_t-tangent) vector fields in $\{\breve{X}, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$ and the $\mathscr{Z}$ derivatives of Ψ, v, and V. Then using (5B.10)–(5B.13), we can express, along Σ_0, the derivatives of μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta^j_{(\mathrm{Small})}$ and Ξ^j with respect to the elements of $\{\breve{X}, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$ in terms of the derivatives of Ψ and v with respect to the elements of $\{\breve{X}, {}^{(2)}\Theta, \dots, {}^{(n)}\Theta\}$. The estimates (5B.6a)–(5B.9b) then follow from these algebraic expressions, the data-size assumptions (5B.2)–(5B.3c), and the standard Sobolev calculus. We stress that the identities (3H.2a)–(3H.2c) show that commutator terms contain a factor involving a differentiation with respect to one of the ${}^{(i)}\Theta$, which, in view of our data-size assumptions from Section 5B2, leads to a gain in $\mathcal{O}(\mathring{\epsilon})$ smallness for all commutator terms. □

5C. ***Bootstrap assumptions.*** In this subsection, we state the bootstrap assumptions that we use to control the solution.

5C1. $T_{(\mathrm{Boot})}$*, the positivity of* μ*, and the diffeomorphism property of* Υ. We now state some basic bootstrap assumptions. We start by fixing a real number $T_{(\mathrm{Boot})}$ with

$$0 < T_{(\mathrm{Boot})} \leq 2\mathring{A}_*^{-1}, \tag{5C.1}$$

where $\mathring{A}_* > 0$ (see Remark 5.3) is the data-dependent parameter from Definition 5.1.

We assume that on the spacetime domain $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$ (see (3A.3f)), we have

$$\mu > 0. \tag{$\boldsymbol{BA}\mu > 0$}$$

Inequality ($\boldsymbol{BA}\mu > 0$) essentially means that no shocks are present in $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$.

We also assume that

$$\begin{gathered}\text{the change of variables map } \Upsilon \text{ from Definition 3.24 is a diffeomorphism from}\\ [0, T_{(\mathrm{Boot})}) \times [0, U_0] \times \mathbb{T}^{n-1} \text{ onto its image.}\end{gathered} \tag{5C.2}$$

5C2. *Fundamental* L^∞ *bootstrap assumptions.* In this section, we state our fundamental L^∞ bootstrap assumptions. We will derive strict improvements of the fundamental bootstrap assumptions in Corollary 8.8, on the basis of a priori energy estimates and Sobolev embedding.

Fundamental bootstrap assumptions for v and V. We assume that the following inequalities hold for $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$ ($\alpha = 0, \dots, n,\ J = 1, \dots, M$):

$$\|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1} v^J\|_{L^\infty(\Sigma_t^u)},\ \|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1} V_\alpha^J\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon, \tag{5C.3}$$

where $\varepsilon > 0$ is a small bootstrap parameter (see Section 5D for discussion on the required smallness).

5C3. *Auxiliary bootstrap assumptions.* In addition to the fundamental bootstrap assumptions, we find it convenient to make auxiliary bootstrap assumptions, which we state in this subsubsection. We will derive strict improvements of the auxiliary bootstrap assumptions in Proposition 6.5.

Auxiliary bootstrap assumptions for Ψ. We assume that the following inequalities hold for $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\|\Psi\|_{L^\infty(\Sigma_t^u)} \leq \mathring{\alpha} + \varepsilon^{1/2}, \tag{5C.4a}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Mid}};]1}\Psi\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2}, \tag{5C.4b}$$

$$\|\breve{X}\Psi\|_{L^\infty(\Sigma_t^u)} \leq \mathring{A} + \varepsilon^{1/2}. \tag{5C.4c}$$

Auxiliary bootstrap assumptions for v and V. We assume that the following inequalities hold for $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\|\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1} v\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2}, \tag{5C.5a}$$

$$\|\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1} V\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2}. \tag{5C.5b}$$

Auxiliary bootstrap assumptions for μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(j)}\Theta^k_{(\mathrm{Small})}$. We assume that the following inequalities hold for $(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]$:

$$\|\mu\|_{L^\infty(\Sigma_t^u)} \le 1 + 2\mathring{A}_*^{-1}\|\mathcal{G}\breve{X}\Psi\|_{L^\infty(\Sigma_0^u)} + \mathring{\alpha}^{1/2} + \varepsilon^{1/2}, \tag{5C.6a}$$

$$\|L\mu\|_{L^\infty(\Sigma_t^u)} \le \|\mathcal{G}\breve{X}\Psi\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \tag{5C.6b}$$

$$\|\mathscr{P}_*^{[1,N_{\mathrm{Mid}}-1]}\mu\|_{L^\infty(\Sigma_t^u)} \le \varepsilon^{1/2}, \tag{5C.6c}$$

where $\mathcal{G}\neq 0$ (see Remark 3.9) is the blowup coefficient from Definition 3.8, $\|\mathcal{G}\breve{X}\Psi\|_{L^\infty(\Sigma_0^u)} \le C_\blacklozenge \mathring{A}$, and $C_\blacklozenge > 0$ is a constant with the parameter-dependence properties described in Section 1H.

Moreover, we assume that

$$\|\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_t^u)} \le \mathring{\alpha}^{1/2} + \mathring{\epsilon}^{1/2}, \tag{5C.7a}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Mid}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_t^u)} \le \varepsilon^{1/2}, \tag{5C.7b}$$

$$\|\breve{X}\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_t^u)} \le \|\breve{X}\xi_j^{(\mathrm{Small})}\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \tag{5C.7c}$$

$$\|\mathscr{Z}^{\le N_{\mathrm{Mid}}-1;1}\,{}^{(j)}\Theta^k_{(\mathrm{Small})}\|_{L^\infty(\Sigma_t^u)} \le \varepsilon^{1/2}. \tag{5C.7d}$$

5D. *Smallness assumptions.* For the remainder of the article, when we say that "statement X holds whenever A is small relative to B", we mean that there is a particular continuous increasing function $f:(0,\infty)\to(0,\infty)$ such that statement X holds whenever $A < f(B)$. The functions f are allowed to vary throughout the article. To avoid lengthening the paper, we often avoid explicitly specifying the form of f.

To ensure that all of the statements needed for our main results hold, we will make the following smallness assumptions, where we will continually adjust the required smallness in order to close our estimates:

- The bootstrap parameter ε and the data smallness parameter $\mathring{\epsilon}$ from Section 5B2. are small relative to 1.
- ε and $\mathring{\epsilon}$ are small relative to $\mathring{A}^{-1}$, where $\mathring{A}$ is the data-size parameter from Section 5B2.
- ε and $\mathring{\epsilon}$ are small relative to the data-size parameter $\mathring{A}_*$ from Definition 5.1.
- The data-size parameter $\mathring{\alpha}$ from Section 5B2 is small relative to 1.
- $\mathring{\epsilon} \le \varepsilon < \mathring{\alpha}$.

The first two assumptions will allow us to treat error terms of size ε and $\varepsilon\mathring{A}$ as small quantities. The third assumption is relevant because the expected blowup time is approximately $\mathring{A}_*^{-1}$, and the assumption will allow us to show that various error products featuring a small factor ε in fact remain small for $t < 2\mathring{A}_*^{-1}$, which is plenty of time for us to show that a shock forms. The smallness assumption on $\mathring{\alpha}$ ensures that the solution remains within the regime of hyperbolicity of the equations and that $\mathcal{G}\neq 0$, where $\mathcal{G}$ is the blowup coefficient from Definition 3.8.

5E. ***Existence of data satisfying the size assumptions.*** We now outline a proof that there exists an open set of data satisfying the size assumptions of Section 5B and the smallness assumptions of Section 5D. Since the assumptions are stable under Sobolev perturbations, it is enough to exhibit data corresponding to plane symmetric solutions, that is, solutions that depend only on t and x^1. This means that along Σ_0, it is enough to exhibit appropriate data that depend only on x^1. To exhibit data for Ψ, we simply let $f(x^1)$ be any smooth nontrivial function that is compactly supported in Σ_0^1, and we set $\Psi(0, x^1, \dots, x^n) := \kappa f(x^1)$, where κ is a real parameter. We then take vanishing data for v, so that, as a consequence of the evolution equation (2A.1b), we have $v \equiv 0$ and $V \equiv 0$. With the help of these facts, it is straightforward to check that by choosing κ to be sufficiently small in magnitude, we can satisfy all of the desired assumptions. More precisely, by construction, we have $\mathring{\epsilon} = 0$, and by choosing $|\kappa|$ to be small, we can ensure that the quantity $\mathring{\alpha} > 0$ on the right-hand side of (5B.3a) is as small as we want.

5F. ***Basic assumptions, facts, and estimates that we use silently.*** In this subsection, we state some basic assumptions and conventions that we silently use throughout the rest of the paper when deriving estimates.

(1) All of the estimates that we derive hold on the bootstrap region $\mathcal{M}_{T_{(\text{Boot})}, U_0}$. Moreover, in deriving estimates, we rely on the data-size assumptions and bootstrap assumptions from Sections 5B–5C, and the smallness assumptions of Section 5D.

(2) All quantities that we estimate can be controlled in terms of the quantities featured in the array $\underline{\gamma}$ from definition (3E.1b), the Cartesian components Ξ^j of the $\mathcal{T}_{t,u}$-tangent vector field Ξ from (3C.7), and the $\mathscr{Z}$-derivatives of these quantities.

(3) We typically use the Leibniz rule for vector field differentiation when deriving pointwise estimates for the $\mathscr{Z}$-derivatives derivatives of products of the schematic form $\prod_{i=1}^{m} p_i$. Our derivative counts are such that after any product is differentiated, all factors except at most one are uniformly bounded in L^∞ on $\mathcal{M}_{T_{(\text{Boot})}, U_0}$.

(4) The constants $C > 0$ in all of our estimates are allowed to depend on the data-size parameters $\mathring{A}$ and $\mathring{A}_*^{-1}$, as we described in Section 1H.

(5) The constants $C_\blacklozenge > 0$ do not depend on $\mathring{A}$ or $\mathring{A}_*$, as we described in Section 1H.

(6) We use the convention for nonsensical terms mentioned in Remark 3.37.

5G. ***Omission of the independent variables in some expressions.*** We use the following notational conventions in the rest of the article:

- Many of our pointwise estimates are stated in the form

$$|f_1| \lesssim F(t)|f_2|$$

for some function F. Unless we otherwise indicate, it is understood that both f_1 and f_2 are evaluated at the point with geometric coordinates (t, u, ϑ).

- Unless we otherwise indicate, in integrals $\int_{\mathcal{T}_{t,u}} f \, d\vartheta$, we view the integrand f as a function of (t, u, ϑ), and ϑ is the integration variable.

- Unless we otherwise indicate, in integrals $\int_{\Sigma_t^u} f \, d\underline{\varpi}$, we view the integrand f as a function of (t, u', ϑ), and (u', ϑ) are the integration variables.
- Unless we otherwise indicate, in integrals $\int_{\mathcal{P}_u^t} f \, d\overline{\varpi}$, we view the integrand f as a function of (t', u, ϑ), and (t', ϑ) are the integration variables.
- Unless we otherwise indicate, in integrals $\int_{\mathcal{M}_{t,u}} f \, d\varpi$, we view the integrand f as a function of (t', u', ϑ), and (t', u', ϑ) are the integration variables.

6. Pointwise estimates and improvements of the auxiliary bootstrap assumptions

In this section, we use the data-size assumptions and bootstrap assumptions of Section 5 to derive pointwise and L^∞ estimates for various quantities. The main result is Proposition 6.5. In particular, the results of this section yield strict improvements of the auxiliary bootstrap assumptions of Section 5C3.

Remark 6.1. Throughout this section, we silently use the conventions described in Section 5F. Moreover, N_{Top} and N_{Mid} denote the integers from Section 5A.

6A. *Commutator estimates.* We start by providing some commutator estimates that we will use throughout the analysis.

Lemma 6.2. *Let $1 \leq N \leq N_{\mathrm{Top}}$ be an integer, let $\vec{I}$ be a multi-index for the set $\mathscr{P}$ of $\mathcal{P}_u$-tangent commutation vector fields such that $|\vec{I}| = N$, and let $\vec{J}$ be any permutation of $\vec{I}$ (in particular, $|\vec{I}| = |\vec{J}| = N \leq N_{\mathrm{Top}}$). Then the following identity for scalar functions f holds*:

$$\mathscr{P}^{\vec{I}} f - \mathscr{P}^{\vec{J}} f = 0. \tag{6A.1}$$

Let $1 \leq N \leq N_{\mathrm{Top}}$ be an integer. Then the following commutator estimate for scalar functions f holds (*see Definition 3.35 regarding the vector field notation*):

$$|[L, \mathscr{Z}^{N;1}] f| \lesssim |\mathscr{P}_*^{[1,N]} f| + \underbrace{|\mathscr{P}_*^{[1,\lfloor N/2 \rfloor]} f| |\mathscr{Z}_*^{[1,N];1} \Psi|}_{\textit{absent if } N=1} + \underbrace{|\mathscr{P}_*^{[1,\lfloor N/2 \rfloor]} f| |\mathscr{P}_*^{[1,N-1]} \underline{\gamma}|}_{\textit{absent if } N=1}. \tag{6A.2}$$

Let $2 \leq N \leq N_{\mathrm{Top}}$ be an integer, let $\vec{I} \in \mathcal{I}_*^{[1,N];1}$ (*see Definition 3.32*), *and let $\vec{J}$ be any permutation of $\vec{I}$. Then the following commutator estimate for scalar functions f holds*:

$$|\mathscr{Z}^{\vec{I}} f - \mathscr{Z}^{\vec{J}} f| \lesssim |\mathscr{P}_*^{[1,N-1]} f| + |\mathscr{P}_*^{[1,\lfloor N/2 \rfloor]} f| |\mathscr{Z}_*^{[1,N-1];1} \gamma| + |\mathscr{P}_*^{[1,\lfloor N/2 \rfloor]} f| |\mathscr{P}_*^{[1,N-1]} \underline{\gamma}|. \tag{6A.3}$$

Proof. Equation (6A.1) is a trivial consequence of the commutation identity (3H.2a).

The estimate (6A.2) is a straightforward consequence of the commutation identities (3H.2a)–(3H.2b) and the bootstrap assumptions.

Similarly, the estimate (6A.3) is a straightforward consequence of the commutation identities (3H.2a)–(3H.2c) and the bootstrap assumptions. ☐

6B. ***Transversal derivatives in terms of tangential derivatives.*** The next lemma, which is algebraic in nature, plays a crucial role in controlling v and V. Roughly, the lemma shows that the $\breve{X}$ derivative of these quantities can be expressed in terms of their $\mathcal{P}_u$-tangential derivatives plus error terms. In particular, this means that we do not have to commute the evolution equations for v and V with $\breve{X}$ in order to control $\breve{X}v$ and $\breve{X}V_\alpha$; we can instead use the equations to algebraically solve for the $\breve{X}$ derivative. This is important because commuting these equations (which must be weighted with μ to avoid singular error terms) with $\breve{X}$ would generate the error term $\breve{X}\mu$, which is uncontrollable based on the degree of $\breve{X}$-differentiability that we have imposed on Ψ.

Lemma 6.3 (algebraic expressions for transversal derivatives of v and V in terms of their tangential derivatives). *There exist smooth functions of γ, schematically denoted by* f, *such that the following algebraic identities hold whenever $|\gamma|$ is sufficiently small* (*where $P \in \mathscr{P}$ and $Z \in \mathscr{Z}$*):

$$\breve{X}v = \mu \mathrm{f}(\gamma)V, \tag{6B.1}$$

$$\breve{X}V_\alpha = \mathrm{f}(\underline{\gamma})PV + \mathrm{f}(\underline{\gamma}, Z\Psi)V. \tag{6B.2}$$

Proof. To prove (6B.1), we first multiply (2A.1b) by μ and use Lemma 3.19 to obtain the following identity, whose right-hand side is written in schematic form: $\mu(A^0+A^a\xi_a)Xv=\mu\mathrm{f}(\gamma)Pv=\mu\mathrm{f}(\gamma)P^\alpha V_\alpha=\mu\mathrm{f}(\gamma)V$. Next, using Definition 3.15, we see that $\mu(A^0+A^a\xi_a)Xv=(A^0-A^1+A_{(\mathrm{Small})})\breve{X}v$, where A^0-A^1 is a matrix whose entries are of the schematic form $\mathrm{f}(\gamma)$ and $A_{(\mathrm{Small})}$ is a matrix whose entries are of the schematic form $\gamma\mathrm{f}(\gamma)$. From these facts and the assumption (2C.1), it follows that whenever $|\gamma|$ is sufficiently small, the matrix $A^0-A^1+A_{(\mathrm{Small})}$ is invertible. From this fact, the desired identity (6B.1) easily follows.

The proof of (6B.2) is based on (2A.5) and is similar but requires one new ingredient: we use Lemma 3.19 to (schematically) express the right-hand side of (2A.5) as $\mathrm{f}(\underline{\gamma}, Z\Psi)V$. □

With the help of Lemmas 6.2 and 6.3, we now derive pointwise estimates showing that the derivatives of v and V involving up to one $\breve{X}$ differentiation can be controlled in terms of quantities that do not depend on the $\breve{X}$ derivatives of v and V.

Lemma 6.4 (pointwise estimates for transversal derivatives of v and V in terms of their tangential derivatives). *The following estimates hold for* $1 \leq N \leq N_{\mathrm{Top}}$:

$$|\mathscr{Z}^{N;1}v| \lesssim |\mathscr{Z}_*^{[1,N-1];1}\Psi| + |\mathscr{P}^{[1,N-1]}v| + |\mathscr{P}^{\leq N-1}V| + \sum_{j=1}^{n}|\mathscr{Z}_*^{[1,N-1];1}\xi_j^{(\mathrm{Small})}| + \sum_{i=2}^{n}\sum_{j=1}^{n}|\mathscr{Z}_*^{[1,N-1];1(i)}\Theta_{(\mathrm{Small})}^j| + |\mathscr{P}_*^{[1,N-1]}\mu|. \tag{6B.3}$$

Moreover, the following estimates hold for $1 \leq N \leq N_{\mathrm{Top}}-1$:

$$|\mathscr{Z}^{N;1}V| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{P}^{[1,N-1]}v| + |\mathscr{P}^{\leq N}V| + \sum_{j=1}^{n}|\mathscr{Z}_*^{[1,N-1];1}\xi_j^{(\mathrm{Small})}| + \sum_{i=2}^{n}\sum_{j=1}^{n}|\mathscr{Z}_*^{[1,N-1];1(i)}\Theta_{(\mathrm{Small})}^j| + |\mathscr{P}_*^{[1,N-1]}\mu|. \tag{6B.4}$$

Proof. We will prove (6B.3)–(6B.4) simultaneously by using induction in N. The base case $N = 1$ can be handled using the same arguments given below and we omit these details. We therefore assume the induction hypothesis that (6B.3)–(6B.4) have been proved with $N-1$ in the role of N; to prove (6B.3)–(6B.4) in the case N, we first consider an order-N operator of the form $\mathscr{P}^{N-1}\breve{X}$. Using (6B.2), we deduce that $\mathscr{P}^{N-1}\breve{X}V_\alpha = \mathscr{P}^{N-1}\{\mathrm{f}(\underline{\gamma})PV + \mathrm{f}(\underline{\gamma}, Z\Psi)V\}$. From this expression and the bootstrap assumptions, we deduce that $|\mathscr{P}^{N-1}\breve{X}V_\alpha| \lesssim$ the right-hand side of (6B.4) as desired. Then using the commutator estimate (6A.3) and the bootstrap assumptions, we can arbitrarily permute the vector field factors in $\mathscr{P}^{N-1}\breve{X}V_\alpha$ up to error terms that are pointwise bounded in magnitude by $\lesssim$ the right-hand side of (6B.4) plus error terms of the form $|\mathscr{Z}_*^{[1,N-1];1}v| + |\mathscr{Z}_*^{[1,N-1];1}V|$, which (by the induction hypothesis) have already been shown to be bounded by $\lesssim$ the right-hand side of (6B.4). We have therefore obtained the desired bounds for V in the case that $\mathscr{Z}^{N;1}$ contains a factor of $\breve{X}$. In the case that the operator $\mathscr{Z}^{N;1}$ contains a factor of $\breve{X}$, the estimate (6B.3) for v follows similarly with the help of (6B.1). To prove (6B.3) in the case that the operator $\mathscr{Z}^{N;1}$ does not contain a factor of $\breve{X}$, that is, that $\mathscr{Z}^{N;1} = \mathscr{P}^N$, we first write $\mathscr{P}^N v = \mathscr{P}^{N-1}(P^\alpha\,\partial_\alpha v) = \mathscr{P}^{N-1}(P^\alpha V_\alpha) = \mathscr{P}^{N-1}(\mathrm{f}(\gamma)V_\alpha)$. From this expression and the bootstrap assumptions, we bound the magnitude of the right-hand side of this equation by $\lesssim$ the right-hand side of (6B.3) as desired. In the case that $\mathscr{Z}^{N;1}$ does not contain a factor of $\breve{X}$, that is, that $\mathscr{Z}^{N;1} = \mathscr{P}^N$, the estimate (6B.4) is trivial. We have therefore closed the induction. We clarify that in the final step, we allow $N = N_{\mathrm{Top}}$ in (6B.3), but not in (6B.4). □

6C. ***Pointwise estimates and improvements of the auxiliary bootstrap assumptions.*** We now state and prove the main result of this section.

Proposition 6.5 (pointwise estimates and improvements of the auxiliary bootstrap assumptions). *Let N_{Top} and N_{Mid} be the integers fixed in Section 5A. If $N \le N_{\mathrm{Top}}$, then the following estimates hold* (*see Section 3N regarding the vector field differential operator notation*).

Pointwise estimates for the commuted evolution equations of Ψ, v and V.

$$|L\mathscr{Z}^{N;1}\Psi| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{Z}_*^{[1,N-1];1}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}|. \tag{6C.1}$$

Similarly, if $1 \le N \le N_{\mathrm{Top}}$, then the following pointwise estimates hold:

$$|\mu A^\alpha\,\partial_\alpha \mathscr{P}^{N-1}v| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{Z}_*^{[1,N-1];1}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}|, \tag{6C.2a}$$

$$|\mu A^\alpha\,\partial_\alpha \mathscr{P}^{N-1}V_\alpha| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{Z}_*^{[1,N-1];1}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}| + |V|. \tag{6C.2b}$$

Pointwise estimates for the commuted evolution equations of $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta_{(\mathrm{Small})}^j$, and μ. If $1 \le N \le N_{\mathrm{Top}}$, then the following estimates hold:

$$|L\mathscr{Z}^{N-1;1}\xi_j^{(\mathrm{Small})}| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{Z}_*^{[1,N-1];1}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}| + |V|, \tag{6C.3a}$$

$$|L\mathscr{Z}^{N-1;1}\,{}^{(i)}\Theta_{(\mathrm{Small})}^j| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{Z}_*^{[1,N-1];1}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}| + |V|. \tag{6C.3b}$$

Furthermore, if $2 \le N \le N_{\mathrm{Top}}$, then the following estimates hold:

$$|L\mathscr{P}^{N-1}\mu| \lesssim |\mathscr{Z}_*^{[1,N];1}\Psi| + |\mathscr{P}^{[1,N-1]}\gamma| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}| + |V|. \tag{6C.4}$$

L^{∞} estimates for Ψ. In addition, the following estimates hold:

$$\|\Psi\|_{L^{\infty}(\Sigma_t^u)} \leq \mathring{\alpha} + C\varepsilon, \tag{6C.5a}$$

$$\|\mathscr{Z}_*^{[1,N_{\text{Mid}}];1}\Psi\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon, \tag{6C.5b}$$

$$\|\breve{X}\Psi\|_{L^{\infty}(\Sigma_t^u)} \leq \mathring{A} + C\varepsilon. \tag{6C.5c}$$

L^{∞} estimates for v and V. Moreover, the following estimates hold:

$$\|\mathscr{Z}^{\leq N_{\text{Mid}}-1;1}v\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon, \tag{6C.6a}$$

$$\|\mathscr{Z}^{\leq N_{\text{Mid}}-1;1}V\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon. \tag{6C.6b}$$

L^{∞} estimates for $\xi_j^{(\text{Small})}$, ${}^{(i)}\Theta_{(\text{Small})}^j$, and μ. The following estimates hold:

$$\|\xi_j^{(\text{Small})}\|_{L^{\infty}(\Sigma_t^u)} \leq C_{\blacklozenge}\mathring{\alpha}\delta_j^1 + C\varepsilon, \tag{6C.7a}$$

$$\|\mathscr{Z}_*^{[1,N_{\text{Mid}}-1];1}\xi_j^{(\text{Small})}\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon, \tag{6C.7b}$$

$$\|\breve{X}\xi_j^{(\text{Small})}\|_{L^{\infty}(\Sigma_t^u)} \leq \|\breve{X}\xi_j^{(\text{Small})}\|_{L^{\infty}(\Sigma_0^u)} + C\varepsilon, \tag{6C.7c}$$

$$\|\mathscr{Z}^{\leq N_{\text{Mid}}-1;1}{}^{(i)}\Theta_{(\text{Small})}^j\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon, \tag{6C.7d}$$

$$\|\mathscr{P}_*^{[1,N_{\text{Mid}}-1]}\mu\|_{L^{\infty}(\Sigma_t^u)} \leq C\varepsilon. \tag{6C.7e}$$

Sharp estimates for μ and $L\mu$. In addition, the following pointwise estimates hold:

$$\begin{aligned} \mu(t,u,\vartheta) &= 1 + t[\mathcal{G}\breve{X}\Psi](0,u,\vartheta) + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon) \\ &= 1 + t[\mathcal{G}\breve{X}\Psi](t,u,\vartheta) + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon), \end{aligned} \tag{6C.8a}$$

$$\begin{aligned} L\mu(t,u,\vartheta) &= [\mathcal{G}\breve{X}\Psi](0,u,\vartheta) + \mathcal{O}(\varepsilon) \\ &= \{\mathcal{G}|_{(\Psi,v)=(0,0)} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\}\breve{X}\Psi(t,u,\vartheta) + \mathcal{O}(\varepsilon), \end{aligned} \tag{6C.8b}$$

where the blowup coefficient $\mathcal{G}$ is defined in Definition 3.8 and, in view of Remark 5.2 and (3G.2b), $\mathcal{G}|_{(\Psi,v)=(0,0)} = -\frac{\partial L^1}{\partial \Psi}\big|_{(\Psi,v)=(0,0)}$.

Moreover,

$$\|\mu\|_{L^{\infty}(\Sigma_t^u)} \leq 1 + 2\mathring{A}_*^{-1}\|\mathcal{G}\breve{X}\Psi\|_{L^{\infty}(\Sigma_0^u)} + C_{\blacklozenge}\mathring{\alpha} + C\varepsilon, \tag{6C.9a}$$

$$\|L\mu\|_{L^{\infty}(\Sigma_t^u)} \leq \|\mathcal{G}\breve{X}\Psi\|_{L^{\infty}(\Sigma_0^u)} + C\varepsilon. \tag{6C.9b}$$

Estimates for Ξ^j. Finally, if $1 \leq N \leq N_{\text{Top}}$, then the following estimates hold for the Cartesian components Ξ^j of the $\mathcal{T}_{t,u}$-tangent vector field Ξ from (3C.7):

$$|L\mathscr{P}^{\leq N-1}\Xi^j| \lesssim |\mathscr{P}^{\leq N-1}\Xi^j| + |\mathscr{Z}_*^{\leq N;1}\Psi| + |\mathscr{P}_*^{[1,N-1]}\underline{\gamma}| + 1, \tag{6C.10a}$$

$$\|\mathscr{P}^{\leq N_{\text{Mid}}-1}\Xi^j\|_{L^{\infty}(\Sigma_t^u)} \lesssim 1. \tag{6C.10b}$$

Remark 6.6 (strict improvements of the auxiliary bootstrap assumptions). The L^{∞} estimates of Proposition 6.5 provide, in particular, strict improvements of the auxiliary bootstrap assumptions of Section 5C3 whenever $\mathring{\alpha}$ and ε are sufficiently small.

Proof of Proposition 6.5. See Section 5F for some comments on the analysis. We start by noting that the order in which we prove estimates is important. Throughout the proof, we use the phrase "conditions on the data" to mean the assumptions from Section 5B2 for the data of Ψ, υ, and V, as well as the estimates from Lemma 5.5 for the data of μ, $\xi_j^{(\text{Small})}$, ${}^{(i)}\Theta^j_{(\text{Small})}$, and Ξ^j. We also silently use the last item on page 491.

Proof of (6C.1): The estimate follows from the evolution equation (2A.1a), the commutator estimate (6A.2), and the bootstrap assumptions.

Proof of (6C.4): We first schematically write (3G.1a) as $L\mu = \mathrm{f}(\gamma)\breve{X}\Psi + \mathrm{f}(\underline{\gamma})L\Psi + \mathrm{f}(\underline{\gamma})V$. Hence, using (6A.1), we deduce $L\mathscr{P}^{N-1}\mu = \mathscr{P}^{N-1}\{\mathrm{f}(\gamma)\breve{X}\Psi + \mathrm{f}(\underline{\gamma})L\Psi + \mathrm{f}(\underline{\gamma})V\}$. The desired bound (6C.4) then follows from this equation and the bootstrap assumptions (we stress that the assumption $N \geq 2$ is needed for this estimate).

Proof of (6C.3a) and (6C.3b): We first schematically write (3G.1b) as $L\xi_j^{(\text{Small})} = \mathrm{f}(\gamma)P\Psi + \mathrm{f}(\gamma)V$. Hence, $L\mathscr{Z}^{N-1;1}\xi_j^{(\text{Small})} = [L, \mathscr{Z}^{N-1;1}]\xi_j^{(\text{Small})} + \mathscr{Z}^{N-1;1}\{\mathrm{f}(\gamma)P\Psi + \mathrm{f}(\gamma)V\}$. To bound the magnitude of the term $\mathscr{Z}^{N-1;1}\{\cdots\}$ by $\lesssim$ the right-hand side of (6C.3a), we use the bootstrap assumptions. To bound the commutator term $[L, \mathscr{Z}^{N-1;1}]\xi_j^{(\text{Small})}$, we also use (6A.2). The estimate (6C.3b) can be proved in the same way as the estimate (6C.3a), since by (3G.6b), ${}^{(i)}\Theta^j_{(\text{Small})}$ obeys a schematically identical evolution equation: $L^{(i)}\Theta^j_{(\text{Small})} = \mathrm{f}(\gamma)P\Psi + \mathrm{f}(\gamma)V$.

Proof of (6C.5b), (6C.7b), (6C.7d), and (6C.7e): We set

$$q = q(t, u, \vartheta) := |\mathscr{Z}_*^{[1,N_{\text{Mid}}];1}\Psi| + \sum_{j=1}^{n} |\mathscr{Z}_*^{[1,N_{\text{Mid}}-1];1}\xi_j^{(\text{Small})}| + \sum_{i=2}^{n}\sum_{j=1}^{n} |\mathscr{Z}^{\leq N_{\text{Mid}}-1;1\,(i)}\Theta^j_{(\text{Small})}| + |\mathscr{P}_*^{[1,N_{\text{Mid}}-1]}\mu|. \tag{6C.11}$$

From (6C.1), (6C.3a)–(6C.4), the pointwise estimates of (6B.3), the fundamental bootstrap assumptions (5C.3), and the fundamental theorem of calculus, we deduce, in view of the fact that $L = \frac{\partial}{\partial t}$, that $|q(t, u, \vartheta)| \leq |q(0, u, \vartheta)| + c\int_{s=0}^{t} |q(s, u, \vartheta)|\,ds + C\varepsilon$. Moreover, the conditions on the data imply that $|q(0, u, \vartheta)| \leq C\varepsilon$. Hence, from Gronwall's inequality, we deduce that $|q(t, u, \vartheta)| \lesssim \varepsilon\exp(ct) \lesssim \varepsilon$, which implies all four of the desired bounds.

Proof of (6C.5a), (6C.5c), (6C.7a), and (6C.7c): To prove (6C.5a), we first use the fundamental theorem of calculus to obtain $|\Psi|(t, u, \vartheta) \leq |\Psi|(0, u, \vartheta) + \int_{s=0}^{t} |L\Psi|(s, u, \vartheta)\,ds$. The estimate (6C.5b) implies that the time integral in the previous inequality is $\lesssim \varepsilon$. In view of the conditions on the data, we conclude (6C.5a). The remaining three estimates can be proved similarly with the help of the estimates (6C.5b) and (6C.7b).

Proof of (6C.6a)–(6C.6b): These estimates follow from the pointwise estimates (6B.3)–(6B.4), the fundamental bootstrap assumptions (5C.3), and the estimates (6C.5b), (6C.7b), (6C.7d), and (6C.7e).

Proof of (6C.2a)–(6C.2b): We first use Lemma 3.19 to deduce the schematic relation

$$\mu\,\partial_\alpha = \mathrm{f}(\gamma)\breve{X} + \mu\mathrm{f}(\gamma)P = \mathrm{f}(\gamma)\breve{X} + \mathrm{f}(\underline{\gamma})P. \tag{6C.12}$$

Next, using (6C.12), the definition $\partial_\alpha v = V_\alpha$, and the fact that for $Z \in \mathscr{Z}$ we have $Z^\alpha = \mathrm{f}(\underline{\gamma})$, we deduce that $\mu \times$ the right-hand side of (2A.5) $= \mathrm{f}(\underline{\gamma}, Z\Psi)V$. Therefore, commuting $\mu \times$ (2A.5) with $\mathscr{P}^{N-1}$, we obtain

$$\mu A^\alpha \, \partial_\alpha \mathscr{P}^{N-1} V_\alpha = [\mathrm{f}(\gamma)\breve{X}, \mathscr{P}^{N-1}]V_\alpha + [\mathrm{f}(\underline{\gamma})P, \mathscr{P}^{N-1}]V_\alpha + \mathscr{P}^{N-1}\{\mathrm{f}(\underline{\gamma}, Z\Psi)V\}. \tag{6C.13}$$

Using the bootstrap assumptions, we deduce that $|\mathscr{P}^{N-1}\{\mathrm{f}(\underline{\gamma}, Z\Psi)V\}| \lesssim$ the right-hand side of (6C.2b) as desired. To bound the commutator term $|[\mathrm{f}(\underline{\gamma})P, \mathscr{P}^{N-1}]V_\alpha|$, we use the bootstrap assumptions and the commutator identity (6A.1). To bound the commutator term $|[\mathrm{f}(\gamma)\breve{X}, \mathscr{P}^{N-1}]V_\alpha|$, we use the bootstrap assumptions, the commutator estimate (6A.3), and the pointwise estimate (6B.4). We have therefore proved (6C.2b). The estimate (6C.2a) can be proved in a similar fashion starting from (2A.1b) and with the help of (6B.3); we omit the details.

Proof of (6C.8b): A special case of (6C.7e) is the estimate $LL\mu(t, u, \vartheta) = \mathcal{O}(\varepsilon)$. From this bound and the fundamental theorem of calculus, we deduce $L\mu(t, u, \vartheta) = L\mu(0, u, \vartheta) + \mathcal{O}(\varepsilon)$. Next, we use the identity $(\breve{X}L^a)\xi_a = -(\breve{X}L^1)\xi_1 + \sum_{a=2}^n (\breve{X}L^a)\xi_a^{(\mathrm{Small})}$, definition (3C.3), and the conditions on the data to decompose (3G.1a) at time 0 as

$$L\mu(0, u, \vartheta) = -[\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \mathcal{O}(\varepsilon). \tag{6C.14}$$

We next note that fundamental theorem of calculus yields

$$[\mathcal{G}\breve{X}\Psi](t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \int_{s=0}^t L[\mathcal{G}\breve{X}\Psi](s, u, \vartheta)\, ds. \tag{6C.15}$$

Since the estimates (6C.5b) and (6C.7b) and the bootstrap assumptions imply that $L[\mathcal{G}\breve{X}\Psi] = \mathcal{O}(\varepsilon)$, we deduce from (6C.15) that $[\mathcal{G}\breve{X}\Psi](t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \mathcal{O}(\varepsilon)$. Moreover, in view of Remark 5.2 and our data assumptions (5B.3a)–(5B.3b), we have, by Taylor expanding, the estimate $\mathcal{G}(0, u, \vartheta) := \mathcal{G}|_{(\Psi(0,u,\vartheta), v(0,u,\vartheta))} = \mathcal{G}|_{(\Psi, v)=(0,0)} + \mathcal{O}_\blacklozenge(\mathring{\alpha}) + \mathcal{O}(\varepsilon)$. Combining these estimates, we arrive at both of the bounds stated in (6C.8b).

Proof of (6C.8a): Using the fundamental theorem of calculus (as in (6C.15)) and the initial condition $\mu|_{\Sigma_0} = 1 + \mathcal{O}_\blacklozenge(\mathring{\alpha}) + \mathcal{O}(\varepsilon)$, which follows from (3B.2) and the conditions on the data, we obtain $\mu(t, u, \vartheta) = 1 + \mathcal{O}_\blacklozenge(\mathring{\alpha}) + \mathcal{O}(\varepsilon) + \int_{s=0}^t L\mu(s, u, \vartheta)\, ds$. Substituting the right-hand side of (6C.8b) (evaluated at (s, u, ϑ)) for the integrand $L\mu(s, u, \vartheta)$, we arrive at the first estimate stated in (6C.8a). To obtain the second estimate stated in (6C.8a), we use the first estimate and the bound $[\mathcal{G}\breve{X}\Psi](t, u, \vartheta) = [\mathcal{G}\breve{X}\Psi](0, u, \vartheta) + \mathcal{O}(\varepsilon)$ noted in the previous paragraph.

Proof of (6C.9a) and (6C.9b): Estimate (6C.9a) follows easily from (6C.8a) and the fact that $0 < t < 2\mathring{A}_*^{-1}$. Similarly, (6C.9b) follows easily from (6C.8b).

Proof of (6C.10a)–(6C.10b): Using (3H.3) and (6A.1), we deduce the following schematic identity: $L\mathscr{P}^{N-1}\Xi^j = \mathscr{P}^{N-1}\{\Xi^a[\mathrm{f}(\gamma)V + \mathrm{f}(\gamma)P\Psi]\} + \mathscr{P}^{N-1}\{\mathrm{f}(\underline{\gamma}, Z\Psi)\}$. From this identity and the bootstrap assumptions, we deduce

$$\max_{1\leq j\leq n}|L\mathscr{P}^{\leq N-1}\Xi^j|\lesssim\max_{1\leq j\leq n}|\mathscr{P}^{\leq N-1}\Xi^j| + \max_{1\leq j\leq n}|\mathscr{P}^{\leq\lfloor(N-1)/2\rfloor}\Xi^j|\{|\mathscr{Z}_*^{\leq N;1}\Psi|+|\mathscr{P}_*^{[1,N-1]}\underline{\gamma}|+1\} + |\mathscr{Z}_*^{\leq N;1}\Psi|+|\mathscr{P}_*^{[1,N-1]}\underline{\gamma}|+1. \quad (6C.16)$$

In particular, from (6C.16) and the bootstrap assumptions, we deduce

$$\max_{1\leq j\leq n}|L\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\Xi^j|\lesssim\max_{1\leq j\leq n}|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\Xi^j|+1. \quad (6C.17)$$

Moreover, from the conditions on the data, we deduce that $\max_{1\leq j\leq n}|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\Xi^j|(0,u,\vartheta)\lesssim 1$. Recalling that $L=\frac{\partial}{\partial t}$, we now use this data bound, (6C.17), and Gronwall's inequality in $\max_{1\leq j\leq n}|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\Xi^j|$ to deduce that $\max_{1\leq j\leq n}\|\mathscr{P}^{\leq N_{\mathrm{Mid}}-1}\Xi^j\|_{L^\infty(\Sigma_t^u)}\lesssim 1$, which is the desired bound (6C.10b). Finally, from (6C.16) and (6C.10b), we conclude (6C.10a). □

6D. ***Estimates closely tied to the formation of the shock.*** In this subsection, we prove a lemma that lies at the heart of showing that μ vanishes in finite time and that its vanishing coincides with the blowup of $\max_{\alpha=0,\dots,n}|\partial_\alpha\Psi|$. Roughly, the lemma shows that when μ is small, $\breve{X}\Psi$ must be quantitatively large in magnitude and that $\breve{X}\Psi$ has a sign that forces μ to continue shrinking (the latter fact is important in that $\breve{X}\Psi$ is the dominant term in the evolution equation (3G.1a) for μ).

We start by defining a quantity that captures the "worst-case" behavior of μ along Σ_t^u.

Definition 6.7. We define the following quantity, where μ is the inverse foliation density in Definition 3.5:

$$\mu_\star(t,u):=\min_{\Sigma_t^u}\mu. \quad (6D.1)$$

We now prove the main result of this subsection.

Lemma 6.8 ($|\breve{X}\Psi|$ is large when μ is small). *The following implication holds*:

$$\mu(t,u,\vartheta)<\tfrac{1}{4}\quad\Longrightarrow\quad[\mathcal{G}\breve{X}\Psi](t,u,\vartheta)<-\tfrac{1}{4}\mathring{A}_*, \quad (6D.2)$$

where the blowup coefficient $\mathcal{G}\neq 0$ (see Remark 3.9) is defined in Definition 3.8 and the data-size parameter $\mathring{A}_*$ is defined in Definition 5.1.

In addition,

$$\mu(t,u,\vartheta)<\tfrac{1}{4}\quad\Longrightarrow\quad|X\Psi|(t,u,\vartheta)>\frac{1}{8|\widetilde{\mathcal{G}}|}\frac{1}{\mu(t,u,\vartheta)}\mathring{A}_*, \quad (6D.3)$$

where the constant $\widetilde{\mathcal{G}}:=\mathcal{G}|_{(\Psi,v)=(0,0)}$ is the blowup coefficient evaluated at the background value of $(\Psi,v)=(0,0)$ (see Remark 5.2 and note that, as we mentioned just below (6C.8b), $\widetilde{\mathcal{G}}=-\frac{\partial L^1}{\partial\Psi}\big|_{(\Psi,v)=(0,0)}$).

Finally, when $U_0=1$, the quantity $\mu_\star$ defined in (6D.1) *satisfies the estimate*

$$\mu_\star(t,1)=1-t\mathring{A}_*+\mathcal{O}_\blacklozenge(\mathring{\alpha})+\mathcal{O}(\varepsilon). \quad (6D.4)$$

Proof. From the second estimate stated in (6C.8a), we deduce that if $\mu(t,u,\vartheta)<\frac{1}{4}$, then $t[\mathcal{G}\breve{X}\Psi](t,u,\vartheta)<-\frac{3}{4}+\mathcal{O}_\blacklozenge(\mathring{\alpha})+\mathcal{O}(\varepsilon)$. From this bound and the fact that $0\leq t<T_{(\mathrm{Boot})}<2\mathring{A}_*^{-1}$, we conclude (6D.2).

To prove (6D.3), we first use the fundamental theorem of calculus to deduce

$$\mathcal{G}(t,u,\vartheta) = \mathcal{G}(0,u,\vartheta) + \int_{s=0}^{t} L\mathcal{G}(s,u,\vartheta)\,ds. \tag{6D.5}$$

Since the estimates (6C.5b) and (6C.7b) and the bootstrap assumptions imply that $L\mathcal{G} = \mathcal{O}(\varepsilon)$, we find from (6D.5) that $\mathcal{G}(t,u,\vartheta) = \mathcal{G}(0,u,\vartheta) + \mathcal{O}(\varepsilon)$. Moreover, in view of Remark 5.2 and our data assumptions (5B.3a)–(5B.3b), we have, by Taylor expanding, the estimate $\mathcal{G}(0,u,\vartheta) := \mathcal{G}|_{(\Psi(0,u,\vartheta),v(0,u,\vartheta))} = \widetilde{\mathcal{G}} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon)$. It follows that $\mathcal{G}(t,u,\vartheta) = \widetilde{\mathcal{G}} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon)$. Using this estimate to substitute for the factor $\mathcal{G}(t,u,\vartheta)$ in the second inequality in (6D.2) and then taking the absolute value of the resulting inequality, we deduce that if $\mu(t,u,\vartheta) < \frac{1}{4}$, then

$$|\breve{X}\Psi|(t,u,\vartheta) > \frac{1}{4\left\{|\widetilde{\mathcal{G}}| + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon)\right\}}\mathring{A}_{*}.$$

Dividing both sides of this inequality by $\mu(t,u,\vartheta)$ and appealing to (3C.4b), we arrive at (6D.3).

To prove (6D.4), we use the first line of (6C.8a) to deduce $\mu(t,u,\vartheta) = 1 + t[\mathcal{G}\breve{X}\Psi](0,u,\vartheta) + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\varepsilon)$. Taking the minimum of both sides of this estimate over $(u,\vartheta) \in [0,1] \times \mathbb{T}^{n-1}$ and appealing to Definitions 5.1 and 6.7, we conclude (6D.4). □

7. Estimates for the change of variables map

In this section, we derive estimates for the change of variables map Υ from Definition 3.24. The main result is Proposition 7.3, which will serve as a technical ingredient in our proof that the solution exists up until the first shock. Roughly, the proposition shows that if μ remains bounded from below strictly away from 0, then Υ can be extended to a diffeomorphism on the closure of the bootstrap domain.

7A. ***Control of the components of the change of variables map.*** In this subsection, we provide two preliminary lemmas that yield estimates for the components of Υ.

Lemma 7.1 (bounds for geometric coordinate partial derivatives of functions in terms of geometric vector field derivatives). *For $K \in \{0,1\}$, the following estimate holds for scalar functions f*:

$$\sum_{i_0+i_1+\cdots+i_n \le 1} \left\| \left(\frac{\partial}{\partial t}\right)^{i_0+K} \left(\frac{\partial}{\partial u}\right)^{i_1} \left(\frac{\partial}{\partial \vartheta^2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial \vartheta^n}\right)^{i_n} f \right\|_{L^\infty(\Sigma_t^u)} \lesssim \|\mathscr{Z}^{\le 1+K;1} f\|_{L^\infty(\Sigma_t^u)}. \tag{7A.1}$$

Proof. From (3C.7) and (3F.1b), the fact that Ξ is $\mathcal{T}_{t,u}$-tangent, and (3C.8c), we deduce the identity

$$\frac{\partial}{\partial u} = \breve{X} + \Xi^a\,\partial_a = \breve{X} + \sum_{i=2}^{n} \Xi^a \mathrm{f}_{ia}(\gamma)\,{}^{(i)}\Theta.$$

From this identity and the L^∞ estimates of Proposition 6.5 (in particular the estimate (6C.10b)), it follows that $\frac{\partial}{\partial u}$ is a linear combination of the elements of $\mathscr{Z}$ with coefficients that are bounded in the norm $\|\cdot\|_{L^\infty(\Sigma_t^u)}$ by $\lesssim 1$. The estimate (7A.1) is a straightforward consequence of this fact and the facts that $L = \frac{\partial}{\partial t} \in \mathscr{Z}$ and ${}^{(i)}\Theta = \frac{\partial}{\partial \vartheta^i} \in \mathscr{Z}$. □

We now show that Υ can be extended to a function defined on the closure of the bootstrap domain that belongs to several function spaces.

Lemma 7.2 (a preliminary extension result for the change of variables map). *The components $\Upsilon^\alpha(t,u,\vartheta)$ of the change of variables map from Definition 3.24 extend to the compact domain $[0,T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$ with the following regularity* ($i=2,\dots,n,\ \alpha=0,\dots,n$):

$$\Upsilon^\alpha,\ \frac{\partial}{\partial\vartheta^i}\Upsilon^\alpha \in \bigcap_{k=0,1} C^k\big([0,T_{(\mathrm{Boot})}], W^{1-k,\infty}([0,U_0]\times\mathbb{T}^{n-1})\big).$$

Moreover, the following estimates[31] *hold for $(t,u)\in[0,T_{(\mathrm{Boot})}]\times[0,U_0]$, where $C=C(\mathring{A})$:*

$$\sum_{i_0+i_1+\cdots+i_n\le 1}\left\|\left(\frac{\partial}{\partial t}\right)^{i_0}\left(\frac{\partial}{\partial u}\right)^{i_1}\left(\frac{\partial}{\partial\vartheta^2}\right)^{i_2}\cdots\left(\frac{\partial}{\partial\vartheta^n}\right)^{i_n}\Upsilon^\alpha\right\|_{L^\infty(\Sigma_t^u)}\le C, \tag{7A.2a}$$

$$\sum_{\substack{i_0+i_1+\cdots+i_n\le 2\\ 1\le i_2+\cdots+i_n}}\left\|\left(\frac{\partial}{\partial t}\right)^{i_0}\left(\frac{\partial}{\partial u}\right)^{i_1}\left(\frac{\partial}{\partial\vartheta^2}\right)^{i_2}\cdots\left(\frac{\partial}{\partial\vartheta^n}\right)^{i_n}\Upsilon^\alpha\right\|_{L^\infty(\Sigma_t^u)}\le C\varepsilon. \tag{7A.2b}$$

Proof. We will show that the following estimates hold for $(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]$:

$$\sum_{K=0}^{1}\sum_{i_0+i_1+\cdots+i_n\le 1}\left\|\left(\frac{\partial}{\partial t}\right)^{i_0+K}\left(\frac{\partial}{\partial u}\right)^{i_1}\left(\frac{\partial}{\partial\vartheta^2}\right)^{i_2}\cdots\left(\frac{\partial}{\partial\vartheta^n}\right)^{i_n}\Upsilon^\alpha\right\|_{L^\infty(\Sigma_t^u)}\lesssim 1, \tag{7A.3}$$

$$\sum_{K=0}^{1}\sum_{\substack{i_0+i_1+\cdots+i_n\le 2\\ 1\le i_2+\cdots+i_n}}\left\|\left(\frac{\partial}{\partial t}\right)^{i_0+K}\left(\frac{\partial}{\partial u}\right)^{i_1}\left(\frac{\partial}{\partial\vartheta^2}\right)^{i_2}\cdots\left(\frac{\partial}{\partial\vartheta^n}\right)^{i_n}\Upsilon^\alpha\right\|_{L^\infty(\Sigma_t^u)}\lesssim \varepsilon. \tag{7A.4}$$

Since $L=\frac{\partial}{\partial t}$ relative to geometric coordinates, all results of the lemma then follow as straightforward consequences of (7A.3)–(7A.4), the fundamental theorem of calculus, and the completeness of the spaces $W^{j,\infty}([0,U_0]\times\mathbb{T}^{n-1})$ for $j=0,1$.

Using (7A.1), we see that to establish (7A.3), it suffices to show that

$$\|\mathscr{Z}^{\le 2;1}\Upsilon^\alpha\|_{L^\infty(\Sigma_t^u)}\lesssim 1. \tag{7A.5}$$

To derive (7A.5), we first clarify that Υ^α can be identified with the Cartesian coordinate x^α, viewed as a function of $(t,u,\vartheta^2,\dots,\vartheta^n)$. To bound x^α, we note that $Lx^\alpha=L^\alpha=\mathrm{f}(\Psi,v)$. Hence, the bootstrap assumptions imply that $\|Lx^\alpha\|_{L^\infty(\Sigma_t^u)}\lesssim 1$. From this estimate and the fundamental theorem of calculus (as in (6C.15)), we conclude (see footnote 31) that $\|x^\alpha\|_{L^\infty(\Sigma_t^u)}\lesssim 1$ as desired. Next, we note that for $P\in\mathscr{P}$, we have $Px^\alpha=P^\alpha=\mathrm{f}(\gamma)$ and $\breve{X}x^\alpha=\breve{X}^\alpha=\mathrm{f}(\underline{\gamma})$. Hence, to complete the proof of (7A.5), we need only to show that $\|\mathscr{P}^{\le 1}\mathrm{f}(\underline{\gamma})\|_{L^\infty(\Sigma_t^u)}\lesssim 1$ and $\|\mathscr{Z}^{\le 1;1}\mathrm{f}(\gamma)\|_{L^\infty(\Sigma_t^u)}\lesssim 1$. These bounds are simple

[31] The L^∞ estimate for the torus coordinates $x^i\in\mathbb{T}$ (where $i=2,\dots,n$) stated in (7A.2a) should be interpreted as the statement that for each fixed $i\in\{2,\dots,n\}$ and $(u,\vartheta)\in[0,U_0]\times\mathbb{T}^{n-1}$, the Euclidean distance traveled by the curves $t\to x^i(t,u,\vartheta),\ t\in[0,T_{(\mathrm{Boot})}]$, in the universal covering space $\mathbb{R}$ of $\mathbb{T}$ is uniformly bounded.

consequences of the bootstrap assumptions. We have therefore proved (7A.3). The estimate (7A.4) can be proved using a similar argument and we omit the details. □

7B. ***The diffeomorphism properties of the change of variables map.*** We now derive the main result of Section 7.

Proposition 7.3 (sufficient conditions for Υ to be a global diffeomorphism). *If*

$$\inf_{(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]} \mu_\star(t,u) > 0, \tag{7B.1}$$

then the change of variables map Υ *extends to a global diffeomorphism from* $[0, T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$ *onto its image with the following regularity* ($i = 2, \dots, n,\ \alpha = 0, \dots, n$):

$$\Upsilon^\alpha,\ {}^{(i)}\Theta\Upsilon^\alpha \in \bigcap_{k=0,1} C^k\big([0,T_{(\mathrm{Boot})}], W^{1-k,\infty}([0,U_0]\times\mathbb{T}^{n-1})\big). \tag{7B.2}$$

Proof. By the bootstrap assumption (5C.2), Υ is a diffeomorphism from $[0,T_{(\mathrm{Boot})})\times[0,U_0]\times\mathbb{T}^{n-1}$ onto its image $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$. In addition, Lemma 7.2 implies that each component Υ^α extends to a function of the geometric coordinates satisfying (7B.2). Next, we use (3I.2), the L^∞ estimates of Proposition 6.5, and the assumption (7B.1) to deduce that the Jacobian determinant of Υ is uniformly bounded in magnitude from above and below away from 0 on $[0,T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$. Hence, from the inverse function theorem, we deduce that Υ extends as a local diffeomorphism from $[0,T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$ onto its image. Therefore, to complete the proof of the lemma, we need only to show that Υ is injective on the domain $[0,T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$. Since Υ is a diffeomorphism on the domain $[0,T_{(\mathrm{Boot})})\times[0,U_0]\times\mathbb{T}^{n-1}$, it suffices to show that $\Upsilon(T_{(\mathrm{Boot})},u_1,\vartheta_1)\neq\Upsilon(T_{(\mathrm{Boot})},u_2,\vartheta_2)$ whenever $(u_i,\vartheta_i)\in[0,U_0]\times\mathbb{T}^{n-1}$ and $(u_1,\vartheta_1)\neq(u_2,\vartheta_2)$.

We first show that if $u_1\neq u_2$, then $\Upsilon(T_{(\mathrm{Boot})},u_1,\vartheta_1)\neq\Upsilon(T_{(\mathrm{Boot})},u_2,\vartheta_2)$. To this end, we observe that from definitions (3C.1b) and (3D.3d), the estimates (6C.9a) and (6C.7a), and the assumption (7B.1), it follows that $\sum_{a=1}^n|\partial_a u|$ is uniformly bounded from above and from below, strictly away from 0. It follows that no two distinct (closed) characteristic hypersurface portions $\mathcal{P}_{u_1}^{T_{(\mathrm{Boot})}}$ and $\mathcal{P}_{u_2}^{T_{(\mathrm{Boot})}}$ can intersect, which yields the desired result.

To finish the proof of the lemma, we must show that $\Upsilon(T_{(\mathrm{Boot})},u,\vartheta_1)\neq\Upsilon(T_{(\mathrm{Boot})},u,\vartheta_2)$ whenever $u\in[0,U_0]$ and $\vartheta_1\neq\vartheta_2$. That is, we must show that for each fixed $u\in[0,U_0]$, the map υ defined by $\upsilon(\vartheta):=\Upsilon(T_{(\mathrm{Boot})},u,\vartheta)$ is an injection from $\mathbb{T}^{n-1}$ onto its image. To this end, for each fixed $u\in[0,U_0]$, we consider the family of t-parametrized maps $\tilde{\upsilon}(t,\cdot)$ (where $t\in[0,T_{(\mathrm{Boot})}]$) defined to be the last $n-1$ components of $\Upsilon(t,u,\cdot)$; that is, $\tilde{\upsilon}(t;\vartheta):=\big(\Upsilon^2(t,u,\cdot),\Upsilon^3(t,u,\cdot),\dots,\Upsilon^n(t,u,\cdot)\big)$ (recall that Υ^i can be identified with the local Cartesian coordinate x^i). Note that $\tilde{\upsilon}(t,\cdot)$ can be viewed as a map from the domain $\mathbb{T}^{n-1}$ (equipped with the geometric coordinates $(\vartheta^2,\dots,\vartheta^n)$) to the target $\mathbb{T}^{n-1}$ (equipped with the Cartesian coordinates $(x^2,\dots,x^n)$). Since Υ is continuous on $[0,T_{(\mathrm{Boot})}]\times[0,U_0]\times\mathbb{T}^{n-1}$, it follows that υ is homotopic to the degree-one[32] map $\tilde{\upsilon}(0,\cdot)$ by the homotopy $\tilde{\upsilon}(t;\vartheta)$. Hence, it is a basic result of degree theory (see, for example, [Lee 2013, Proposition 17.36]) that $\tilde{\upsilon}(t,\cdot)$ is also a degree-one map. In

[32] $\tilde{\upsilon}(0,\cdot)$ is degree-one because $x^i(0,u,\vartheta^2,\dots,\vartheta^n)=\vartheta^i$ for $i=2,\dots,n$ by construction.

particular, $\upsilon(\cdot) = \tilde{\upsilon}(T_{(\mathrm{Boot})}, \cdot)$ is degree-one. Next, we note that Lemma 7.2 implies that $\Upsilon^j(T_{(\mathrm{Boot})}, u, \cdot)$ can be viewed as a C^1 function of $(\vartheta^2, \dots, \vartheta^n) \in \mathbb{T}^{n-1}$ and that by (3D.3c) and (7A.2b), for $i, j = 2, \dots, n$, we have ${}^{(i)}\Theta\Upsilon^j(T_{(\mathrm{Boot})}, u, \vartheta^2, \dots, \vartheta^n) = \delta^{ij} + {}^{(i)}\Theta^j_{(\mathrm{Small})}(T_{(\mathrm{Boot})}, u, \vartheta^2, \dots, \vartheta^n) = \delta^{ij} + \mathcal{O}(\varepsilon)$, where δ^{ij} is the standard Kronecker delta. From this estimate and the degree-one property of $\upsilon(\cdot) = \tilde{\upsilon}(T_{(\mathrm{Boot})}, \cdot)$, we deduce[33] that for sufficiently small ε, $\upsilon(\cdot)$ is a bijection[34] from $\mathbb{T}^{n-1}$ to $\mathbb{T}^{n-1}$. In particular, υ is injective, which is the desired result. □

8. Energy estimates and strict improvements of the fundamental bootstrap assumptions

In this section, we derive the main estimates of the paper: a priori energy estimates that hold up to top order on the bootstrap region. The main ingredients in the proofs are the energy identities of Section 4 and the pointwise estimates of Proposition 6.5. As a corollary, we also derive strict improvements of the fundamental L^∞ bootstrap assumptions of Section 5C2.

8A. ***Definition of the fundamental L^2-controlling quantity.*** We start by defining the coercive quantity that we use to control the solution in L^2 up to top order.

Definition 8.1 (the main coercive L^2-controlling quantity). In terms of the energy-characteristic flux quantities of Definition 4.1 and the multi-index set $\mathcal{I}_*^{[1,N_{\mathrm{Top}}];1}$ of Definition 3.32, we define

$$\mathbb{Q}(t,u) := \sup_{(t',u')\in[0,t]\times[0,u]} \max\Big\{ \max_{\vec{I}\in\mathcal{I}_*^{[1,N_{\mathrm{Top}}];1}} \mathbb{E}^{(\mathrm{Shock})}[\mathscr{Z}^{\vec{I}}\Psi](t',u'), \max_{\substack{|\vec{I}|\le N_{\mathrm{Top}}-1 \\ f\in\{v^J\}_{1\le J\le M}\cup\{V^J_\alpha\}_{0\le\alpha\le n;1\le J\le M}}} \big\{\mathbb{E}^{(\mathrm{Regular})}[\mathscr{P}^{\vec{I}} f](t',u') + \mathbb{F}^{(\mathrm{Regular})}[\mathscr{P}^{\vec{I}} f](t',u')\big\}\Big\}. \quad (8A.1)$$

8B. ***Coerciveness of the fundamental L^2-controlling quantity.*** In the next lemma, we exhibit the coerciveness properties of $\mathbb{Q}(t,u)$.

Lemma 8.2 (coerciveness of $\mathbb{Q}(t,u)$). *The following estimates hold*:

$$\sup_{(t',u')\in[0,t]\times[0,u]} \|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}];1}\Psi\|_{L^2(\Sigma_{t'}^{u'})} \le \mathbb{Q}^{1/2}(t,u), \quad (8B.1)$$

$$\sup_{(t',u')\in[0,t]\times[0,u]} \|\sqrt{\mu}\mathscr{P}^{\le N_{\mathrm{Top}}-1} v\|_{L^2(\Sigma_{t'}^{u'})} \le C\mathbb{Q}^{1/2}(t,u), \quad (8B.2a)$$

$$\sup_{(t',u')\in[0,t]\times[0,u]} \|\sqrt{\mu}\mathscr{P}^{\le N_{\mathrm{Top}}-1} V\|_{L^2(\Sigma_{t'}^{u'})} \le C\mathbb{Q}^{1/2}(t,u), \quad (8B.2b)$$

---

[33] Recall that if $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ is a C^1 surjective map without critical points, then f is degree-one if for $p, q \in \mathbb{T}^{n-1}$, $1 = \sum_{p\in f^{-1}(q)} (\operatorname{sign}\det df(p))$, where $df(p)$ denotes the differential of f at p and the $df(p)$ are computed relative to an atlas corresponding to the smooth orientation on $\mathbb{T}^{n-1}$ chosen at the beginning of the article. It is a basic fact of degree theory (see, for example, [Lee 2013, Theorem 17.35]) that the sum is independent of q. Note that in the context of the present argument, the components of the $(n-1)\times(n-1)$ matrix $df(\cdot)$ are ${}^{(i)}\Theta\Upsilon^j(T_{(\mathrm{Boot})}, u, \cdot)$ $(i, j = 2, 3, \dots, n)$.

[34] The surjective property of this map is easy to deduce.

$$\sup_{(t',u')\in[0,t]\times[0,u]} \|\mathscr{P}^{\le N_{\text{Top}}-1} v\|_{L^2(\mathcal{P}_{u'}^{t'})} \le C\mathbb{Q}^{1/2}(t,u), \tag{8B.3a}$$

$$\sup_{(t',u')\in[0,t]\times[0,u]} \|\mathscr{P}^{\le N_{\text{Top}}-1} V\|_{L^2(\mathcal{P}_{u'}^{t'})} \le C\mathbb{Q}^{1/2}(t,u). \tag{8B.3b}$$

Proof. Lemma 8.2 follows from Definition 8.1, Definition 4.1, Lemma 4.2, and the L^∞ estimates of Proposition 6.5 (which provide the smallness of γ that is assumed, for example, in the hypotheses of Lemma 4.2). □

8C. *Sobolev embedding.* The main result of this subsection is Lemma 8.4, a Sobolev embedding result which shows that the norm $\|\cdot\|_{L^\infty(\Sigma_t^u)}$ of v and V and their $\mathcal{P}_u$-tangential derivatives up to mid-order is controlled by $\mathbb{Q}$. In Corollary 8.8, we will use the lemma as an ingredient in our derivation of strict improvements of the fundamental L^∞ bootstrap assumptions. As a preliminary step, we provide the following lemma, in which we derive some L^2 estimates for v, V, and their derivatives along the codimension-two tori $\mathcal{T}_{t,u}$.

Lemma 8.3 (L^2 control of the non-shock-forming variables on $\mathcal{T}_{t,u}$). *The following estimates hold for* $0 \le \alpha \le n$ *and* $1 \le J \le M$:

$$\|\mathscr{P}^{\le N_{\text{Top}}-2} v^J\|_{L^2(\mathcal{T}_{t,u})},\ \|\mathscr{P}^{\le N_{\text{Top}}-2} V_\alpha^J\|_{L^2(\mathcal{T}_{t,u})} \le C\mathring{\epsilon} + C\mathbb{Q}^{1/2}(t,u). \tag{8C.1}$$

Proof. We first note the following estimate for scalar functions f, which follows from differentiating under the integral and using Young's inequality:

$$\frac{\partial}{\partial t}\|f\|^2_{L^2(\mathcal{T}_{t,u})} = 2\int_{\mathcal{T}_{t,u}} f Lf \, d\vartheta \le \|f\|^2_{L^2(\mathcal{T}_{t,u})} + \|Lf\|^2_{L^2(\mathcal{T}_{t,u})}. \tag{8C.2}$$

Integrating (8C.2) from time 0 to time t, we find that

$$\|f\|^2_{L^2(\mathcal{T}_{t,u})} \le \|f\|^2_{L^2(\mathcal{T}_{0,u})} + \int_{s=0}^t \|f\|^2_{L^2(\mathcal{T}_{s,u})}\, ds + \|Lf\|^2_{L^2(\mathcal{P}_u^t)}. \tag{8C.3}$$

From (8C.3) and Gronwall's inequality, we deduce that

$$\|f\|^2_{L^2(\mathcal{T}_{t,u})} \le C\|f\|^2_{L^2(\mathcal{T}_{0,u})} + C\|Lf\|^2_{L^2(\mathcal{P}_u^t)}. \tag{8C.4}$$

We now apply (8C.4) with the role of f played by $\mathscr{P}^{\le N_{\text{Top}}-2} v^J$ and $\mathscr{P}^{\le N_{\text{Top}}-2} V_\alpha^J$. In view of the data-size assumptions (5B.5) and the bounds $\|L\mathscr{P}^{\le N_{\text{Top}}-2} v\|^2_{L^2(\mathcal{P}_u^t)} \lesssim \mathbb{Q}(t,u)$ and $\|L\mathscr{P}^{\le N_{\text{Top}}-2} V\|^2_{L^2(\mathcal{P}_u^t)} \lesssim \mathbb{Q}(t,u)$, which follow from (8B.3a)–(8B.3b), we arrive at the desired estimate (8C.1). □

We now prove the main result of this subsection.

Lemma 8.4 (L^∞ control of the non-shock-forming variables up to mid-order in terms of $\mathbb{Q}$). *The following estimates hold*:

$$\|\mathscr{P}^{\le N_{\text{Mid}}-1} v\|_{L^\infty(\Sigma_t^u)},\ \|\mathscr{P}^{\le N_{\text{Mid}}-1} V_\alpha\|_{L^\infty(\Sigma_t^u)} \le C\mathring{\epsilon} + C\mathbb{Q}^{1/2}(t,u). \tag{8C.5}$$

Proof. Standard Sobolev embedding on $\mathbb{T}^{n-1}$ yields the following estimate for scalar functions f:

$$\|f\|_{L^\infty(\mathcal{T}_{t,u})} \lesssim \|f\|_{L^2(\mathcal{T}_{t,u})} + \sum_{K=1}^{\lfloor (n+1)/2 \rfloor} \sum_{Y_{(1)},\dots,Y_{(K)} \in \{{}^{(i)}\Theta\}_{i=2,3,\dots,n}} \|Y_{(1)}\cdots Y_{(K)} f\|_{L^2(\mathcal{T}_{t,u})}. \tag{8C.6}$$

The desired estimate (8C.5) now follows from (8C.6), (8C.1), and (5A.1), where the last of these equations in particular implies that $N_{\mathrm{Mid}} - 1 + \lfloor (n+1)/2 \rfloor \le N_{\mathrm{Top}} - 2$. □

8D. ***Preliminary L^2 estimates for μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(i)}\Theta_{(\mathrm{Small})}^j$.*** In the next lemma, we bound the L^2 norms of the derivatives of the quantities μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(i)}\Theta_{(\mathrm{Small})}^j$ in terms of $\mathbb{Q}$. This serves as a preliminary step for our forthcoming derivation of L^2 estimates for Ψ, v, and V, since μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(i)}\Theta_{(\mathrm{Small})}^j$ appear as source terms in their commuted evolution equations (as is shown by the right-hand sides of (6C.1)–(6C.2b)).

Lemma 8.5 (L^2 estimates for μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(i)}\Theta_{(\mathrm{Small})}^j$ in terms of $\mathbb{Q}$). *The following estimates hold for* $2 \le i \le n$, $1 \le j \le n$, *and* $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$, *where $\mathbb{Q}$ is defined in Definition 8.1*:

$$\|\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu\|_{L^2(\Sigma_t^u)} \le C\mathring{\epsilon} + C\mathbb{Q}^{1/2}(t,u), \tag{8D.1a}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^2(\Sigma_t^u)} \le C\mathring{\epsilon} + C\mathbb{Q}^{1/2}(t,u), \tag{8D.1b}$$

$$\|\mathscr{Z}^{[1,N_{\mathrm{Top}}-1];1\,(i)}\Theta_{(\mathrm{Small})}^j\|_{L^2(\Sigma_t^u)} \le C\mathring{\epsilon} + C\mathbb{Q}^{1/2}(t,u). \tag{8D.1c}$$

Proof. See Section 5F for some comments on the analysis. We set

$$q = q(t,u) := \|\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu\|_{L^2(\Sigma_t^u)}^2 + \sum_{j=1}^n \|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^2(\Sigma_t^u)}^2 + \sum_{i=2}^n \sum_{j=1}^n \|\mathscr{Z}^{[1,N_{\mathrm{Top}}-1];1\,(i)}\Theta_{(\mathrm{Small})}^j\|_{L^2(\Sigma_t^u)}^2. \tag{8D.2}$$

The estimates from Lemma 5.5 for the data of μ, $\xi_j^{(\mathrm{Small})}$, and ${}^{(i)}\Theta_{(\mathrm{Small})}^j$ imply that $q(0,u) \le C\mathring{\epsilon}^2$. Hence, from the pointwise estimates (6C.3a)–(6C.3b) and (6C.4), the pointwise estimates (6B.3)–(6B.4), Definition 3.16, Young's inequality, the energy identity (4B.2), and Lemma 8.2, we deduce that

$$\begin{aligned}
q(t,u) &\le C\mathring{\epsilon}^2 + C\sum_{j=1}^n \int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}|^2\,d\varpi \\
&\quad + C\sum_{i=2}^n\sum_{j=1}^n \int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1\,(i)}\Theta_{(\mathrm{Small})}^j|^2\,d\varpi \\
&\quad + C\int_{\mathcal{M}_{t,u}} |\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu|^2\,d\varpi + C\int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}];1}\Psi|^2\,d\varpi \\
&\quad + C\int_{\mathcal{M}_{t,u}} |\mathscr{P}^{\le N_{\mathrm{Top}}-1}v|^2\,d\varpi + C\int_{\mathcal{M}_{t,u}} |\mathscr{P}^{\le N_{\mathrm{Top}}-1}V|^2\,d\varpi \\
&\le C\mathring{\epsilon}^2 + C\int_{s=0}^t q(s,u)\,ds + C\int_{s=0}^t \mathbb{Q}(s,u)\,ds + C\int_{u'=0}^u \mathbb{Q}(t,u')\,du' \\
&\le C\mathring{\epsilon}^2 + C\int_{s=0}^t q(s,u)\,ds + C\mathbb{Q}(t,u).
\end{aligned} \tag{8D.3}$$

From (8D.3) and Gronwall's inequality, we conclude the bound $q(t,u) \le C\mathring{\epsilon}^2 + C\mathbb{Q}(t,u)$, from which the estimates (8D.1a)–(8D.1c) easily follow. □

8E. ***The main a priori estimates.*** In the next proposition, we derive our main a priori energy estimates.

Proposition 8.6 (the main a priori estimates). *There exists a constant $C > 0$ such that under the data-size assumptions of Section 5B2, the bootstrap assumptions of Section 5C2, and the smallness assumptions of Section 5D, the following estimates hold for $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:*

$$\mathbb{Q}(t,u) \leq C\mathring{\epsilon}^2 + C\int_{s=0}^{t} \mathbb{Q}(s,u)\,ds + C\int_{u'=0}^{u} \mathbb{Q}(t,u')\,du'. \tag{8E.1}$$

Moreover, as a consequence of (8E.1), *the following estimate holds for $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, 1]$:*

$$\mathbb{Q}(t,u) \leq C\mathring{\epsilon}^2. \tag{8E.2}$$

Remark 8.7 (a top-order L^2 estimate for v). From the pointwise estimate (6B.3), the bootstrap assumptions, Lemma 8.5, and (8E.2), one can easily obtain the bound $\|\mathscr{Z}^{\leq N_{\mathrm{Top}};1} v\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon}$, which is a gain of one derivative for v compared to what is directly implied by (8E.2). Similarly, we could gain a derivative for v in the L^∞ estimate (8E.8) below. However, we have no need for these gains of a derivative, so we will ignore them for the remainder of the paper.

Proof of Proposition 8.6. Proof of (8E.1): We first derive energy inequalities for Ψ and its derivatives. Let $\vec{I} \in \mathcal{I}_*^{[1,N_{\mathrm{Top}}];1}$ (see Definition 3.32). From definitions (3E.1a)–(3E.1b), the energy identity (4B.2), the data-size assumption (5B.2), the pointwise estimate (6C.1), the estimates (6B.3)–(6B.4), and Young's inequality, we deduce

$$\begin{aligned}
\mathbb{E}^{(\mathrm{Shock})}[\mathscr{Z}^{\vec{I}}\Psi](t,u) \leq C\mathring{\epsilon}^2 &+ C\int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}];1}\Psi|^2\,d\varpi + C\int_{\mathcal{M}_{t,u}} |\mathscr{P}^{\leq N_{\mathrm{Top}}-1} v|^2\,d\varpi \\
&+ C\int_{\mathcal{M}_{t,u}} |\mathscr{P}^{\leq N_{\mathrm{Top}}-1} V|^2\,d\varpi + C\int_{\mathcal{M}_{t,u}} |\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu|^2\,d\varpi \\
&+ C\sum_{j=1}^{n}\int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}|^2\,d\varpi \\
&+ C\sum_{i=2}^{n}\sum_{j=1}^{n}\int_{\mathcal{M}_{t,u}} |\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}{}^{(i)}\Theta_{(\mathrm{Small})}^{j}|^2\,d\varpi.
\end{aligned} \tag{8E.3}$$

From Lemmas 8.2 and 8.5, and (8E.3), we deduce

$$\mathbb{E}^{(\mathrm{Shock})}[\mathscr{Z}^{\vec{I}}\Psi](t,u) \leq C\mathring{\epsilon}^2 + C\int_{s=0}^{t} \mathbb{Q}(s,u)\,ds + C\int_{u'=0}^{u} \mathbb{Q}(t,u')\,du'. \tag{8E.4}$$

We now derive a similar energy inequality for v, V, and their derivatives. Specifically, using definitions (3E.1a)–(3E.1b), the energy-characteristic flux identity (4B.4), the data-size assumptions (5B.2) and (5B.4), the pointwise estimates (6C.2a)–(6C.2b), the estimates (6B.3)–(6B.4), Lemmas 8.2 and 8.5, and the L^∞ estimates of Proposition 6.5, we deduce that for $|\vec{I}| \leq N_{\mathrm{Top}} - 1$, we have, for any $f \in$

$\{v^J\}_{1 \le J \le M} \cup \{V^J_\alpha\}_{0 \le \alpha \le n; 1 \le J \le M}$, the estimate

$$\mathbb{E}^{(\mathrm{Regular})}[\mathscr{P}^{\vec{I}} f](t,u) + \mathbb{F}^{(\mathrm{Regular})}[\mathscr{P}^{\vec{I}} f](t,u) \le C\mathring{\epsilon}^2 + C\int_{s=0}^{t} \mathbb{Q}(s,u)\,ds + C\int_{u'=0}^{u} \mathbb{Q}(t,u')\,du'. \quad (8E.5)$$

From (8E.4), (8E.5), and Definition 8.1, we conclude the desired bound (8E.1).

Proof of (8E.2): With $c > 0$ a real parameter to be chosen below, we define

$$\mathbb{Q}_c(t,u) := \sup_{(\hat{t},\hat{u}) \in [0,t]\times[0,u]} \{\exp(-c\hat{t})\exp(-c\hat{u})\mathbb{Q}(\hat{t},\hat{u})\}. \quad (8E.6)$$

Using (8E.1) and the simple inequality $\int_{y'=0}^{y} \exp(cy')\,dy' \le (1/c)\exp(cy)$, we deduce that for $(\hat{t},\hat{u}) \in [0,t]\times[0,u] \subset [0,T_{(\mathrm{Boot})})\times[0,U_0]$, the following estimate holds:

$$\begin{aligned}
\exp(-c\hat{t})\exp(-c\hat{u})\mathbb{Q}(\hat{t},\hat{u}) &\le C\exp(-c\hat{t})\exp(-c\hat{u})\mathring{\epsilon}^2 \\
&\quad + C\exp(-c\hat{t})\exp(-c\hat{u}) \times \Big\{\sup_{t' \in [0,\hat{t}]} \exp(-ct')\mathbb{Q}(t',\hat{u})\Big\} \times \int_{t'=0}^{\hat{t}} \exp(ct')\,dt' \\
&\quad + C\exp(-c\hat{t})\exp(-c\hat{u}) \times \Big\{\sup_{u' \in [0,\hat{u}]} \exp(-cu')\mathbb{Q}(\hat{t},u')\Big\} \times \int_{u'=0}^{\hat{u}} \exp(cu')\,du' \\
&\le C\mathring{\epsilon}^2 + \frac{2C}{c} \sup_{(t',u') \in [0,\hat{t}]\times[0,\hat{u}]} \{\exp(-ct')\exp(-cu')\mathbb{Q}(t',u')\},
\end{aligned} \quad (8E.7)$$

where the constant C on the right-hand side of (8E.7) can be chosen to be independent of $c > 0$. From (8E.7) and definition (8E.6), we deduce that $\mathbb{Q}_c(t,u) \le C\mathring{\epsilon}^2 + (2C/c)\mathbb{Q}_c(t,u)$. Hence, fixing $c := c' > 2C$, we deduce that $\mathbb{Q}_{c'}(t,u) \le C'\mathring{\epsilon}^2$. From this bound and the definition of $\mathbb{Q}_{c'}$, it follows that for $(t,u) \in [0,T_{(\mathrm{Boot})})\times[0,U_0]$, we have $\mathbb{Q}(t,u) \le C'\exp(c't)\exp(c'u)\mathring{\epsilon}^2 \le C''\mathring{\epsilon}^2$, where C'' depends on C', c', and $\mathring{A}_*^{-1}$ (in view of the bootstrap assumption (5C.1)). This is precisely the desired bound (8E.2). □

Corollary 8.8 (improvement of the fundamental L^∞ bootstrap assumptions). *For $0 \le \alpha \le n$ and $1 \le J \le M$, the following estimates hold*:

$$\|\mathscr{P}^{\le N_{\mathrm{Mid}}-1} v^J\|_{L^\infty(\Sigma_t^u)},\ \|\mathscr{P}^{\le N_{\mathrm{Mid}}-1} V^J_\alpha\|_{L^\infty(\Sigma_t^u)} \le C\mathring{\epsilon}. \quad (8E.8)$$

In particular, if $C\mathring{\epsilon} < \varepsilon$, then the estimate (8E.8) *is a strict improvement of the fundamental bootstrap assumption* (5C.3).

Proof. Estimate (8E.8) follows from the energy estimate (8E.2) and the Sobolev embedding result (8C.5). □

9. Continuation criteria

In this section, we provide a proposition that yields continuation criteria. We will use the proposition during the proof of the main theorem (Theorem 10.1), specifically as an ingredient in showing that the solution survives until the shock.

Proposition 9.1 (continuation criteria). *Let* $(\Psi, v^1, \dots, v^M)$ *be a smooth solution to the system* (2A.1a)–(2A.1b) *satisfying the size assumptions*[35] *on* Σ_0^1 *and* $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$ *stated in Section 5B as well as the smallness assumptions stated in Section 5D. Let* $T_{(\mathrm{Local})} \in (0, 2\mathring{A}_*^{-1})$ *and* $U_0 \in (0, 1]$, *and assume that the solution exists classically on the* (*"open-at-the-top"*) *spacetime region* $\mathcal{M}_{T_{(\mathrm{Local})}, U_0}$ (*where* $\mathcal{M}_{T_{(\mathrm{Local})}, U_0}$ *is defined in* (3A.3f)) *that is completely determined by the data on* $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{A}_*^{-1}}$ (*see Figure 2 on page 472*). *Let u be the eikonal function that satisfies the eikonal equation initial value problem* (3A.1), *let* μ *be the inverse foliation density of the characteristics* $\mathcal{P}_u$ *defined in* (3B.1), *and let* $\lambda_\alpha = \mu\, \partial_\alpha u$ (*as in* (3C.1a)). *Assume that* $\mu > 0$ *on* $\mathcal{M}_{T_{(\mathrm{Local})}, U_0}$ *and that the change of variables map* Υ *from geometric to Cartesian coordinates* (*see Definition 3.24*) *is a diffeomorphism from* $[0, T_{(\mathrm{Local})}) \times [0, U_0] \times \mathbb{T}^{n-1}$ *onto* $\mathcal{M}_{T_{(\mathrm{Local})}, U_0}$ *such that for* $i = 2, \dots, n$ *and* $\alpha = 0, \dots, n$, *we have*

$$\Upsilon^\alpha,\ {}^{(i)}\Theta\Upsilon^\alpha \in \bigcap_{k=0,1} C^k\big([0, T_{(\mathrm{Local})}), W^{1-k,\infty}([0, U_0] \times \mathbb{T}^{n-1})\big). \tag{9.1}$$

Let $\mathcal{H} \subset \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^{1+n}$ *be the set of arrays* $(\widetilde{\Psi}, \tilde{v}, \tilde{\lambda})$ *such that the following two conditions hold*:

- *The Cartesian components* $L^i(\Psi, v)$ $(i = 1, \dots, n)$ *and the* $M \times M$ *matrices* $A^\alpha(\Psi, v)$ $(\alpha = 0, \dots, n)$ *are smooth functions for* (Ψ, v) *belonging to a neighborhood of* $(\widetilde{\Psi}, \tilde{v})$.
- $A^0(\Psi, v)$ *and* $A^\alpha(\Psi, v)\lambda_\alpha$ *are positive definite matrices for* (Ψ, v, λ) *belonging to a neighborhood of* $(\widetilde{\Psi}, \tilde{v}, \tilde{\lambda})$.

Assume that none of the following four breakdown scenarios occur:

(1) $\inf_{\mathcal{M}_{T_{(\mathrm{Local})}, U_0}} \mu = 0$.

(2) $\sup_{\mathcal{M}_{T_{(\mathrm{Local})}, U_0}} \mu = \infty$.

(3) *There exists a sequence* $p_n \in \mathcal{M}_{T_{(\mathrm{Local})}, U_0}$ *such that* $(\Psi(p_n), v(p_n), \lambda(p_n))$ *escapes every compact subset of* $\mathcal{H}$ *as* $n \to \infty$.

(4) $\sup_{\mathcal{M}_{T_{(\mathrm{Local})}, U_0}} \max_{\alpha=0,1,\dots,n}\{|\partial_\alpha\Psi| + |V_\alpha|\} = \infty$, *where* $V_\alpha^J = \partial_\alpha v^J$.

In addition, assume that the following condition is satisfied:

(5) *The change of variables map* Υ *extends to the compact set* $[0, T_{(\mathrm{Local})}] \times [0, U_0] \times \mathbb{T}^{n-1}$ *as a diffeomorphism onto its image that enjoys the regularity properties* (9.1) *with* $[0, T_{(\mathrm{Local})})$ *replaced by* $[0, T_{(\mathrm{Local})}]$.

Then there exists a $\Delta > 0$ *such that* Ψ, v, V, u, μ, λ, *and all of the other geometric quantities defined throughout the article can be uniquely extended* (*where* Ψ, v, u, *and* μ *are smooth solutions to their evolutions equations*) *to a strictly larger region of the form* $\mathcal{M}_{T_{(\mathrm{Local})}+\Delta, U_0}$ *into which their Sobolev regularity along* $\Sigma_0^{U_0}$ *and* $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$ (*described in Section 5B*) *is propagated.*[36] *Moreover, if* Δ *is sufficiently small, then none of the four breakdown scenarios occur in the larger region, and* Υ *extends to*

[35]Recall that even though we make size assumptions only for certain Sobolev norms, for technical convenience, we have assumed that the data on Σ_0^1 and $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$ are C^∞.

[36]Put differently, the same norms that are finite along $\Sigma_0^{U_0}$ and $\mathcal{P}_0^{2\mathring{A}_*^{-1}}$ (as stated in Section 5B) are also finite along Σ_t^u and $\mathcal{P}_u^t$ for $(t, u) \in [0, T_{(\mathrm{Local})} + \Delta] \times [0, U_0]$.

$[0, T_{(\mathrm{Local})} + \Delta] \times [0, U_0] \times \mathbb{T}^{n-1}$ *as a diffeomorphism onto its image that enjoys the regularity properties* (9.1) *with* $[0, T_{(\mathrm{Local})})$ *replaced by* $[0, T_{(\mathrm{Local})} + \Delta]$.

Discussion of proof. The proof of Proposition 9.1 is mostly standard. A sketch of a similar result was provided in [Speck 2016, Proposition 21.1.1], so here, we only mention the main ideas. Criterion (3) is connected to avoiding a breakdown in hyperbolicity of the equation. Criterion (4) is a standard criterion used to locally continue the solution relative to the Cartesian coordinates. Criteria (1) and (2) and the assumption (5) for Υ are connected to ruling out the blowup of u, degeneracy of the change of variables map, and degeneracy of the region $\mathcal{M}_{T_{(\mathrm{Local})},U_0}$. In particular, criteria (1) and (2) play a role in a proving that $\sum_{a=1}^{n} |\partial_a u|$ is uniformly bounded from above and strictly from below away from 0 on $\mathcal{M}_{T_{(\mathrm{Local})},U_0}$ (the proof was essentially given in the proof of Proposition 7.3). □

10. The main theorem

We now prove the main result of the paper.

Theorem 10.1 (stable shock formation). *Let n denote the number of spatial dimensions, let N_{Top} and N_{Mid} be positive integers satisfying* (5A.1), *and let $\mathring{\alpha} > 0$, $\mathring{\epsilon} \geq 0$, $\mathring{A} > 0$, and $\mathring{A}_* > 0$ be the data-size parameters from Section 5B. For each $U_0 \in (0, 1]$ (as in* (3A.2)), *let*

$$T_{(\mathrm{Lifespan});U_0} := \sup\bigl\{t \in [0,\infty) \bigm| \text{the solution exists classically on } \mathcal{M}_{t;U_0} \text{ and } \Upsilon \text{ is a diffeomorphism from } [0,t) \times [0,U_0] \times \mathbb{T}^{n-1} \text{ onto its image}\bigr\},$$

where Υ is the change of variables map from Definition 3.24. If $\mathring{\alpha}$ is sufficiently small relative to 1 *and if $\mathring{\epsilon}$ is sufficiently small relative to* 1, $\mathring{A}^{-1}$, *and $\mathring{A}_*$ in the sense explained in Section 5D, then the following conclusions hold, where all constants can be chosen to be independent of U_0 (see Section 1H for our conventions regarding the dependence of constants on the various parameters).*

<u>Dichotomy of possibilities</u>. *One of the following mutually disjoint possibilities must occur, where $\mu_\star(t,u) = \min_{\Sigma_t^u} \mu$ (as in* (6D.1)) *and μ is the inverse foliation density of the transport characteristics $\mathcal{P}_u$ from Definition 3.5*:

(I) $T_{(\mathrm{Lifespan});U_0} > 2\mathring{A}_*^{-1}$. *In particular, the solution exists classically on the spacetime region* $\mathrm{cl}\,\mathcal{M}_{2\mathring{A}_*^{-1},U_0}$, *where* cl *denotes closure. Furthermore,* $\inf\{\mu_\star(s,U_0) \mid s \in [0, 2\mathring{A}_*^{-1}]\} > 0$.

(II) $0 < T_{(\mathrm{Lifespan});U_0} \leq 2\mathring{A}_*^{-1}$, *and*

$$T_{(\mathrm{Lifespan});U_0} = \sup\bigl\{t \in [0, 2\mathring{A}_*^{-1}) \bigm| \inf\{\mu_\star(s,U_0) \mid s \in [0,t)\} > 0\bigr\}. \tag{10.1}$$

In addition, case (II) *occurs when $U_0 = 1$, and we have the estimate*[37]

$$T_{(\mathrm{Lifespan});1} = \{1 + \mathcal{O}_\blacklozenge(\mathring{\alpha}) + \mathcal{O}(\mathring{\epsilon})\}\mathring{A}_*^{-1}. \tag{10.2}$$

<u>Case (I)</u>. *The energy estimates of Proposition 8.6 and the L^∞ estimates of Corollary 8.8 hold on* $\mathrm{cl}\,\mathcal{M}_{2\mathring{A}_*^{-1},U_0}$. *The same is true for the estimates of Lemma 6.4 and Proposition 6.5, but with all factors ε*

[37] See Section 1H regarding our use of the symbol $\mathcal{O}_\blacklozenge$.

on the right-hand side of all inequalities replaced by $C\mathring{\epsilon}$. Moreover, for μ and the quantities from Definition 3.15, the following estimates hold for $2 \leq i \leq n$, $1 \leq j \leq n$, and $(t, u) \in [0, 2\mathring{A}_*^{-1}] \times [0, U_0]$ (see Section 3N regarding the differential operator notation):

$$\|\mathscr{P}_*^{[1,N_{\mathrm{Top}}-1]}\mu\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon}, \tag{10.3a}$$

$$\|\mathscr{Z}_*^{[1,N_{\mathrm{Top}}-1];1}\xi_j^{(\mathrm{Small})}\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon}, \tag{10.3b}$$

$$\|\mathscr{Z}^{[1,N_{\mathrm{Top}}-1];1\,(i)}\Theta_{(\mathrm{Small})}^j\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon}. \tag{10.3c}$$

Case (II). *The energy estimates of Proposition 8.6 and the L^∞ estimates of Corollary 8.8 hold on $\mathcal{M}_{T_{(\mathrm{Lifespan});U_0},U_0}$, as do the estimates of Lemma 6.4 and Proposition 6.5 with all factors ε on the right-hand side of all inequalities replaced by $C\mathring{\epsilon}$. Moreover, the estimates (10.3a)–(10.3c) hold for $(t, u) \in [0, T_{(\mathrm{Lifespan});U_0}) \times [0, U_0]$. In addition, the scalar functions $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-1;1}\Psi$, $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1}v^J$, $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1}V_\alpha^J$, $\mathscr{P}^{\leq N_{\mathrm{Mid}}-2}\mu$, $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1}\xi_j$, $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1\,(i)}\Theta^j$, $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1}L^i$, $\mathscr{P}^{\leq N_{\mathrm{Mid}}-2}\breve{X}^i$, and $\mathscr{Z}^{\leq N_{\mathrm{Mid}}-2;1}X^i$ extend to $\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0}$ as functions of the geometric coordinates (t, u, ϑ) belonging to the space $C\big([0, T_{(\mathrm{Lifespan});U_0}], L^\infty([0, U_0]\times\mathbb{T}^{n-1})\big)$.*

Moreover, let $\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0;(\mathrm{Blowup})}$ be the subset of $\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0}$ defined by

$$\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0;(\mathrm{Blowup})} := \{(T_{(\mathrm{Lifespan});U_0}, u, \vartheta) \mid \mu(T_{(\mathrm{Lifespan});U_0}, u, \vartheta) = 0\}. \tag{10.4}$$

Then for each point $(T_{(\mathrm{Lifespan});U_0}, u, \vartheta) \in \Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0;(\mathrm{Blowup})}$, there exists a past neighborhood[38] *containing it such that the following lower bound holds in the neighborhood:*

$$|X\Psi(t, u, \vartheta)| \geq \frac{1}{8\mathring{A}_*}\frac{1}{|\widetilde{\mathcal{G}}|\mu(t, u, \vartheta)}, \tag{10.5}$$

*where $\widetilde{\mathcal{G}} := \mathcal{G}|_{(\Psi,v)=(0,0)}$ is the blowup coefficient of Definition 3.8, evaluated at the background value of $(\Psi, v) = (0, 0)$ (see Remark 5.2 and note that, as we mentioned just below (6C.8b), $\widetilde{\mathcal{G}} = -\frac{\partial L^1}{\partial \Psi}\big|_{(\Psi,v)=(0,0)}$). In (10.5), $1/(8|\widetilde{\mathcal{G}}|\mathring{A}_*)$ is a **positive**[39] data-dependent constant, and the $\mathcal{T}_{t,u}$-transversal, Σ_t-tangent vector field X is of order-unity Euclidean length: $C^{-1} \leq \delta_{ab}X^aX^b \leq C$, where δ_{ij} is the standard Kronecker delta. In particular, $X\Psi$ blows up like $1/\mu$ at all points in $\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0;(\mathrm{Blowup})}$. Conversely, at all points $(T_{(\mathrm{Lifespan});U_0}, u, \vartheta) \in \Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0}\backslash\Sigma_{T_{(\mathrm{Lifespan});U_0}}^{U_0;(\mathrm{Blowup})}$, we have*

$$|X\Psi(T_{(\mathrm{Lifespan});U_0}, u, \vartheta)| < \infty. \tag{10.6}$$

Proof. Let $C' > 1$ be a constant. We will enlarge C' as needed throughout the proof. We define

$$T_{(\mathrm{Max});U_0} := \text{the supremum of the set of times } T_{(\mathrm{Boot})} \in [0, 2\mathring{A}_*^{-1}] \text{ such that:} \tag{10.7}$$

- Ψ, v^J, V_α^J, u, μ, $\xi_j^{(\mathrm{Small})}$, ${}^{(i)}\Theta_{(\mathrm{Small})}^j$, and all of the other quantities defined throughout the article exist classically on $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$.

[38] By a past neighborhood, we mean an open set of points (t, u, ϑ) intersected with the slab $[0, T_{(\mathrm{Lifespan});U_0}] \times \mathbb{R} \times \mathbb{T}^{n-1}$.

[39] See Remarks 3.9 and 5.3.

- The change of variables map Υ from Definition 3.24 is a (global) diffeomorphism from $[0, T_{(\text{Boot})}) \times [0, U_0] \times \mathbb{T}^{n-1}$ onto its image $\mathcal{M}_{T_{(\text{Boot})},U_0}$ satisfying
$$\Upsilon^\alpha,\ \frac{\partial}{\partial \vartheta^i}\Upsilon^\alpha \in \bigcap_{k=0,1} C^k\big([0, T_{(\text{Boot})}), W^{1-k,\infty}([0, U_0] \times \mathbb{T}^{n-1})\big).$$
- $\inf\{\mu_\star(t, U_0) \mid t \in [0, T_{(\text{Boot})})\} > 0$ (see Definition 6.7).
- The fundamental L^∞ bootstrap assumptions (5C.3) hold with $\varepsilon := C'\mathring{\epsilon}$ for $(t, u) \in \times[0, T_{(\text{Boot})}) \times [0, U_0]$.

By standard local well-posedness for quasilinear hyperbolic systems (see, for example, [Ringström 2009, Part I]), if $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small in the sense explained in Section 5D and C' is sufficiently large, then $T_{(\text{Max});U_0} > 0$. Under the same smallness/largeness assumptions, by Corollary 8.8, the bootstrap assumptions (5C.3) are not saturated for $(t, u) \in [0, T_{(\text{Max});U_0}) \times [0, U_0]$. For this reason, all estimates proved throughout the article on the basis of the bootstrap assumptions in fact hold on $\mathcal{M}_{T_{(\text{Boot})},U_0}$ with ε replaced by $C\mathring{\epsilon}$. We use this fact throughout the remainder of the proof without further remark. In particular, the estimates of Proposition 6.5 hold for $(t, u) \in [0, T_{(\text{Max});U_0}) \times [0, U_0]$ with all factors ε on the right-hand side of all inequalities replaced by $C\mathring{\epsilon}$. Moreover, by inserting the energy estimates of Proposition 8.6 into the right-hand sides of the estimates of Lemma 8.5, we conclude that the estimates (10.3a)–(10.3c) hold for $(t, u) \in [0, T_{(\text{Max});U_0}) \times [0, U_0]$.

We now establish the dichotomy of possibilities. We first show that if
$$\inf\{\mu_\star(t, U_0) \mid t \in [0, T_{(\text{Max});U_0})\} > 0, \tag{10.8}$$
then $T_{(\text{Max});U_0} = 2\mathring{A}_*^{-1}$. To proceed, we assume for the sake of deriving a contradiction that (10.8) holds but that $T_{(\text{Max});U_0} < 2\mathring{A}_*^{-1}$. Then from (10.8) and Proposition 7.3, we see that if $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small, then Υ extends to a global diffeomorphism from $[0, T_{(\text{Max});U_0}] \times [0, U_0] \times \mathbb{T}$ onto its image that enjoys the regularity (7B.2) (with $T_{(\text{Boot})}$ replaced by $T_{(\text{Max});U_0}$ in (7B.2)). Also using the assumption (2C.1), Definition 3.6, definition (3D.3d), and the estimates of Proposition 6.5, we see that none of the four breakdown scenarios of Proposition 9.1 occur on $\mathcal{M}_{T_{(\text{Max});U_0},U_0}$. Hence, by Proposition 9.1, we can classically extend the solution to a region of the form $\mathcal{M}_{T_{(\text{Max});U_0}+\Delta,U_0}$, with $\Delta > 0$ and $T_{(\text{Max});U_0} + \Delta < 2\mathring{A}_*^{-1}$, such that all of the properties defining $T_{(\text{Max});U_0}$ hold for the larger time $T_{(\text{Max});U_0} + \Delta$. This contradicts the definition of $T_{(\text{Max});U_0}$ and in fact implies that if (10.8) holds and if $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small, then (I) $T_{(\text{Max});U_0} = 2\mathring{A}_*^{-1}$ and $T_{(\text{Lifespan});U_0} > 2\mathring{A}_*^{-1}$. The only other possibility is: (II) $\inf\{\mu_\star(t, U_0) \mid t \in [0, T_{(\text{Max});U_0})\} = 0$.

We now aim to show that case (II) corresponds to the formation of a shock singularity in the constant-time hypersurface subset $\Sigma^{U_0}_{T_{(\text{Max});U_0}}$. We first derive the statements regarding the quantities that extend to $\Sigma^{U_0}_{T_{(\text{Lifespan});U_0}}$ as elements of the space $C\big([0, T_{(\text{Lifespan});U_0}], L^\infty([0, U_0] \times \mathbb{T})\big)$. Here we will prove the desired results with $T_{(\text{Max});U_0}$ in place of $T_{(\text{Lifespan});U_0}$; in the next paragraph, we will show that $T_{(\text{Max});U_0} = T_{(\text{Lifespan});U_0}$. Let q denote any of the quantities $\mathscr{Z}^{\leq N_{\text{Mid}}-1;1}\Psi, \ldots, \mathscr{Z}^{\leq N_{\text{Mid}}-2;1}X^i$ that, in the theorem, are stated to extend. From the estimates of Lemma 6.4 and Proposition 6.5, we deduce that $\|Lq\|_{L^\infty(\Sigma_t^{U_0})}$ is uniformly bounded for $0 \leq t < T_{(\text{Max});U_0}$. Using this fact, the fact that $L = \frac{\partial}{\partial t}$, the fundamental theorem of calculus, and the completeness of the space $L^\infty([0,U_0]\times\mathbb{T})$, we conclude that q extends to $\Sigma^{U_0}_{T_{(\text{Max});U_0}}$ as a function of the geometric coordinates (t,u,ϑ) belonging to the space $C\big([0,T_{(\text{Max});U_0}],L^\infty([0,U_0]\times\mathbb{T})\big)$, as desired.

We now show that the classical lifespan is characterized by (10.1) and that $T_{(\text{Max});U_0}=T_{(\text{Lifespan});U_0}$. To this end, we first use (6D.3) and the continuous extension properties proved in the previous paragraph to deduce (10.5). Also using Definition 3.15, the schematic relation $X^j_{(\text{Small})}=\gamma\mathrm{f}(\gamma)$, and the L^∞ estimates of Proposition 6.5, we deduce that $C^{-1}\le\delta_{ab}X^aX^b\le C$. That is, the vector field X is of order-unity Euclidean length. From this estimate and (10.5), we deduce that at points in $\Sigma^{U_0}_{T_{(\text{Max});U_0}}$ where μ vanishes, $|X\Psi|$ blows up like $1/\mu$. Hence, $T_{(\text{Max});U_0}$ is the classical lifespan. That is, we conclude that $T_{(\text{Max});U_0}=T_{(\text{Lifespan});U_0}$, and we obtain the characterization (10.1) of the classical lifespan. The estimate (10.6) follows from the estimate (6C.5c), the fact that $\breve{X}=\mu X$, and the continuous extension properties proved in the previous paragraph.

Finally, to obtain (10.2), we use (6D.4) to conclude that $\mu_\star(t,1)$ vanishes for the first time when $t=\{1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha})+\mathcal{O}(\mathring{\epsilon})\}\mathring{A}_*^{-1}$. We have therefore proved the theorem. □

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References

[Alinhac 1999a] S. Alinhac, "Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, II", *Acta Math.* **182**:1 (1999), 1–23. MR Zbl

[Alinhac 1999b] S. Alinhac, "Blowup of small data solutions for a quasilinear wave equation in two space dimensions", *Ann. of Math.* (2) **149**:1 (1999), 97–127. MR Zbl

[Alinhac 2001] S. Alinhac, "The null condition for quasilinear wave equations in two space dimensions, II", *Amer. J. Math.* **123**:6 (2001), 1071–1101. MR Zbl

[Christodoulou 2000] D. Christodoulou, *The action principle and partial differential equations*, Annals of Math. Studies **146**, Princeton Univ. Press, 2000. MR Zbl

[Christodoulou 2007] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*, Eur. Math. Soc., Zürich, 2007. MR Zbl

[Christodoulou 2009] D. Christodoulou, *The formation of black holes in general relativity*, Eur. Math. Soc., Zürich, 2009. MR Zbl

[Christodoulou 2019] D. Christodoulou, *The shock development problem*, Eur. Math. Soc., Zürich, 2019. MR Zbl

[Christodoulou and Klainerman 1993] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Math. Series **41**, Princeton Univ. Press, 1993. MR Zbl

[Christodoulou and Lisibach 2016] D. Christodoulou and A. Lisibach, "Shock development in spherical symmetry", *Ann. PDE* **2**:1 (2016), art. id. 3. MR Zbl

[Christodoulou and Miao 2014] D. Christodoulou and S. Miao, *Compressible flow and Euler's equations*, Surveys of Modern Math. **9**, Int. Press, Somerville, MA, 2014. MR Zbl

[Christodoulou and Perez 2016] D. Christodoulou and D. R. Perez, "On the formation of shocks of electromagnetic plane waves in non-linear crystals", *J. Math. Phys.* **57**:8 (2016), art. id. 081506. MR Zbl

[Dafermos 2010] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd ed., Grundlehren der Math. Wissenschaften **325**, Springer, 2010. MR Zbl

[Holzegel et al. 2016] G. Holzegel, S. Klainerman, J. Speck, and W. W.-Y. Wong, "Small-data shock formation in solutions to 3D quasilinear wave equations: an overview", *J. Hyperbolic Differ. Equ.* **13**:1 (2016), 1–105. MR Zbl

[John 1974] F. John, "Formation of singularities in one-dimensional nonlinear wave propagation", *Comm. Pure Appl. Math.* **27** (1974), 377–405. MR Zbl

[John 1981] F. John, "Blow-up for quasilinear wave equations in three space dimensions", *Comm. Pure Appl. Math.* **34**:1 (1981), 29–51. MR Zbl

[Klainerman 1985] S. Klainerman, "Uniform decay estimates and the Lorentz invariance of the classical wave equation", *Comm. Pure Appl. Math.* **38**:3 (1985), 321–332. MR Zbl

[Klainerman 1986] S. Klainerman, "The null condition and global existence to nonlinear wave equations", pp. 293–326 in *Nonlinear systems of partial differential equations in applied mathematics, I* (Santa Fe, NM, 1984), edited by B. Nicolaenko et al., Lectures in Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Klainerman and Rodnianski 2003] S. Klainerman and I. Rodnianski, "Improved local well-posedness for quasilinear wave equations in dimension three", *Duke Math. J.* **117**:1 (2003), 1–124. MR Zbl

[Klainerman and Rodnianski 2005] S. Klainerman and I. Rodnianski, "Rough solutions of the Einstein-vacuum equations", *Ann. of Math.* (2) **161**:3 (2005), 1143–1193. MR Zbl

[Klainerman et al. 2015] S. Klainerman, I. Rodnianski, and J. Szeftel, "The bounded L^2 curvature conjecture", *Invent. Math.* **202**:1 (2015), 91–216. MR Zbl

[Lax 1964] P. D. Lax, "Development of singularities of solutions of nonlinear hyperbolic partial differential equations", *J. Math. Phys.* **5** (1964), 611–613. MR Zbl

[Lax 1972] P. D. Lax, "The formation and decay of shock waves", *Amer. Math. Monthly* **79**:3 (1972), 227–241. MR Zbl

[Lax 1973] P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, CBMS Regional Conf. Series Appl. Math. **11**, Soc. Indust. Appl. Math., Philadelphia, 1973. MR Zbl

[Lee 2013] J. M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Math. **218**, Springer, 2013. MR Zbl

[Lindblad and Rodnianski 2010] H. Lindblad and I. Rodnianski, "The global stability of Minkowski space-time in harmonic gauge", *Ann. of Math.* (2) **171**:3 (2010), 1401–1477. MR Zbl

[Luk and Speck 2016] J. Luk and J. Speck, "The hidden null structure of the compressible Euler equations and a prelude to applications", 2016. To appear in *J. Hyperbolic Differ. Equ.* arXiv

[Luk and Speck 2018] J. Luk and J. Speck, "Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity", *Invent. Math.* **214**:1 (2018), 1–169. MR Zbl

[Miao 2018] S. Miao, "On the formation of shock for quasilinear wave equations with weak intensity pulse", *Ann. PDE* **4**:1 (2018), art. id. 10. MR Zbl

[Miao and Yu 2017] S. Miao and P. Yu, "On the formation of shocks for quasilinear wave equations", *Invent. Math.* **207**:2 (2017), 697–831. MR Zbl

[Rauch 1986] J. Rauch, "BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one", *Comm. Math. Phys.* **106**:3 (1986), 481–484. MR Zbl

[Riemann 1860] B. Riemann, "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite", *Abh. Königlichen Ges. Wiss. Göttingen* **8** (1860), 43–66.

[Ringström 2009] H. Ringström, *The Cauchy problem in general relativity*, Eur. Math. Soc., Zürich, 2009. MR Zbl

[Sbierski 2016] J. Sbierski, "On the existence of a maximal Cauchy development for the Einstein equations: a dezornification", *Ann. Henri Poincaré* **17**:2 (2016), 301–329. MR Zbl

[Sideris 1984] T. C. Sideris, "Formation of singularities in solutions to nonlinear hyperbolic equations", *Arch. Ration. Mech. Anal.* **86**:4 (1984), 369–381. MR Zbl

[Sideris 1985] T. C. Sideris, "Formation of singularities in three-dimensional compressible fluids", *Comm. Math. Phys.* **101**:4 (1985), 475–485. MR Zbl

[Smith and Tataru 2005] H. F. Smith and D. Tataru, “Sharp local well-posedness results for the nonlinear wave equation”, *Ann. of Math.* (2) **162**:1 (2005), 291–366. MR Zbl

[Speck 2016] J. Speck, *Shock formation in small-data solutions to 3D quasilinear wave equations*, Math. Surveys and Monographs **214**, Amer. Math. Soc., Providence, RI, 2016. MR Zbl

[Speck 2017] J. Speck, “A new formulation of the $3D$ compressible Euler equations with dynamic entropy: remarkable null structures and regularity properties”, preprint, 2017. To appear in *Arch. Ration. Mech. Anal.* arXiv

[Speck 2018] J. Speck, “Shock formation for $2D$ quasilinear wave systems featuring multiple speeds: blowup for the fastest wave, with non-trivial interactions up to the singularity”, *Ann. PDE* **4**:1 (2018), art. id. 6. MR Zbl

[Speck et al. 2016] J. Speck, G. Holzegel, J. Luk, and W. Wong, “Stable shock formation for nearly simple outgoing plane symmetric waves”, *Ann. PDE* **2**:2 (2016), art. id. 10. MR Zbl

[Wong 2013] W. W.-Y. Wong, “A comment on the construction of the maximal globally hyperbolic Cauchy development”, *J. Math. Phys.* **54**:11 (2013), art. id. 113511. MR Zbl

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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