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ON GEOMETRIC AND ANALYTIC MIXING SCALES: COMPARABILITY AND CONVERGENCE RATES FOR TRANSPORT PROBLEMS

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We are interested in the geometric and analytic mixing scales of solutions to passive scalar problems. Here, we show that both notions are comparable after possibly removing large-scale projections. In order to discuss our techniques in a transparent way, we further introduce a dyadic model problem.

In a second part of our article we consider the question of sharp decay rates for both scales for Sobolev regular initial data when evolving under the transport equation and related active and passive scalar equations. Here, we show that slightly faster rates than the expected algebraic decay rates are optimal.

1. Introduction and main results

We are interested in the mixing behavior of passive scalar problems

$$\begin{aligned} \partial_t \rho + v \cdot \nabla \rho &= 0, \\ \rho|_{t=0} &= \rho_0, \end{aligned} \tag{1}$$

where $v(t)$ is a given divergence-free vector field on $\mathbb{R}^n$ or $\mathbb{T}^n \times \mathbb{R}^n$. Assuming sufficient regularity of v, the flow preserves all L^p norms; i.e., $\|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p}$ for all $p \in [1, \infty]$ and all $t > 0$. On the other hand, if the flow is, for instance, ergodic, then ρ weakly converges to its average $\langle \rho_0 \rangle$ and, in particular, $\|\langle \rho_0 \rangle\|_{L^p} \leq \|\rho_0\|_{L^p}$ with generically a strict decrease for $p \neq 1$. The solution is *mixed* as $t \to \infty$.

In order to quantify this limiting behavior, one commonly considers two different functionals:

Definition 1.1 (mixing scales; [Thiffeault 2012]). Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a given measurable function. Then we call $\|\rho\|_{H^{-1}}$ the *analytic mixing scale*.

Furthermore, for given $r > 0$, we define the *geometric mixing functionals*

$$\mathfrak{g}_r[\rho] := \sup_{B_R(\xi);\, R \geq r} \frac{1}{|B_R|} \left| \int_{B_R(\xi)} \rho \right|. \tag{2}$$

If further $\rho \in L^\infty$, then for each $\kappa \in (0, 1)$ we define the *geometric mixing scale* as

$$\mathcal{G}_\kappa[\rho] := \inf\{r : \mathfrak{g}_r[\rho] \leq \kappa \|\rho\|_{L^\infty}\}. \tag{3}$$

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As one of the main results of this article, we show that while both notions are not equivalent, they are comparable in the sense that smallness of one implies smallness of the other. A control in one direction has, for instance, been used in [Iyer et al. 2014] to establish lower bounds on decay rates of $\|\rho(t)\|_{H^{-1}}$ (see Section 5). Furthermore, in [Lunasin et al. 2012] the authors construct several examples exhibiting obstructions to comparability. As we discuss in Section 2, the present article suggests to instead view these obstructions as contributions at large scales that need to be projected out.

Theorem 1.2 (comparison of mixing scales). *Let $\rho \in L^2(\mathbb{R}^n)$ and $\|\rho\|_{L^2} \leq 1$. Then for all $0 < \epsilon \leq 1$:*

(1) *There exists a constant $C > 0$, depending only on the dimension n and $\alpha = 2/(n+2)$ and $\beta = 2/(n+4)$, such that if $\|\rho\|_{H^{-1}} \leq \epsilon$ and ρ is supported in B_1, then also $\mathfrak{g}_{\epsilon'}[\rho] \leq C\epsilon'$ for all $\epsilon' \geq \epsilon^\alpha$ and $\mathfrak{g}_{\tilde{\epsilon}}[\rho] \leq C$ for all $\tilde{\epsilon} \geq \epsilon^\beta$.*

In particular, supposing additionally that $\|\rho\|_{L^\infty} = 1$, it follows that

$$\mathcal{G}_C[\rho] \leq \tilde{\epsilon},$$
$$\mathcal{G}_{C\epsilon'}[\rho] \leq \epsilon'.$$

(2) *If $\mathfrak{g}_\epsilon[\rho] \leq \epsilon$ and ρ is supported in a compact set K, then also $\|\rho\|_{H^{-1}} \leq C_K \epsilon$.*

These estimates are optimal in the powers of ϵ.

In order to introduce our methods, we construct a dyadic *Walsh–Fourier model* on $L^2(\mathbb{T})$ in Section 3, where we introduce new dyadic analogues of both scales and show them to be equivalent when restricted to appropriate subspaces E_j. In particular, in that setting optimality of estimates is transparent. Subsequently, we discuss the continuous case as stated in Theorem 1.2 in Section 4.

A natural question here, of course, is whether

$$\mathfrak{g}_r[\rho] \leq Cr \tag{4}$$

can be assumed in applications. Indeed, if $\rho(t)$ solves the passive scalar problem (1) and asymptotically converges weakly to a nontrivial state ρ_∞, then we can generally not expect better control than

$$\mathfrak{g}_r[\rho(t)] \leq \|\rho_\infty\|_{L^\infty}.$$

However, as we discuss in Section 2, upon removing large-scale projections (corresponding to asymptotic states) this assumption is natural and comparability holds in the above sense.

As a second part of our article, in Section 5, we consider the evolution of the mixing scales under transport-type equations and are interested in (sharp) upper and lower bounds on decay rates of the scales. As a first model problem we consider the case of $\rho(t)$ evolving under the free transport equation on $\mathbb{T}^n \times \mathbb{R}^n$:

$$\begin{aligned} \partial_t \rho + y\partial_x \rho &= 0 \quad \text{for } t \in (0,\infty),\ x \in \mathbb{T}^n,\ y\mathbb{R}^n, \\ \rho_{t=0} &= \rho_0 \quad \text{for } x \in \mathbb{T}^n,\ y \in \mathbb{R}^n. \end{aligned} \tag{5}$$

Here, we show that if the initial data is normalized in a Sobolev space H^s, $0 \leq s \leq 1$, with respect to y, then the first expected decay rates of t^{-s} turn out to be slightly suboptimal and instead decay rates of $t^{-s}o(1)$ are achieved.

Theorem 1.3. *In the following, let* $0 < s \leq 1$, $\rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$, *with* $\int_{\mathbb{T}^n} \rho_0(x, y)\, dx = 0$, *and let*

$$\rho(t, x, y) = \rho_0(t, x - ty, y)$$

be the solution of the free transport problem. For $\sigma, s \in \mathbb{R}$ *let* $H^\sigma H^s = H^\sigma(\mathbb{T}^n; H^s(\mathbb{R}^n))$ *denote the Hilbert space with norm*

$$\|\rho\|^2_{H^\sigma H^s} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2\sigma} \int_{\mathbb{R}^n} \langle \eta \rangle^{2s} |\tilde{\rho}(k, \eta)|^2\, d\eta.$$

(1) *There exists* $C_s > 1$ *such that for all* $t \geq 1$ *and all initial data*

$$\|\rho(t)\|_{L^2 H^{-1}} \leq C t^{-s} \|\rho_0\|_{H^{-s} H^s}.$$

(2) *Let* $\alpha_j > 0$ *with* $\|(\alpha_j)_j\|_{l^2} = 1$. *Then there exist* $c > 0$, *a sequence of times* $t_j \to \infty$ *and initial data* ρ_0 *such that*

$$\|\rho(t_j)\|_{L^2 H^{-1}} \geq c \alpha_j t_j^{-s} \|\rho_0\|_{H^{-s} H^s}.$$

(3) *There exists no nontrivial initial data* $\rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ *such that*

$$\|\rho(t_j)\|_{L^2 H^{-1}} \geq c t_j^{-s} \|\rho_0\|_{H^{-s} H^s}$$

along some sequence $t_j \to \infty$.

In the second statement, t_j can always be chosen larger and more rapidly increasing. For instance, we may chose $t_j = \exp(\exp(\cdots \exp(j)))$ and $\alpha_j = 1/j = \ln(\ln(\cdots \ln(t_j)))$ as iterated exponentials and logarithms. Informally stated, the theorem hence shows that algebraic decay rates can be achieved along a subsequence up to an arbitrarily small loss. Conversely, the third statement shows that this loss is necessary and that the lower estimate is sharp in this sense.

We remark that in several works on (linear) inviscid damping, [Wei et al. 2018; Coti Zelati and Zillinger 2019; Zillinger 2016; Bedrossian and Masmoudi 2015] or (linear) Landau damping [Bedrossian et al. 2016], it is shown that perturbations to the Euler equations or Vlasov–Poisson equations scatter to solutions of the free transport problem as $t \to \infty$. As a corollary, we hence obtain the optimality of the decay rates for these equations as well.

Corollary 1.4. *Let* $U(y)$ *be bi-Lipschitz and* $U'' \in W^{2,\infty}$ *with* $\|U''\|_{W^{2,\infty}}$ *sufficiently small. Then for any* $s \in (0, 1)$ *and any* $\omega_0 \in H^s(\mathbb{T} \times \mathbb{R})$ *the solution* ω *of the linearized Euler equations*

$$\partial_t \omega + U(y)\partial_x \omega - U''(y)\partial_x \Delta^{-1} \omega = 0,$$
$$\omega|_{t=0} = \omega_0$$

satisfies

$$\|\omega(t)\|_{L^2 H^{-1}} \leq C t^{-s} \|\omega_0\|_{H^{-s} H^s}$$

and there exists no nontrivial initial data $\omega_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ *such that*

$$\|\omega_0(t_j)\|_{L^2 H^{-1}} \geq c t_j^{-s} \|\omega_0\|_{H^{-s} H^s}$$

along some sequence $t_j \to \infty$.

We remark that we do not require U to be bounded and for instance allow U to be (a perturbation of) an affine function, which is of interest in the study of the free transport equation and similar passive scalar problems.

Proof. In [Zillinger 2016], we have shown that solutions to the linearized Euler equations around such a shear flow $U(y)$ are stable and scatter in H^s. That is, for any initial datum $\omega_0 \in H^s$ there exists a solution $\omega(t, x, y)$ and the associated (finite-time) *scattering profile*

$$W(t, x, y) = \omega(t, x + tU(y), y)$$

satisfies

$$C^{-1}\|\omega_0\|_{H^\sigma H^s} \leq \|W(t)\|_{H^s} \leq C\|\omega_0\|_{H^\sigma H^s}$$

for all $t \geq 0$, $\sigma \in \mathbb{R}$ and $\sigma \in \left[0, \frac{3}{2}\right)$. Furthermore, there exists W_∞ such that $W(t) \to W_\infty$ in $H^\sigma H^s$ as $t \to \infty$. Since $U(y)$ is bi-Lipschitz, we may change coordinates to $(x, z = U(y))$ and note that the Sobolev spaces $H^\sigma H^s$ with respect to (x, y) and (x, z) are equivalent. Thus, with slight abuse of notation

$$\omega(t, x, z) = W(t, x - tz, z)$$

can be considered as a solution of the free transport problem at time t with initial data $W(t)$. Hence, by Theorem 1.3, it follows that

$$\|\omega(t)\|_{L^2H^{-1}} \leq Ct^{-s}\|W(t)\|_{H^{-s}H^s} \leq Ct^{-s}\|\omega_0\|_{H^{-s}H^s}.$$

We prove the second claim by contradiction and suppose there exists such a nontrivial ω_0. Then, by the (asymptotic) scattering result, there also exists a nontrivial W_∞ such that $W(t) \to W_\infty$ and hence

$$\|\omega(t, x, z) - W_\infty(t, x + tz, z)\|_{L^2H^{-1}} \leq Ct^{-s}\|W(t) - W_\infty\|_{H^{-s}H^s} = o(t^{-s}).$$

But this would then imply

$$\begin{aligned}\|W_\infty(x + t_j z, z)\|_{L^2H^{-1}} &\geq \|\omega(t_j)\|_{L^2H^{-1}} - \|\omega(t_j, x, z) - W_\infty(t_j, x + t_j z, z)\|_{L^2H^{-1}} \\ &\geq ct_j - o(t_j) \geq \frac{c}{2}t_j\end{aligned}$$

as $j \to \infty$. This contradicts the third statement of Theorem 1.3. □

Concerning more general passive scalar problems, a recent active area of research [Alberti et al. 2014; Seis 2017; Crippa and Schulze 2017; Iyer et al. 2014] is given by the study of upper and lower bounds on decay rates of mixing scales for solutions of (1),

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$

where v may be chosen arbitrarily under given constraints such as $\|v(t)\|_{W^{1,p}} \leq 1$. While in the present article we establish no improved bound, the comparability results allow us to connect existing results in the literature established for either mixing scale [Crippa and De Lellis 2008; Seis 2013; Bruè and Nguyen 2018] (see Corollary 5.6). For instance, the following corollary establishes *analytic mixing cost estimates* [Iyer et al. 2014, Theorem 1.1] as a corollary of *geometric mixing cost estimates* [Crippa and De Lellis 2008].

Corollary 1.5. *Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(\mathbb{T}^2)$ and suppose that for $\frac{1}{4} > \epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution ρ of*

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$
$$\nabla \cdot \rho = 0$$

satisfies

$$\left\| \rho|_{t=1} - \tfrac{1}{2} \right\|_{H^{-1}} \leq \epsilon.$$

Then for $p > 1$ the velocity field v satisfies

$$\int_0^1 \|\nabla v\|_{L^p} \, dt \geq C |\log(\epsilon)|.$$

The remainder of the article is organized as follows:

- In Section 2, we discuss to which extent both mixing scales might possibly be comparable by considering two prototypical examples and also discuss the role of large-scale asymptotic profiles.
- In Section 3, we introduce a new dyadic Walsh–Fourier model of mixing scales. Due to improved orthogonality properties here we can establish our estimates in a transparent, accessible way.
- Subsequently, in Section 4 we show that most properties and estimates persist in the continuous setting despite the loss of beneficial additional structure.
- Finally, Section 5 considers (sharp) decay rates of mixing scales under passive scalar problems. Here, we first establish optimal rates for the free transport problem and then discuss more general dynamics.

1A. ***Notation.*** Throughout this article we will consider $x \in \mathbb{T}^n$ and $y \in \mathbb{R}^n$ unless otherwise specified. For a given function $\rho(x, y)$, we denote by $\hat{\rho}(k, y)$ its partial Fourier transform with respect to x and by $\tilde{\rho}(k, \eta)$ its Fourier transform with respect to both x and y. Constants $C, c > 0$ may change from line to line and may depend on the dimension but are independent of the choice of function ρ or initial data ρ_0.

2. Preliminaries and prototypical examples

In [Lunasin et al. 2012] two families of functions are constructed to highlight the differences of the analytic mixing scale (4) and the geometric mixing functionals (2). In order to introduce our ideas, we recall their construction and show that after removing the large-scale weak limit of the second family both notions are comparable and thus motivate our choice of spaces and estimates.

2A. ***The analytic mixing scale controls the geometric mixing scale.*** We briefly recall the construction from [Lunasin et al. 2012, Section IV.B]. As a building block consider the "hat" function

$$v(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1, \\ 0 & \text{else.} \end{cases}$$

Let $\epsilon = 2^{-n}$ and let $\alpha \in \mathbb{N}$ with $\alpha < 2^{n-1}$. We then build an odd function u on $[-1, 1]$ such that for $x > 0$

$$u(x) = \alpha \epsilon v\left(\frac{x}{\alpha \epsilon}\right) + \sum_{j=1}^{2^n - 2\alpha} \epsilon v\left(\frac{x - (2\alpha + 2j)\epsilon}{\epsilon}\right).$$

This function is a sawtooth function with one large tooth on the interval $(0, 2\alpha\epsilon)$ and smaller teeth of width 2ϵ on the remainder of $(0, 1)$. Furthermore, u is Lipschitz and $\rho = u' \in \{-1, 1\}$ satisfies

$$\|\rho\|_{H^{-1}}^2 = \|u\|_{L^2}^2 = \epsilon^2\left(\tfrac{1}{3} - \tfrac{2}{3}\alpha\epsilon\right) + \tfrac{2}{3}\alpha^3\epsilon^3 \approx \epsilon^2$$

if ϵ is small. Taking $\alpha \approx \epsilon^{-1/3}$, we thus obtain that $\|\rho\|_{H^{-1}} \approx \epsilon$ and that $\rho = 1$ on $B_{\epsilon^{2/3}}(0)$ and is hence geometrically mixed at most at scale $\epsilon^{2/3}$.

Additionally, we compute that if we consider larger radii $r \geq \epsilon^{2/5}$, then

$$\frac{1}{|B_r|}\left|\int_{B_r(\xi)} \rho\, dx\right| \leq Cr.$$

This example hence shows that an estimate of the type $\epsilon \rightsquigarrow \epsilon$ is not possible, but $\epsilon \rightsquigarrow \epsilon^\alpha$ is for this case. In Sections 3 and 4 we show that such an estimate indeed holds for general functions and in higher dimensions and that our exponents of ϵ are optimal. Roughly speaking, the loss of power in ϵ here is due to the L^1 normalization of the characteristic functions and equivalence constants of (weighted) l^2 and l^∞ norms on finite-dimensional vector spaces (see Section 3) and agrees with scaling (see the proof of Theorem 4.3).

2B. ***The geometric mixing scale and weak limits.*** Consider a periodic characteristic function $u \in L^2(\mathbb{T})$ with $\int_{\mathbb{T}} u\, dx = \frac{1}{2}$ and for $k \in \mathbb{N}$ define

$$\rho_k(x) = \operatorname{sgn}(x)\, u(k|x|) \in L^2((-1, 1)).$$

We note that $\int \rho_k\, dx = 0,\ \|\rho_k\|_{L^\infty} = 1$ and

$$\rho_k \xrightarrow{L^2} \tfrac{1}{2}\operatorname{sgn}(x).$$

Hence, for any given ball $B_r(\xi) \subset (-1, 1)$,

$$\frac{1}{|B_r|}\left|\int_{B_r(\xi)} \rho_k\, dx\right| \to \frac{1}{|B_r|}\left|\int_{B_r(\xi)} \tfrac{1}{2}\operatorname{sgn}(x)\, dx\right| \leq \tfrac{1}{2}$$

as $k \to \infty$. Using the periodicity to obtain more quantitative estimates, we obtain that for any $\kappa > \frac{1}{2}$ and any $r > 0$, we can achieve

$$\frac{1}{|B_r|}\left|\int_{B_r(\xi)} \rho_k\, dx\right| \leq \kappa\|\rho_k\|_{L^\infty}$$

provided k is sufficiently large and that thus $\mathcal{G}_\kappa[\rho_k] \to 0$ as $k \to \infty$.

However, this fails for $\kappa < \frac{1}{2}$ and by lower semicontinuity,

$$\liminf_{k\to\infty} \|\rho_k\|_{H^{-1}} \geq \left\|\tfrac{1}{2}\operatorname{sgn}(x)\right\|_{H^{-1}} > 0.$$

Hence, this at first suggests that the geometric mixing scale is distinct from the analytic mixing scale. In view of our comparison result, Theorem 1.3, we instead suggest to interpret this example as

$$\|\rho_k\|_{H^{-1}} \leq C\max(\kappa, r)$$

and note that the lower bound on $\kappa \geq \frac{1}{2}$ here is due to large-scale structures in the weak limit. Indeed, consider the functions v_k obtained by projecting out large scales:

$$v_k(x) = \rho_k(x) - \tfrac{1}{2}\operatorname{sgn}(x).$$

Then it holds that

$$\frac{1}{|B_r|}\left|\int_{B_r(\xi)} v_k \, dx\right| \leq \min(2\kappa, Cr) \quad \text{for } r \geq \frac{c}{k},$$

$$\|v_k\|_{L^\infty} \geq \kappa, \qquad \|v_k\|_{H^{-1}} \leq \frac{C}{k}, \qquad v_k \xrightarrow{L^2} 0 \quad \text{as } k \to \infty.$$

Remark 2.1. When considering $\rho_k = \rho(t_k)$ for $\rho(t)$ evolving under ergodic dynamics, the weak limit is given by a constant function. In that setting, we may assume that constant to be zero after normalization and hence, there without loss of generality both notions of geometric mixing coincide, i.e., $v_k = \rho_k$.

More generally, we don't need to know a candidate for the weak limit (or even have a sequence), but rather consider the following setting:

If $\rho \in L^2 \cap L^\infty \cap H^{-1}$ is such that for all $r \geq r_0$ and all $x_0 \in \mathbb{R}^n$ it holds that

$$\frac{1}{|B_r|}\left|\int_{B_r(x_0)} \rho \, dx\right| \leq \kappa,$$

then we claim that (see Lemma 4.8)

$$\rho_{r_0} = \rho - \left(\frac{1}{|B_{r_0}|} 1_{B_{r_0}(\xi)} * \rho\right)$$

satisfies

$$\|\rho - \rho_{r_0}\|_{L^\infty} \leq \kappa, \quad \frac{1}{|B_r|}\left|\int_{B_r(\xi)} \rho_{r_0} \, dx\right| \leq Cr, \quad \|\rho_{r_0}\|_{H^{-1}} \leq Cr_0\|\rho\|_{L^2}$$

for all $r \geq r_0$. Here, we stress that, while

$$\frac{1}{|B_{r_0}|} 1_{B_{r_0}(\xi)} * \rho \to \rho$$

as $r_0 \downarrow 0$ for fixed ρ, this is much more subtle for sequences ρ depending on r_0. Indeed, letting $u_k(x)$ be as above, we obtain

$$\frac{1}{|B_{1/k}|} 1_{B_{1/k}} * u_k(x) = \tfrac{1}{2}\operatorname{sgn}(x)$$

for all x with $\operatorname{dist}(x, \{0, \pi, -\pi\}) \geq 2/k$.

3. A Walsh–Fourier model

In order to introduce our ideas and establish sharpness of estimates, we first discuss and compare both mixing scales in a dyadic model setting. Here, we consider averages over dyadic intervals and replace the sine basis of $L^2(\mathbb{T})$ by functions that are constantly $+1$ or -1 on dyadic intervals. This setting is known in harmonic analysis as a Walsh–Fourier setting and associated with a "tile" characterization and Haar wavelet expansions [Muscalu et al. 2004; Thiele 2000a; 2000b]. In the following we briefly provide some definitions and statements. For a more in-depth introduction we refer the interested reader to [Thiele 2006]. We remark that, for simplicity of notation and estimates, we here consider the setting of $L^2([0, 1))$ instead of $L^2(\mathbb{R})$. The dyadic setting has the benefit of greatly simplifying estimates due to orthogonality and allows for explicit computations of newly introduced analogues of the mixing scales as Besov-type norms in terms of certain L^2 bases. Hence, here it is transparent what estimates are possible and whether they are optimal. In Section 4 we show that, with minor modifications, these results also extend to the continuous Sobolev setting.

3A. *Definitions, tiles and bases.*

Definition 3.1. Let $[0, 1)$ be the half-open unit interval. Then for each $j \in \mathbb{N}_+$, we define the set of *dyadic intervals at scale* 2^{-j} by

$$\mathcal{D}_j = \{I_{k,j} := 2^{-j}[k, k+1) : k \in \{0, \dots, 2^j - 1\}\}.$$

Associated with this partition of $[0, 1)$, we introduce the L^2-normalized characteristic functions

$$\chi_I = \frac{1}{\sqrt{|I|}} 1_I \in L^2([0, 1)).$$

We note that, if $I, I' \in \mathcal{D}_j$, then either $I = I'$ or the intervals are disjoint. If the intervals are not of the same size, that is, $I \in \mathcal{D}_j$ and $I' \in \mathcal{D}_{j'}$ with $j \neq j'$, they are either disjoint or one is contained in the other.

In addition to the (normalized) characteristic functions, χ_I, the following definition introduces a large family of oscillating L^2 normalized functions, which we use to define (fractional) Sobolev-type spaces.

Definition 3.2. A *tile* p is a dyadic rectangle of area 1 in $[0, 1) \times [0, \infty)$. That is,

$$p := I \times \omega = [2^{-j}k, 2^{-j}(k+1)) \times [2^j l, 2^j(l+1)),$$

where $k \in \{0, 2^j - 1\}$, $j \in \mathbb{N}_0$, $l \in \mathbb{N}_0$. If $l = 0$, we define the *wave-packet* $\phi_p := (1/\sqrt{|I|})1_I$. For $l > 0$, we define ϕ_p recursively. That is, if p is a tile at level l and scale 2^j, we can express it as either the upper or lower half of the union of two tiles p_l, p_r at level $\lfloor l/2 \rfloor$ and scale 2^{-j-1}. We thus define

$$\phi_p = \tfrac{1}{\sqrt{2}}(\phi_{p_l} + \phi_{p_r}) \quad \text{if } l \text{ is even},$$
$$\phi_p = \tfrac{1}{\sqrt{2}}(\phi_{p_l} - \phi_{p_r}) \quad \text{if } l \text{ is odd}.$$

This definition allows us to consider questions of orthogonality and basis expansions in a graphical way, a so-called *cartoon* (see Figure 1). The following lemma summarizes some of the main properties we use in the following.

Lemma 3.3 [Thiele 2000a, Lemma 2.9]. *For any two tiles p, p' we have*

$$\left| \int_{[0,1)} \phi_p \phi_{p'} \right| = \sqrt{|p \cap p'|}.$$

In particular two wave packets are L^2-orthogonal if and only if the underlying tiles have empty intersection.

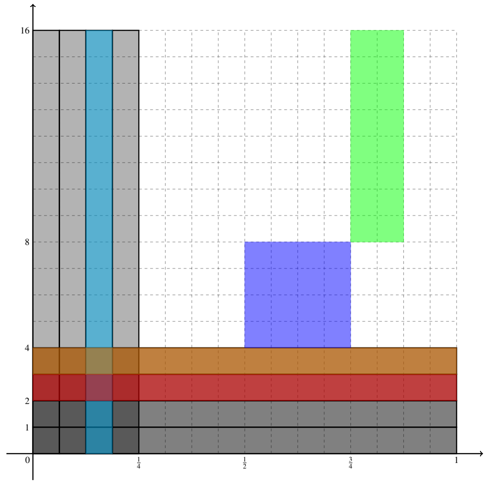

Figure 1. Various tiles down to scale 2^{-4}. The vertical gray tiles correspond to characteristic functions χ_I. The horizontal gray lines correspond to our replacement of a Fourier basis, ϕ_{p_l}. The tiles in green and blue in the upper right corner are at level $l = 1$. Plots of the corresponding wave packets of all colored tiles are given in Figure 2. By Lemma 3.3 wave packs $\phi_p, \phi_{p'}$ are L^2 orthogonal if and only if they are disjoint.

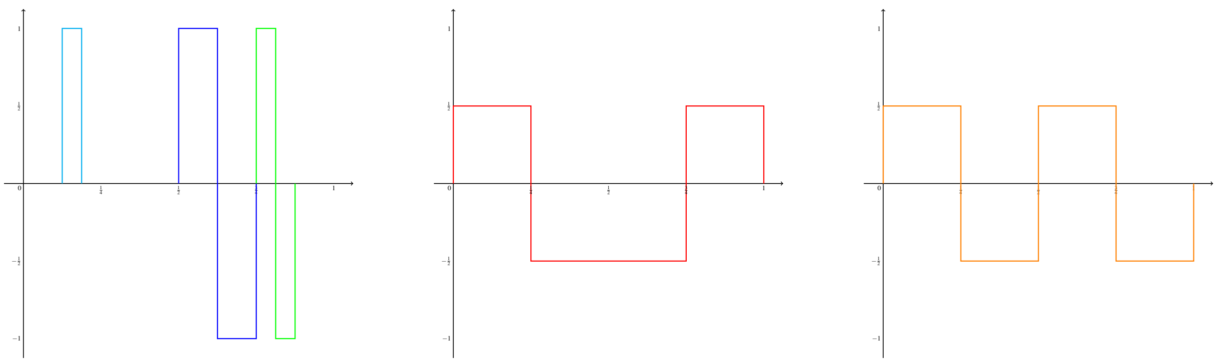

Figure 2. Walsh wave packets associated with the tiles of Figure 1.

Corollary 3.4 [Thiele 2000a, Corollary 2.7]. *Furthermore, two families of tiles P, P' cover the same region in* $[0, 1) \times [0, \infty)$ *if and only if the spans of* $\{\phi_p : p \in P\}$ *and* $\{\phi_p : p \in P'\}$ *are identical. In particular, setting* $p_l = [0, 1) \times [l, l+1)$, *we obtain that* $\{\chi_I\}_{I \in \mathcal{D}_j}$ *and* $\{\phi_{p_l} : l \in \{0, 2^j - 1\}\}$ *are both orthonormal bases of the same space, which we denote by* E_j. *We further introduce the* L^2 *orthogonal* projection operators P_j *onto* E_j.

Definition 3.5 (mixing scales). Given $\rho \in L^2([0, 1))$, we introduce the *geometric mixing seminorms* at scale j as

$$\mathfrak{g}_j[\rho] = \sup\left\{\frac{1}{I}\left|\int_I \rho\right| : I \in \mathcal{D}_j\right\} = \sqrt{2^j}\,\sup\{|\langle \chi_I, \rho\rangle| : I \in \mathcal{D}_j\}.$$

Furthermore, for $s \in [-1, 1]$ we define *analytic mixing seminorms* up to scale j by

$$\|\rho\|_{h_j^s}^2 = \sum_{l=0}^{2^j-1} (1+l^2)^2 |\langle \rho, \phi_{p_l} \rangle|^2.$$

We further define $\|\rho\|_{h^s} = \sup_j \|\rho\|_{h_j^s}$.

3B. *Estimates.* We note that the geometric mixing seminorms can be expressed as

$$\mathfrak{g}_j[\rho] = \sup_{I \in \mathcal{D}_j} \sqrt{2^j} |\langle \chi_I, \rho \rangle|$$

and are hence indeed norms on the space E_j. Likewise the analytic seminorms are weighted l^2 norms on the same space E_j when expressed in the orthonormal basis $\{\phi_{p_l}\}$. Hence, both mixing seminorms are equivalent, with constants depending on j.

The following theorem establishes the corresponding estimates as well as related estimates between different scales with uniform constants.

Theorem 3.6. *Let $\rho \in L^2((0, 1))$. Then for all $j \in \mathbb{N}$*

$$\begin{aligned} \|\rho\|_{h_j^{-1}} &\leq \mathfrak{g}_j[\rho], \\ \|\rho\|_{h^{-1}} &\leq \mathfrak{g}_j[\rho] + 2^{-j} \|(1-P_j)\rho\|_{L^2}, \\ \mathfrak{g}_j[\rho] &\leq 2^{(3/2)j} \|\rho\|_{h_j^{-1}}. \end{aligned}$$

Furthermore, both seminorms depend on ρ only via its projection; that is, $\|\rho\|_{h_j^{-1}} = \|P_j\rho\|_{h_j^{-1}}$ and $\mathfrak{g}_j[\rho] = \mathfrak{g}_j[P_j\rho]$.

We remark that the loss of the factor $2^{(3/2)j}$ here can be interpreted as corresponding to the embedding $H^{-1} \subset H^{1/2} \subset L^\infty$. Indeed, if $\rho \in E_j$, then ρ is constant on each interval $I \in \mathcal{D}_j$ and thus $\mathfrak{g}_j[\rho] = \|\rho\|_{L^\infty}$.

Corollary 3.7. *Let $\rho \in L^2((0, 1))$. Then*:

(1) *If $\|\rho\|_{h_{j_0}^{-1}} \leq 2^{-j_0}$, then $\mathfrak{g}_j[\rho] \leq C 2^{(3/2)j - j_0}$. In particular, if $j \leq \frac{2}{3} j_0$, then ρ is geometrically mixed at scale 2^{-j}.*

(2) *Furthermore, for $j \leq \frac{2}{5} j_0$, we obtain that $\mathfrak{g}_j[\rho] \leq 2^{-j}$.*

(3) *Conversely, if $\mathfrak{g}_j[\rho] \leq 2^{-j}$, then*

$$\|\rho\|_{h_j^{-1}} \leq \mathfrak{g}_j[\rho] \leq 2^{-j}.$$

If we additionally assume that $\|(1-P_j)\rho\|_{L^2} \leq 1$, then furthermore

$$\|\rho\|_{h^{-1}} \leq \mathfrak{g}_j[\rho] \leq 2^{-j} + 2^{-j} \|(1-P_j)\rho\|_{L^2} \leq C 2^{-j}.$$

Remark 3.8. Setting $2^{-j_0} = \epsilon$, the above results show that $\|\rho\|_{h^{-1}} \leq \epsilon$ implies $\mathfrak{g}_{\log(\epsilon^{2/3})}[\rho] \leq C$ and $\mathfrak{g}_{\log(\epsilon')}[\rho] \leq \epsilon'$ for $\epsilon' \geq \epsilon^{2/5}$.

Conversely, if $\mathfrak{g}_{\log(\epsilon)}[\rho] \leq \epsilon$ and we control $\|\rho\|_{L^2}$, then also $\|\rho\|_{h^{-1}} \leq \epsilon$.

The analytic and geometric mixing scales are hence comparable with a loss in the exponent in one direction (thus we do not use the word equivalent).

Furthermore, we show in Lemma 3.9 that this loss is optimal.

As mentioned in the introductory Section 2, in this article we hence stress the viewpoint that the examples constructed in [Lunasin et al. 2012] should instead be interpreted as showing the necessity of the control of $\mathfrak{g}_{\log(\epsilon)}[\rho] \leq \epsilon$ instead of $\mathfrak{g}_{\log(\epsilon)}[\rho] \leq \kappa$ and of the loss $\epsilon \to \epsilon^{2/3}$.

We discuss this interpretation, scaling and the constructions further in Section 4.

Proof of Theorem 3.6. Let $\rho \in L^2$ be given and consider the basis expansions of $P_j \rho \in E_j$:

$$P_j \rho = \sum_I d_I \chi_I = \sum_l c_l \phi_{p_l}.$$

Since both χ_I and ϕ_{p_l} are orthonormal bases of E_j, it follows that

$$\|d_I\|_{l^2} = \|c_l\|_{l^2}.$$

Then we may estimate

$$\|\rho\|_{h_j^{-1}} = \|\langle l \rangle^{-1} c_l\|_{l^2} \leq \|c_l\|_{l^2} = \|d_I\|_{l^2} \leq \sqrt{2^j} \max |d_I|.$$

On the other hand, the normalization of the geometric mixing functionals is such that

$$\mathfrak{g}_j[\rho] = \max \frac{1}{|I|} \langle \chi_I, \rho \rangle = \sqrt{2^j} \max |d_I|.$$

For the converse estimate, we note that

$$\begin{aligned} \mathfrak{g}_j[\rho] &= \sqrt{2^j} \max |d_I| \leq \sqrt{2^j} \|d_I\|_{l^2} \\ &= \sqrt{2^j} \|c_l\|_{l^2} \leq \sqrt{2^j} 2^j \|\langle l \rangle^{-1} c_l\|_{l^2} \leq 2^{(3/2)j} \|\rho\|_{h_j^{-1}}. \end{aligned}$$

If $\rho \notin E_j$, we note that by definition

$$\|\rho\|_{h_j^{-1}}^2 = \sum_{l=0}^{2^j-1} \frac{1}{1+l^2} |\langle \rho, \phi_{p_l} \rangle|^2 \leq \sum_{l=0}^{\infty} \frac{1}{1+l^2} |\langle \rho, \phi_{p_l} \rangle|^2 = \|\rho\|_{h^{-1}}^2.$$

For the estimate of $\|\rho\|_{h^{-1}}$, we thus split ρ using P_j, $1 - P_j$ and obtain

$$\|\rho\|_{h^{-1}}^2 = \|\rho\|_{h_j^{-1}}^2 + \sum_{l \geq 2^j} \frac{1}{1+l^2} |\langle \rho, \phi_{p_l} \rangle|^2 \leq \mathfrak{g}_j[\rho]^2 + 2^{-2j} \|\rho\|_{L^2}^2. \qquad \square$$

Proof of Corollary 3.7. We apply Theorem 3.6 to obtain

$$\mathfrak{g}_j[\rho] \leq C 2^{(3/2)j - j_0}.$$

The first statements hence follow by noting that

$$\tfrac{3}{2} j - j_0 \leq 0 \quad \iff \quad j \leq \tfrac{2}{3} j_0,$$

and the second from

$$\tfrac{3}{2} j - j_0 \leq -j \quad \iff \quad j \leq \tfrac{2}{5} j_0.$$

The last statement similarly follows as a direct corollary of Theorem 3.6. $\square$

The following lemma shows that these restrictions on j are optimal.

Lemma 3.9. *There exists a family of functions $\rho = \rho(j_0)$ such that $\|\rho\|_{h^{-1}_{j_0}} \leq C2^{-j_0}$ and which satisfies the following properties*:

(1) *For any $\alpha < \frac{2}{3}$ and $j = \lfloor \alpha j_0 \rfloor$, it holds that $\mathfrak{g}_j[\rho] = o(1)$ as $j_0 \to \infty$. If instead $\alpha > \frac{2}{3}$, then $\mathfrak{g}_j[\rho] \to \infty$.*

(2) *For any $\alpha < \frac{2}{5}$ and $j = \lfloor \alpha j_0 \rfloor$, it holds that $2^j \mathfrak{g}_j[\rho] = o(1)$ as $j_0 \to \infty$. If instead $\alpha > \frac{2}{5}$, then $2^j \mathfrak{g}_j[\rho] \to \infty$.*

Proof. As shown in the preceding Theorem 3.6 and Corollary 3.7, we have

$$\mathfrak{g}_j[\rho] = \sqrt{2^j}\|d_I\|_{l^\infty} \leq \sqrt{2^j}\|d_I\|_{l^2} \leq 2^{(3/2)j}\|\rho\|_{H^{-1}} \leq 2^{(3/2)j - j_0}.$$

Hence, we note that $\frac{3}{2}j - j_0 \leq 0 \iff j \leq \frac{2}{3}j_0$ and that $\frac{3}{2}j - j_0 \leq j \iff j \leq \frac{2}{5}$.

Thus it only remains to show that these estimates are indeed sharp in the thresholds in j. For this purpose, consider

$$\rho = \sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{j_0} - 1} \phi_{p_l},$$

with $\alpha \in (0, 1)$ to be chosen later (see also Remark 3.10). Then we can compute

$$\|\rho\|_{h^{-1}_{j_0}} = \left(\sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{j_0} - 1} l^{-2} \right)^{1/2} \approx 2^{-j_0 \alpha/2}.$$

On the other hand, averaging over the interval $[0, 2^{-j})$, wave packets ϕ_{p_l} with $l > 2^{-j}$ are orthogonal, while wave packets with $l \leq 2^j$ are constantly equal to 1 when restricted to this interval. Hence, we obtain that for any $j \leq j_0$

$$\mathfrak{g}_j[\rho] = \sum_{2^{j_0 \alpha} - 1 \leq l \leq 2^{j-1}} 1 \approx 2^j$$

if $j > j_0 \alpha$. If we now multiply ρ by $2^{-j_0(1-\alpha/2)}$, we are exactly in the setting described. That is,

$$\begin{aligned}\|2^{-j_0(1-\alpha/2)}\rho\|_{H^{-1}} &\approx 2^{-j_0(1-\alpha/2)}2^{-j_0\alpha/2} = 2^{-j_0},\\ \mathfrak{g}_j[2^{-j_0(1-\alpha/2)}\rho] &\approx 2^{j - j_0(1-\alpha/2)}\end{aligned}$$

for any j with $\alpha j_0 \leq j \leq j_0$. Since the exponent is monotone in j, we only need to consider the case when $\alpha j_0 = j$ and thus the behavior of $(\alpha - (1 - \alpha/2))$. This exponent is less than or equal to zero if and only if $\alpha \leq \frac{2}{3}$ and less than or equal to $-\alpha$ if and only if $\alpha \leq \frac{2}{5}$. The thresholds in j, respectively α, are thus indeed optimal. □

Remark 3.10. We remark that $\langle \chi_{[0,2^{-j})}, \phi_{p_l} \rangle = 2^{-j}$ for all $j = 0, \dots, 2^{j-1}$. Hence, $\sum_{0 \leq l \leq 2^j - 1} \phi_{p_l} = 2^j 1_{[0,2^{-j})}$. Thus, if αj_0 is an integer, we obtain

$$\sum_{2^{\alpha j_0} \leq l \leq 2^{j_0} - 1} \phi_{p_l} = 2^{j_0} 1_{[0,2^{-j_0})} - 2^{\alpha j_0} 1_{[0,2^{-\alpha j_0})},$$

which provides a more immediate view of the geometric mixing size. However, this explicit characterization is much less simple if $2^{\alpha j_0}$ is not a power of 2 and also not transparent in terms of the H^{-1} norm.

4. The continuous setting

In the following we show that, with minor modifications, the estimates of the dyadic setting of Section 3 persist in the continuous setting. Here, additional key challenges are given by the lack of orthogonality and thus nonexistence of spaces like E_j.

4A. *Definition of mixing scales.*

Definition 4.1. If $\rho \in \dot{H}^{-1}(\mathbb{R}^n)$, we call $\|\rho\|_{\dot{H}^{-1}}$ the *analytic mixing scale*.

Let $\phi \in L^1(\mathbb{R}^n)$ with $\phi \geq 0$ and $\|\phi\|_{L^1} = 1$ and define $\phi_r(x) := \phi(rx)/r^n$. Then for any $\rho \in L^1_{\text{loc}}(\mathbb{R}^n)$ and every $\epsilon_0 > 0$, we introduce the (nonlinear) functionals

$$\mathfrak{g}_{\epsilon_0}[\rho] := \|\phi_r * \rho\|_{L^\infty}.$$

Here, the most common choice is given by $\phi = (1/|B_1|)1_{B_1}$, in which case

$$\mathfrak{g}_{\epsilon_0}[\rho] = \sup_{r>\epsilon_0, x\in\mathbb{R}^n} |B_r(x)|^{-1} \left| \int_{B_r(x)} \rho(y)\, dy \right|.$$

We say that a function $\rho \in L^\infty(\Omega) \cap L^1_{\text{loc}}(\Omega)$ is *geometrically mixed* by a factor $\kappa \in (0, 1)$ up to scale $\epsilon_0 > 0$ if

$$\mathfrak{g}_{\epsilon_0}[\rho] \leq \kappa \|\rho\|_{L^\infty}.$$

For a given κ, we define

$$\mathcal{G}_\kappa[\rho] = \inf\{\epsilon_0 > 0 : \mathfrak{g}_{\epsilon_0}[\rho] \leq \kappa \|\rho\|_{L^\infty}\},$$

where the infimum is over all such ϵ_0, and call it the *geometric mixing scale*.

We remark that, by Hölder's inequality,

$$\mathfrak{g}_{\epsilon_0}[\rho] \leq \|\rho\|_{L^\infty} \|\phi_r\|_{L^1} = \|\rho\|_{L^\infty}$$

and that by Lebesgue integration theory

$$\lim_{\epsilon_0 \downarrow 0} \mathfrak{g}_{\epsilon_0}[\rho] = \|\rho\|_{L^\infty}.$$

The functionals $\mathfrak{g}$ and the geometric mixing scale $\mathcal{G}$ thus describe the competition between cancellations in Hölder's inequality and convergence of Dirac sequences.

Remark 4.2. The reason for our more general formulation in terms of $\phi \in L^1$ is that in later estimates optimality is easier to phrase and establish if we additionally require that $\phi \in H^1$. In particular, by duality

$$\sup_{\rho \in H^{-s} : \|\rho\|_{H^{-s}} \leq 1} \mathfrak{g}_1[\rho] = \|\phi\|_{H^s}$$

and hence an estimate like (6) is not possible unless ϕ is sufficiently regular. However, for most estimates this only poses technical challenges (see Lemma 4.10) in terms of the control of certain Fourier projections. In the dyadic setting of Section 3 these complications could be avoided by using orthogonality properties.

4B. ***Comparison estimates.*** Our main results are given by the following theorems and corollaries.

Theorem 4.3 (estimates). *Let $\rho \in L^2 \cap \dot{H}^{-1}(\mathbb{R}^n)$ and suppose that $\phi \in H^\lambda \cap L^1$ for some $\lambda \in [0, 1]$. Then for any $\epsilon_0 > 0$ it holds that*

$$\mathfrak{g}_{\epsilon_0}[\rho] \leq C \frac{\|\phi\|_{H^\lambda}}{\|\phi\|_{L^1}} \epsilon_0^{-n/2-\lambda} \|\rho\|_{H^{-\lambda}}.$$

As a consequence, the geometric mixing scale of ρ can be estimated by

$$\mathcal{G}_\kappa(\rho) \leq \left(\frac{C_{\lambda,\phi} \|\rho\|_{H^{-\lambda}}}{\kappa \|\rho\|_{L^\infty}} \right)^{1/(n/2+\lambda)}.$$

In particular, if $\lambda = 1$, then

$$\begin{aligned} \mathfrak{g}_{\epsilon_0}[\rho] &\leq C \epsilon_0^{-n/2-1} \|\rho\|_{H^{-1}}, \\ \mathcal{G}_\kappa(\rho) &\leq C \left(\frac{\|\rho\|_{H^{-1}}}{\kappa \|\rho\|_{L^\infty}} \right)^{1/(n/2+1)}. \end{aligned} \tag{6}$$

Conversely, if ρ is compactly supported and C denotes the measure of a 1-neighborhood of the support, then for every $\epsilon_0 \leq 1$ it holds that

$$\|\rho\|_{H^{-1}} \leq C \mathfrak{g}_{\epsilon_0}[\rho] + C \epsilon_0 \|\rho\|_{L^2}. \tag{7}$$

We note that in the estimate (7), assuming $\mathcal{G}_\kappa[\rho] \leq \epsilon_0$ only yields a bound of $\|\rho\|_{H^{-1}} \leq \kappa$. Indeed, as explored in Section 2B for fixed κ it is possible to find a sequence ρ_n such that

$$\mathcal{G}_\kappa[\rho_n] \to 0, \quad \|\rho_n\|_{H^{-1}} \geq \kappa,$$

where the failure of decay of $\|\rho_n\|_{H^{-1}}$ was due to the persistence of structures at scale κ (see also Theorem 1.2 and the remarks thereafter).

As discussed in Remark 4.2, if $\phi = c 1_{B_1}$, we cannot choose $\lambda = 1$ since $\phi \notin H^1$. However, we may recover this estimate upon imposing further conditions on ρ (see Lemma 4.10).

As an application of the above estimates we derive comparability of both mixing scales. That is, while not equivalent (semi-)norms, smallness of one scale implies smallness of the other with a necessary loss in the exponents. For easier reference, we restate Theorem 1.2.

Theorem 4.4 (comparison of mixing scales). *Let $\rho \in L^2(\mathbb{R}^n)$ and $\|\rho\|_{L^2} \leq 1$ and let $\phi \in H^1$. Then for all $0 < \epsilon \leq 1$ it holds that:*

(1) *If $\mathfrak{g}_\epsilon[\rho] \leq \epsilon$ and ρ is supported in B_1, then also $\|\rho\|_{H^{-1}} \leq C\epsilon$.*

(2) *If $\|\rho\|_{H^{-1}} \leq \epsilon$, then also $\mathfrak{g}_{\tilde{\epsilon}}[\rho] \leq C$ for all $\tilde{\epsilon} \geq \epsilon^\alpha$ and $\mathfrak{g}_{\epsilon'}[\rho] \leq C\epsilon'$ for all $\epsilon' \geq \epsilon^\beta$, where $\alpha = 2/(n+2)$ and $\beta = 2/(n+4)$ depend only on the dimension. In particular, supposing additionally that $\|\rho\|_{L^\infty} = 1$, it follows that*

$$\begin{aligned} \mathcal{G}_C[\rho] &\leq \tilde{\epsilon}, \\ \mathcal{G}_{C\epsilon'}[\rho] &\leq \epsilon'. \end{aligned}$$

These estimates are optimal in the powers of ϵ.

In Section 3 we have seen that the loss of exponents is caused by the (L^1, L^∞) normalization in the geometric scale instead of L^2 normalization for the analytic scale and can also be seen as being due to the Sobolev embedding into L^∞. In this continuous setting, this is much less transparent due to the lack of spaces E_j. The necessity of the loss is established in Lemma 4.7 in analogy with Lemma 3.9 of the dyadic case.

Corollary 4.5. *Let $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\|\rho(t)\|_{H^{-1}} \geq Ce^{-Ct}.$$

Then, it follows that

$$\mathfrak{g}_{e^{-Ct}}[\rho(t)] \geq Ce^{-Ct}.$$

Remark 4.6. For example $u(t)$ may be given by the solution of a passive or active scalar problem, as in [Crippa et al. 2019]. As noted in Section 2, if one were instead to consider $\mathfrak{g}_\kappa[\rho(t)]$ for fixed κ, this functional is not lower semicontinuous and there is no reason to expect any lower bound.

Proof of Theorem 4.3. As a first consistency check, we verify that the estimates of Theorem 4.3 scale correctly.

Let thus $\delta > 0$ and consider $\rho_\delta(x) = \rho(\delta x)$. For simplicity of notation, we consider $\lambda = \frac{1}{2}$. Then it holds that

$$\begin{aligned}\mathfrak{g}_{\epsilon_0}[\rho_\delta] = \mathfrak{g}_{\delta\epsilon_0}[\rho] &\leq C(\delta\epsilon_0)^{-n/2-1/4}\|\rho\|_{L^2}^{1/2}\|\rho\|_{H^{-1}}^{1/2} \\ &= C\delta^{-n/2-1/4}\epsilon_0^{-n/2-1/4}\|\rho\|_{L^2}^{1/2}\|\rho\|_{H^{-1}}^{1/2}.\end{aligned}$$

On the other hand, estimating directly, we obtain

$$\begin{aligned}\mathfrak{g}_{\epsilon_0}[\rho_\delta] &\leq C\epsilon_0^{-n/2-1/4}\|\rho_\delta\|_{L^2}^{1/2}\|\rho_\delta\|_{H^{-1}}^{1/2} \\ &= C\epsilon_0^{-n/2-1/4}(\delta^{-n/2}\|\rho\|_{L^2})^{1/2}(\delta^{-n/2-1/2}\|\rho\|_{L^2})^{1/2} \\ &= C\delta^{-n/2-1/4}\epsilon_0^{-n/2-1/4}\|\rho\|_{L^2}^{1/2}\|\rho\|_{H^{-1}}^{1/2}.\end{aligned}$$

Note that this estimate is further invariant under replacing $\rho(x)$ by $\mu\rho(x)$ for any $\mu > 0$. Hence, we may in addition choose $\mu = \delta^{n/2}$ to ensure L^2 normalization.

Let us now consider the proof of the theorem and let $\lambda \in [0, 1]$ and $\phi \in H^\lambda$ be given. Then we may use duality to estimate

$$\mathfrak{g}_r[\rho] \leq \frac{\|\phi_r\|_{H^\lambda}}{\|\phi_r\|_{L^1}}\|\rho\|_{H^{-\lambda}}$$

and use interpolation to control

$$\|\rho\|_{H^{-\lambda}} \leq C_\lambda\|\rho\|_{L^2}^{1-\lambda}\|\rho\|_{H^{-1}}^{\lambda}.$$

We further note that by scaling

$$\frac{\|\phi_r\|_{H^\lambda}}{\|\phi_r\|_{L^1}} \leq Cr^{-n/2-\lambda}\frac{\|\phi\|_{H^\lambda}}{\|\phi\|_{L^1}}$$

for $r \leq 1$ (for the homogeneous Sobolev norms we would have equality).

Combining both estimates, we obtain

$$\mathfrak{g}_r[\rho] \leq C \frac{\|\phi\|_{H^\lambda}}{\|\phi\|_{L^1}} r^{-n/2-\lambda} \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^{\lambda}.$$

For the estimate on the geometric mixing scale, we have to show that for given κ and all $\epsilon_0 \geq \mathcal{G}[\rho]$

$$\mathfrak{g}_{\epsilon_0}[\rho] \leq \kappa \|\rho\|_{L^\infty}.$$

In view of the previous calculation this is implied by showing that

$$C_{\lambda,\phi} \epsilon_0^{-n/2-\lambda/2} \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^{\lambda} \leq \kappa \|\rho\|_{L^\infty},$$

with $C_{\lambda,\phi} = C(\|\phi\|_{H^\lambda}/\|\phi\|_{L^1})$. Dividing by $\kappa\|\rho\|_{L^\infty} > 0$ and taking a power $1/(n/2+\lambda/2)$, we obtain that this holds if

$$\left(\frac{C_\lambda \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^{\lambda}}{\kappa \|\rho\|_{L^\infty}} \right)^{1/(n/2+\lambda/2)} \leq \epsilon_0.$$

We thus obtain an upper bound on the geometric mixing scale by the left-hand-side, which concludes the proof. □

Proof of Theorem 1.2. We proceed as in the proof of Corollary 3.7 and consider $\epsilon_0 = \epsilon^t$ for $t \in (0, 1)$ to be determined. Then the estimate (6) of Theorem 4.3 implies

$$\mathfrak{g}_{\epsilon^t} \leq C \epsilon^{t(-n/2-1)+1},$$

which yields the critical cases

$$t\Big(-\frac{n}{2} - 1\Big) + 1 = 0 \iff t = \frac{2}{n+2},$$
$$t\Big(-\frac{n}{2} - 1\Big) + 1 = t \iff t = \frac{2}{n+4}.$$ □

The following lemmata consider questions of optimality and the removal of small scales discussed in Section 2B.

Lemma 4.7 (counterexample in the continuous setting). *There exists a sequence $\epsilon \downarrow 0$ and $\rho = \rho(\epsilon) \in L^2(\mathbb{R})$ with*

$$\|\rho\|_{H^{-1}} \leq \epsilon,$$

but such that for every $\alpha < \frac{2}{3}$, it holds that

$$\mathfrak{g}_{\epsilon^\alpha}[\rho] \to \infty.$$

as $\epsilon \downarrow 0$ and such that for all $\beta < \frac{2}{5}$

$$\epsilon^{-\beta} \mathfrak{g}_{\epsilon^\beta}[\rho] \to \infty.$$

That is, the exponents in Theorem 1.2 are optimal.

Proof of Lemma 4.7. We follow a similar strategy as in the proof of Lemma 3.9 in the dyadic setting. Let $\epsilon = 2^{-j_0}$ and consider

$$\rho = 2^{j_0} 1_{[0,2^{-j_0}]} - 2^{j_1} 1_{[0,2^{-j_1}]},$$

with $j_1 = \alpha j_0$, $\alpha \in (0,1)$. Then for any $j_1 \le j \le j_0$, we obtain

$$2^j \int_0^{2^{-j}} \rho\, dx = 2^j (2^{j_0} \min(2^{-j}, 2^{-j_0}) + 2^{j_1} \min(2^{-j}, 2^{-j_1})) = 2^j - 2^{j_1},$$

which is comparable to 2^j as long as $j > j_1$.

We further make the following *claim*:

$$\|\rho\|_{H^{-1}} \approx 2^{-j_1/2}. \tag{8}$$

Suppose that this claim holds and consider

$$\rho' := 2^{-j_0(1-\alpha/2)} \rho,$$

which satisfies

$$\begin{aligned} &\|\rho'\|_{H^{-1}} \approx 2^{-j_0}, \\ &\mathfrak{g}_{2^{-j}}[\rho'] \approx 2^{j - j_0(1-\alpha/2)} = 2^{-j_0(1-(3/2)\alpha)}. \end{aligned}$$

We hence conclude as in the proof of Lemma 3.9.

Thus it remains to show the claim. We directly compute

$$\hat{\rho}(\xi) = \int_{\mathbb{R}} e^{i\xi x} 2^{j_0} 1_{[0,2^{-j_0}]} - 2^{j^1} 1_{[0,2^{-j_1}]}\, dx = \frac{e^{i\xi 2^{-j_0}} - 1}{i\xi 2^{-j_0}} - \frac{e^{i\xi 2^{-j_1}} - 1}{i\xi 2^{-j_1}}.$$

Both difference quotients are uniformly bounded by 1 and we distinguish the regions based on the size of $\xi 2^{-j_0}$ and $\xi 2^{-j_1}$.

If $|\xi| > c2^{j_0}$, we may roughly estimate

$$\int_{\{\xi : |\xi| > c2^{j_0}\}} \frac{|\hat{\rho}(\xi)|^2}{|\xi|^2} \le \int_{c2^{j_0}}^{\infty} \frac{4}{|\xi|^2}\, d\xi \le C2^{-j_0},$$

which is a very small contribution.

If $|\xi| < c2^{j_1}$ with c small, we may use a Taylor expansion to estimate the error of the difference quotient:

$$\frac{e^{i\xi 2^{-j_0}} - 1}{i\xi 2^{-j_0}} - \frac{e^{i\xi 2^{-j_1}} - 1}{i\xi 2^{-j_1}} = 1 + \mathcal{O}(\xi 2^{-j_0}) - 1 + \mathcal{O}(\xi 2^{-j_1}) = \mathcal{O}(\xi 2^{-j_1}).$$

The H^{-1} energy for this segment can hence be estimated by

$$\int_{\{\xi : |\xi| < c2^{j_1}\}} \frac{|\hat{\rho}(\xi)|^2}{|\xi|^2} \le C \int_{\{\xi : |\xi| < c2^{j_1}\}} 2^{-2j_1} \le C2^{-j_1}.$$

Finally, if $cj_1 \le j \le cj_0$, one difference quotient is about 1, while the other oscillates, but is bounded by 1. Thus the contribution can be estimated as

$$\int_{\{\xi : c2^{j_1} \le |\xi| < c2^{j_0}\}} \frac{|\hat{\rho}(\xi)|^2}{|\xi|^2} \approx \int_{c2^{j_1}}^{c2^{j_0}} \frac{1}{|\xi|^2}\, d\xi \approx 2^{-j_1}. \qquad \square$$

Lemma 4.8. *Let $\rho \in L^2(\mathbb{R}^n)$ and $r > 0$ and define $\rho_r := (1/|B_r|)1_{B_r} * \rho$. Then $\rho - \rho_r \in H^{-1}$ and there exists $C > 0$ depending only on the dimension n such that*

$$\|\rho - \rho_r\|_{H^{-1}} \leq Cr\|\rho\|_{L^2}.$$

Proof of Lemma 4.8. We consider the Fourier transform of ϕ_r. Let thus $r > 0$ and $\xi \in \mathbb{R}^n$ be given and consider

$$\frac{c_n}{r^n}\int_{B_r} e^{ix\cdot\xi}\,dx = \frac{1}{r^n}\int_{B_r} e^{ix_1|\xi|} = \frac{c_n}{|r\xi|^n}\int_{B_{r|\xi|}} e^{ix_1}\,dx =: \psi(|r\xi|).$$

We remark that ψ can be explicitly computed in terms of Bessel functions (see the proof of Theorem 5.2). Since $\psi(\,\cdot\,)$ is an average of e^{ix_1} it follows that $|\psi| \leq 1$. Furthermore, by continuity of e^{ix_1}

$$|\psi(|r\xi|) - 1| \leq \frac{c_n}{|r\xi|^n}\int_{B_{r|\xi|}} |e^{ix_1} - 1|\,dx \leq Cr|\xi|$$

as $r|\xi| \downarrow 0$.

Hence, we can control

$$|\mathcal{F}(\rho_r - \rho)|^2 = |(\psi(r|\xi|) - 1)\hat{\rho}(\xi)|^2 \leq \min(2, Cr|\xi|)|\hat{\rho}|^2$$

and can estimate the H^{-1} energy of $\rho - \phi_r * \rho$ by

$$\int \frac{\min(C^2|\xi r|^2, 4)}{|\xi|^2}|\hat{\rho}(\xi)|^2 \leq C^2r^2\int |\hat{\rho}(\xi)|^2 = C^2r^2\|\rho\|_{L^2}^2. \qquad \square$$

Lemma 4.9. *Let $\rho \in L^2(\mathbb{R}^n)$ with $\|\rho\|_{L^2} \leq 1$ be supported in $B_1(0)$ such that*

$$\left\|\frac{1}{c_n\epsilon^n}1_{B_\epsilon} * \rho\right\|_{L^\infty} \leq \epsilon$$

for some $0 < \epsilon < 1$. Then there exists C depending only on the dimension n such that the analytic mixing scale satisfies

$$\|\rho\|_{H^{-1}} \leq C\epsilon.$$

Proof of Lemma 4.9. By the triangle inequality

$$\|\rho\|_{H^{-1}} \leq \|\rho_\epsilon\|_{H^{-1}} + \|\rho - \rho_\epsilon\|_{H^{-1}}.$$

The second term can be estimated by $C\epsilon\|\rho\|_{L^2} \leq C\epsilon$ using Lemma 4.8, while for the first term we control

$$\|\rho_\epsilon\|_{H^{-1}} \leq \|\rho_\epsilon\|_{L^2} \leq |\operatorname{supp}(\rho_\epsilon)|\|\rho_\epsilon\|_{L^\infty} \leq 2^n\epsilon,$$

where we estimated the support of ρ_ϵ by $B_{1+\epsilon} \subset B_2$. $\square$

We remark that since the definition of $\mathfrak{g}$ is given in terms of local Lebesgue spaces, some support or decay condition is necessary.

Indeed, consider $\rho \in L^2(\mathbb{R})$, which is compactly supported in $(0, 1)$. Then for any $N \in \mathbb{N}$ we can define $\sigma(x) := \sum_{j=0}^{N} \rho(x+2j)$. Due to the disjoint supports for all $\epsilon < \frac{1}{2}$, it holds that

$$\begin{aligned} \|\sigma_\epsilon\|_{L^\infty} &= \|\rho_\epsilon\|_{L^\infty}, \\ \|\sigma\|_{L^2} &= \sqrt{N}\|\rho\|_{L^2}. \end{aligned}$$

Furthermore, while H^{-1} is nonlocal, we obtain that $\|\sigma\|_{H^{-1}} \approx \sqrt{N}\|\rho\|_{H^{-1}}$ for N large.

The following lemma establishes the converse control of the geometric scale by the analytic scale. As noted in Remark 4.2, here regularity of ϕ allows for easier proofs. However, under additional assumptions, ϕ can also be chosen less regular, such as 1_{B_1}.

Lemma 4.10. *Let $\rho \in H^{-1}(\mathbb{R}^n)$ with $\operatorname{supp}(\hat{\rho}) \subset B_{r^{-1}}(0)$ and let $\phi \in L^1$. Then there exists a constant C depending on ρ such that*

$$\mathfrak{g}_{\epsilon_0}[\rho] \leq C r^{-n/2-1}\|\rho\|_{H^{-1}}.$$

If we require that $\phi \in H^1$, then the support assumption can be omitted.

Proof of Lemma 4.10. Using Plancherel's theorem we compute

$$\phi_r * \rho(x) = \int \phi_r(x-y)\rho(y)\,dy = \int e^{ix\xi}\bar{\hat{\phi}}_r(\xi)\hat{\rho}(\xi)\,d\xi.$$

Now recall that $\phi_r(x) = \phi(rx)/r^n$ has constant L^1 norm and thus $\|\hat{\phi}_r\|_{L^\infty} \leq \|\phi_r\|_{L^1}$. We may hence control further by

$$\|\hat{\phi}_r\|_{L^\infty}\|\xi\|_{L^2(\operatorname{supp}(\hat{\rho}))}\left\|\frac{\hat{\rho}}{\xi}\right\|_{L^2} \leq r^{-n/2-1}\|\rho\|_{H^{-1}},$$

where we used the support of $\hat{\rho}$ and that $\|\hat{\phi}_r\|_{L^\infty} \leq \|\phi_r\|_{L^1} = 1$.

If $\phi \in H^1$, we can instead directly estimate

$$\begin{aligned} \|\phi_r * \rho\|_{L^\infty} &\leq \|\phi_r\|_{H^1}\|\rho\|_{H^{-1}} \\ &= r^{-n/2-1}\|\phi\|_{H^1}\|\rho\|_{H^{-1}}. \end{aligned}$$ ☐

Hence, the failure of estimates with $s \geq \frac{1}{2}$ is due to the interaction of the "tail" of $\hat{\rho}$ for $|\xi| \geq r^{-1}$. We remark that in the dyadic setting of Section 3 this complication does not arise, since our seminorms project on spaces E_j of lower frequency.

5. Damping rates in transport-type equations

In this second part of our article, we are interested in the time dependence of mixing scales when $\rho(t)$ evolves under the passive scalar equation

$$\partial_t \rho + v \cdot \nabla \rho = 0$$

for a given divergence-free velocity field v. In particular, we are interested optimal decay rates of $\|\rho(t)\|_{H^{-1}}$ and $\mathcal{G}_{\kappa(t)}[\rho(t)]$.

In Section 5A, we consider the special case when $\rho(t)$ is advected by a given specific, regular, incompressible velocity field. This study is motivated by recent work of Crippa, Lucà and Schulze [Crippa et al. 2019], who studied the time behavior of both mixing scales under the evolution

$$\rho(t, r, \theta) = \rho_0(r, \theta - tr),$$

where r, θ are polar coordinates on $\mathbb{R}^2 \setminus \{0\}$, $\rho_0 \in L^1 \cap L^\infty$ and the angular averages $\langle \rho_0 \rangle_\theta = 0$ identically vanish. Adapting conformal polar coordinates $\theta, e^s = r$, this setting shares strong similarities with problems of inviscid damping in fluid dynamics.

Further examples of interest here are given by:

- Perturbations around shear flow solutions of Euler's equations on $\mathbb{T} \times \mathbb{R}$. In [Zillinger 2016; 2017a] we showed that if $U(y)$ is, roughly speaking, close to affine, the linearized Euler equations in vorticity formulation asymptotically scatter in H^s to the transport problem with $v = (U(y), 0)$. Using different, spectral methods [Wei et al. 2018] have further shown similar results under weaker conditions.
- When considering circular flows [Zillinger 2017b; Coti Zelati and Zillinger 2019] and when $v = u(r)e_\theta$ is an annular region or on $\mathbb{R}^2$, we similarly obtain stability, damping and scattering in weighted spaces and for more degenerate profiles.
- In the setting of Landau damping [Bedrossian et al. 2016] similarly one observes scattering to a transport problem.

The following results on the free transport equation

$$\partial_t \rho(t, x, y) - y \partial_x \rho = 0, \quad (t, x, y) \in (0, \infty) \times \mathbb{T}^n \times \mathbb{R}^n,$$

hence also extend by scattering to asymptotics for further equations exhibiting *phase-mixing*.

Finally, we discuss optimal mixing and stirring for more general passive scalar problems. Here, a recent active area of research, [Alberti et al. 2014; Seis 2017; Crippa and Schulze 2017; Bressan 2003; Crippa and De Lellis 2008] is given by the study of upper and lower bounds on decay rates of mixing scales for solutions of (1)

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$

where v may be chosen arbitrarily under given constraints such as $\|v(t)\|_{W^{1,p}} \leq 1$. Using our comparison estimates of Theorem 1.2, we discuss implications of some known results.

5A. ***On sharp decay rates for the free transport equation.*** Our main results for the analytic mixing scale are summarized in Theorem 1.3, which we restate in the following for easier reference. Using the estimates of Theorem 4.3 we also obtain control of the geometric scale, which we study in Theorem 5.2.

Theorem 5.1. *Let $H^\sigma H^s = H^\sigma(\mathbb{T}^n; H^s(\mathbb{R}^n))$ denote the Hilbert space with norm*

$$\|\rho\|^2_{H^\sigma_x H^s_y} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2\sigma} \int_{\mathbb{R}^n} \langle \eta \rangle^{2s} |\tilde{\rho}(k, \eta)|^2 \, d\eta.$$

In the following, let $0 < s \leq 1,\ \rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ *with* $\int_{\mathbb{T}^n} \rho_0(x, y)\, dx = 0$, *and let*

$$\rho(t, x, y) = \rho_0(t, x - ty, y)$$

be the solution of the free transport problem. Then:

(1) *There exists* $C_s > 1$ *such that for all* $t \geq 1$ *and all initial data*

$$\|\rho(t)\|_{H^{-1}} \leq Ct^{-s}\|\rho_0\|_{H^{-s}(\mathbb{T}^n; H^s(\mathbb{R}^n))}.$$

(2) *Let* $\alpha_j > 0$ *with* $\|(\alpha_j)_j\|_{l^2} = 1$. *Then there exist* $c > 0$, *a sequence of times* $t_j \to \infty$ *and initial data* u_0 *such that*

$$\|\rho(t_j)\|_{H^{-1}} \geq c\alpha_j t_j^{-s}\|\rho_0\|_{H^{-s}(\mathbb{T}^n; H^s(\mathbb{R}^n))}.$$

(3) *There exists no nontrivial initial data* $\rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ *such that*

$$\|\rho(t_j)\|_{H^{-1}} \geq ct_j^{-s}\|\rho_0\|_{H^{-s}(\mathbb{T}^n; H^s(\mathbb{R}^n))}$$

along some sequence $t_j \to \infty$.

In the second statement, t_j can always be chosen larger and more rapidly increasing. For instance, we may chose $t_j = \exp(\exp(\cdots \exp(j)))$ and $\alpha_j = 1/j = \ln(\ln(\cdots \ln(t_j)))$ as iterated exponentials and logarithms. Informally stated, the theorem hence shows that algebraic decay rates can be achieved along a subsequence up to an arbitrarily small loss. Conversely, the third statement shows that this loss is necessary and that the lower estimate is sharp in this sense.

Proof of Theorem 1.3. We note that for all t the map $L^2 \ni \rho_0 \mapsto \rho(t) \in L^2$ is unitary and thus the statement holds for $s = 0$. Furthermore, we may use the explicit solution of the free transport problem and Plancherel's identity with respect to x to obtain

$$\begin{aligned}
\|\rho(t)\|_{H^{-1}} &= \sup_{\phi:\|\phi\|_{H^1}\leq 1}\left|\sum_{k\neq 0}\int \bar{\hat{\phi}}(k, y)e^{ikty}\hat{\rho}_0(k, y)\, dy\right| \\
&= \sup_{\phi:\|\phi\|_{H^1}\leq 1}\left|\sum_{k\neq 0}\int e^{ikty}\partial_y\frac{\bar{\hat{\phi}}(k, y)\hat{\rho}_0(k, y)}{ikt}\right| \\
&\leq t^{-1}\|\rho_0\|_{H^{-1}(\mathbb{T}^n; H^1(\mathbb{R}^n))},
\end{aligned}$$

and thus establish the result for $s = 1$. The result for $0 < s < 1$ then is obtained by interpolation.

For the second statement, we make use of resonant times and frequencies. Roughly speaking, if ρ_0 is frequency localized at (k, η) (with respect to x and y), then free transport in physical space is also a transport equation in Fourier space and $\rho(t)$ will be frequency localized near $(k, \eta + kt)$. Hence, if $k \neq 0$, η and k are parallel and $t = -\eta/k$, the frequency localization is near zero and hence any H^σ norm is equivalent to the L^2 norm for such a function.

Let thus $\phi \in C_c^\infty$ be supported in a ball of radius 2 and L^2 normalized and let $(\alpha_j)_j \in l^2$ with $\|(\alpha_j)_j\|_{l^2} = 1$. Suppose further that t_j, to be determined precisely later, satisfies $\min_{j_1\neq j_2}|t_{j_1} - t_{j_2}| > 4$

and $\min |t_j| > 4$. Then we define the function $\rho_0 \in H^s(\mathbb{T}^n \times \mathbb{R}^n)$ by its Fourier transform:

$$\tilde{\rho}_0(k,\eta) = \delta_{k=e_1} \sum_{j\in\mathbb{N}} \alpha_j \langle t_j\rangle^{-s} \phi(\eta - t_j e_1).$$

We remark that the Dirac in k corresponds to assuming periodicity in x. This construction also readily extends to the whole space case, $\mathbb{R}^n \times \mathbb{R}^n$, if $\delta_{k=e_1}\phi(\eta - t_j e_1)$ is replaced by a bump function $\phi(10k - e_1)\phi(\eta - kt_j)$.

By our assumption on $t_{j_1} - t_{j_2}$, the functions $\phi(\cdot - t_j)$ are disjointly supported and thus

$$\|\rho_0\|_{H^s}^2 = \sum_j |\alpha_j|^2 \left\| \frac{\langle \cdot \rangle^{2s}}{\langle t_j\rangle^{2s}} \phi(\cdot - t_j e_1) \right\|_{L^2(B_2(0))}^2 = \sum_j |\alpha_j|^2 \left\| \frac{\langle \cdot - t_j e_1\rangle^{2s}}{\langle t_j\rangle^{2s}} \phi(\cdot) \right\|_{L^2(B_2(0))}^2.$$

Similarly, for any $t \in \mathbb{R}$, it holds that

$$\|\rho(t)\|_{H^s}^2 = \sum_j |\alpha_j|^2 \left\| \frac{\langle (t_j - t)e_1 + \cdot \rangle^{2s}}{\langle t_j\rangle^{2s}} \phi(\cdot) \right\|_{L^2(B_2(0))}^2.$$

By our assumptions on t_j and ϕ, in the first sum $|\eta| \leq 2 \leq |t_j|/2$, and thus $\|u_0\|_{H^s}^2$ is comparable (within a factor $4^{\pm s}$) to $\sum_j |\alpha_j|^2 = 1$. One the other hand, for $t = t_j$, the second sum is bounded from below by

$$|\alpha_j|^2 \left\| \frac{\langle \cdot \rangle^{2s}}{\langle t_j\rangle^{2s}} \phi(\cdot) \right\|_{L^2(B_2(0))}^2 \geq \frac{|\alpha_j|^2}{\langle t_j\rangle^{2s}}.$$

This concludes the proof of the second statement. We note that a similar lower bound can also be obtained for the homogeneous Sobolev spaces by choosing $t = t_j + 4$ instead.

Finally, suppose that there exists ρ_0 attaining the algebraic decay rates. Expressed in terms of ρ_0 this implies that for a sequence $t_j \to \infty$

$$\left\| \frac{t_j^s}{\langle \eta - t_j k\rangle^1 \langle \eta\rangle^s} \langle \eta\rangle^s \langle k\rangle^{-s} \tilde{u}_0 \right\|_{l^2 L^2}^2 \geq C \|\langle \eta\rangle^s \langle k\rangle^{-s} \tilde{u}_0\|_{l^2 L^2}^2. \tag{9}$$

Since the equation decouples with respect to k and for easier notation, in the following we consider k arbitrary but fixed and omit the factors $\langle k\rangle^{-s}$. The result of the theorem then follows by multiplying by $\langle k\rangle^{-s}$ and summing in k.

Now let $t = t_j$ and consider the sets

$$\Omega_{C,t} = \left\{ (k,\eta) : \frac{|t^s|}{\langle \eta - tk\rangle^1 \langle \eta\rangle^s} < \sqrt{\frac{C}{2}} \right\}$$

and $A_{C,t}$, their complements. Then the inequality (9) implies

$$\left\| \frac{t^s}{\langle \eta - t_j k\rangle^1 \langle \eta\rangle^s} \langle \eta\rangle^s \tilde{u}_0 \right\|_{L^2(A_{C,t})}^2 + \frac{C}{2} \|\langle \eta\rangle^s \tilde{u}_0\|_{L^2(\Omega_{C,t})}^2 \geq C \|\langle \eta\rangle^s \tilde{u}_0\|_{L^2(\Omega_{C,t})}^2$$

$$\implies \left\| \frac{t^s}{\langle \eta - k t_j\rangle^1 \langle \eta\rangle^s} \langle \eta\rangle^s \tilde{\rho}_0 \right\|_{L^2(A_{C,t})}^2 \geq \frac{C}{2} \|\langle \eta\rangle^s \tilde{\rho}_0\|_{L^2}^2. \tag{10}$$

On the other hand

$$\frac{t^s}{\langle \eta - kt_j\rangle^1 \langle \eta\rangle^s} \le \max\left(\frac{t^s}{\langle kt_j/2\rangle^1 1}, \frac{t^s}{1\langle kt_j/2\rangle^s}\right) \le 2^s,$$

by considering $|\eta| \le t_j/2$ and $|\eta| > t_j/2$ and using that $0 \le s \le 1$.

Hence, it follows that, for a constant depending on s, but independent of t,

$$\|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2(A_{C,t})} \ge C_s \|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2}.$$

By assumption, this holds for a sequence $t_j \to \infty$. Upon passing to a subsequence, the sets A_{C,t_j} can be ensured to be mutually disjoint. Hence, by orthogonality

$$\|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2} \ge \sum_j \|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2(A_{C,t_j})} \ge C_s \sum_j \|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2} = \infty \|\langle \eta\rangle^s \tilde{\rho}_0\|^2_{L^2}.$$

This is a contradiction unless $\rho_0 = 0$. □

As a consequence of our comparison result, Theorem 1.2, we also obtain lower bounds on the decay of the geometric mixing scale. The following theorem instead provides a direct construction of a lower bound, where averages are taken over the scale j/t_j instead.

Theorem 5.2. *Let $0 \le s < \frac{1}{2}$; then there exists initial data $\rho_0 \in L^2_x H^s_y(\mathbb{T} \times [0, 2\pi])$ so that along a sequence of times $t_j = 2^{100+j}$ the solution u of the free transport problem satisfies*

$$\mathfrak{g}_{j/t_j}[\rho] \ge \frac{\|\rho_0\|_{L^2 H^s}}{j\langle t_j\rangle^s}.$$

That is, at scale $r = j/t_j$ we have a lower bound by $1/(jt_j^s)$.

We remark that as in the previous Theorem 1.3, t_j can be chosen to be increasing more rapidly and thus $1/j = o(1)$ as $t_j \to \infty$ can be chosen with very slow decay. Furthermore, the construction of our proof also extends to the n-dimensional transport equations by extending constantly in the other directions.

Proof of Theorem 5.2. Consider the function

$$\rho_0(x, y) = ce^{ix} \sum_{j=1}^{\infty} \frac{1}{j} \frac{e^{it_j y}}{\langle t_j\rangle^s}$$

as a functions on $\mathbb{T} \times [0, 2\pi]$, where $c \in \left(\frac{1}{100}, 100\right)$ can be chosen such that this function is normalized since $1/j \in l^2$.

Then, the solution ρ of the free transport problem is explicitly given by

$$\rho(t, x, y) = ce^{ix} \sum_{j=1}^{\infty} \frac{1}{j} \frac{e^{i(t_j - t)y}}{\langle t_j\rangle^s}.$$

For simplicity of notation and presentation we first consider averages over squares instead of balls, which allows for a simpler straightforward calculation. An extension to the latter setting is given at the end of this proof. Let thus $t = t_{j_0}$ and consider a square $S = I_x \times I_y$ of side length $\frac{1}{100} > d > j_0/t_{j_0}$, which is centered close to a point where $e^{ix}=1$. Then the integral of ρ over this square decouples by Fubini's theorem and we may compute that

$$\frac{1}{d}\int_{I_x} e^{ix}\,dx \approx 1$$

and that

$$\frac{1}{d}\int_{I_y}\sum_{j=1}^{\infty}\frac{1}{j}\frac{\cos((t_j-t)y)}{\langle t_j\rangle^s}\,dy=\frac{1}{d}\int_{I_y}\frac{1}{j_0\langle t_j\rangle^s}+\frac{1}{d}\sum_{j\neq j_0}\int_{I_y}\frac{1}{j}\frac{e^{i(t_j-t_{j_0})y)}}{\langle t_j\rangle^s}.$$

We note that as an average over a constant function, the first term equals

$$\frac{1}{j_0\langle t_j\rangle^s}\approx\frac{1}{j_0 2^{j_0 s}}. \tag{11}$$

On the other hand, by the construction of $t_j=2^{100+j}$, for each $j\neq j_0$ we have $|t_j-t_{j_0}|\geq\frac{1}{2}\max(t_j,t_{j_0})$ and thus all further integrands are highly oscillatory. In particular,

$$\left|\int_{S_y}\frac{1}{j}\frac{\cos((t_j-t_{j_0})y)}{\langle t_j\rangle^s}\right|\leq\frac{1}{j\langle t_j\rangle^s|t_j-t_{j0)}|}\leq\frac{1}{j2^{js}|2^j-2^{j_0}|},$$

where we used integration by parts. Considering the sum in j, we split into

$$\sum_{j<j_0}\frac{1}{j2^{js}|2^j-2^{j_0}|}\leq\frac{2}{2^{j_0}}\sum_j\frac{1}{j2^{js}}\leq\frac{c_s}{2^{j_0}}$$

and

$$\sum_{j>j_0}\frac{1}{j2^{js}|2^j-2^{j_0}|}\leq 2^{-j_0 s}\sum_{j>j_0}\frac{2}{j2^j}\leq 2^{-j_0 s}2^{-j_0}.$$

Dividing by $d>j2^{-j_0}/(100c_s)$, both terms will be smaller than the term in (11) by a large factor and hence

$$\frac{1}{|S|}\int_S\rho(t_j)\geq c\frac{1}{j_0\langle t_j\rangle^s},$$

as claimed.

Let us next consider the original problem of averages over balls. In this case the integrals

$$\frac{1}{j\langle t_j\rangle^s}\frac{1}{|B_d|}\int_{B_d}e^{ix}e^{i(t_j-t)y}$$

can be explicitly computed in terms of Bessel functions. That is, if the center of the ball is the point (ξ_1,ξ_2), then after translating in x and y, we obtain an exponential factor $e^{i\xi_1+i(t_j-t)\xi_2}$ and an integral over a ball centered in $(0,0)$. We hence need to compute

$$\int_{B_d}e^{ix(1,t_j-t)}\,dx=d^n\int_{B_1}e^{ix(d,d(t_j-t))}\,dx.$$

That is, the Fourier transform of the indicator function of a ball.

Using the rotation-invariance of B_1, we compute

$$\int_{B_1} e^{ix\xi}\,dx = \int_{B_1} e^{ix_1|\xi|}\,dx_1\,dx_2 = \int_{-1}^{1} \sqrt{1-x_1^2}e^{ix_1|\xi|}\,dx_1 = c\frac{J_1(|\xi|)}{|\xi|},$$

where J_1 denotes the Bessel function of the first kind and $c \le 10$. It hence follows that

$$d^{-n}\int_{B_d} e^{ix(1,t_j-t)}\,dx \le C_d \min\left(1, \frac{C}{d\sqrt{1+|t_j-t|^2}}\right) \quad \text{and} \quad d^{-n}\int_{B_d} e^{ix(1,t_j-t)}\,dx \approx 1$$

if $t_j - t$ is small. Thus, the above estimates for squares extend to this case in a straightforward way. □

5B. ***On lower bounds for mixing costs.*** Consider again the passive scalar problem

$$\begin{aligned}\partial_t\rho + v\cdot\nabla\rho &= 0,\\ \nabla\cdot\rho &= 0.\end{aligned} \tag{12}$$

In the previous section we considered v as given and asked about decay rates of mixing scales for $\rho_0 \in H^s$ to be chosen freely.

As a related and in a sense inverse problem, one can ask about *mixing costs*. That is, you are given an explicit initial datum $\rho_0 \in H^s$ and want it to be mixed to scale ϵ by time 1. What kind of lower bound does this imply on Sobolev norms of v in space and time?

More precisely, the aim is to a establish a lower bound of the type

$$\int_0^1 \|v\|_{W^{1,p}}\,dt \ge C_p|\log(\epsilon)|,$$

when $\rho(1)$ is geometrically mixed to scale ϵ. The case $p > 1$ was established in [Crippa and De Lellis 2008] and the case $p = 1$ is a conjecture in [Bressan 2003].

As an application of our comparison results, we consider the simplest case of $p = \infty$, following the proof in [Crippa and De Lellis 2008] via Gronwall's estimate.

Lemma 5.3. *Let* $1 > \epsilon > 0$ *and* ρ *be a solution of* (12) *on* R^n *such that*

$$\|\rho|_{t=0}\|_{H^{-1}} = 1, \quad \|\rho|_{t=1}\|_{H^{-1}} = \epsilon > 0,$$

with $v \in W^{1,\infty}$. *Then it holds that*

$$\int_0^1 \|\partial_i v_j + \partial_j v_i\|_{L^\infty}\,dt \ge C|\log(\epsilon)|.$$

Proof. Since v is divergence-free, the solution operator $S(t_2, t_1)$ mapping $\rho|_{t_1}$ to $\rho|_{t_2}$ is unitary. Characterizing the H^{-1} norm via duality we hence obtain

$$\begin{aligned}\|\rho|_{t=0}\|_{H^{-1}} &= \sup_{\|\psi\|_{H^1}\le 1}\int \psi\rho_{t=0} = \sup_{\|\psi\|_{H^1}\le}\int \psi\rho_{t=0}\\ &= \sup_{\|\psi\|_{H^1}\le 1}\int (S(1,0)\psi)\rho_{t=1} \le \|S(1,0)\|_{H^1\to H^1\|}\|\rho_{t=1}\|_{H^{-1}},\end{aligned}$$

and thus

$$\|S(1,0)\|_{H^1\to H^1} \geq \frac{1}{\epsilon}. \tag{13}$$

On the other hand, $S(1,0)$ conserves the L^2 norm and $\partial_j \rho$ satisfies

$$\partial_t \partial_j \rho + v \cdot \nabla \partial_j \rho + (\partial_j v) \cdot \nabla \rho = 0.$$

Testing against ρ_j we thus obtain

$$\frac{d}{dt}\|\nabla \rho\|_{L^2}^2 \leq 2\sum_{i,j} \int (\partial_j \rho)(\partial_j v_i)(\partial_i \rho) = \sum_{i,j} \int (\partial_j \rho)(\partial_j v_i + \partial_i v_j)(\partial_i \rho) \leq \|\partial_j v_i + \partial_i v_j\|_{L^\infty} \|\nabla \rho\|_{L^2}^2.$$

Gronwall's inequality thus implies

$$\|S(1,0)\|_{H^1\to H^1} \leq \exp\biggl(\int_0^1 \|\partial_j v_i + \partial_i v_j\|_{L^\infty}\, dt\biggr),$$

which in combination with the inequality (13) concludes the proof. □

As a corollary we obtain a lower bound on the geometric scale. While our comparison estimates of Section 4 cannot be expected to be optimal due the different time dependencies, we remark that lower bounds in terms of powers of ϵ yield the same logarithmic lower bounds. Hence, we may consider the assumptions of the following corollary to be equivalent to those of Lemma 5.3 for our purposes.

Corollary 5.4. *Let $1 > \epsilon > 0$ and ρ be a solution of* (12) *on $\mathbb{R}^n$ with $v \in W^{1,\infty}(\mathbb{R}^n)$. Suppose that $\|\rho|_{t=0}\|_{H^{-1}} = 1$ and that $\rho|_{t=1}$ is supported in B_1 and*

$$\mathcal{G}_\epsilon[\rho|_{t=1}] \leq \epsilon.$$

Then it follows that

$$\int_0^1 \|\partial_i v_j + \partial_j v_i\|_{L^\infty}\, dt \geq C|\log(\epsilon)|.$$

Proof. By Theorem 1.2, ρ also satisfies the assumptions of Lemma 5.3, which implies the result. □

For the case $p > 1$, Crippa and De Lellis [2008] obtained the following mixing cost result; see also [Seis 2013; Bruè and Nguyen 2018; Iyer et al. 2014]. Unlike the setting $p = \infty$ this seminal result requires considerable effort to prove. In subsequent works we intend to study whether the comparability can be used to simplify steps of this proof.

Theorem 5.5 [Crippa and De Lellis 2008, Theorem 6.2]. *Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(\mathbb{T}^2)$ and suppose that for $\epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution of* (12) *satisfies*

$$\kappa \leq \frac{1}{B_\epsilon} \int_{B_\epsilon} \rho|_{t=1} \leq 1 - \kappa.$$

Then there exits a constant C such that

$$\int_0^1 \|\nabla v\|_{L^p}\, dt \geq C|\log(\epsilon)| \tag{14}$$

for every $0 < \epsilon < \frac{1}{4}$.

The following corollary establishes a similar result for analytic mixing costs as consequence of this result on geometric mixing costs by using comparability. Such an estimate was also obtained in [Iyer et al. 2014] by more direct methods. We thus do not claim novelty of this estimate, but highlight that comparability allows us to relate existing results available for either scale.

Corollary 5.6. *Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(\mathbb{T}^2)$ and suppose that for $\epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution of* (12) *satisfies*

$$\left\| \rho|_{t=1} - \tfrac{1}{2} \right\|_{H^{-1}} \leq \epsilon.$$

Then inequality (14) *holds.*

Proof. Theorem 4.3 implies that for $0 < \alpha < \frac{1}{2}$

$$\left| \frac{1}{B_{\epsilon^\alpha}} \int_{B_{\epsilon^\alpha}} \rho|_{t=1} - \tfrac{1}{2} \right| \leq C\epsilon^{1/2-\alpha},$$

where the upper bound on α is due to the regularity of 1_{B_1} as discussed in Remark 4.2. Defining $\delta := C\epsilon^{1/2-\alpha}$ and adding $\frac{1}{2}$, we thus obtain

$$\tfrac{1}{2} - \delta \leq \frac{1}{B_{\epsilon^\alpha}} \int_{B_{\epsilon^\alpha}} \rho|_{t=1} \leq \tfrac{1}{2} + \delta.$$

Thus, we may apply the theorem of Crippa and De Lellis with $\kappa \leq \frac{1}{2} - \delta$ and ϵ^α to obtain

$$\int_0^1 \|\nabla v\|_{L^p}\, dt \geq C|\log(\epsilon^\alpha)| = C\alpha|\log(\epsilon^\alpha)|. \qquad \square$$

References

[Alberti et al. 2014] G. Alberti, G. Crippa, and A. L. Mazzucato, "Exponential self-similar mixing and loss of regularity for continuity equations", *C. R. Math. Acad. Sci. Paris* **352**:11 (2014), 901–906. MR Zbl

[Bedrossian and Masmoudi 2015] J. Bedrossian and N. Masmoudi, "Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations", *Publ. Math. Inst. Hautes Études Sci.* **122**:1 (2015), 195–300. MR Zbl

[Bedrossian et al. 2016] J. Bedrossian, N. Masmoudi, and C. Mouhot, "Landau damping: paraproducts and Gevrey regularity", *Ann. PDE* **2**:1 (2016), art. id. 4. MR Zbl

[Bressan 2003] A. Bressan, "A lemma and a conjecture on the cost of rearrangements", *Rend. Sem. Mat. Univ. Padova* **110** (2003), 97–102. MR Zbl

[Bruè and Nguyen 2018] E. Bruè and Q.-H. Nguyen, "Sharp regularity estimates for solutions of the continuity equation drifted by Sobolev vector fields", preprint, 2018. arXiv

[Coti Zelati and Zillinger 2019] M. Coti Zelati and C. Zillinger, "On degenerate circular and shear flows: the point vortex and power law circular flows", *Comm. Partial Differential Equations* **44**:2 (2019), 110–155. MR Zbl

[Crippa and De Lellis 2008] G. Crippa and C. De Lellis, "Estimates and regularity results for the DiPerna–Lions flow", *J. Reine Angew. Math.* **616** (2008), 15–46. MR Zbl

[Crippa and Schulze 2017] G. Crippa and C. Schulze, "Cellular mixing with bounded palenstrophy", *Math. Models Methods Appl. Sci.* **27**:12 (2017), 2297–2320. MR Zbl

[Crippa et al. 2019] G. Crippa, R. Lucà, and C. Schulze, "Polynomial mixing under a certain stationary Euler flow", *Phys. D* **394** (2019), 44–55. MR

[Iyer et al. 2014] G. Iyer, A. Kiselev, and X. Xu, "Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows", *Nonlinearity* **27**:5 (2014), 973–985. MR Zbl

[Lunasin et al. 2012] E. Lunasin, Z. Lin, A. Novikov, A. Mazzucato, and C. R. Doering, "Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows", *J. Math. Phys.* **53**:11 (2012), art. id. 115611. MR Zbl

[Muscalu et al. 2004] C. Muscalu, T. Tao, and C. Thiele, "L^p estimates for the biest, I: The Walsh case", *Math. Ann.* **329**:3 (2004), 401–426. MR Zbl

[Seis 2013] C. Seis, "Maximal mixing by incompressible fluid flows", *Nonlinearity* **26**:12 (2013), 3279–3289. MR Zbl

[Seis 2017] C. Seis, "A quantitative theory for the continuity equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34**:7 (2017), 1837–1850. MR Zbl

[Thiele 2000a] C. Thiele, "The quartile operator and pointwise convergence of Walsh series", *Trans. Amer. Math. Soc.* **352**:12 (2000), 5745–5766. MR Zbl

[Thiele 2000b] C. M. Thiele, "Time-frequency analysis in the discrete phase plane", pp. 99–152 in *Topics in analysis and its applications: selected theses*, edited by R. Coifman, World Sci., River Edge, NJ, 2000. MR Zbl

[Thiele 2006] C. Thiele, *Wave packet analysis*, CBMS Regional Conf. Series Math. **105**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Thiffeault 2012] J.-L. Thiffeault, "Using multiscale norms to quantify mixing and transport", *Nonlinearity* **25**:2 (2012), R1–R44. MR Zbl

[Wei et al. 2018] D. Wei, Z. Zhang, and W. Zhao, "Linear inviscid damping for a class of monotone shear flow in Sobolev spaces", *Comm. Pure Appl. Math.* **71**:4 (2018), 617–687. MR Zbl

[Zillinger 2016] C. Zillinger, "Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity", *Arch. Ration. Mech. Anal.* **221**:3 (2016), 1449–1509. MR Zbl

[Zillinger 2017a] C. Zillinger, "Linear inviscid damping for monotone shear flows", *Trans. Amer. Math. Soc.* **369**:12 (2017), 8799–8855. MR Zbl

[Zillinger 2017b] C. Zillinger, "On circular flows: linear stability and damping", *J. Differential Equations* **263**:11 (2017), 7856–7899. MR Zbl

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