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ENHANCED CONVERGENCE RATES AND ASYMPTOTICS FOR A DISPERSIVE BOUSSINESQ-TYPE SYSTEM WITH LARGE ILL-PREPARED DATA

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We obtain, for a stratified, rotating, incompressible Navier–Stokes system, generalized asymptotics as the Rossby number ε goes to zero (without assumptions on the diffusion coefficients). For ill-prepared, less regular initial data with large blowing-up norm in terms of ε, we show global well-posedness and improved convergence rates (as a power of ε) towards the solution of the limit system, called the 3-dimensional quasigeostrophic system. Aiming for significant improvements required us to avoid as much as possible resorting to classical energy estimates involving oscillations. Our approach relies on the use of structures and symmetries of the limit system, and of highly improved Strichartz-type estimates.

1. Introduction

1.1. ***Geophysical fluids.*** The primitive system (also called primitive equations; see for example [Chemin 1997; Babin et al. 2001]) is a rotating Boussinesq-type system used to describe geophysical fluids located at the surface of the Earth (in a large physical extent) under the assumption that the vertical motion is much smaller than the horizontal one. Two phenomena exert a crucial influence on geophysical fluids: the Coriolis force induced by the rotation of the Earth around its axis and the vertical stratification of the density induced by gravity. The former induces a vertical rigidity in the fluid velocity as described by the Taylor–Proudman theorem, and the latter induces a horizontal rigidity to the fluid density: heavier masses lie under lighter ones.

In order to measure the importance of these two concurrent phenomena, physicists defined two numbers: the Rossby number Ro and the Froude number Fr. We refer to the introduction of [Charve 2005; 2018a; 2018b] for more details and to [Bourgeois and Beale 1994; Cushman-Roisin 1994; Bougeault and Sadourny 1998; Pedlosky 1979] for an in-depth presentation.

The smaller these numbers, the more important these two phenomena become and we will consider the primitive equations in the whole space, under the Boussinesq approximation and when both phenomena are of the same scale, i.e., $\mathrm{Ro} = \varepsilon$ and $\mathrm{Fr} = \varepsilon F$ with $F > 0$. In what follows ε will be called the Rossby number and F the Froude number. The system is then written as follows (we refer to [Chemin 1997;

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Babin et al. 2001] for the model):

$$\begin{cases} \partial_t U_\varepsilon + v_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon}\mathcal{A}U_\varepsilon = \frac{1}{\varepsilon}(-\nabla\Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_{\varepsilon|t=0} = U_{0,\varepsilon}. \end{cases} \tag{PE$_\varepsilon$}$$

The unknowns are $U_\varepsilon = (v_\varepsilon, \theta_\varepsilon) = (v_\varepsilon^1, v_\varepsilon^2, v_\varepsilon^3, \theta_\varepsilon)$ (where v_ε denotes the velocity of the fluid and θ_ε the scalar potential temperature) and Φ_ε which is called the geopotential and gathers the pressure term and centrifugal force. The diffusion operator L is defined by

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu\Delta v_\varepsilon, \nu'\Delta\theta_\varepsilon),$$

where $\nu, \nu' > 0$ are the kinematic viscosity and the thermal diffusivity. The matrix $\mathcal{A}$ is defined by

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

We will also make precise later the properties satisfied by the sequence of initial data $U_{0,\varepsilon}$ (as ε goes to zero). Let us now state some remarks about this system (we refer to the introductions of [Charve 2005; 2018a; 2018b; Charve and Ngo 2011] for more details):

- This system generalizes the well-known rotating fluids system (see [Chemin et al. 2000; 2002; 2006]). The penalized terms (which are divided by the small parameter ε), namely $\mathcal{A}U_\varepsilon$ and the geopotential, will impose a special structure to the limit when ε goes to zero.

- As $\mathcal{A}$ is skew-symmetric, thanks to the incompressibility, any energy method (that is based on L^2 or $H^s/\dot{H}^s$-inner products) will not "see" these penalized terms and will work as for the classical incompressible Navier–Stokes system ($\mathcal{A}U \cdot U = 0$ and $(\nabla\Phi_\varepsilon, v_\varepsilon)_{H^s/\dot{H}^s} = 0$). Therefore the Leray and Fujita–Kato theorems provide global-in-time weak solutions if $U_{0,\varepsilon} \in L^2$ and local-in-time unique strong solutions if $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}$ (global for small initial data).

- There are two distinct regimes: $F \neq 1$ or $F = 1$. The first one features very important dispersive properties. In the second case, the operators are simpler but we cannot rely on Strichartz estimates and the methods are completely different (see [Chemin 1997; Charve 2018b]). *In the present article we focus on the case $F \neq 1$.*

1.2. ***Strong solutions.*** As explained before, thanks to the skew-symmetry of matrix $\mathcal{A}$, any computation involving L^2 or Sobolev inner products will be the same as for the Navier–Stokes system. So given the regularity of the initial data (even if some norms can blow up in ε), we can adapt the Leray and Fujita–Kato theorems as well as the classical weak-strong uniqueness results: for a fixed $\varepsilon > 0$, if $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, we denote by U_ε the unique strong solution of system (PE$_\varepsilon$), defined on $[0, T]$ for all $0 < T < T_\varepsilon^*$. In addition, if the lifespan T_ε^* is finite then we have (blow up criterion)

$$\int_0^{T_\varepsilon^*} \|\nabla U_\varepsilon(\tau)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)}\, d\tau = \infty. \tag{1-1}$$

Moreover, if $U_{0,\varepsilon} \in \dot{H}^s$ then we also can propagate the regularity as done for the Navier–Stokes system.

1.3. *The limit system, the QG/osc decomposition.* We are interested in the asymptotics, as the small parameter ε goes to zero. Let us recall that the limit system is a transport-diffusion system coupled with a Biot–Savart inversion law and is called the 3-dimensional quasigeostrophic system:

$$\begin{cases} \partial_t \widetilde{\Omega}_{\mathrm{QG}} + \tilde{v}_{\mathrm{QG}} \cdot \nabla \widetilde{\Omega}_{\mathrm{QG}} - \Gamma \widetilde{\Omega}_{\mathrm{QG}} = 0, \\ \widetilde{U}_{\mathrm{QG}} = (\tilde{v}_{\mathrm{QG}}, \tilde{\theta}_{\mathrm{QG}}) = (-\partial_2, \partial_1, 0, -F\partial_3)\Delta_F^{-1}\widetilde{\Omega}_{\mathrm{QG}}, \end{cases} \tag{QG}$$

where the operator Γ is defined by

$$\Gamma \stackrel{\text{def}}{=} \Delta\Delta_F^{-1}(\nu\partial_1^2 + \nu\partial_2^2 + \nu' F^2\partial_3^2),$$

with $\Delta_F = \partial_1^2 + \partial_2^2 + F^2\partial_3^2$. Moreover we also have the relation

$$\widetilde{\Omega}_{\mathrm{QG}} = \partial_1 \widetilde{U}^2_{\mathrm{QG}} - \partial_2 \widetilde{U}^1_{\mathrm{QG}} - F\partial_3 \widetilde{U}^4_{\mathrm{QG}} = \partial_1 \tilde{v}^2_{\mathrm{QG}} - \partial_2 \tilde{v}^1_{\mathrm{QG}} - F\partial_3 \tilde{\theta}_{\mathrm{QG}}.$$

Remark 1. The operator Δ_F is a simple anisotropic Laplacian but Γ is in general a tricky nonlocal diffusion operator of order 2 (except in the case $F = 1$ where $\Delta_F = \Delta$ and $\Gamma = \nu\partial_1^2 + \nu\partial_2^2 + \nu'\partial_3^2$, or in the case $\nu = \nu'$ where $\Gamma = \nu\Delta$). We refer to [Charve 2016; 2018a] for an in-depth study of Γ in the general case (neither its Fourier kernel nor singular integral kernel have a constant sign and no classical result can be used).

This limit system is first formally derived, and then rigorously justified (see [Chemin 1997; Charve 2005]). Led by the limit system we introduce the following decomposition: for any 4-dimensional vector field $U = (v, \theta)$ we define its potential vorticity $\Omega(U)$ by

$$\Omega(U) \stackrel{\text{def}}{=} \partial_1 v^2 - \partial_2 v^1 - F\partial_3\theta$$

and its quasigeostrophic and oscillating (or oscillatory) parts by

$$U_{\mathrm{QG}} = \mathcal{Q}(U) \stackrel{\text{def}}{=} \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1}\Omega(U) \quad \text{and} \quad U_{\mathrm{osc}} = \mathcal{P}(U) \stackrel{\text{def}}{=} U - U_{\mathrm{QG}}. \tag{1-2}$$

As emphasized in [Charve 2005; Charve and Ngo 2011] this is an orthogonal decomposition of 4-dimensional vector fields (similar to the Leray orthogonal decomposition into divergence-free and gradient vector fields) and if $\mathcal{Q}$ and $\mathcal{P}$ are the associated orthogonal projectors on the quasigeostrophic or oscillating fields, they satisfy (see [Chemin 1997; Charve 2004; 2005; 2018a; 2018b]):

Proposition 2. *For any function $U = (v, \theta) \in \dot{H}^s$ (for some s) we have*:

(1) *$\mathcal{P}$ and $\mathcal{Q}$ are pseudodifferential operators of order* 0.

(2) *For any $s \in \mathbb{R}$, we have $(\mathcal{P}(U) \mid \mathcal{Q}(U))_{H^s/\dot{H}^s} = (\mathcal{A}U \mid \mathcal{P}(U))_{H^s/\dot{H}^s} = 0$ (when defined).*

(3) $\mathcal{P}(U)=U \iff \mathcal{Q}(U)=0 \iff \Omega(U)=0$.

(4) $\mathcal{Q}(U)=U \iff \mathcal{P}(U)=0 \iff$ *there exists a scalar function Φ such that $U = (-\partial_2, \partial_1, 0, -F\partial_3)\Phi$. Such a vector field is said to be quasigeostrophic (or QG) and is also divergence-free.*

(5) *If $U = (v, \theta)$ is a quasigeostrophic vector field, then $v \cdot \nabla\Omega(U) = \Omega(v \cdot \nabla U)$ and $\Gamma U = \mathcal{Q}(LU)$.*

(6) *Denoting by $\mathbb{P}$ the Leray orthogonal projector on divergence-free vector fields, $\mathbb{P}\mathcal{P} = \mathcal{P}\mathbb{P}$ and $\mathbb{P}\mathcal{Q} = \mathcal{Q}\mathbb{P} = \mathcal{Q}$.*

Thanks to this, system (QG) can for example be rewritten in one of the equivalent following velocity formulations:

$$\begin{cases} \partial_t \widetilde{U}_{\mathrm{QG}} + \mathcal{Q}(\tilde{v}_{\mathrm{QG}} \cdot \nabla \widetilde{U}_{\mathrm{QG}}) - \Gamma \widetilde{U}_{\mathrm{QG}} = 0, \\ \widetilde{U}_{\mathrm{QG}} = \mathcal{Q}(\widetilde{U}_{\mathrm{QG}}), \quad \text{(or equivalently } \mathcal{P}(\widetilde{U}_{\mathrm{QG}}) = 0), \\ \widetilde{U}_{\mathrm{QG}|t=0} = \widetilde{U}_{0,\mathrm{QG}}, \end{cases} \tag{QG}$$

or

$$\begin{cases} \partial_t \widetilde{U}_{\mathrm{QG}} + \tilde{v}_{\mathrm{QG}} \cdot \nabla \widetilde{U}_{\mathrm{QG}} - L\widetilde{U}_{\mathrm{QG}} = \mathcal{P}\widetilde{\Phi}_{\mathrm{QG}}, \\ \widetilde{U}_{\mathrm{QG}} = \mathcal{Q}(\widetilde{U}_{\mathrm{QG}}), \\ \widetilde{U}_{\mathrm{QG}|t=0} = \widetilde{U}_{0,\mathrm{QG}}. \end{cases} \tag{QG}$$

Remark 3. We recall that Theorem 2 from [Charve 2004] claims that if $\widetilde{U}_{0,\mathrm{QG}} \in H^1$ then system (QG) has a unique global solution $\widetilde{U}_{\mathrm{QG}} \in \dot{E}^0 \cap \dot{E}^1$ (see below for the space notation). We refer to [Charve 2004; 2008] and to the next sections for more details.

Remark 4. It is natural to investigate the link between the quasigeostrophic/oscillating parts decomposition of the initial data and the asymptotics when ε goes to zero. This leads to the notion of well-prepared/ill-prepared initial data depending on whether or not the initial data is already close to the quasigeostrophic structure, i.e., when the initial oscillating part is small/large (or going to zero/blowing up as ε goes to zero). In the present article we consider large and ill-prepared initial data with very large oscillating parts depending on ε.

Going back to system (PE$_\varepsilon$), we introduce $\Omega_\varepsilon = \Omega(U_\varepsilon)$, $U_{\varepsilon,\mathrm{QG}} = \mathcal{Q}(U_\varepsilon)$ and $U_{\varepsilon,\mathrm{osc}} = \mathcal{P}(U_\varepsilon)$. We showed in [Charve 2005] that for an initial data in L^2 (independent of ε), the oscillating part $U_{\varepsilon,\mathrm{osc}}$ of a weak global Leray solution U_ε goes to zero in $L^2_{\mathrm{loc}}(\mathbb{R}_+, L^q(\mathbb{R}^3))$ ($q \in]2, 6[$), and the quasigeostrophic part $U_{\varepsilon,\mathrm{QG}}$ goes to a solution of system (QG) (with the QG-part of U_0 as initial data). This requires the study of system (A-79), and its associated matrix in the Fourier space: As explained in detail in Proposition 43, when $\nu \neq \nu'$ there are four distinct eigenvalues (it is necessary to perform frequency truncations to obtain their expression). The first one is explicit but discarded as its associated eigenvector is not divergence-free, the second one is real (and mainly linked to the quasigeostrophic part). The last two are nonreal and mainly linked to the oscillating part.

Let us denote by $\mathbb{P}_i$ ($i \in \{2, 3, 4\}$) the associated projectors. When $\nu = \nu'$, many simplifications arise (see Remark 44). Unfortunately none of these simplifications are true anymore in general (when $\nu \neq \nu'$)

but we are able to bound their operator norms and prove that the $\mathbb{P}_2$-part of an oscillating divergence-free vector field is small (we refer to [Charve 2005; 2006]; see also Proposition 43).

Moreover we are able to obtain Strichartz estimates for the last two projections $\mathbb{P}_{3+4}$. In [Charve 2005] we obtained the following Strichartz estimate upon which the main result depended:

$$\|\mathbb{P}_{3+4}\mathcal{P}_{r,R}f\|_{L^4L^\infty} \le C_{r,R}\varepsilon^{\frac{1}{4}}(\|\mathcal{P}_{r,R}f_0\|_{L^2} + \|\mathcal{P}_{r,R}F\|_{L^2}).$$

In [Charve 2004] we focused on strong solutions. We first proved that if the initial QG-part $U_{0,\mathrm{QG}}$ is H^1 then the limit system has a unique global solution $\widetilde{U}_{\mathrm{QG}}$. We proved that if $U_{0,\mathrm{osc}} \in \dot{H}^{\frac{1}{2}}$ then U_ε is global if ε is small enough. For this we filtered some waves: we constructed a solution W_ε^T of (A-79) with a particular external force term (constructed from $\widetilde{U}_{\mathrm{QG}}$) and proved that $U_\varepsilon - \widetilde{U}_{\mathrm{QG}} - W_\varepsilon^T$ goes to zero thanks to a generalization of the previous Strichartz estimates (allowing different regularities for the external force term):

$$\|\mathbb{P}_{3+4}\mathcal{P}_{r,R}f\|_{L^2L^\infty} \le C_{r,R}\varepsilon^{\frac{1}{4}}(\|\mathcal{P}_{r,R}f_0\|_{L^2} + \|\mathcal{P}_{r,R}F^b\|_{L^1L^2} + \|\mathcal{P}_{r,R}F^l\|_{L^2L^2}).$$

In [Charve 2006] we generalized the previous result for initial data depending on ε and with large oscillating part (bounded by $|\ln|\ln\varepsilon||$ in the general case and $|\ln\varepsilon|$ when $\nu = \nu'$) considering frequency truncations $\mathcal{P}_{r_\varepsilon,R_\varepsilon}$ with radii depending on ε allowing us to exhibit explicit convergence rates. In this work we distinguished the case $\nu = \nu'$ for which we were able to produce Strichartz estimates *without* frequency truncations in *inhomogeneous* spaces:

$$\|W_\varepsilon\|_{L^2B^s_{\infty,q}} \le C\varepsilon^{\frac{1}{8}}(\|f_0\|_{B^{s+3/4}_{2,q}} + \|G\|_{L^1(B^{s+3/4}_{2,q})}).$$

In the second part of [Charve 2006], inspired by the work of Dutrifoy [2004] on vortex patches in the inviscid case and by the work of Hmidi [2005] for Navier–Stokes vortex patches, we investigated the case of initial potential vorticity which is a regularized patch, and very large initial oscillating part (regular but bounded by a negative power of ε) when $\nu = \nu'$. This work was recently generalized in the case $\nu \neq \nu'$ in [Charve 2016; 2018a] where we deeply studied the limit quasigeostrophic operator Γ which is nonlocal and nonradial. In this setting, the fact that $\nu \neq \nu'$ highly complicates every computation.

Let us also mention that in [Charve 2008] we obtained global existence when the initial QG-part is only $H^{\frac{1}{2}+\eta}$. This required real interpolation methods (inspired by [Gallagher and Planchon 2002]) in order to obtain economic estimates for the limit system (see (1-12)). In [Charve and Ngo 2011] with V. S. Ngo we studied the asymptotics in the case of evanescent viscosities (as a power of ε) and for simplified oscillating initial data (as the initial QG part is zero, the limit is also zero).

Let us now give a survey of other results on this system. In the nondispersive setting $F = 1$ there are few works: let us mention the seminal work [Chemin 1997] (that we recently generalized in [Charve 2018b]) and [Iftimie 1999a] in the inviscid case.

In [Koba et al. 2012] the authors distinguish the rotation and stratification effects, in the case $\nu = \nu'$ for initial data in $\dot{H}^{\frac{1}{2}} \cap \dot{H}^1$ and for a special condition $\partial_2 u_0^1 - \partial_1 u_0^2 = 0$ (the initial potential vorticity only depends on the temperature), and they obtain existence of a unique global solution to (PE$_\varepsilon$) in $\mathcal{C}(\mathbb{R}_0, \dot{H}^1)$

for strong enough rotation and stratification. If the initial data is small in $\dot H^{\frac12}$ they manage to obtain that $\nabla U_\varepsilon \in L^2 \dot H^{\frac12}$.

Lee and Takada [2017] studied global well-posedness in the case of stratification only (no rotational effects) when $\nu=\nu'$ and for large initial oscillating part (independent of ε). They first give global existence of a unique mild solution in $L^4(\mathbb{R}_+, \dot W^{\frac12,3}(\mathbb{R}^3))$ for large initial oscillating part in $\dot H^s$ ($s\in \left]\frac12,\frac58\right]$, there is a kind of smallness condition; see Remark 17) and small QG-part in $\dot H^{\frac12}$. Then they show global well-posedness in the case $s=\frac12$ and for any initial oscillating part and small QG-part, of a unique mild solution in $\mathcal{C}(\mathbb{R}_+, \dot H^{\frac12})\cap L^4(\mathbb{R}_+, \dot W^{\frac12,3}(\mathbb{R}^3))$.

These results are adapted to the primitive system in [Iwabuchi et al. 2017]. Iwabuchi, Mahalov and Takada focused on the case $\nu=\nu'$ and obtained (through stationary phase methods) the following Strichartz estimates, which we state with our notation:

Proposition 5 [Iwabuchi et al. 2017, Theorem 1.1 and Corollary 1.2]. *Assume $F\neq 1$. If $r\in]2,4[$ and $p\in]2,\infty[\,\cap \left[1/\left(2\left(\frac12-\frac1r\right)\right), 2/\left(3\left(\frac12-\frac1r\right)\right)\right]$, there exists a constant $C=C_{F,\nu,p,r}$ such that, if f solves the homogeneous* (A-79),

$$\|f\|_{L^p(\mathbb{R}_+,L^r)} \le C\varepsilon^{\frac1p-\frac32\left(\frac12-\frac1r\right)}\|f_0\|_{L^2}.$$

If $s\in \left]\frac12,\frac58\right]$, there exists a constant $C=C(F,s,\nu)$ such that

$$\|f\|_{L^4(\mathbb{R}_+,\dot W^{s,6/(1+2s)})} \le C\varepsilon^{\frac12\left(s-\frac12\right)}\|f_0\|_{\dot H^s}.$$

From this they are able to obtain through a fixed point argument the following global well-posedness results for initial data (independent of ε) with small quasigeostrophic part (assume $\nu=\nu'$ and $F\neq 1$):

- If $s\in \left]\frac12,\frac58\right]$, there exist $\delta_1,\delta_2>0$ (depending on ν, F, s) such that for any $\varepsilon>0$ and any initial data $U_0=U_{0,\mathrm{QG}}+U_{0,\mathrm{osc}}$ with $(U_{0,\mathrm{QG}},U_{0,\mathrm{osc}})\in \dot H^{\frac12}\times \dot H^s$ and

$$\begin{cases} \|U_{0,\mathrm{QG}}\|_{\dot H^{1/2}}\le \delta_1,\\ \|U_{0,\mathrm{osc}}\|_{\dot H^s}\le \delta_2\varepsilon^{-\frac12\left(s-\frac12\right)}, \end{cases} \tag{1-3}$$

there exists a unique global mild solution in $L^4(\mathbb{R}_+, \dot W^{s,\frac13}(\mathbb{R}^3))$.

- There exists $\delta>0$ such that for any initial data $U_0=U_{0,\mathrm{QG}}+U_{0,\mathrm{osc}}\in \dot H^{\frac12}$ with $\|U_{0,\mathrm{QG}}\|_{\dot H^{1/2}}\le\delta$, there exists $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$, system ($\mathrm{PE}_\varepsilon$) has a unique global mild solution in $\mathcal{C}(\mathbb{R}_+,\dot H^{\frac12})\cap L^4(\mathbb{R}_+, \dot W^{s,\frac13}(\mathbb{R}^3))$.

Let us also mention works in the periodic case where resonances have to be studied (see for example [Gallagher 1998; Ngo 2009; Ngo and Scrobogna 2018; Scrobogna 2018]), in the rotating fluids system case (see [Chemin et al. 2000; 2002; 2006; Giga et al. 2008; Hieber and Shibata 2010; Koh et al. 2014a]) or in the inviscid case (see [Dutrifoy 2004; 2005; Koh et al. 2014b; Takada 2016; Widmayer 2018]).

In the present article we wish to generalize our results from [Charve 2004; 2006; 2008] and motivated by the very interesting results in [Iwabuchi et al. 2017] we want to obtain full asymptotics (as in [Charve and Ngo 2011; Charve 2018a]) for very large ill-prepared initial data (less regular, depending on ε and bounded by a negative power of ε). In our work we will provide global well-posedness results but also precise convergence rates as ε goes to zero. We also generalize [Iwabuchi et al. 2017] in the sense that we consider initial data with *large quasigeostrophic part* (with low-frequency assumptions) and provide solutions in *homogeneous* energy spaces $\dot{E}^s$ both in the particular case $\nu = \nu'$ *and* in the general case $\nu \neq \nu'$. Let us also mention that our methods closely rely on the special structures and properties of the 3-dimensional quasigeostrophic system.

1.3.1. *Statement of the results.* We will consider general ill-prepared initial data $U_{0,\varepsilon} = U_{0,\varepsilon,\text{osc}} + U_{0,\varepsilon,\text{QG}}$, whose QG-part converges to some $\widetilde{U}_{0,\text{QG}}$ (without any smallness condition), and whose oscillating part is very large (see below for details).

The aim of the present article is to generalize Theorem 3 from [Charve 2004], Theorems 1.2 and 1.3 from [Charve 2006] and Theorem 4 from [Charve 2008] with the least possible extra regularity for the initial data and the biggest possible blowing-up initial oscillatory part (as a negative power of ε). The energy methods used in [Charve 2004; 2005; 2006] would only allow at best an initial blow-up of $U_{0,\varepsilon,\text{osc}}$ as $|\ln \varepsilon|^\beta$. Indeed, these methods require the use of energy estimates for the oscillations W_ε and W_ε^T and produce large terms involving $\exp(\|U_{0,\varepsilon,\text{osc}}\|^2)$ that can only be balanced thanks to ε^γ provided by the Strichartz estimates. We need to change our point of view and try to not resort to energy estimates for these oscillations. This will require us to make more flexible dispersive estimates so that the oscillations can be estimated with minimal use of their energy (the only term where it was unavoidable is F_8; see below for details). We will here state only the new results. Let us define (in the whole space $\mathbb{R}^3$) the family of spaces $\dot{E}_T^s$ for $s \in \mathbb{R}$,

$$\dot{E}_T^s = \mathcal{C}_T(\dot{H}^s) \cap L_T^2(\dot{H}^{s+1}),$$

endowed with the following norm (where we define $\nu_0 = \min(\nu, \nu')$; see the Appendix for other notation):

$$\|f\|_{\dot{E}_T^s}^2 \overset{\text{def}}{=} \|f\|_{L_T^\infty \dot{H}^s}^2 + \nu_0 \int_0^T \|f(\tau)\|_{\dot{H}^{s+1}}^2 \, d\tau.$$

When $T = \infty$ we write $\dot{E}^s$ and the corresponding norm is over $\mathbb{R}_+$ in time. Let us now state the main result of this article (we *do not* assume $\nu = \nu'$).

Theorem 6. *Assume $F \neq 1$. For any $\mathbb{C}_0 \geq 1$, $\delta \in \left]0, \frac{1}{10}\right]$, and $\alpha_0 > 0$, there exist five constants $\varepsilon_0, \eta, \mathbb{B}_0, \kappa, \beta > 0$ (depending on $F, \nu, \nu', \mathbb{C}_0, \alpha_0$) such that for all $\varepsilon \in]0, \varepsilon_0]$ and all divergence-free initial data $U_{0,\varepsilon} = U_{0,\varepsilon,\text{QG}} + U_{0,\varepsilon,\text{osc}}$ satisfying*

(1) *$U_{0,\varepsilon,\text{QG}}$ converges towards some quasigeostrophic vector field $\widetilde{U}_{0,\text{QG}} \in H^{\frac{1}{2}+\delta}$ with*

$$\begin{cases} \|U_{0,\varepsilon,\text{QG}} - \widetilde{U}_{0,\text{QG}}\|_{H^{1/2+\delta}} \leq \mathbb{C}_0 \varepsilon^{\alpha_0}, \\ \|\widetilde{U}_{0,\text{QG}}\|_{H^{1/2+\delta}} \leq \mathbb{C}_0, \end{cases} \tag{1-4}$$

(2) *$\|U_{0,\varepsilon,\text{osc}}\|_{\dot{F}_\delta} \leq \mathbb{C}_0 \varepsilon^{-\kappa\delta}$, where the space $\dot{F}_\delta$ is defined as follows $\left(q = \frac{2}{1+\delta}\right)$:*

$$\dot{F}_\delta = \begin{cases} \dot{H}^{\frac{1}{2}-\delta} \cap \dot{H}^{\frac{1}{2}+\delta} & \text{if } \nu = \nu', \\ \dot{B}_{q,q}^{\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}+\delta} & \text{if } \nu \neq \nu', \end{cases}$$

then system (QG) *has a unique global solution* $\widetilde{U}_{\mathrm{QG}} \in \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$, *and system* ($\mathrm{PE}_\varepsilon$) *has a unique global solution* $U_\varepsilon \in \dot{E}^s$ *for all* $s \in \left[\frac{1}{2} - \eta\delta, \frac{1}{2} + \eta\delta\right]$, *which converges towards* $\widetilde{U}_{\mathrm{QG}}$ *with the estimate*

$$\|U_\varepsilon - \widetilde{U}_{\mathrm{QG}}\|_{L^2 L^\infty} \le \mathbb{B}_0 \varepsilon^{\min(\alpha_0, \delta\beta)}.$$

Remark 7. In the general case, κ is small $\left(\text{less than } \frac{1}{4000}\right)$, whereas in the case $\nu = \nu'$, $\kappa < \frac{1}{2}$ (and is as close to $\frac{1}{2}$ as we want). We refer to the next section for a more precise statement of this theorem.

Remark 8. It is interesting to adapt these results to the case with only stratification.

1.4. ***Precise statement of the main results.*** This section is devoted to giving the precise statement of Theorem 6, which will be split into two formulations based on whether we have $\nu = \nu'$ or $\nu \neq \nu'$. This statement requires us to introduce auxiliary systems, which is the subject of the first two subsections, and state additional regularity properties for the solution of the limit system (we refer to the third subsection). Then we will state the results we will prove in this article.

1.4.1. *Auxiliary systems in the general case* $\nu \neq \nu'$.

Remark 9. In what follows, we will systematically write, for $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f \cdot \nabla f = \sum_{i=1}^3 f_i \partial_i f$.

Following [Charve 2004] we rewrite the primitive system, projecting onto the divergence-free vector fields ($\mathbb{P}$ is the classical Leray projector):

$$\begin{cases} \partial_t U_\varepsilon - L U_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A} U_\varepsilon = -\mathbb{P}(U_\varepsilon \cdot \nabla U_\varepsilon). \\ U_{\varepsilon|t=0} = U_{0,\varepsilon}. \end{cases} \tag{1-5}$$

Notice that we can rewrite (QG) as follows (we also refer to [Charve 2004] where it was first used):

$$\begin{cases} \partial_t \widetilde{U}_{\mathrm{QG}} - L U_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A} \widetilde{U}_{\mathrm{QG}} = -\mathbb{P}(\widetilde{U}_{\mathrm{QG}} \cdot \nabla \widetilde{U}_{\mathrm{QG}}) + G, \\ \widetilde{U}_{QG|t=0} = \widetilde{U}_{0,\mathrm{QG}}. \end{cases} \tag{QG}$$

where

$$G = G^b + G^l \overset{\text{def}}{=} \mathbb{P}\mathcal{P}(\widetilde{U}_{\mathrm{QG}} \cdot \nabla \widetilde{U}_{\mathrm{QG}}) - F(\nu - \nu')\Delta \Delta_F^{-2} \begin{pmatrix} -F\partial_2 \partial_3^2 \\ F\partial_1 \partial_3^2 \\ 0 \\ (\partial_1^2 + \partial_2^2)\partial_3 \end{pmatrix} \widetilde{\Omega}_{\mathrm{QG}}. \tag{1-6}$$

Remark 10. It is important to notice that G is the sum of two terms, both divergence-free and whose potential vorticity is zero, which is crucial to fully take advantage of (A-84). We refer to [Charve 2004; 2008] for more details.

As explained in [Charve 2004; 2005; 2006; 2008; 2018a; Charve and Ngo 2011], in the case $F \neq 1$ the oscillatory part enjoys dispersive properties that allow us to obtain Strichartz-type estimates. More precisely the oscillatory part satisfies system (A-79) (we refer to the Appendix for details), and in all the cited articles, we used that the frequency-truncated third and fourth projections of the oscillatory part satisfy Strichartz-type estimates as given by Proposition 46. As in [Charve 2004; 2008; Charve and Ngo 2011], in the present article we will consider some particular oscillatory terms whose existence is devoted to absorbing some constant terms in order to get the desired convergence rate for the asymptotics as ε goes to zero.

More precisely, we introduce the following linear system (we refer to the Appendix for the notation r_ε, R_ε and $\mathcal{P}_{r_\varepsilon,R_\varepsilon}$):

$$\begin{cases} \partial_t W_\varepsilon^T - LW_\varepsilon^T + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}W_\varepsilon^T = -\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_{3+4}G, \\ W_\varepsilon^T{}_{|t=0} = \mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_{3+4}U_{0,\varepsilon,\mathrm{osc}}. \end{cases} \tag{1-7}$$

Remark 11. We recall that it would be useless to consider the free system: indeed the system satisfied by $U_\varepsilon - \widetilde{U}_{\mathrm{QG}}$ features G as an external force term which is independent of ε and blocks any convergence. It is then necessary to absorb a large part of this term, which is the reason why we considered such an external force term in system (1-7). In other words, W_ε^T is small due to dispersive properties, but still it allows us to "eat" a large part of G. We refer to [Charve 2004] for more details.

Finally we define $\delta_\varepsilon = U_\varepsilon - \widetilde{U}_{\mathrm{QG}} - W_\varepsilon^T$, which satisfies the following system (see [Charve 2004] for details):

$$\begin{cases} \partial_t \delta_\varepsilon - L\delta_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\delta_\varepsilon = \sum_{i=1}^8 F_i + f^b + f^l, \\ \delta_{\varepsilon|t=0} = (U_{0,\varepsilon,\mathrm{QG}} - \widetilde{U}_{0,\mathrm{QG}}) + (\mathrm{Id} - \mathcal{P}_{r_\varepsilon,R_\varepsilon})U_{0,\varepsilon,\mathrm{osc}} + \mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2 U_{0,\varepsilon,\mathrm{osc}}, \end{cases} \tag{1-8}$$

where we define

$$\begin{aligned} &F_1 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon\cdot\nabla\delta_\varepsilon), \quad F_2 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon\cdot\nabla\widetilde{U}_{\mathrm{QG}}), \quad F_3 \stackrel{\text{def}}{=} -\mathbb{P}(\widetilde{U}_{\mathrm{QG}}\cdot\nabla\delta_\varepsilon), \quad F_4 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon\cdot\nabla W_\varepsilon^T), \\ &F_5 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon^T\cdot\nabla\delta_\varepsilon), \quad F_6 \stackrel{\text{def}}{=} -\mathbb{P}(\widetilde{U}_{\mathrm{QG}}\cdot\nabla W_\varepsilon^T), \quad F_7 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon^T\cdot\nabla\widetilde{U}_{\mathrm{QG}}), \quad F_8 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon^T\cdot\nabla W_\varepsilon^T), \\ &\qquad f^b \stackrel{\text{def}}{=} -(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})G^b - \mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2 G^b, \\ &\qquad f^l \stackrel{\text{def}}{=} -(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})G^l - \mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2 G^l. \end{aligned} \tag{1-9}$$

1.4.2. *Auxiliary systems in the special case* $\nu = \nu'$. In this case, many simplifications arise in the computations of the eigenvalues and eigenvectors of system (A-79) (see Remark 44). In this case, as used in the first part of [Charve 2006], we can use the following system instead of (1-7):

$$\begin{cases} \partial_t W_\varepsilon - LW_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}W_\varepsilon = -G^b, \\ W_{\varepsilon|t=0} = U_{0,\varepsilon,\mathrm{osc}}. \end{cases} \tag{1-10}$$

We will be able in the present article to provide for this system much more accurate Strichartz estimates without any frequency restrictions (generalizing the ones obtained in [Charve 2006]). If we write $\delta_\varepsilon = U_\varepsilon - \widetilde{U}_{\mathrm{QG}} - W_\varepsilon$, it satisfies the system

$$\begin{cases} \partial_t \delta_\varepsilon - L\delta_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\delta_\varepsilon = \sum_{i=1}^8 F_i, \\ \delta_{\varepsilon|t=0} = U_{0,\varepsilon,\mathrm{QG}} - \widetilde{U}_{0,\mathrm{QG}}. \end{cases} \tag{1-11}$$

Remark 12. We choose here to use the same notation as in the general case; the only difference is that W_ε^T has to be replaced by W_ε.

1.4.3. *The limit system.* Let us recall that Theorem 2 from [Charve 2004] states that when the initial data $\widetilde{U}_{0,\mathrm{QG}}$ is in the *inhomogeneous* Sobolev space H^1, system (QG) has a unique global solution $\widetilde{U}_{\mathrm{QG}} \in \dot{E}^0 \cap \dot{E}^1$; moreover there exists a constant $C = C(\delta) > 0$ such that for all $s \in [0,1]$ and all $t \in \mathbb{R}_+$

(and denoting as usual $\nu_0 = \min(\nu, \nu') > 0$)

$$\|\widetilde{U}_{\mathrm{QG}}\|^2_{L^\infty_t \dot{H}^s} + \nu_0 \int_0^t \|\nabla \widetilde{U}_{\mathrm{QG}}(\tau)\|^2_{\dot{H}^s}\, d\tau \le C(\|\widetilde{U}_{0,\mathrm{QG}}\|^{1-s}_{L^2} \|\widetilde{U}_{0,\mathrm{QG}}\|^s_{\dot{H}^1})^2 \le C\|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^1}.$$

In [Charve 2008] we used real interpolation methods from [Gallagher and Planchon 2002] (we also refer to [Calderón 1990]) to obtain a much more accurate estimate, which allowed us to bound the energy in $\dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$ only with the $H^{\frac{1}{2}+\delta}$ initial norm instead of the full H^1-norm (we refer to Lemma 2.1 in [Charve 2008]; our aim was to consider less regular initial data): for any $\delta > 0$ there exists a constant $C = C_{\delta,\nu_0} > 0$ such that for all $t \in \mathbb{R}_+$

$$\|\widetilde{U}_{\mathrm{QG}}\|^2_{L^\infty_t H^{1/2+\delta}} + \nu_0 \int_0^t \|\nabla \widetilde{U}_{\mathrm{QG}}(\tau)\|^2_{H^{1/2+\delta}}\, d\tau \le C_{\delta,\nu_0} \|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^{1/2+\delta}} \max(1, \|\widetilde{U}_{0,\mathrm{QG}}\|^{\frac{1}{\delta}}_{H^{1/2+\delta}}). \tag{1-12}$$

Remark 13. The reader may wonder why the right-hand side is not simply $C_{\delta,\nu_0}\|\widetilde{U}_{0,\mathrm{QG}}\|^{2+\frac{1}{\delta}}_{H^{1/2+\delta}}$ as stated in [Charve 2008; Gallagher and Planchon 2002]. This is the right formulation when $\|\widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}}$ is large (in [Charve 2008] we implicitly focused on large initial QG part). When it is small, the right-hand side is even simpler: $C_{\delta,\nu_0}\|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^{1/2+\delta}}$. In the proof in [Charve 2008] of (1-12) it is crucial to use Lemma 4.3 from [Gallagher and Planchon 2002], and for this some threshold $j_0 \ge 1$ has to be defined:

- either $\|\widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}} > \frac{2}{3} c\nu_0 2^{2\delta}$ (where c is the smallness constant from the Fujita–Kato theorem), and we can define the threshold j_0 as stated in [Charve 2008] so that the right-hand side of (1-12) is

$$C_0(1-2^{-4\delta})^{-2}\left(\frac{3}{2c\nu_0}\right)^{\frac{1}{\delta}} \|\widetilde{U}_{0,\mathrm{QG}}\|^{2+\frac{1}{\delta}}_{H^{1/2+\delta}}$$

(C_0 is a universal constant),

- or $\|\widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}} \le \frac{2}{3} c\nu_0 2^{2\delta}$ and then we can simply choose the threshold $j_0 = 1$ and obtain (1-12) with right-hand side that can be simplified into $C_0(1-2^{-4\delta})^{-2}\|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^{1/2+\delta}}$.

In other words, the right-hand side of (1-12) is in general

$$C_0(1-2^{-4\delta})^{-2\delta}\|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^{1/2+\delta}} \max\left(1, \frac{1}{4}\left(\frac{3}{2c\nu_0}\|\widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}}\right)^{\frac{1}{\delta}}\right).$$

Our first result is devoted to the limit system and generalizes Theorem 2 from [Charve 2004] using the precise estimates obtained in [Charve 2008]:

Theorem 14. *Let $\delta > 0$ and $\widetilde{U}_{0,\mathrm{QG}} \in H^{\frac{1}{2}+\delta}$ be a quasigeostrophic vector field* (*that is, $\widetilde{U}_{0,\mathrm{QG}} = \mathcal{Q}\widetilde{U}_{0,\mathrm{QG}}$*). *Then system* (QG) *has a unique global solution in $E^{\frac{1}{2}+\delta} = \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$ and the previous estimates hold true.*

1.4.4. *Statement in the case $\nu = \nu'$.*

Theorem 15. *Assume $F \neq 1$. For any $\mathbb{C}_0 \ge 1$, $\delta \in \left]0, \frac{1}{10}\right]$, $\gamma \in \left]0, \frac{\delta}{2}\right[$, and any $\alpha_0 > 0$, if we define $\eta > 0$ such that*

$$\gamma = (1-2\eta)\frac{\delta}{2} \quad \left(\textit{that is, } \eta = \tfrac{1}{2}\left(1 - \tfrac{2\gamma}{2}\right)\right),$$

there exist $\varepsilon_0, \mathbb{B}_0 > 0$ (*both of them depending on* $F, \nu, \mathbb{C}_0, \delta, \gamma, \alpha_0$) *such that for all* $\varepsilon \in]0, \varepsilon_0]$ *and all divergence-free initial data* $U_{0,\varepsilon} = U_{0,\varepsilon,\mathrm{QG}} + U_{0,\varepsilon,\mathrm{osc}}$ *satisfying*

(1) *there exists a quasigeostrophic vector field* $\widetilde{U}_{0,\mathrm{QG}} \in H^{\frac{1}{2}+\delta}$ *such that*

$$\begin{cases} \|U_{0,\varepsilon,\mathrm{QG}} - \widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}} \leq \mathbb{C}_0 \varepsilon^{\alpha_0}, \\ \|\widetilde{U}_{0,\mathrm{QG}}\|_{H^{1/2+\delta}} \leq \mathbb{C}_0, \end{cases} \tag{1-13}$$

(2) $U_{0,\varepsilon,\mathrm{osc}} \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}+\delta}$ *with* $\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2} \cap \dot{H}^{1/2+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}$,

then system (PE_ε) *has a unique global solution* $U_\varepsilon \in \dot{E}^s$ *for all* $s \in \left[\frac{1}{2}, \frac{1}{2} + \eta\delta\right]$, *and if we define*

- $\widetilde{U}_{\mathrm{QG}}$ *as the unique global solution of* (QG) *in* $\dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$,
- W_ε *as the unique global solution of* (1-10) *in* $\dot{E}^{\frac{1}{2}} \cap \dot{E}^{\frac{1}{2}+\delta}$,
- $\delta_\varepsilon = U_\varepsilon - \widetilde{U}_{\mathrm{QG}} - W_\varepsilon$,

then for all $s \in \left[\frac{1}{2}, \frac{1}{2} + \eta\delta\right]$

$$\|\delta_\varepsilon\|_{\dot{E}^s} \leq \mathbb{B}_0 \varepsilon^{\min(\alpha_0, \frac{\delta\eta}{2})}. \tag{1-14}$$

Moreover if we ask for more low-frequency regularity for the initial oscillating part, *that is*, $U_{0,\varepsilon,\mathrm{osc}} \in \dot{H}^{\frac{1}{2}-\delta} \cap \dot{H}^{\frac{1}{2}+\delta}$ *with* $\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2-\delta} \cap \dot{H}^{1/2+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}$ *then* (1-14) *is true for all* $s \in \left[\frac{1}{2} - \eta\delta, \frac{1}{2} + \eta\delta\right]$ *and we also can get rid of the oscillations* W_ε *and obtain that*

$$\|U_\varepsilon - \widetilde{U}_{\mathrm{QG}}\|_{L^2 L^\infty} \leq \mathbb{B}_0 \varepsilon^{\min(\alpha_0, \frac{\delta\eta}{2})}.$$

Remark 16. Compared to Theorem 1.3 from [Charve 2006] we highly reduced the regularity of the initial data, only the quasigeostrophic part lies in a inhomogeneous space, and we allow a far greater blowup in ε for the oscillating part, keeping a satisfying convergence rate as a power of ε (in accordance with physicists) for any size of the initial quasigeostrophic part.

Remark 17. In [Lee and Takada 2017; Iwabuchi et al. 2017] there is a smallness condition for the initial quasigeostrophic part (and also for the oscillating part in some sense). Their result states there exist $\delta_{1,2} > 0$ such that for any initial data satisfying (1-3), there exists a global unique mild solution for any $\varepsilon > 0$. This has to be compared with our formulation, where we prove that for any size $\mathbb{C}_0$ and any initial data with $\|U_{0,\varepsilon,\mathrm{QG}}\| \leq \mathbb{C}_0$ and $\|U_{0,\varepsilon,\mathrm{osc}}\| \leq \mathbb{C}_0 \varepsilon^{-\gamma}$, there exists a unique global solution when $\varepsilon \leq \varepsilon_0$.

Remark 18. Compared to the assumptions in [Iwabuchi et al. 2017, Theorems 1.3 and 1.5], we reach the same regularity for the oscillating part, we ask more regularity for the initial QG-part, and we ask more low-frequency regularity for both of them (we have to assume $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}$ as we need to consider Fujita–Kato strong solutions)

$$\begin{cases} U_{0,\varepsilon,\mathrm{osc}} \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}+\delta} \quad (\dot{H}^{\frac{1}{2}+\delta} \text{ in [Iwabuchi et al. 2017]}), \\ U_{0,\varepsilon,\mathrm{QG}} \in H^{\frac{1}{2}+\delta} \quad (\dot{H}^{\frac{1}{2}} \text{ in [Iwabuchi et al. 2017]}), \end{cases}$$

but *we do not ask any smallness to the initial quasigeostrophic part*, and we provide global strong solutions *in the energy spaces* $\dot{E}^s$ for any $s \in \left[\frac{1}{2} - \eta\delta, \frac{1}{2} + \eta\delta\right]$ (compared to mild solutions in $L^4(\mathbb{R}_+, \dot{W}^{\frac{1}{2},3})$).

Remark 19. At first sight our blow-up rate seems slightly less general than the one from [Iwabuchi et al. 2017] $\big($in that work they require $\varepsilon^{\frac{\delta}{2}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2+\delta}}$ be smaller than some $\delta_2>0$, and in the present work, we choose any $\mathbb{C}_0$ and require $\varepsilon^{\gamma}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2}\cap\dot H^{1/2+\delta}}\le\mathbb{C}_0$ for any $\gamma<\frac{\delta}{2}\big)$ but in our result we look for explicit rates of convergence as powers of ε. We refer to Remark 32 for more details.

Remark 20. We refer to Remark 48 for a comparison of the Strichartz estimates we use and the ones from [Iwabuchi et al. 2017].

1.4.5. *Statement in the general case $\nu\neq\nu'$.*

Theorem 21. *Assume $F\neq 1$. Let $\delta\in\left]0,\frac12\right]$, $q=\frac{2}{1+\delta}$, $\alpha_0>0$, $m\in\left]0,\frac{1}{100}\right]$, and $M,\eta>0$ such that*

$$0<2\eta\le\frac{M}{m}\le\frac12\frac{1}{5+\delta},$$

and let $\gamma_0\in\left]0,\frac{M\delta}{4}\right]$. If we define $R_\varepsilon=\varepsilon^{-M}$ and $r_\varepsilon=\varepsilon^{m}$ then for all $\mathbb{C}_0\ge1$ there exist ε_0, $\mathbb{B}_0$ (all of them depending on $F,\nu,\nu',\mathbb{C}_0,\delta,\gamma,\alpha_0$) such that for all initial data $U_{0,\varepsilon}=U_{0,\varepsilon,\mathrm{osc}}+U_{0,\varepsilon,\mathrm{QG}}$ satisfying

(1) *there exists a quasigeostrophic vector field $\widetilde U_{0,\mathrm{QG}}\in H^{\frac12+\delta}$ such that*

$$\begin{cases}\|U_{0,\varepsilon,\mathrm{QG}}-\widetilde U_{0,\mathrm{QG}}\|_{H^{1/2+\delta}}\le\mathbb{C}_0\varepsilon^{\alpha_0},\\ \|\widetilde U_{0,\mathrm{QG}}\|_{H^{1/2+\delta}}\le\mathbb{C}_0,\end{cases}\tag{1-15}$$

(2) $U_{0,\varepsilon,\mathrm{osc}}\in\dot B^{\frac12}_{q,q}\cap\dot H^{\frac12+\delta}$ *with* $\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot B^{1/2}_{q,q}\cap\dot H^{1/2+\delta}}\le\mathbb{C}_0\varepsilon^{-\gamma}$,

system (PE$_\varepsilon$) *has a unique global solution $U_\varepsilon\in\dot E^s$ for all $s\in\left[\frac12-\eta\delta,\frac12+\eta\delta\right]$. Moreover, with the same notation as in Theorem 14 (replacing W_ε by W_ε^T, which involves m,M),*

$$\|\delta_\varepsilon\|_{\dot E^s}\le\mathbb{B}_0\varepsilon^{\min(\alpha_0,\frac{M\delta}{4})},\tag{1-16}$$

and finally, thanks to the Strichartz estimates, we can get rid of the oscillations W_ε^T and obtain

$$\|U_\varepsilon-\widetilde U_{\mathrm{QG}}\|_{L^2(\mathbb{R}_+,L^\infty)}\le\mathbb{B}_0\varepsilon^{\min(\alpha_0,\frac{M\delta}{4})}.$$

Remark 22. This generalizes the first result from [Charve 2006]: in the present work we reduced the assumptions on high and low frequencies for the initial oscillating part, and the choice for r_ε and R_ε now correctly fits the power of ε provided by the Strichartz estimates, which produces a convergence rate as a power of ε without any assumption on the viscosities.

Remark 23. The low-frequencies assumption $U_{0,\varepsilon,\mathrm{osc}}\in\dot B^{\frac12}_{q,q}$ is mainly needed to produce a positive power of ε when estimating $\|\chi(|D|/R_\varepsilon)\chi(|D_3|/r_\varepsilon)U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^s}$ $\big($the other need is to reach regularities less than $\frac12\big)$, and the high-frequencies assumption $U_{0,\varepsilon,\mathrm{osc}}\in\dot H^{\frac12+\delta}$ helps to estimate $\|(1-\chi(|D|/R_\varepsilon))U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^s}$.

Remark 24. The classical Bernstein estimate ensures that $\dot B^{\frac12}_{q,q}\hookrightarrow\dot H^{\frac12-\frac32\delta}$ so that $U_{0,\varepsilon,\mathrm{osc}}\in\dot H^s$ for all $s\in\left[\frac12-\frac32\delta,\frac12+\delta\right]$.

The rest of this article is structured as follows: we will first prove Theorem 14, then turn to the proof of Theorem 15 in the case $\nu = \nu'$ (much easier computations to obtain the eigenvalues and vectors, but needs more careful use for the Strichartz estimates as W_ε is not frequency-truncated) and we will finish with the proof of Theorem 21 (the eigenvectors are not mutually orthogonal anymore, and care is needed for the frequency-truncated terms). We end the article with an Appendix gathering results on Sobolev and Besov spaces, the process of diagonalization of system (A-79), and the new Strichartz estimates that allow us to reach this level of precision.

2. Proof of the results

2.1. ***The limit system.*** If $\widetilde{U}_{0,\mathrm{QG}}$ is as described in Theorem 14, we regularize it by introducing for $\lambda > 0$ (where χ is the smooth cut-off function introduced in the Appendix)

$$\widetilde{U}^{\lambda}_{0,\mathrm{QG}} \stackrel{\text{def}}{=} \chi\left(\frac{|D|}{\lambda}\right)\widetilde{U}_{0,\mathrm{QG}}.$$

Then $\widetilde{U}^{\lambda}_{0,\mathrm{QG}} \in H^1$ and applying Theorem 2 from [Charve 2004] there exists a unique global solution $\widetilde{U}^{\lambda}_{\mathrm{QG}} \in \dot{E}^0 \cap \dot{E}^1$ to system (QG) and thanks to Lemma 2.1 from [Charve 2008] we apply (1-12) to $\widetilde{U}^{\lambda}_{\mathrm{QG}}$ and for all $t \in \mathbb{R}_+$ (taking $\mathbb{C}_0 = \max(1, \|\widetilde{U}_{0,\mathrm{QG}}\|_{\dot{H}^{1/2+\delta}})$)

$$\begin{aligned}\|\widetilde{U}^{\lambda}_{\mathrm{QG}}\|^2_{L^\infty_t H^{1/2+\delta}} + \min(\nu,\nu')\int_0^t \|\nabla \widetilde{U}^{\lambda}_{\mathrm{QG}}(\tau)\|^2_{H^{1/2+\delta}}\,d\tau &\\ \leq C_{\delta,\nu_0}\left\|\chi\left(\frac{|D|}{\lambda}\right)\widetilde{U}_{0,\mathrm{QG}}\right\|^2_{H^{1/2+\delta}} \max\left(1, \left\|\chi\left(\frac{|D|}{\lambda}\right)\widetilde{U}_{0,\mathrm{QG}}\right\|^{\frac{1}{\delta}}_{H^{1/2+\delta}}\right)&\\ \leq C_{\delta,\nu_0}\|\widetilde{U}_{0,\mathrm{QG}}\|^2_{H^{1/2+\delta}}\max(1, \|\widetilde{U}_{0,\mathrm{QG}}\|^{\frac{1}{\delta}}_{H^{1/2+\delta}}) \leq C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}.& \qquad (2\text{-}17)\end{aligned}$$

Then (taking $\lambda = n$) we prove that $(\widetilde{U}^{n}_{\mathrm{QG}})_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $E^{\frac{1}{2}+\delta} = \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$. For $n \geq m$, let us define $\tilde{\delta}_{n,m} = \widetilde{U}^{n}_{\mathrm{QG}} - \widetilde{U}^{m}_{\mathrm{QG}}$, which satisfies the system

$$\begin{cases} \partial_t \tilde{\delta}_{n,m} - \Gamma\tilde{\delta}_{n,m} = -\mathcal{Q}(\widetilde{U}^{n}_{\mathrm{QG}} \cdot \nabla\tilde{\delta}_{n,m} + \tilde{\delta}_{n,m}\cdot\nabla\widetilde{U}^{m}_{\mathrm{QG}}), \\ \tilde{\delta}_{n,m|t=0} = \left(\chi\left(\frac{|D|}{n}\right) - \chi\left(\frac{|D|}{m}\right)\right)\widetilde{U}_{0,\mathrm{QG}}. \end{cases} \qquad (2\text{-}18)$$

For any $s \in \left[0, \frac{1}{2}+\delta\right]$, taking the $\dot{H}^s$-inner product and then using the classical Sobolev product laws (see Proposition 38), we get $\left((s_1,s_2) \in \left\{\left(1, s-\frac{1}{2}\right), \left(s, \frac{1}{2}\right)\right\}\right)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{\delta}_{n,m}\|^2_{\dot{H}^s} &+ \nu_0\|\nabla\tilde{\delta}_{n,m}\|^2_{\dot{H}^s}\\ &\leq C\|\widetilde{U}^{n}_{\mathrm{QG}}\cdot\nabla\tilde{\delta}_{n,m} + \tilde{\delta}_{n,m}\cdot\nabla\widetilde{U}^{m}_{\mathrm{QG}}\|_{\dot{H}^{s-1}}\|\tilde{\delta}_{n,m}\|_{\dot{H}^{s+1}}\\ &\leq C(\|\widetilde{U}^{n}_{\mathrm{QG}}\|_{\dot{H}^1}\|\tilde{\delta}_{n,m}\|^{\frac{1}{2}}_{\dot{H}^s}\|\tilde{\delta}_{n,m}\|^{\frac{3}{2}}_{\dot{H}^{s+1}} + \|\widetilde{U}^{m}_{\mathrm{QG}}\|_{\dot{H}^{3/2}}\|\tilde{\delta}_{n,m}\|_{\dot{H}^s}\|\tilde{\delta}_{n,m}\|_{\dot{H}^{s+1}})\\ &\leq \frac{\nu_0}{2}\|\nabla\tilde{\delta}_{n,m}\|^2_{\dot{H}^s} + \frac{C}{\nu_0}\|\tilde{\delta}_{n,m}\|^2_{\dot{H}^s}\left(\|\nabla\widetilde{U}^{m}_{\mathrm{QG}}\|^2_{\dot{H}^{1/2}} + \frac{1}{\nu_0^2}\|\widetilde{U}^{n}_{\mathrm{QG}}\|^2_{\dot{H}^{1/2}}\|\nabla\widetilde{U}^{n}_{\mathrm{QG}}\|^2_{\dot{H}^{1/2}}\right). \qquad (2\text{-}19)\end{aligned}$$

Thanks to the Gronwall lemma and using (2-17), we obtain that

$$\|\tilde{\delta}_{n,m}\|^2_{E^{1/2+\delta}} \leq \|\tilde{\delta}_{n,m}(0)\|^2_{H^{1/2+\delta}} e^{(C_{\delta,\nu_0}/\nu_0^2)\mathbb{C}_0^{2+1/\delta}(1+(C_{\delta,\nu_0}/\nu_0^2)\mathbb{C}_0^{2+1/\delta})}.$$

As $\|\tilde{\delta}_{n,m}(0)\|_{H^{1/2+\delta}}$ goes to zero when $m=\min(n,m)$ goes to infinity, the sequence is Cauchy and if we denote by $\widetilde{U}_{\mathrm{QG}}$ its limit in $E^{\frac{1}{2}+\delta}$, we immediately get that it solves system (QG) and satisfies the expected estimates. □

As an immediate consequence we easily bound $G^{b,l}$ (introduced with the auxiliary systems) as follows:

Proposition 25. *There exists a constant $C_F>0$ such that, for all $\delta\in\left]0,\frac{1}{2}\right]$ and $s\in\left[0,\frac{1}{2}+\delta\right]$,*

$$\begin{aligned} \int_0^\infty \|G^b(\tau)\|_{\dot{H}^s}\,d\tau &\leq \frac{C_F}{\nu_0} C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}, \\ \int_0^\infty \|G^l(\tau)\|^2_{\dot{H}^{s-1}}\,d\tau &\leq C_F\frac{|\nu-\nu'|^2}{\nu_0} C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}. \end{aligned} \tag{2-20}$$

Remark 26. In [Charve 2004] the previous terms were estimated for any $s\in[0,1]$ with $\|\widetilde{U}_{0,\mathrm{QG}}\|_{H^1}$.

Proof of Proposition 25. G^l is estimated as in [Charve 2004], and for G^b, as we wish to use only $\frac{1}{2}+\delta$ derivatives on $\widetilde{U}_{0,\mathrm{QG}}$, a much better way than in that work is to write (thanks to the Bony decomposition; see the Appendix for details)

$$\begin{aligned} \|G^b\|_{\dot{H}^s} &\leq C_F\|\widetilde{U}_{\mathrm{QG}}\cdot\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s} \leq C_F\|\operatorname{div}(\widetilde{U}_{\mathrm{QG}}\otimes\widetilde{U}_{\mathrm{QG}})\|_{\dot{H}^s} \\ &\leq C_F(2\|T_{\widetilde{U}_{\mathrm{QG}}}\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s+1}}+\|R(\widetilde{U}_{\mathrm{QG}},\widetilde{U}_{\mathrm{QG}})\|_{\dot{H}^{s+1}}) \\ &\leq C_F(2\|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty}+\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{B}^0_{\infty,\infty}})\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s+1}}. \end{aligned} \tag{2-21}$$

Then using the injection $\dot{B}^0_{\infty,1}\hookrightarrow L^\infty$ together with the Bernstein lemma and Lemma 27 below (whose proof is close to Lemma 5 from [Charve 2016]), we obtain that

$$2\|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty}+\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{B}^0_{\infty,\infty}} \leq 3\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{B}^{3/2}_{2,1}} \leq C\|\widetilde{U}_{\mathrm{QG}}\|^{\frac{1}{2}}_{\dot{H}^{3/2-\delta}}\|\widetilde{U}_{\mathrm{QG}}\|^{\frac{1}{2}}_{\dot{H}^{3/2+\delta}}, \tag{2-22}$$

and we end up with (using also (1-12))

$$\begin{aligned} \int_0^\infty \|G^b\|_{\dot{H}^s}\,d\tau &\leq C_F\|\nabla\widetilde{U}_{\mathrm{QG}}\|^{\frac{1}{2}}_{L^2\dot{H}^{1/2-\delta}}\|\nabla\widetilde{U}_{\mathrm{QG}}\|^{\frac{1}{2}}_{L^2\dot{H}^{1/2+\delta}}\|\nabla\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^s} \\ &\leq \frac{C_F}{\nu_0}C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}. \end{aligned} \tag{2-23}$$

This completes the proof of the proposition. □

Lemma 27. *For any $\alpha,\beta>0$ there exists a constant $C_{\alpha,\beta}>0$ such that for any $u\in\dot{H}^{s-\alpha}\cap\dot{H}^{s+\beta}$ we have $u\in\dot{B}^s_{2,1}$ and*

$$\|u\|_{\dot{B}^s_{2,1}} \leq C_{\alpha,\beta}\|u\|^{\frac{\beta}{\alpha+\beta}}_{\dot{H}^{s-\alpha}}\|u\|^{\frac{\alpha}{\alpha+\beta}}_{\dot{H}^{s+\beta}}. \tag{2-24}$$

2.2. *The case $\nu = \nu'$.*

2.2.1. *Estimates for W_ε.* Let us first focus on the linear system (1-10). Let us recall that thanks to Proposition 25 we obtain that (see [Charve 2004] for details) for any $s \in \left[\frac{1}{2}, \frac{1}{2} + \delta\right]$

$$\|W_\varepsilon\|_{\dot{E}^s}^2 \le \left(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}^2 + \frac{1}{2}\int_0^t \|G^b(\tau)\|_{\dot{H}^s}\right) e^{\frac{1}{2}\int_0^t \|G^b(\tau)\|_{\dot{H}^s}} \le \mathbb{D}_0(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}^2 + 1), \tag{2-25}$$

with

$$\mathbb{D}_0 \overset{\text{def}}{=} \frac{C_F}{\nu_0} C_{\delta,\nu_0} \mathbb{C}_0^{2+\frac{1}{\delta}} e^{\frac{C_F}{\nu_0} C_{\delta,\nu_0} \mathbb{C}_0^{2+1/\delta}}.$$

One of the main ingredients is to provide a generalization of the Strichartz estimates obtained in [Charve 2006]. Our new Strichartz estimates are much more flexible and we refer to the Appendix for the most general formulation (see Propositions 47 and 51). We also postpone to the end of the next section the precise statement of the Strichartz estimates we will use.

2.2.2. *Energy estimates.* As explained in Section 1.3, we already have a local strong solution U_ε whose lifespan is denoted by T_ε^*. As seen in the previous section $\widetilde{U}_{\mathrm{QG}}$ and W_ε exist globally, and δ_ε is well-defined in $\dot{E}_T^{\frac{1}{2}} \cap \dot{E}_T^{\frac{1}{2}+\delta}$ for all $T < T_\varepsilon^*$ and we can perform for any $s \in \left[\frac{1}{2}, \frac{1}{2} + \eta\delta\right]$ the inner product in $\dot{H}^s$ of system (1-11) with δ_ε. We have to bound each term from the right-hand side.

Let us begin with the easiest terms, namely F_1, F_2 and F_3: thanks to the classical Sobolev product laws ($(s_1, s_2) = \left(\frac{1}{2}, s\right)$; see Proposition 38), we obtain that

$$|(F_1 \mid \delta_\varepsilon)_{\dot{H}^s}| \le \|\delta_\varepsilon \cdot \nabla\delta_\varepsilon\|_{\dot{H}^{s-1}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \le C\|\delta_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}}^2. \tag{2-26}$$

Similarly we obtain that

$$\begin{aligned} |(F_2 \mid \delta_\varepsilon)_{\dot{H}^s}| &\le C\|\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{\frac{1}{2}}} \|\delta_\varepsilon\|_{\dot{H}^s} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \le \frac{\nu}{16}\|\delta_\varepsilon\|_{\dot{H}^{s+1}}^2 + \frac{C}{\nu}\|\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{1/2}}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_3 \mid \delta_\varepsilon)_{\dot{H}^s}| &\le C\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^1} \|\delta_\varepsilon\|_{\dot{H}^{s+1/2}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \le \frac{\nu}{16}\|\delta_\varepsilon\|_{\dot{H}^{s+1}}^2 + \frac{C}{\nu^3}\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^1}^4 \|\delta_\varepsilon\|_{\dot{H}^s}^2. \end{aligned} \tag{2-27}$$

Compared to [Charve 2004; 2006] we cannot use for the other F_i the same methods because they would produce (after using the Gronwall lemma) a coefficient of the form $e^{\|W_\varepsilon\|_{\dot{E}^s}}$ which would ruin our efforts to allow large initial blow up for the oscillating part (which could only be of size $(-\ln\varepsilon)^\beta$). We need to estimate carefully these terms and especially use as much as possible the new Strichartz estimates (giving positive powers of ε thanks to Proposition 47) and the least possible basic energy estimates on W_ε (that produce $\varepsilon^{-\gamma}$ from (2-25)).

The most obvious way would be to use the paraproduct and remainder laws (see the Appendix). For example with F_7, as $s - 1 < 0$, we have

$$\begin{aligned} |(F_7 \mid \delta_\varepsilon)_{\dot{H}^s}| &\le \|W_\varepsilon \cdot \nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-1}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \\ &\le C(\|T_{W_\varepsilon}\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-1}} + \|T_{\nabla\widetilde{U}_{\mathrm{QG}}} W_\varepsilon\|_{\dot{H}^{s-1}} + \|\operatorname{div}(R(W_\varepsilon, \widetilde{U}_{\mathrm{QG}}))\|_{\dot{H}^{s-1}})\|\delta_\varepsilon\|_{\dot{H}^{s+1}} \\ &\le C(\|W_\varepsilon\|_{L^\infty}\|\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-1}} + \|\nabla\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-1}}\|W_\varepsilon\|_{\dot{B}^0_{\infty,\infty}} + \|W_\varepsilon\|_{\dot{B}^0_{\infty,\infty}}\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s})\|\delta_\varepsilon\|_{\dot{H}^{s+1}} \\ &\le C\|W_\varepsilon\|_{\dot{B}^0_{\infty,1}}\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s}\|\delta_\varepsilon\|_{\dot{H}^{s+1}} \le \frac{\nu}{16}\|\delta_\varepsilon\|_{\dot{H}^{s+1}}^2 + \frac{C}{\nu}\|W_\varepsilon\|_{\dot{B}^0_{\infty,1}}^2\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s}^2. \end{aligned} \tag{2-28}$$

This result could be also usable for F_5 but to deal with $\|W_\varepsilon\|_{L^p\dot{B}^0_{\infty,1}}$ from Proposition 47 we would have to use Lemma 27 which would force us to have a slightly smaller range for γ. More important, for F_8 this method would force us to require $\gamma < \frac{\delta}{4}$, which is clearly not optimal.

Finally, the most important problem is that the previous estimates cannot be used to estimate F_4 and F_6. Indeed for instance if we wish to estimate F_6 the same way

$$\|F_6\|_{\dot{H}^{s-1}} \leq C(\|T_{\widetilde{U}_{\mathrm{QG}}}\nabla W_\varepsilon\|_{\dot{H}^{s-1}} + \|T_{\nabla W_\varepsilon}\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-1}} + \|\operatorname{div}(R(\widetilde{U}_{\mathrm{QG}}, W_\varepsilon))\|_{\dot{H}^{s-1}}),$$

and the first paraproduct (see the Appendix for the Bony decomposition) leads to an obstruction as the only possibilities to estimate it are (for $\beta > s$)

$$\|T_{\widetilde{U}_{\mathrm{QG}}}\nabla W_\varepsilon\|_{\dot{H}^{s-1}} \leq C \begin{cases} \|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty}\|W_\varepsilon\|_{\dot{H}^s}, \\ \|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s-\beta}}\|W_\varepsilon\|_{\dot{B}^\beta_{\infty,\infty}}, \end{cases} \tag{2-29}$$

In the first estimate each term is well-defined but the $\dot{H}^s$-norm of W_ε produces negative powers of ε, and in the second one the first term is not defined ($\widetilde{U}_{\mathrm{QG}}$ is not defined for negative regularities). It is possible to deal with this term using the same idea as in [Charve 2004] (with $a, b \geq 1$ so that $\frac{1}{a} + \frac{1}{b} = 1$),

$$\begin{aligned} \int_0^t \|\widetilde{U}_{\mathrm{QG}} \cdot \nabla W_\varepsilon\|^2_{\dot{H}^{s-1}}\, d\tau &\leq C \int_0^t \|\widetilde{U}_{\mathrm{QG}} \cdot \nabla W_\varepsilon\|_{L^2}\|\widetilde{U}_{\mathrm{QG}} \cdot \nabla W_\varepsilon\|_{\dot{H}^{2(s-1)}}\, d\tau \\ &\leq \|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty L^2}\|\nabla W_\varepsilon\|_{L^a L^\infty}\|\widetilde{U}_{\mathrm{QG}}\|_{L^b \dot{H}^{s+1/2}}\|W_\varepsilon\|_{L^\infty \dot{H}^s}, \end{aligned} \tag{2-30}$$

and due to the gradient pounding on W_ε, the most interesting use of Proposition 47 consists in choosing a as close as possible to 1, which implies that b is very large. As $s + \frac{1}{2} \geq 1$, this forces us to use (1-12) for regularity index close to 1 (in this case it would be necessary to require that $\widetilde{U}_{0,\mathrm{QG}} \in H^s$ with s close to 1), which was something we wished to avoid as we only consider indices $s \leq \frac{1}{2} + \delta$. Moreover it would also produce a clearly nonoptimal decrease in ε.

Finally both of these two methods fail for F_4: the former for the same reason as for F_6, and the latter as we cannot consider $\|\delta_\varepsilon\|_{L^2}$: there is a lack of derivatives pounding on δ_ε.

To overcome this lack of derivatives, we will distribute them differently among the whole $\dot{H}^s$-inner product. We will do this for all the last five external force terms and the idea will be to do as in [Charve 2016; 2018a] and deal with the nonlocal operator $|D|^s$ applied to a product and dispatch s derivatives on δ_ε and obtain something close to the second line of (2-29). More precisely, we directly deal with the inner product as follows:

$$|(F_4 \mid \delta_\varepsilon)_{\dot{H}^s}| = |(\operatorname{div}(\delta_\varepsilon \otimes W_\varepsilon) \mid \delta_\varepsilon)_{\dot{H}^s}| = |(|D|^s(\delta_\varepsilon \cdot W_\varepsilon) \mid |D|^s \nabla \delta_\varepsilon)_{L_2}|. \tag{2-31}$$

The nonlocal operator $|D|^s$ can be written as a singular principal value integral (we refer to [Stein 1970; Córdoba and Córdoba 2004; Hmidi and Keraani 2007; Abidi and Hmidi 2008; Charve 2016; 2018a]) and when the index s lies in $]0, 1[$ (which is the case here as s is close to $\frac{1}{2}$) it is a classical singular integral:

$$|D|^s f(x) = C_s \int_{\mathbb{R}^3} \frac{f(x) - f(y)}{|x-y|^{3+s}}\, dy = C_s \int_{\mathbb{R}^3} \frac{f(x) - f(x-y)}{|y|^{3+s}}\, dy.$$

Let us recall that an equivalent formulation of the Besov norm involves translations as stated in the following result:

Theorem 28 [Bahouri et al. 2011, 2.36]. *Let $s \in]0,1[$ and $p, r \in [1,\infty]$. There exists a constant C such that for any $u \in \dot{B}^s_{p,r}$*

$$C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leq \Big\|\frac{\|u(\cdot-y)-u(\cdot)\|_{L^p}}{|y|^s}\Big\|_{L^r(\mathbb{R}^d;\frac{dy}{|y|^d})} \leq C\|u\|_{\dot{B}^s_{p,r}}.$$

From this we can prove exactly as in [Charve 2018a] (see Section A.3.1 there) the following result:

Proposition 29. *For any $s \in]0,1[$ and any smooth functions f, g we can write*

$$|D|^s(fg) = (|D|^s f)g + f|D|^s g + M_s(f,g),$$

where the bilinear operator M_s is defined for all $x \in \mathbb{R}^3$ by

$$M_s(f,g)(x) = \int_{\mathbb{R}^3} \frac{(f(x)-f(x-y))(g(x)-g(x-y))}{|y|^{3+s}}\,dy. \tag{2-32}$$

Moreover there exists a constant C_s such that for all f, g and all $p, p_1, p_2, r_1, r_2 \in [1,\infty]$ and $s_1, s_2 > 0$ satisfying

$$\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}, \quad 1 = \frac{1}{r_1}+\frac{1}{r_2}, \quad s_1+s_2 = s,$$

we have

$$\|M_s(f,g)\|_{L^p} \leq C_s\|f\|_{\dot{B}^{s_1}_{p_1,r_1}}\|g\|_{\dot{B}^{s_2}_{p_2,r_2}}. \tag{2-33}$$

Remark 30. The additional term M_s allows us to freely dispatch the derivatives as desired provided that $s_1, s_2 > 0$, which will force us to spend a small extra amount of derivative in order to meet these conditions. So even if it is not possible to use Proposition 29 for $(s_1, s_2) = (s, 0)$, our method will enable us to do nearly as if we could estimate $\|M_s(\delta_\varepsilon, W_\varepsilon)\|_{L^2}$ by $\|\delta_\varepsilon\|_{\dot{H}^{1/2}}\||D|^s W_\varepsilon\|_{L^6}$.

More precisely for a small $\alpha_1 > 0$, instead of (2-31), we will write (also using the Sobolev injections):

$$\begin{aligned}
&|(F_6\,|\,\delta_\varepsilon)_{\dot{H}^s}| \\
&= |(\mathrm{div}(\widetilde{U}_{\mathrm{QG}}\otimes W_\varepsilon)\,|\,\delta_\varepsilon)_{\dot{H}^s}| = |(|D|^{s+\alpha_1}(\widetilde{U}_{\mathrm{QG}}\cdot W_\varepsilon)\,|\,|D|^{s-\alpha_1}\nabla\delta_\varepsilon)_{L_2}| \\
&\leq \|(|D|^{s+\alpha_1}\widetilde{U}_{\mathrm{QG}})\cdot W_\varepsilon + \widetilde{U}_{\mathrm{QG}}\cdot|D|^{s+\alpha_1}W_\varepsilon + M_{s+\alpha_1}(\widetilde{U}_{\mathrm{QG}}, W_\varepsilon)\|_{L^{6/(3+2\alpha_1)}}\cdot\||D|^{s-\alpha_1}\nabla\delta_\varepsilon\|_{L^{6/(3-2\alpha_1)}} \\
&\leq C(\||D|^{s+\alpha_1}\widetilde{U}_{\mathrm{QG}}\|_{L^2}\|W_\varepsilon\|_{L^{3/\alpha_1}} + \|\widetilde{U}_{\mathrm{QG}}\|_{L^3}\||D|^{s+\alpha_1}W_\varepsilon\|_{L^{6/(1+2\alpha_1)}} + \|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s}\|W_\varepsilon\|_{\dot{B}^{\alpha_1}_{3/\alpha_1,2}}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\||D|^{s-\alpha_1}\nabla\delta_\varepsilon\|_{\dot{H}^{\alpha_1}} \\
&\leq C(\|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{s+\alpha_1}}\|W_\varepsilon\|_{L^{3/\alpha_1}} + \|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^{1/2}}\||D|^{s+\alpha_1}W_\varepsilon\|_{L^{6/(1+2\alpha_1)}} + \|\widetilde{U}_{\mathrm{QG}}\|_{\dot{H}^s}\|W_\varepsilon\|_{\dot{B}^{\alpha_1}_{3/\alpha_1,2}})\cdot\|\delta_\varepsilon\|_{\dot{H}^{s+1}} \\
&\leq \frac{\nu}{16}\|\delta_\varepsilon\|^2_{\dot{H}^{s+1}} + \frac{C}{\nu}\big(\|\widetilde{U}_{\mathrm{QG}}\|^{2(1-\alpha_1)}_{\dot{H}^s}\|\widetilde{U}_{\mathrm{QG}}\|^{2\alpha_1}_{\dot{H}^{s+1}}\|W_\varepsilon\|^2_{L^{3/\alpha_1}} \\
&\qquad\qquad + \|\widetilde{U}_{\mathrm{QG}}\|^2_{\dot{H}^{1/2}}\||D|^{s+\alpha_1}W_\varepsilon\|^2_{L^{6/(1+2\alpha_1)}} + \|\widetilde{U}_{\mathrm{QG}}\|^2_{\dot{H}^s}\|W_\varepsilon\|^2_{\dot{B}^{\alpha_1}_{3/\alpha_1,2}}\big).
\end{aligned} \tag{2-34}$$

Remark 31. Notice that as $\delta_\varepsilon, W_\varepsilon, \widetilde{U}_{\mathrm{QG}}$ are divergence-free, we will systematically (thanks to integration by parts) transfer the divergence as a gradient on the right-hand part of the inner product, and as a consequence the computations are the same respectively for F_4 and F_5 and for F_6 and F_7.

Let us continue with F_4, by the classical Sobolev interpolation and Young estimates, we can write that (for $\alpha_2 > 0$ small)

$$\begin{aligned}
&|(F_4 \,|\, \delta_\varepsilon)_{\dot H^s}|\\
&= |(\operatorname{div}(\delta_\varepsilon \otimes W_\varepsilon) \,|\, \delta_\varepsilon)_{\dot H^s}| = |(|D|^{s+\alpha_2}(\delta_\varepsilon \cdot W_\varepsilon) \,|\, |D|^{s-\alpha_2}\nabla\delta_\varepsilon)_{L_2}|\\
&\le C\,\|(|D|^{s+\alpha_2})\cdot W_\varepsilon + \delta_\varepsilon \cdot |D|^{s+\alpha_2} W_\varepsilon + M_{s+\alpha_2}(\delta_\varepsilon, W_\varepsilon)\|_{L^{6/(3+2\alpha_2)}} \cdot \||D|^{s-\alpha_2}\nabla\delta_\varepsilon\|_{L^{6/(3-2\alpha_2)}}\\
&\le C\big(\||D|^{s+\alpha_2}\delta_\varepsilon\|_{L^2}\|W_\varepsilon\|_{L^{3/\alpha_2}} + \|\delta_\varepsilon\|_{L^3}\||D|^{s+\alpha_2} W_\varepsilon\|_{L^{6/(1+2\alpha_2)}} + \|\delta_\varepsilon\|_{\dot H^s}\|W_\varepsilon\|_{\dot B^{\alpha_2}_{3/\alpha_2,2}}\big)\\
&\hspace{25em} \times \||D|^{s-\alpha_2}\nabla\delta_\varepsilon\|_{\dot H^{\alpha_2}}\\
&\le C\,\|\delta_\varepsilon\|_{\dot H^s}^{1-\alpha_2}\|\delta_\varepsilon\|_{\dot H^{s+1}}^{1+\alpha_2}\|W_\varepsilon\|_{L^{3/\alpha_2}} + C\big(\|\delta_\varepsilon\|_{L^3}\||D|^{s+\alpha_2} W_\varepsilon\|_{L^{6/(1+2\alpha_2)}} + \|\delta_\varepsilon\|_{\dot H^s}\|W_\varepsilon\|_{\dot B^{\alpha_2}_{3/\alpha_2,2}}\big)\|\delta_\varepsilon\|_{\dot H^{s+1}}\\
&\le \frac{\nu}{16}\|\delta_\varepsilon\|_{\dot H^{s+1}}^2\\
&\qquad + C\,\|\delta_\varepsilon\|_{\dot H^s}^2\bigg(\frac{1}{\nu^{\frac{1+\alpha_2}{1-\alpha_2}}}\|W_\varepsilon\|_{L^{3/\alpha_2}}^{\frac{2}{1-\alpha_2}} + \frac{1}{\nu}\|W_\varepsilon\|_{\dot B^{\alpha_2}_{3/\alpha_2,2}}^2\bigg) + \frac{C}{\nu}\|\delta_\varepsilon\|_{\dot H^{1/2}}^2\||D|^{s+\alpha_2} W_\varepsilon\|_{L^{6/(1+2\alpha_2)}}^2.
\end{aligned} \tag{2-35}$$

Finally we estimate F_8 with the same method, but the term $M_{s+\alpha_3}(W_\varepsilon, W_\varepsilon)$ has to be estimated differently (otherwise we end up with the same problem as explained in the beginning of this section): instead of estimating it as for the other terms by

$$\|W_\varepsilon\|_{\dot H^s}\|W_\varepsilon\|_{\dot B^{\alpha_3}_{3/\alpha_3,2}}$$

(the first term being L^∞, and the second L^2 in time), we will estimate it by

$$\|W_\varepsilon\|_{\dot H^{s+\alpha_3-\beta\delta}}\|W_\varepsilon\|_{\dot B^{\beta\delta}_{3/\alpha_3,2}},$$

for small enough $\alpha_3, \beta > 0$ so that the first term keeps L^∞ in time and the second one is L^2 (we try to be as close as possible to the forbidden choice $\beta = 0$). As we will make precise below, dealing with

$$\|W_\varepsilon\|_{L^\infty \dot H^s}^{2(1-\alpha_3)}\|W_\varepsilon\|_{L^2 \dot H^s}^{2\alpha_3}\|W_\varepsilon\|_{L^2 L^{3/\alpha_3}}^2$$

(for the first term) will only lead to $\gamma < \frac{\delta}{4}$, whereas

$$\|W_\varepsilon\|_{L^\infty \dot H^{s+\alpha_3}}^2\|W_\varepsilon\|_{L^2 L^{3/\alpha_3}}^2$$

will allow us to reach $\gamma < \frac{\delta}{2}$. For the same reason we will estimate the other term by

$$\|W_\varepsilon\|_{L^2 \dot B^{\beta\delta}_{3/\alpha_3,2}}$$

instead of

$$\|W_\varepsilon\|_{L^{2/(1-\alpha_3)} \dot B^{\beta\delta}_{3/\alpha_3,2}}.$$

Although this choice seems very close to the other, it allows us to use a smaller p in the Strichartz estimates, which allows a slightly wider range for θ helping us to reach $\gamma < \frac{\delta}{2}$ instead of $\gamma < \frac{\delta}{4}$. Once more, we try to obtain as close as possible to what we would get if it Proposition 29 could be applied for $s_1 = s + \alpha_3$ and $s_2 = 0$:

$$\begin{aligned} |(F_8 \mid \delta_\varepsilon)_{\dot H^s}| &= \||D|^{s+\alpha_3}(W_\varepsilon \otimes W_\varepsilon)\|_{L^{6/(3+2\alpha_3)}} \||D|^{s-\alpha_3}\nabla\delta_\varepsilon\|_{L^{6/(3-2\alpha_3)}} \\ &\le (2\||D|^{s+\alpha_3}W_\varepsilon\|_{L^2}\|W_\varepsilon\|_{L^{3/\alpha_3}} + \|W_\varepsilon\|_{\dot H^{s+\alpha_3-\beta\delta}}\|W_\varepsilon\|_{\dot B^{\beta\delta}_{3/\alpha_3,2}}) \cdot \||D|^{s-\alpha_3}\nabla\delta_\varepsilon\|_{\dot H^{\alpha_3}} \\ &\le \frac{\nu}{16}\|\delta_\varepsilon\|^2_{\dot H^{s+1}} + \frac{C}{\nu}(\|W_\varepsilon\|^2_{\dot H^{s+\alpha_3}}\|W_\varepsilon\|^2_{L^{3/\alpha_3}} + \|W_\varepsilon\|^2_{\dot H^{s+\alpha_3-\beta\delta}}\|W_\varepsilon\|^2_{\dot B^{\beta\delta}_{3/\alpha_3,2}}). \end{aligned} \tag{2-36}$$

We can now gather all the external force term estimates (2-26), (2-27), (2-35), (2-34), (2-36) and taking the $\dot H^s$-inner product of system (1-11) with δ_ε, we obtain that for all $s \in \left[\frac{1}{2}, \frac{1}{2} + \eta\delta\right]$ and all $t < T^*_\varepsilon$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\delta_\varepsilon\|^2_{\dot H^s} + \nu\|\nabla\delta_\varepsilon\|^2_{\dot H^s} \\ &\quad \le \left(C\|\delta_\varepsilon\|_{\dot H^{1/2}} + 8\frac{\nu}{16}\right)\|\delta_\varepsilon\|^2_{\dot H^s} \\ &\qquad + \frac{C}{\nu}\|\delta_\varepsilon\|^2_{\dot H^s}\left\{\|\nabla\widetilde U_{\mathrm{QG}}\|^2_{\dot H^{1/2}}\left(1 + \frac{1}{\nu^2}\|\widetilde U_{\mathrm{QG}}\|^2_{\dot H^{1/2}}\right) + \frac{1}{\nu^{\frac{2\alpha_2}{1-\alpha_2}}}\|W_\varepsilon\|^{\frac{2}{1-\alpha_2}}_{L^{3/\alpha_2}} + \|W_\varepsilon\|^2_{\dot B^{\alpha_2}_{3/\alpha_2,2}}\right\} \\ &\qquad + \frac{C}{\nu}\Big[\|\widetilde U_{\mathrm{QG}}\|^{2(1-\alpha_1)}_{\dot H^s}\|\widetilde U_{\mathrm{QG}}\|^{2\alpha_1}_{\dot H^{s+1}}\|W_\varepsilon\|^2_{L^{3/\alpha_1}} + \|\widetilde U_{\mathrm{QG}}\|^2_{\dot H^{1/2}}\||D|^{s+\alpha_1}W_\varepsilon\|^2_{L^{6/(1+2\alpha_1)}} \\ &\qquad\qquad + \|\widetilde U_{\mathrm{QG}}\|^2_{\dot H^s}\|W_\varepsilon\|^2_{\dot B^{\alpha_1}_{3/\alpha_1,2}} + \|\delta_\varepsilon\|^2_{\dot H^{1/2}}\||D|^{s+\alpha_2}W_\varepsilon\|^2_{L^{6/(1+2\alpha_2)}} \\ &\qquad\qquad\qquad + \|W_\varepsilon\|^2_{\dot H^{s+\alpha_3}}\|W_\varepsilon\|^2_{L^{3/\alpha_3}} + \|W_\varepsilon\|^2_{\dot H^{s+\alpha_3-\beta\delta}}\|W_\varepsilon\|^2_{\dot B^{\beta\delta}_{3/\alpha_3,2}}\Big]. \end{aligned} \tag{2-37}$$

In order to perform the bootstrap argument (we refer to in [Charve 2004; 2006]), let us now define

$$T_\varepsilon \overset{\text{def}}{=} \sup\left\{t \in [0, T^*_\varepsilon[: \text{for all } t' \le t,\ \|\delta_\varepsilon(t')\|_{\dot H^{1/2}} \le \frac{\nu}{4C}\right\}. \tag{2-38}$$

Due to the assumptions, $\|\delta_\varepsilon(0)\|_{H^{1/2+\delta}} \le \mathbb{C}_0\varepsilon^{\alpha_0}$ so that we are sure that $T_\varepsilon > 0$ if

$$\varepsilon \le \left(\frac{\nu}{8C\mathbb{C}_0}\right)^{\frac{1}{\alpha_0}}.$$

Thanks to the Gronwall and Young estimates, and estimating the first terms in the last block as

$$\begin{aligned} &\int_0^\infty \|\widetilde U_{\mathrm{QG}}\|^{2(1-\alpha_1)}_{\dot H^s}\|\widetilde U_{\mathrm{QG}}\|^{2\alpha_1}_{\dot H^{s+1}}\|W_\varepsilon\|^2_{L^{3/\alpha_1}}\,d\tau \\ &\qquad \le \left(\int_0^\infty \|\widetilde U_{\mathrm{QG}}\|^2_{\dot H^{s+1}}\,d\tau\right)^{\alpha_1}\left(\int_0^\infty \|W_\varepsilon\|^{\frac{2}{1-\alpha_1}}_{L^{3/\alpha_1}}\|\widetilde U_{\mathrm{QG}}\|^2_{\dot H^s}\,d\tau\right)^{1-\alpha_1}, \end{aligned} \tag{2-39}$$

we can now state that for all $s \in \left[\frac{1}{2}, \frac{1}{2} + \eta\delta\right]$ and all $t \le T_\varepsilon$, we have (as W_ε and $\widetilde U_{\mathrm{QG}}$ are globally defined, each time integral in the right-hand side is over $\mathbb{R}_+$)

$$
\begin{aligned}
&\|\delta_\varepsilon(t)\|_{\dot H^s}^2+\frac{\nu}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|_{\dot H^s}^2\,d\tau\\
&\leq\Bigg[\|\delta_\varepsilon(0)\|_{\dot H^s}^2+\frac{C}{\nu}\big(\|\widetilde U_{\mathrm{QG}}\|_{L^\infty\dot H^s}^{2(1-\alpha_1)}\|\widetilde U_{\mathrm{QG}}\|_{L^2\dot H^{s+1}}^{2\alpha_1}\|W_\varepsilon\|_{L^{2/(1-\alpha_1)}L^{3/\alpha_1}}^2\\
&\qquad+\|\widetilde U_{\mathrm{QG}}\|_{L^\infty\dot H^{1/2}}^2\||D|^{s+\alpha_1}W_\varepsilon\|_{L^2L^{6/(1+2\alpha_1)}}^2+\|\widetilde U_{\mathrm{QG}}\|_{L^\infty\dot H^s}^2\|W_\varepsilon\|_{L^2\dot B^{\alpha_1}_{3/\alpha_1,2}}^2\\
&\qquad+\||D|^{s+\alpha_2}W_\varepsilon\|_{L^2L^{6/(1+2\alpha_2)}}^2+\|W_\varepsilon\|_{L^\infty\dot H^{s+\alpha_3}}^2\|W_\varepsilon\|_{L^2L^{3/\alpha_3}}^2\\
&\qquad+\|W_\varepsilon\|_{L^\infty\dot H^{s+\alpha_3-\beta\delta}}^2\|W_\varepsilon\|_{L^2\dot B^{\beta\delta}_{3/\alpha_3,2}}^2\big)\Bigg]\\
&\quad\times\exp\frac{C}{\nu}\Bigg\{\|\nabla\widetilde U_{\mathrm{QG}}\|_{L^2\dot H^{1/2}}^2\Big(1+\frac{1}{\nu^2}\|\widetilde U_{\mathrm{QG}}\|_{L^\infty\dot H^{1/2}}^2\Big)\\
&\qquad+\frac{1}{\nu^{\frac{2\alpha_2}{1-\alpha_2}}}\|W_\varepsilon\|_{L^{2/(1-\alpha_2)}L^{3/\alpha_2}}^{\frac{2}{1-\alpha_2}}+\|W_\varepsilon\|_{L^2\dot B^{\alpha_2}_{3/\alpha_2,2}}^2\Bigg\}.
\end{aligned}\tag{2-40}
$$

It is now time to properly use the new Strichartz estimates we proved in the present article (see the Appendix for Proposition 47 and its proof).

Let us begin with the case $(d,p,r,q)=\big(s+\alpha,2,\frac{6}{1+2\alpha},2\big)$, for all $\theta\in\big]0,\frac{1-\alpha}{1-4\alpha}\big[\cap]0,1]=]0,1]$. Thanks to Proposition 40 (for simplicity we will not track the dependency in ν),

$$
\begin{aligned}
&\||D|^{s+\alpha}W_\varepsilon\|_{L^2_tL^{6/(1+2\alpha)}}\\
&\qquad\leq C\||D|^{s+\alpha}W_\varepsilon\|_{\widetilde L^2_t\dot B^0_{6/(1+2\alpha),2}}\\
&\qquad\leq C_{F,\nu,p,\theta,\alpha}\varepsilon^{\frac{\theta}{12}(1-4\alpha)}\bigg(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{s+(\theta/6)(1-4\alpha)}}+\int_0^t\|G^b(\tau)\|_{\dot H^{s+(\theta/6)(1-4\alpha)}}\,d\tau\bigg),
\end{aligned}\tag{2-41}
$$

and if we choose $\alpha\in\big]0,\frac14\big[$, and

$$\theta=\frac{6(\delta+\frac12-s)}{1-4\alpha}$$

$\big($which is in $]0,1]$ if $\delta\leq s-\frac13-\frac{2\alpha}{3}$, recall that $s\sim\frac12\big)$, then we obtain (thanks to Proposition 25)

$$
\begin{aligned}
\||D|^{s+\alpha}W_\varepsilon\|_{L^2_tL^{6/(1+2\alpha)}}&\leq C\||D|^{s+\alpha}W_\varepsilon\|_{\widetilde L^2_t\dot B^0_{6/(1+2\alpha),2}}\\
&\leq C_{F,\nu,s,\delta,\alpha}\varepsilon^{\frac12(\delta+\frac12-s)}\bigg(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2+\delta}}+\int_0^t\|G^b(\tau)\|_{\dot H^{1/2+\delta}}\,d\tau\bigg)\\
&\leq C_{F,\nu,s,\delta,\alpha}\varepsilon^{\frac12(\delta+\frac12-s)}\mathbb{D}_0(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2+\delta}}+1).
\end{aligned}\tag{2-42}
$$

Let us continue with the case $(d,p,r,q)=\big(\alpha,2,\frac{3}{\alpha},2\big)$, for all

$$\theta\in\bigg]0,\frac{\frac12-\frac{\alpha}{3}}{1-\frac{4\alpha}{3}}\bigg[,$$

if we assume $\alpha\in]0,\frac34[$, and choose $\theta=\frac{6\delta}{3-4\alpha}$, then

$$\|W_\varepsilon\|_{\widetilde L^2_t\dot B^\alpha_{3/\alpha,2}}\leq C_{F,\nu,\delta,\alpha}\varepsilon^{\frac{\delta}{2}}\mathbb{D}_0(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2+\delta}}+1).\tag{2-43}$$

For the case $(d,p,r,q)=\big(0,\frac{2}{1-\alpha},\frac{3}{\alpha},2\big)$, for all

$$\theta\in\Bigg]0,\frac{\frac{1}{2}-\frac{\alpha}{3}}{1-\frac{4\alpha}{3}}\Bigg[,$$

if $\alpha\in\big]0,\frac{3}{4}\big[$, and if we choose $\theta=\frac{6\delta}{3-4\alpha}$, then

$$\|W_\varepsilon\|_{L_t^{2/(1-\alpha)}L^{3/\alpha}}\le C\|W_\varepsilon\|_{\widetilde{L}_t^{2/(1-\alpha)}\dot{B}^0_{3/\alpha,2}}\le C_{F,\nu,\delta,\alpha}\varepsilon^{\frac{\delta}{2}}\mathbb{D}_0(\|U_{0,\varepsilon,\text{osc}}\|_{\dot{H}^{1/2+\delta}}+1). \tag{2-44}$$

All these estimates are verified for $\alpha_1=\alpha_2=\alpha=\frac{1}{16}$ if $\delta\le\frac{1}{8}$. Then we turn to the last two terms from (2-36), let us begin with the first one: as announced, due to the first factor (estimated thanks to (2-25)), doing as before will only allow us to get $\varepsilon^{\frac{\delta}{2}}\mathbb{D}_0(\|U_{0,\varepsilon,\text{osc}}\|_{\dot{H}^{1/2+\delta}}+1)^2$, which leads to $\gamma<\frac{\delta}{4}$. In order to reach the announced bound $\frac{\delta}{2}$, we will try to take a slightly smaller p which will allow us to widen the range for θ. But taking $p=2$ instead of $\frac{2}{1-\alpha}$ requires that $\|W_\varepsilon\|_{\dot{H}^{s+\alpha_3}}$ is L^∞; that is, we need that $s+\alpha_3\le\frac{1}{2}+\delta$. More precisely with $(d,p,r,q)=\big(0,2,\frac{3}{\alpha},2\big)$, we have

$$\|W_\varepsilon\|_{L_t^2L^{3/\alpha_3}}\le C\|W_\varepsilon\|_{\widetilde{L}_t^2\dot{B}^0_{3/\alpha_3,2}}\le C_{F,\nu,\theta,s}\varepsilon^{\frac{\theta}{4}(1-\frac{4\alpha_3}{3})}\mathbb{D}_0(\|U_{0,\varepsilon,\text{osc}}\|_{\dot{H}^{1/2-\alpha_3+(\theta/2)(1-4\alpha_3/3)}}+1) \tag{2-45}$$

and as we want

$$\alpha_3+s=\tfrac{1}{2}+\delta=\tfrac{1}{2}-\alpha_3+\tfrac{\theta}{2}\big(1-\tfrac{4\alpha_3}{3}\big)$$

we choose

$$(\alpha_3,\theta)=\Bigg(\delta+\frac{1}{2}-s,\frac{2(\delta+\alpha_3)}{1-\frac{4\alpha_3}{3}}\Bigg).$$

This is possible (according to the condition from Proposition 47) when

$$\theta<\frac{\frac{1}{2}-\frac{\alpha_3}{3}}{1-\frac{4\alpha_3}{3}},$$

that is if

$$\delta<\tfrac{7s-2}{13}, \tag{2-46}$$

which is realized $\big($recall that $s\in\big[\frac{1}{2},\frac{1}{2}+\eta\delta\big]\big)$ when $\delta\le\frac{1}{10}<\frac{3}{26}$. Then we have

$$\|W_\varepsilon\|_{L_t^2L^{3/\alpha_3}}\le C\|W_\varepsilon\|_{\widetilde{L}_t^2\dot{B}^0_{3/\alpha_3,2}}\le C_{F,\nu,\delta,s}\varepsilon^{\frac{1}{2}(2\delta+\frac{1}{2}-s)}\mathbb{D}_0(\|U_{0,\varepsilon,\text{osc}}\|_{\dot{H}^{1/2+\delta}}+1). \tag{2-47}$$

Now, for the last term, α_3 is fixed and we will adjust θ and β. For $(d,p,r,q)=\big(\beta\delta,2,\frac{3}{\alpha_3},2\big)$, we choose θ so that the corresponding σ (see Proposition 47) is equal to $\frac{1}{2}+\delta$; that is

$$\tfrac{\theta}{2}\big(1-\tfrac{4\alpha_3}{3}\big)=(2-\beta)\delta+\tfrac{1}{2}-s,$$

which is possible when

$$\theta\in\Bigg]0,\frac{\frac{1}{2}-\frac{\alpha_3}{3}}{1-\frac{4\alpha_3}{3}}\Bigg[,$$

that is $\delta < \frac{7s-2}{13-6\beta}$, which is realized when (2-46) is true (when $\beta \in]0,1[$). In this case, we end up with

$$\|W_\varepsilon\|_{\widetilde{L}_t^2 \dot{B}^{\beta\delta}_{3/\alpha_3,2}} \leq C_{F,\nu,\alpha,\delta,s}\varepsilon^{\frac{1}{2}((2-\beta)\delta+\frac{1}{2}-s)}\mathbb{D}_0(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}}+1). \tag{2-48}$$

Combining (2-40) with all these Strichartz estimates, namely (2-42), (2-43), (2-44), (2-47) and (2-48), we end up with for all $s \in \left[\frac{1}{2}, \frac{1}{2}+\eta\delta\right]$, all $\beta > 0$ small and all $t \leq T_\varepsilon$

$$\begin{aligned}
&\|\delta_\varepsilon(t)\|^2_{\dot{H}^s} + \frac{\nu}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|^2_{\dot{H}^s}\,d\tau \\
&\leq \Big[\|\delta_\varepsilon(0)\|^2_{\dot{H}^s} + \mathbb{D}_0\big((\varepsilon^{\delta+\frac{1}{2}-s}+\varepsilon^\delta)(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}}+1)^2 \\
&\qquad\qquad + (\varepsilon^{2\delta+\frac{1}{2}-s}+\varepsilon^{(2-\beta)\delta+\frac{1}{2}-s})(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}}+1)^4\big)\Big] \\
&\qquad\qquad\qquad \times \exp\{\mathbb{D}_0(1+\varepsilon^\delta(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}}+1)^2)\} \\
&\leq \mathbb{D}_0[\varepsilon^{2\alpha_0} + (\varepsilon^{\delta+\frac{1}{2}-s}+\varepsilon^\delta)\|U_{0,\varepsilon,\mathrm{osc}}\|^2_{\dot{H}^{1/2+\delta}} + (\varepsilon^{2\delta+\frac{1}{2}-s}+\varepsilon^{(2-\beta)\delta+\frac{1}{2}-s})\|U_{0,\varepsilon,\mathrm{osc}}\|^4_{\dot{H}^{1/2+\delta}}] \\
&\qquad\qquad \times \exp\{\mathbb{D}_0(1+\varepsilon^\delta\|U_{0,\varepsilon,\mathrm{osc}}\|^2_{\dot{H}^{1/2+\delta}} + (\varepsilon^{\frac{\delta}{2}}\|U_{0,\varepsilon,\mathrm{osc}}\|^2_{\dot{H}^{1/2+\delta}})^{\frac{2}{1-\alpha_2}})\}.
\end{aligned} \tag{2-49}$$

As $s \in \left[\frac{1}{2}, \frac{1}{2}+\eta\delta\right]$, we can write that

$$\begin{aligned}
&\|\delta_\varepsilon(t)\|^2_{\dot{H}^s} + \frac{\nu}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|^2_{\dot{H}^s}\,d\tau \\
&\qquad \leq \mathbb{D}_0[\varepsilon^{2\alpha_0} + \varepsilon^{(1-\eta)\delta-2\gamma} + \varepsilon^{(2-\eta)\delta-4\gamma} + \varepsilon^{(2-\eta-\beta)\delta-4\gamma}]e^{\mathbb{D}_0(1+\varepsilon^{\delta-2\gamma})},
\end{aligned} \tag{2-50}$$

so that we need

$$\gamma < \min\big((1-\eta)\tfrac{\delta}{2}, \big(1-\tfrac{\eta}{2}\big)\tfrac{\delta}{2}, \big(1-\tfrac{\beta+\eta}{2}\big)\tfrac{\delta}{2}\big).$$

If we fix $\beta = \eta$, the condition is reduced to $\gamma < (1-\eta)\frac{\delta}{2}$, so that if $0 < \gamma < \frac{\delta}{2}$, we define $\eta = \frac{1}{2}\big(1-\frac{2\gamma}{\delta}\big)$ $\big($or equivalently $\gamma = (1-2\eta)\frac{\delta}{2}\big)$; then with $\beta = \eta$, for all $s \in \left[\frac{1}{2}, \frac{1}{2}+\eta\delta\right]$ and $t \leq T_\varepsilon$, we end up with (as soon as $\varepsilon \leq 1$)

$$\|\delta_\varepsilon(t)\|^2_{\dot{H}^s} + \frac{\nu}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|^2_{\dot{H}^s}\,d\tau \leq \mathbb{D}_0 e^{2\mathbb{D}_0}\varepsilon^{2\min(\alpha_0,\frac{\eta\delta}{2})}. \tag{2-51}$$

We can now conclude the bootstrap argument: there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the previous quantity is bounded by $\big(\frac{\nu}{8C}\big)^2$, so that $\big($in particular for $s = \frac{1}{2}\big)$ if we assume by contradiction that $T_\varepsilon < T_\varepsilon^*$, then

$$\|\delta_\varepsilon\|_{L^\infty_{T_\varepsilon}\dot{H}^{1/2}} \leq \frac{\nu}{8C},$$

which contradicts the maximality of T_ε $\big($in this case, we would have $\|\delta_\varepsilon(T_\varepsilon)\|_{\dot{H}^{1/2}} = \frac{\nu}{4C}\big)$. Then $T_\varepsilon = T_\varepsilon^*$ and the previous estimates hold true for any $t < T_\varepsilon^*$, so that by the blowup criterion $T_\varepsilon = T_\varepsilon^* = \infty$ and the previous estimate is true for all $t \geq 0$ and all $s \in \left[\frac{1}{2}, \frac{1}{2}+\eta\delta\right]$:

$$\|\delta_\varepsilon\|_{\dot{E}^s} \leq \mathbb{B}_0\varepsilon^{\min(\alpha_0,\frac{\eta\delta}{2})}.$$

Finally, to prove the last part of the theorem, we only have to remark that the previous argument is then true for any $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$ when we require $\delta < \frac{3}{26+14\eta}$ (instead of $\delta < \frac{3}{26}$; see (2-46)), and use

Lemma 27:

$$\|\delta_\varepsilon\|_{L^2L^\infty} \le \|\delta_\varepsilon\|_{L^2\dot B^0_{\infty,1}} \le (\|\delta_\varepsilon\|_{L^2\dot H^{3/2-\eta\delta}}\|\delta_\varepsilon\|_{L^2\dot H^{3/2+\eta\delta}})^{\frac12} \le \mathbb{B}_0\varepsilon^{\min(\alpha_0,\frac{\eta\delta}{2})}.$$

For $(d,p,r,q)=(0,2,\infty,1)$ and for all $\theta\in\left]0,\frac12\right[$, from Proposition 47,

$$\|W_\varepsilon\|_{L^2L^\infty} \le \mathbb{C}_0\varepsilon^{\frac{\theta}{4}}\bigg(\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot B^{1/2+\theta/2}_{2,1}} + \int_0^\infty \|G^b(\tau)\|_{\dot B^{1/2+\theta/2}_{2,1}}\,d\tau\bigg).$$

Using Lemma 27 with $(\alpha,\beta)=\left(\frac{\theta}{2},k\frac{\theta}{2}\right)$, and if $\theta=\frac{2\delta}{1+k}$ (for some small $k>0$),

$$\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot B^{1/2+\theta/2}_{2,1}} \le \|U_{0,\varepsilon,\mathrm{osc}}\|^{\frac{k}{1+k}}_{\dot H^{1/2}}\|U_{0,\varepsilon,\mathrm{osc}}\|^{\frac{1}{1+k}}_{\dot H^{1/2+(1+k)(\theta/2)}} \le \mathbb{C}_0\varepsilon^{-\gamma}. \tag{2-52}$$

Choosing $k=\frac{\eta}{1-\eta}$, we get

$$\|W_\varepsilon\|_{L^2L^\infty} \le \mathbb{D}_0\varepsilon^{\frac{\delta}{2}(\frac{1}{1+k}-(1-2\eta))} = \mathbb{D}_0\varepsilon^{\frac{\eta\delta}{2}},$$

and the conclusion follows from the fact that $U_\varepsilon-\widetilde U_{\mathrm{QG}}=\delta_\varepsilon+W_\varepsilon$. □

Remark 32. Going back to (2-49), in the case $s=\frac12$ if we only seek for global well-posedness, we retrieve here the same condition as in [Iwabuchi et al. 2017], except for the last term because Proposition 29 imposes $\beta>0$, so that the condition for global well-posedness is still $\gamma<\frac{\delta}{2}$. If β could reach zero, the conditions would be:

- $\|U_{0,\varepsilon,\mathrm{osc}}\|^2_{\dot H^{1/2}\cap\dot H^{1/2+\delta}}\varepsilon^\delta\le c$, with c some fixed small constant, if we want global well-posedness.
- $\|U_{0,\varepsilon,\mathrm{osc}}\|^2_{\dot H^{1/2}\cap\dot H^{1/2+\delta}}\varepsilon^\delta\to 0$ as $\varepsilon\to 0$ if we want δ_ε to go to zero.
- $\gamma<\frac{\delta}{2}$, if we want δ_ε to go to zero as a positive power of ε (which is what we originally searched for).

In our case, due to this $\beta>0$ these three conditions coincide.

2.3. *The general case.*

2.3.1. *Estimates on W_ε^T.* Let us begin by recalling the energy estimates for W_ε^T (we refer to Theorem 21 for m, M).

Proposition 33. *Assume $M<\frac12$, there exist $\varepsilon_0=\varepsilon_0(\nu,\nu',M)>0$, and $\mathbb{B}_0=\mathbb{B}_0(\mathbb{C}_0,\nu,\nu',F,\delta)\ge 1$ such that for any $0<\varepsilon\le\varepsilon_0$ and $s\in\left[\frac12-\frac32\delta,\frac12+\delta\right]$, we have*

$$\|W_\varepsilon^T\|^2_{L^\infty(\mathbb{R}_+,\dot H^s)} + \nu_0\|W_\varepsilon^T\|^2_{L^2(\mathbb{R}_+,\dot H^{s+1})} \le \mathbb{B}_0(\varepsilon^{-2\gamma}+1). \tag{2-53}$$

Proof. We know from [Charve 2004] that there exists a constant $C_F>0$ such that for any $s\in[0,1]$ and $t\in\mathbb{R}_+$, we have

$$\|W_\varepsilon^T\|_{\dot E^s_t} \le e^{\int_0^t\|G^b(\tau)\|_{\dot H^s}\,d\tau} \times\bigg(\|W_\varepsilon^T(0)\|^2_{\dot H^s}+C_F(1+\varepsilon R_\varepsilon^2|\nu-\nu'|)^2\int_0^t\bigg(\|G^b(\tau)\|_{\dot H^s}+\frac{1}{\nu_0}\|G^l(\tau)\|_{\dot H^{s-1}}\bigg)d\tau\bigg). \tag{2-54}$$

Combined with (A-86), Proposition 25 allows us to obtain that when $s \in \left[\frac{1}{2}-\frac{3}{2}\delta, \frac{1}{2}+\delta\right]$

$$\|W_\varepsilon^T\|_{\dot{E}_t^s} \le C_F(1+|\nu-\nu'|\varepsilon R_\varepsilon^2)^2 e^{\frac{C_F}{\nu_0} C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}} \times \left(\|\mathcal{P}_{r_\varepsilon,R_\varepsilon} U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}^2 + \left(\frac{1}{\nu_0}+\frac{|\nu-\nu'|^2}{\nu_0^2}\right) C_{\delta,\nu_0}\mathbb{C}_0^{2+\frac{1}{\delta}}\right). \tag{2-55}$$

We have $|\nu-\nu'|\varepsilon R_\varepsilon^2 \le 1$ as soon as $M > \frac{1}{2}$ and $\varepsilon \le \varepsilon_0 = |\nu-\nu'|^{\frac{-1}{1-2M}}$, which leads to (2-53). □

2.3.2. *Estimates on δ_ε.* As explained in the previous section (see also [Charve 2004; 2018a]), as $U_{0,\varepsilon} \in \dot{H}^s$ for all $s \in \left[\frac{1}{2}-\frac{3}{2}\delta, \frac{1}{2}+\delta\right]$, in particular it lies in $\dot{H}^{\frac{1}{2}}$ and thanks to the Fujita–Kato theorem there exists a unique local strong solution $U_\varepsilon \in L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}$ for all $0 < T < T_\varepsilon^*$, where $T_\varepsilon^* > 0$ denotes the maximal lifespan. In addition, if T_ε^* is finite then we have

$$\int_0^{T_\varepsilon^*} \|\nabla U_\varepsilon(\tau)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 \, d\tau = \infty.$$

Moreover, as our initial data enjoys additional regularity properties, they are transmitted to the solution: for all $s \in \left[\frac{1}{2}-\frac{3}{2}\delta, \frac{1}{2}+\delta\right]$ and $T < T_\varepsilon^*$, we have $U_\varepsilon \in L_T^\infty \dot{H}^s \cap L_T^2 \dot{H}^{s+1}$. As before, with a view to a bootstrap argument, let us now define

$$T_\varepsilon \overset{\text{def}}{=} \sup\left\{ t \in [0, T_\varepsilon^*[: \text{for all } t' \le t,\ \|\delta_\varepsilon(t')\|_{\dot{H}^{1/2}} \le \frac{\nu}{4C} \right\}. \tag{2-56}$$

Thanks to (2-59), we are sure that $\|\delta_\varepsilon(0)\|_{\dot{H}^1/2} \le \frac{\nu}{8C}$ (and then $T_\varepsilon > 0$) if ε is small enough. Assuming that $T_\varepsilon < T_\varepsilon^*$, the computations from the previous case imply that, for all $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$, and all $t \le T_\varepsilon$,

$$\begin{aligned}
&\|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \frac{\nu_0}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|_{\dot{H}^s}^2\, d\tau \\
&\le \Big[\|\delta_\varepsilon(0)\|_{\dot{H}^s}^2 + \frac{C}{\nu_0}\big(\nu_0\|f^b\|_{L^1\dot{H}^s} + \|f^l\|_{L^2\dot{H}^{s-1}}^2 + \|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty\dot{H}^s}^{2(1-\alpha_1)} \|\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^{s+1}}^{2\alpha_1} \|W_\varepsilon^T\|_{L^{2/(1-\alpha_1)}L^{3/\alpha_1}}^2 \\
&\qquad + \|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty\dot{H}^{1/2}}^2 \||D|^{s+\alpha_1} W_\varepsilon^T\|_{L^2L^{6/(1+2\alpha_1)}}^2 + \|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty\dot{H}^s}^2 \|W_\varepsilon^T\|_{L^2\dot{B}^{\alpha_1}_{3/\alpha_1,2}}^2 \\
&\qquad + \||D|^{s+\alpha_2} W_\varepsilon^T\|_{L^2L^{6/(1+2\alpha_2)}}^2 + \|W_\varepsilon^T\|_{L^\infty\dot{H}^s}^{2(1-\alpha_3)} \|W_\varepsilon^T\|_{L^2\dot{H}^{s+1}}^{2\alpha_3} \|W_\varepsilon^T\|_{L^{2/(1-\alpha_3)}L^{3/\alpha_3}}^2 \\
&\qquad + \|W_\varepsilon^T\|_{L^\infty\dot{H}^s}^2 \|W_\varepsilon^T\|_{L^2\dot{B}^{\alpha_3}_{3/\alpha_3,2}}^2 \big)\Big] \\
&\quad \times \exp\frac{C}{\nu_0}\Big\{ \nu_0\|f^b\|_{L^1\dot{H}^s} + \|\nabla\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^{1/2}}^2 \Big(1+\frac{1}{\nu_0^2}\|\widetilde{U}_{\mathrm{QG}}\|_{L^\infty\dot{H}^{1/2}}^2\Big) \\
&\qquad + \frac{1}{\nu_0^{\frac{2\alpha_2}{1-\alpha_2}}} \|W_\varepsilon^T\|_{L^{2/(1-\alpha_2)}L^{3/\alpha_2}}^{\frac{2}{1-\alpha_2}} + \|W_\varepsilon^T\|_{L^2\dot{B}^{\alpha_2}_{3/\alpha_2,2}}^2 \Big\}.
\end{aligned} \tag{2-57}$$

Compared to (2-40), the only differences are:

- The force terms $f^{b,l}$ (dealt with as in [Charve 2004; 2006]).

• The simpler estimates for F_8: as precision will be imposed by the truncated terms, we only write

$$|(F_8|\delta_\varepsilon)_{\dot H^s}| \le \frac{\nu_0}{16}\|\delta_\varepsilon\|^2_{\dot H^{s+1}} + \frac{C}{\nu_0}\big(\|W_\varepsilon^T\|^{2(1-\alpha_3)}_{L^\infty \dot H^s}\|W_\varepsilon^T\|^{2\alpha_3}_{L^2\dot H^{s+1}}\|W_\varepsilon\|^2_{L^{2/(1-\alpha_3)}L^{3/\alpha_3}} + \|W_\varepsilon\|^2_{L^\infty \dot H^s}\|W_\varepsilon\|^2_{L^2 \dot B^{\alpha_3}_{3/\alpha_3,2}}\big). \tag{2-58}$$

2.3.3. *Estimates for the truncated quantities.* We will now bound much more precisely than in [Charve 2004; 2006] the external force terms and initial data (see (1-9)):

Proposition 34. *There exists a constant $\mathbb{B}_0 \ge 1$ such that for all $s \in \big[\frac12 - \eta\delta, \frac12 + \eta\delta\big]$,*

$$\begin{aligned} \|f^b\|_{L^1\dot H^s} &\le \mathbb{B}_0\big(\varepsilon^{1-2M} + \varepsilon^{M(1-\eta)\delta} + \varepsilon^{\frac{m}{6} - M(\frac56 + \eta\delta)}\big),\\ \|f^l\|_{L^2\dot H^{s-1}} &\le \mathbb{B}_0\big(\varepsilon^{1-2M} + \varepsilon^{M(1-\eta)\delta} + \varepsilon^{m(\frac12 - \eta\delta)}\big),\\ \|\delta_\varepsilon(0)\|_{\dot H^s} &\le \mathbb{B}_0\big(\varepsilon^{\alpha_0} + \varepsilon^{1-2M-\gamma} + \varepsilon^{\delta(M-\eta m)-\gamma} + \varepsilon^{\delta((\frac12-\eta)m - M)-\gamma}\big).\end{aligned} \tag{2-59}$$

Remark 35. Note that as we want positive powers of ε, the previous estimates imply the conditions

$$\begin{aligned} M, \eta, \eta\delta &\in \big]0, \tfrac12\big[,\\ \eta < \tfrac{M}{m} &< \min\big(\tfrac{1}{5+6\eta\delta}, \tfrac12 - \eta\big),\\ \gamma &< \min\big(1-2M, \delta(M-\eta m), \delta\big(\big(\tfrac12 - \eta\big)m - M\big)\big).\end{aligned} \tag{2-60}$$

Proof of Proposition 34. Let us begin with the terms involving G: thanks to (A-84), and Remark 10 and Proposition 25, we immediately obtain that there exists a constant $\mathbb{B}_0$ (only depending on $\mathbb{C}_0$, ν, ν' and F) such that for all $s \in \big[\frac12 - \eta\delta, \frac12 + \eta\delta\big]$

$$\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2 G^b\|_{L^1\dot H^s} + \|\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2 G^l\|_{L^2\dot H^{s-1}} \le \mathbb{B}_0 \varepsilon R_\varepsilon^2.$$

Thanks to Lemma 42 (see the Appendix), Proposition 25 and (2-17), the second term in f_1 can be bounded $\big($for all $s \in \big[\frac12 - \eta\delta, \frac12 + \eta\delta\big]\big)$ according to

$$\begin{aligned} \|(\mathrm{Id} - \mathcal{P}_{r_\varepsilon,R_\varepsilon})G^b\|_{L^1\dot H^s} &\le \left\|\left(\mathrm{Id} - \chi\left(\frac{|D|}{R_\varepsilon}\right)\right)G^b\right\|_{L^1\dot H^s} + \left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)G^b\right\|_{L^1\dot H^s}\\ &\le \frac{1}{R_\varepsilon^{\frac12+\delta-s}}\left\|\left(\mathrm{Id} - \chi\left(\frac{|D|}{R_\varepsilon}\right)\right)G^b\right\|_{L^1\dot H^{1/2+\delta}} + R_\varepsilon^s\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)G^b\right\|_{L^1L^2}\\ &\le \frac{1}{R_\varepsilon^{\frac12+\delta-s}}\|G^b\|_{L^1\dot H^{1/2+\delta}} + R_\varepsilon^s(R_\varepsilon^2 r_\varepsilon)^{\frac23-\frac12}\|\widetilde U_{\mathrm{QG}}\cdot\nabla\widetilde U_{\mathrm{QG}}\|_{L^1L^{3/2}}\\ &\le \frac{1}{R_\varepsilon^{\frac12+\delta-s}}\|G^b\|_{L^1\dot H^{1/2+\delta}} + R_\varepsilon^{s+\frac13} r_\varepsilon^{\frac16}\int_0^\infty \|\widetilde U_{\mathrm{QG}}(\tau)\|_{L^6}\|\nabla\widetilde U_{\mathrm{QG}}(\tau)\|_{L^2}\,d\tau\\ &\le \mathbb{B}_0\left(\frac{1}{R_\varepsilon^{\frac12+\delta-s}} + R_\varepsilon^{s+\frac13} r_\varepsilon^{\frac16}\right),\end{aligned} \tag{2-61}$$

which implies the first estimates in (2-59) for all $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$. Similarly, we have

$$\left\|\left(\mathrm{Id}-\chi\left(\frac{|D|}{R_\varepsilon}\right)\right)G^l\right\|_{L^2\dot{H}^{s-1}}^2 \leq \frac{\mathbb{B}_0}{R_\varepsilon^{2(\frac{1}{2}+\delta-s)}}, \tag{2-62}$$

and using that the expression of G^l (see (1-9)) features some derivative ∂_3, for all $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$ we have

$$\begin{aligned}\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)G^l\right\|_{L^2\dot{H}^{s-1}} &\leq C_F|\nu-\nu'|\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)\partial_3\nabla\widetilde{U}_{\mathrm{QG}}\right\|_{L^2\dot{H}^{s-1}}\\ &\leq C_F|\nu-\nu'|r_\varepsilon^s\|\partial_3^{1-s}\nabla\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^{s-1}}\\ &\leq C_F|\nu-\nu'|r_\varepsilon^s\|\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^1}.\end{aligned} \tag{2-63}$$

Let us now turn to bound the initial data $\delta_\varepsilon(0)$:

$$\begin{aligned}\|\delta_\varepsilon(0)\|_{\dot{H}^s} &\leq \|U_{0,\varepsilon,\mathrm{QG}}-\widetilde{U}_{0,\mathrm{QG}}\|_{\dot{H}^s}+\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}+\|(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}\\ &\leq \mathbb{C}_0\varepsilon^{\alpha_0}+\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_2U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}\\ &\qquad+\left\|\left(\mathrm{Id}-\chi\left(\frac{|D|}{R_\varepsilon}\right)\right)U_{0,\varepsilon,\mathrm{osc}}\right\|_{\dot{H}^s}+\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)U_{0,\varepsilon,\mathrm{osc}}\right\|_{\dot{H}^s}.\end{aligned} \tag{2-64}$$

As before, we easily estimate the second and third terms for all $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$ by

$$C_F|\nu-\nu'|\varepsilon R_\varepsilon^2\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s}+\frac{C_F}{R_\varepsilon^{\frac{1}{2}+\delta-s}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}} \leq \mathbb{B}_0\varepsilon^{-\gamma}\left[\varepsilon R_\varepsilon^2+\frac{1}{R_\varepsilon^{\frac{1}{2}+\delta-s}}\right]. \tag{2-65}$$

It is here that the $\dot{B}_{q,q}^{\frac{1}{2}}$-assumption on the initial data will be specifically used (everywhere else we only use the fact that this space is embedded in $\dot{H}^{\frac{1}{2}-\frac{3}{2}\delta}$). To bound the last term, thanks to Proposition 40 let us write that (we recall that $q=\frac{2}{1+\delta}<2$)

$$\begin{aligned}\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)U_{0,\varepsilon,\mathrm{osc}}\right\|_{\dot{H}^s} &= \left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)|D|^sU_{0,\varepsilon,\mathrm{osc}}\right\|_{L^2}\\ &\leq C(R_\varepsilon^2r_\varepsilon)^{\frac{1}{q}-\frac{1}{2}}R_\varepsilon^{s-\frac{1}{2}}\left\|\chi\left(\frac{|D|}{R_\varepsilon}\right)\chi\left(\frac{|D_3|}{r_\varepsilon}\right)|D|^{\frac{1}{2}}U_{0,\varepsilon,\mathrm{osc}}\right\|_{L^q}\\ &\leq CR_\varepsilon^{\delta+s-\frac{1}{2}}r_\varepsilon^{\frac{\delta}{2}}\||D|^{\frac{1}{2}}U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{B}_{q,q}^0}\\ &\leq CR_\varepsilon^{\delta+s-\frac{1}{2}}r_\varepsilon^{\frac{\delta}{2}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{B}_{q,q}^{1/2}}.\end{aligned} \tag{2-66}$$

Note that this can be done only if $s \geq \frac{1}{2}$. In the case $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}\right[$, we simply go back to (2-64) and write that (taking advantage of the frequency localization)

$$\begin{aligned}\|(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^s} &\leq \frac{1}{r_\varepsilon^{\frac{1}{2}-s}}\|(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2}}\\ &\leq \frac{C_F}{r_\varepsilon^{\frac{1}{2}-s}}\left(\frac{1}{R_\varepsilon^\delta}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{1/2+\delta}}+R_\varepsilon^\delta r_\varepsilon^{\frac{\delta}{2}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{B}_{q,q}^{1/2}}\right).\end{aligned} \tag{2-67}$$

We can sum up as follows: for all $s \in \left[\frac{1}{2}-\eta\delta, \frac{1}{2}+\eta\delta\right]$

$$
\begin{aligned}
\|(\mathrm{Id}-\mathcal{P}_{r_\varepsilon,R_\varepsilon})U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^s} &\le \mathbb{C}_0\varepsilon^{-\gamma}\times\begin{cases}\frac{1}{R_\varepsilon^{(1-\eta)\delta}}+R_\varepsilon^{(1+\eta)\delta}r_\varepsilon^{\frac{\delta}{2}} & \text{if } s\in\left[\frac12,\frac12+\eta\delta\right],\\ \frac{1}{r_\varepsilon^{\eta\delta}}\left(\frac{1}{R_\varepsilon^\delta}+R_\varepsilon^\delta r_\varepsilon^{\frac{\delta}{2}}\right) & \text{if } s\in\left[\frac12-\eta\delta,\frac12\right]\end{cases}\\
&\le \mathbb{C}_0\varepsilon^{-\gamma}\times\begin{cases}\varepsilon^{M\delta(1-\eta)}+\varepsilon^{\delta(\frac{m}{2}-(1+\eta)M)} & \text{if } s\in\left[\frac12,\frac12+\eta\delta\right],\\ \varepsilon^{\delta(M-m\eta)}+\varepsilon^{\delta((\frac12-\eta)m-M)} & \text{if } s\in\left[\frac12-\eta\delta,\frac12\right].\end{cases}
\end{aligned}
\tag{2-68}
$$

As

$$
M(1-\eta)-(M-m\eta)=\eta(m-M),\quad \text{and}\quad \left(\tfrac{m}{2}-(1+\eta)M\right)-\left(\left(\tfrac12-\eta\right)m-M\right)=\eta(m-M),
$$

and as $m>M$ (see (2-60)), we obtain the announced result. □

2.3.4. *Strichartz estimates for W_ε^T.* We will need the following Strichartz estimates to complete our bootstrap argument:

Proposition 36. *There exist $\varepsilon_0, \mathbb{B}_0>0$ such that for any $\alpha>0$ and $\varepsilon<\varepsilon_0$, W_ε^T satisfies*

$$
\begin{aligned}
\|W_\varepsilon^T\|_{\widetilde L^2\dot B^\alpha_{3/\alpha,2}} &\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-M(\frac92-4\alpha-\delta)-m(\frac92-2\alpha)}\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-\frac92(M+m)},\\
\|W_\varepsilon^T\|_{L^{2/(1-\alpha)}L^{3/\alpha}} &\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-M(\frac92-3\alpha-\delta)-m(\frac92-3\alpha)}\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-\frac92(M+m)},\\
\||D|^{s+\alpha}W_\varepsilon^T\|_{L^2L^{6/(1+2\alpha)}} &\le \mathbb{B}_0\varepsilon^{\frac{1}{12}-\frac{\alpha}{3}-M(\frac72-3\alpha)-m(\frac72-2\alpha)}\le \mathbb{B}_0\varepsilon^{\frac{1}{12}-\frac{\alpha}{3}-\frac72(M+m)}.
\end{aligned}
\tag{2-69}
$$

Proof. Using Proposition 51 in the case $(d,p,r,q)=\left(\alpha,2,\frac{3}{\alpha},2\right)$, we obtain that

$$
\begin{aligned}
&\|W_\varepsilon^T\|_{\widetilde L^2\dot B^\alpha_{3/\alpha,2}}\\
&\le \mathbb{B}_0\varepsilon^{\frac14(1-\frac{4\alpha}{3})}\frac{R_\varepsilon^{4-3\alpha}}{r_\varepsilon^{\frac72-2\alpha}}\left(\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^\alpha}+\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}G^b\|_{L^1\dot H^\alpha}+\frac{1}{\nu_0^{\frac12}r_\varepsilon}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}G^l\|_{L^2\dot H^\alpha}\right)\\
&\le \mathbb{B}_0\varepsilon^{\frac14(1-\frac{4\alpha}{3})}\frac{R_\varepsilon^{4-3\alpha}}{r_\varepsilon^{\frac72-2\alpha}}\times\left(\frac{1}{r_\varepsilon^{\frac12-\frac{3\delta}{2}-\alpha}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2-3\delta/2}}+\|G^b\|_{L^1\dot H^\alpha}+\frac{1}{\nu_0^{\frac12}r_\varepsilon}R_\varepsilon^{\frac12-\delta-\alpha}\|G^l\|_{L^2\dot H^{1/2+\delta}}\right)\\
&\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-M(4-3\alpha)-m(\frac72-2\alpha)}\left(\varepsilon^{-\gamma-m(\frac12-\frac{3\delta}{2}-\alpha)}+\varepsilon^{-m-M(\frac12-\delta-\alpha)}\right).
\end{aligned}
\tag{2-70}
$$

From (2-60), we know that $\gamma<\delta M$ so that

$$
m+M\left(\tfrac12-\delta-\alpha\right)-\left(\gamma+m\left(\tfrac12-\tfrac{3\delta}{2}-\alpha\right)\right)=M\left(\tfrac12-\delta-\alpha\right)+m\left(\tfrac12+\tfrac{3\delta}{2}+\alpha\right)-\gamma>0,
$$

which leads to the first estimate. Similarly, considering Proposition 51 in the case $(d,p,r,q)=\left(0,\frac{2}{1-\alpha},\frac{3}{\alpha},2\right)$, we get (thanks to Proposition 40)

$$
\begin{aligned}
\|W_\varepsilon^T\|_{L^{2/(1-\alpha)}L^{3/\alpha}}\le \|W_\varepsilon^T\|_{\widetilde L^2\dot B^0_{3/\alpha,2}}
&\le \mathbb{B}_0\varepsilon^{\frac14(1-\frac{4\alpha}{3})}\frac{R_\varepsilon^{4-3\alpha}}{r_\varepsilon^{\frac72-3\alpha}}\left(\frac{1}{r_\varepsilon^{\frac12-\frac{3\delta}{2}}}\|U_{0,\varepsilon,\mathrm{osc}}\|_{\dot H^{1/2-3\delta/2}}+\|G^b\|_{L^1L^2}\right.\\
&\qquad\left.+\frac{1}{\nu_0^{\frac12}r_\varepsilon}R_\varepsilon^{\frac12-\delta}\|G^l\|_{L^2\dot H^{1/2+\delta}}\right)\\
&\le \mathbb{B}_0\varepsilon^{\frac14-\frac{\alpha}{3}-M(4-3\alpha)-m(\frac72-3\alpha)}\left(\varepsilon^{-\gamma-m(\frac12-\frac{3\delta}{2})}+\varepsilon^{-m-M(\frac12-\delta)}\right),
\end{aligned}
\tag{2-71}
$$

which leads to the second estimate. In the case $(d,p,r,q)=\big(s+\alpha,2,\frac{6}{1+2\alpha},2\big)$, we obtain that (provided $0<\alpha<\delta+\frac12-s$)

$$\begin{aligned}
\||D|^{s+\alpha}W_\varepsilon^T\|_{L^2L^{6/(1+2\alpha)}} &\le \|W_\varepsilon^T\|_{\widetilde{L}^2\dot{B}^{s+\alpha}_{6/(1+2\alpha),2}}\\
&\le \mathbb{B}_0\varepsilon^{\frac{1}{12}(1-4\alpha)}\frac{R_\varepsilon^{\frac52-3\alpha}}{r_\varepsilon^{\frac52-2\alpha}}\bigg(\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}U_{0,\varepsilon,\mathrm{osc}}\|_{\dot{H}^{s+\alpha}}+\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}G^b\|_{L^1\dot{H}^{s+\alpha}}\\
&\qquad\qquad+\frac{1}{\nu_0^{\frac12}r_\varepsilon}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}G^l\|_{L^2\dot{H}^{s+\alpha}}\bigg)\\
&\le \mathbb{B}_0\varepsilon^{\frac{1}{12}(1-4\alpha)}\frac{R_\varepsilon^{\frac52-3\alpha}}{r_\varepsilon^{\frac52-2\alpha}}\Big(\varepsilon^{-\gamma}+1+\frac{1}{r_\varepsilon}R_\varepsilon^{s+\alpha+\frac12-\delta}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}\widetilde{U}_{\mathrm{QG}}\|_{L^2\dot{H}^{3/2+\delta}}\Big)\\
&\le \mathbb{B}_0\varepsilon^{\frac{1}{12}-\frac{\alpha}{3}-M(\frac52-3\alpha)-m(\frac52-2\alpha)}(\varepsilon^{-\gamma}+\varepsilon^{-m-M}),
\end{aligned}\tag{2-72}$$

which concludes the proof. □

2.3.5. *Bootstrap.* We are now able to conclude the bootstrap argument (see the previous section and [Charve 2004; 2006]). Gathering (2-57), (2-59) and (2-69), we obtain that for all $t\le T_\varepsilon$

$$\begin{aligned}
&\|\delta_\varepsilon(t)\|^2_{\dot{H}^s}+\frac{\nu_0}{2}\int_0^t\|\nabla\delta_\varepsilon(\tau)\|^2_{\dot{H}^s}\,d\tau\\
&\quad\le \mathbb{D}_0\big[\varepsilon^{2\alpha_0}+\varepsilon^{2(1-2M-\gamma)}+\varepsilon^{2(\delta(M-\eta m)-\gamma)}+\varepsilon^{2(\delta((\frac12-\eta)m-M)-\gamma)}+\varepsilon^{1-2M}\\
&\qquad\qquad+\varepsilon^{M(1-\eta)\delta}+\varepsilon^{\frac{m}{6}-M(\frac56+\eta\delta)}+\varepsilon^{2(1-2M)}+\varepsilon^{2M(1-\eta)\delta}\\
&\qquad\qquad\qquad+\varepsilon^{2m(\frac12-\eta\delta)}+\varepsilon^{\frac14-\frac{\alpha}{3}-\frac92(M+m)-\gamma}+\varepsilon^{\frac{1}{12}-\frac{\alpha}{3}-\frac72(M+m)}\big]\\
&\qquad\times\exp\frac{C}{\nu_0}\{1+\varepsilon^{1-2M}+\varepsilon^{M(1-\eta)\delta}+\varepsilon^{\frac{m}{6}-M(\frac56+\eta\delta)}+\varepsilon^{\min(2,\frac{2}{1-\alpha})(\frac14-\frac{\alpha}{3}-\frac92(M+m))}\}.
\end{aligned}\tag{2-73}$$

For simplicity we will require, instead of the second condition from (2-60), that

$$2\eta\le\tfrac{M}{m}\le\tfrac12\min\big(\tfrac{1}{5+6\eta\delta},\tfrac12-\eta\big).$$

This obviously implies that $\eta\le\frac{1}{10}$, so we will finally ask that

$$\begin{gathered}
M\in\,]0,\tfrac14[,\quad \eta\in\,]0,\tfrac{1}{10}[,\\
2\eta\le\tfrac{M}{m}\le\tfrac12\tfrac{1}{5+\delta},\\
\gamma<\min\big(\tfrac12(1-2M),\tfrac12\delta(M-\eta m),\tfrac12\delta\big(\big(\tfrac12-\eta\big)m-M\big)\big).
\end{gathered}\tag{2-74}$$

Moreover, if we take $\alpha=\gamma$ and ask that

$$\begin{aligned}
&\tfrac92(M+m)\le\tfrac18 \quad\text{and}\quad \tfrac43\delta\le\tfrac12\big(\tfrac14-\tfrac92(M+m)\big),\\
&\tfrac72(M+m)\le\tfrac{1}{24} \quad\text{and}\quad \tfrac{\delta}{3}\le\tfrac12\big(\tfrac{1}{12}-\tfrac72(M+m)\big).
\end{aligned}\tag{2-75}$$

As $M\le\frac{m}{10}$, this is realized when

$$m\in\,]0,\tfrac{1}{100}[,\quad 2\eta\le\tfrac{M}{m}\le\tfrac12\tfrac{1}{5+\delta}.$$

When

$$\gamma \le \min\left(\tfrac{M\delta}{4}, \tfrac{m\delta}{16}, \tfrac{m}{12}, \tfrac{1}{32}\right) = \tfrac{M\delta}{4},$$

we obtain that all powers of ε in the exponential are positive so that for small enough ε, we get that for all $s \in \left[\frac{1}{2} - \eta\delta, \frac{1}{2} + \eta\delta\right]$ and $t \le T_\varepsilon$

$$\|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \frac{\nu_0}{2}\int_0^t \|\nabla\delta_\varepsilon(\tau)\|_{\dot{H}^s}^2\, d\tau \le \mathbb{D}_0 e^{2\mathbb{D}_0}\varepsilon^{\min(2\alpha_0, \frac{M\delta}{2})}, \tag{2-76}$$

so that we finally end up with (for small enough ε), $\delta_\varepsilon(T_\varepsilon) \le \frac{\nu_0}{8C}$, which clearly contradicts the maximality of T_ε. We can conclude that $T_\varepsilon = T_\varepsilon^*$ and then the previous estimate is valid for all $t < T_\varepsilon^*$, which implies for $s = \frac{1}{2}$ that the integral in (1-1) is finite. Therefore $T_\varepsilon^* = \infty$ and (2-76) is then valid for all $t \ge 0$. The rest of the theorem is done as for the case $\nu = \nu'$. □

Appendix

A.1. ***Notation and Sobolev spaces.*** For $s \in \mathbb{R}$, $\dot{H}^s$ and H^s are the classical homogeneous/inhomogeneous Sobolev spaces in $\mathbb{R}^3$ endowed with the norms

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|\, d\xi \quad \text{and} \quad \|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (1+|\xi|^2)^s|\hat{u}(\xi)|\, d\xi.$$

We also use the following notation: if E is a Banach space and $T > 0$,

$$\mathcal{C}_T E = \mathcal{C}([0,T], E) \quad \text{and} \quad L_T^p E = L^p([0,T], E).$$

Let us recall the Sobolev injections, and product laws:

Proposition 37. *There exists a constant $C > 0$ such that if $s < \frac{3}{2}$, then for any $u \in \dot{H}^s(\mathbb{R}^3)$, we have $u \in L^p(\mathbb{R}^3)$ with $p = \frac{6}{3-2s}$ and*

$$\|u\|_{L^p} \le C\|u\|_{\dot{H}^s}.$$

Proposition 38 [Bahouri et al. 2011, Chapter 2]. *There exists a constant C such that for any $(u, v) \in \dot{H}^{s_1}(\mathbb{R}^3) \times \dot{H}^{s_2}(\mathbb{R}^3)$, if $s_1, s_2 \in \left]-\frac{3}{2}, \frac{3}{2}\right[$ and $s_1 + s_2 > 0$ then $uv \in \dot{H}^{s_1+s_2-\frac{3}{2}}(\mathbb{R}^3)$ and we have*

$$\|uv\|_{\dot{H}^{s_1+s_2-3/2}} \le C\|u\|_{\dot{H}^{s_1}}\|v\|_{\dot{H}^{s_2}}.$$

A.2. ***Besov spaces.*** We refer to Chapter 2 from [Bahouri et al. 2011] for an in-depth presentation of the classical homogeneous and inhomogeneous Besov and Sobolev spaces. We also refer to the appendix of [Charve 2018a] for a quick presentation.

Let us just recall that ψ is a smooth radial function supported in the ball $B\left(0, \frac{4}{3}\right)$, equal to 1 in a neighborhood of $B\left(0, \frac{3}{4}\right)$ and such that $r \mapsto \psi(r \cdot e_r)$ is nonincreasing over $\mathbb{R}_+$. If we set $\varphi(\xi) = \psi\left(\frac{\xi}{2}\right) - \psi(\xi)$, then φ is compactly supported in the annulus $\mathcal{C} = \left\{\xi \in \mathbb{R}^d : c_0 = \frac{3}{4} \le |\xi| \le C_0 = \frac{8}{3}\right\}$ and we define the homogeneous dyadic blocks: for all $j \in \mathbb{Z}$,

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u = 2^{jd}h(2^j\,\cdot) * u, \quad \text{with } h = \mathcal{F}^{-1}\varphi.$$

We recall that $\widehat{\phi(D)u}(\xi) = \phi(\xi)\hat{u}(\xi)$ and we can define the homogeneous Besov norms and spaces:

Definition 39. For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, we set

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(\sum_{l\in\mathbb{Z}} 2^{rls}\|\dot{\Delta}_l u\|^r_{L^p}\Big)^{\frac{1}{r}} \quad \text{if } r<\infty \qquad \text{and} \qquad \|u\|_{\dot{B}^s_{p,\infty}} := \sup_l 2^{ls}\|\dot{\Delta}_l u\|_{L^p}.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions such that

$$\lim_{j\to-\infty}\|\dot{S}_j u\|_{L^\infty} = 0$$

and $\|u\|_{\dot{B}^s_{p,r}}$ is finite (where $\dot{S}_j u = \sum_{l\le j-1}\dot{\Delta}_l u = \psi(2^{-j}D)u$).

- The space $\dot{B}^s_{p,r}$ is complete whenever $s < d/p$, or $s \le d/p$ and $r = 1$.
- For any $p \in [1,\infty]$, we have the continuous embedding $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.
- If $\sigma \in \mathbb{R}$, $1 \le p_1 \le p_2 \le \infty$, and $1 \le r_1 \le r_2 \le \infty$, then we have $\dot{B}^\sigma_{p_1,r_1} \hookrightarrow \dot{B}^{\sigma-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.
- The space $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p<\infty$).
- $\dot{H}^s = \dot{B}^s_{2,2}$.
- Interpolation: if $1 \le p, r_1, r_2, r \le \infty$, $\sigma_1 \ne \sigma_2$, and $\theta \in (0,1)$,

$$\|f\|_{\dot{B}^{\theta\sigma_2+(1-\theta)\sigma_1}_{p,r}} \lesssim \|f\|^{1-\theta}_{\dot{B}^{\sigma_1}_{p,r_1}} \|f\|^{\theta}_{\dot{B}^{\sigma_2}_{p,r_2}}. \tag{A-77}$$

Proposition 40 [Bahouri et al. 2011]. *We have the following continuous injections*:

- $\dot{B}^0_{p,1} \hookrightarrow L^p$ *for any* $p \ge 1$.
- $\dot{B}^0_{p,2} \hookrightarrow L^p$ *for any* $p \in [2,\infty[$.
- $\dot{B}^0_{p,p} \hookrightarrow L^p$ *for any* $p \in [1,2]$.

Let us then define the spaces $\widetilde{L}^\rho_T \dot{B}^s_{p,r}$ from the following norm:

Definition 41. For $T > 0$, $s \in \mathbb{R}$, and $1 \le r, \rho \le \infty$, we set

$$\|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} := \big\|2^{js}\|\dot{\Delta}_q u\|_{L^\rho_T L^p}\big\|_{\ell^r(\mathbb{Z})}.$$

Any product of two distributions u and v may be formally written through the Bony decomposition:

$$uv = T_u v + T_v u + R(u,v), \tag{A-78}$$

where

$$T_u v := \sum_l \dot{S}_{l-1} u \dot{\Delta}_l v, \quad T_v u := \sum_l \dot{S}_{l-1} v \dot{\Delta}_l u \quad \text{and} \quad R(u,v) := \sum_l \sum_{|l'-l|\le 1} \dot{\Delta}_l u\, \dot{\Delta}_{l'} v.$$

The above operator T is called a "paraproduct", whereas R is called a "remainder". We refer to [Bahouri et al. 2011] for general properties and for paraproduct and remainder estimates but we can recall that $\big($if $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\big)$:

- For any $s \in \mathbb{R}$, we have $\|T_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}$.

- For any $(s,t) \in \mathbb{R}_-^* \times \mathbb{R}$, we have $\|T_u v\|_{\dot{B}^{s+t}_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p_1,r_1}} \|v\|_{\dot{B}^t_{p_2,r_2}}$.
- For any $s,t \in \mathbb{R}$ with $s+t>0$, we have $\|R(u,v)\|_{\dot{B}^{s+t}_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p_1,r_1}} \|v\|_{\dot{B}^t_{p_2,r_2}}$.

A.3. ***Dispersion and Strichartz estimates.*** Consider the system

$$\begin{cases} \partial_t f - \left(L - \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\right) f = F_{\text{ext}}, \\ f_{|t=0} = f_0. \end{cases} \tag{A-79}$$

If we apply the Fourier transform, the equation becomes (see [Charve 2005] for details)

$$\partial_t \hat{f} - \mathbb{B}(\xi,\varepsilon)\hat{f} = \widehat{F}_{\text{ext}},$$

where

$$\mathbb{B}(\xi,\varepsilon) = \widehat{L - \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}} = \begin{pmatrix} -\nu|\xi|^2 + \xi_1\xi_2/(\varepsilon|\xi|^2) & (\xi_2^2+\xi_3^2)/(\varepsilon|\xi|^2) & 0 & \xi_1\xi_3/(\varepsilon F|\xi|^2) \\ -(\xi_1^2+\xi_3^2)/(\varepsilon|\xi|^2) & -\nu|\xi|^2 - \xi_1\xi_2/(\varepsilon|\xi|^2) & 0 & \xi_2\xi_3/(\varepsilon F|\xi|^2) \\ \xi_2\xi_3/(\varepsilon|\xi|^2) & -\xi_1\xi_3/(\varepsilon|\xi|^2) & -\nu|\xi|^2 & -(\xi_1^2+\xi_2^2)/(\varepsilon F|\xi|^2) \\ 0 & 0 & 1/(\varepsilon F) & -\nu'|\xi|^2 \end{pmatrix}.$$

For $0<r<R$ we will denote by $\mathcal{C}_{r,R}$ the set

$$\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}^3 : |\xi| \le R \text{ and } |\xi_3| \ge r\}.$$

We also introduce the following frequency truncation operator on $\mathcal{C}_{r,R}$:

$$\mathcal{P}_{r,R} = \chi\left(\frac{|D|}{R}\right)\left(1 - \chi\left(\frac{|D_3|}{r}\right)\right), \tag{A-80}$$

where χ is the smooth cut-off function introduced before and ($\mathcal{F}^{-1}$ is the inverse Fourier transform)

$$\chi\left(\frac{|D|}{R}\right) f = \mathcal{F}^{-1}\left(\chi\left(\frac{|\xi|}{R}\right)\hat{f}(\xi)\right) \quad \text{and} \quad \chi\left(\frac{|D_3|}{r}\right) f = \mathcal{F}^{-1}\left(\chi\left(\frac{|\xi_3|}{r}\right)\hat{f}(\xi)\right),$$

and $|D|^s$ the classical derivation operator $|D|^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))$.

In what follows we will use it for particular radii $r_\varepsilon = \varepsilon^m$ and $R_\varepsilon = \varepsilon^{-M}$, where m and M will be made precise later. Let us end with the following anisotropic Bernstein-type result (we refer to [Charve 2005], and to [Iftimie 1999b] for more general anisotropic estimates):

Lemma 42. *There exists a constant $C>0$ such that for any function f, $\alpha>0$, $1 \le q \le p \le \infty$, and all $0<r<R$, we have*

$$\begin{aligned} \left\| \chi\left(\frac{|D|}{R}\right)\chi\left(\frac{|D_3|}{r}\right) f \right\|_{L^p} &\le C\|f\|_{L^p}, \\ \left\| \chi\left(\frac{|D|}{R}\right)\chi\left(\frac{|D_3|}{r}\right) f \right\|_{L^p} &\le C(R^2 r)^{\frac{1}{q}-\frac{1}{p}} \left\| \chi\left(\frac{|D|}{R}\right)\chi\left(\frac{|D_3|}{r}\right) f \right\|_{L^q}. \end{aligned} \tag{A-81}$$

Moreover if f has its frequencies located in $\mathcal{C}_{r,R}$, then

$$\||D|^\alpha f\|_{L^p} \le CR^\alpha \|f\|_{L^p}. \qquad \square$$

A.3.1. *Eigenvalues, projectors.* We begin with the eigenvalues and eigenvectors of matrix the $\mathbb{B}(\xi,\varepsilon)$. We refer to [Charve 2004; 2005; 2006; 2018a; Charve and Ngo 2011] for details about the following proposition. We will only state the results and skip details as the proof is an adaptation of Proposition 3.1 from [Charve and Ngo 2011] (there in the anisotropic case).

Proposition 43. *If $\nu \neq \nu'$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, for all $r_\varepsilon = \varepsilon^m$, and $R_\varepsilon = \varepsilon^{-M}$, with $M < \frac{1}{4}$ and $3M + m < 1$, and for all $\xi \in \mathcal{C}_{r_\varepsilon, R_\varepsilon}$, the matrix $\mathbb{B}(\xi,\varepsilon) = \widehat{L - \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}}$ is diagonalizable and its eigenvalues have the following asymptotic expansions with respect to ε*:

$$\begin{aligned} \mu_0 &= -\nu|\xi|^2, & \lambda &= -\tau(\xi)|\xi|^2 + i\frac{|\xi|_F}{\varepsilon F|\xi|} + \varepsilon E(\xi,\varepsilon), \\ \mu &= -(\nu\xi_1^2 + \nu\xi_2^2 + \nu' F^2\xi_3^2)\frac{|\xi|^2}{|\xi|_F^2} + \varepsilon^2 D(\xi,\varepsilon), & \bar{\lambda} &= -\tau(\xi)|\xi|^2 - i\frac{|\xi|_F}{\varepsilon F|\xi|} + \varepsilon \bar{E}(\xi,\varepsilon), \end{aligned} \tag{A-82}$$

where $|\xi|_F^2 = \xi_1^2 + \xi_2^2 + F^2\xi_3^3$, and D, E denote remainder terms satisfying for all $\xi \in \mathcal{C}_{r_\varepsilon, R_\varepsilon}$

$$\begin{aligned} \varepsilon^2|D(\xi,\varepsilon)| &\le C_F|\nu-\nu'|^3\varepsilon^2|\xi|^6 \le C_F|\nu-\nu'|^3\varepsilon^{2-6M} \ll 1, \\ \varepsilon|E(\xi,\varepsilon)| &\le C_F|\nu-\nu'|^2\varepsilon|\xi|^4 \le C_F|\nu-\nu'|^2\varepsilon^{1-4M} \ll 1, \\ \varepsilon|\partial_{\xi_2}E(\xi,\varepsilon)| &\le C_F|\nu-\nu'|^2\varepsilon|\xi|^3 \le C_F|\nu-\nu'|^2\varepsilon^{1-3M} \ll 1, \end{aligned}$$

and

$$\tau(\xi) = \frac{\nu}{2}\left(1 + \frac{F^2\xi_3^2}{|\xi|_F^2}\right) + \frac{\nu'}{2}\left(1 - \frac{F^2\xi_3^2}{|\xi|_F^2}\right) \ge \min(\nu,\nu') > 0.$$

Moreover, if we denote by $\mathcal{P}_i(\xi,\varepsilon)$ the projectors onto the eigenspaces corresponding to μ, λ and $\bar{\lambda}$ ($i \in \{2,3,4\}$), and set

$$\mathbb{P}_i(u) = \mathcal{F}^{-1}(\mathcal{P}_i(\xi,\varepsilon)(\hat{u}(\xi))), \tag{A-83}$$

then for any divergence-free vector field f whose Fourier transform is supported in $\mathcal{C}_{r_\varepsilon, R_\varepsilon}$ and $s \in \mathbb{R}$, we have the estimates:

$$\|\mathbb{P}_2 f\|_{\dot{H}^s} \le C_F\|f\|_{\dot{H}^s} \times \begin{cases} 1 & \text{if } \Omega(f) \neq 0, \\ |\nu-\nu'|\varepsilon R_\varepsilon^2 = |\nu-\nu'|\varepsilon^{1-2M} & \text{if } \Omega(f) = 0, \end{cases} \tag{A-84}$$

and, for $i = 3, 4$,

$$\|\mathbb{P}_i f\|_{\dot{H}^s} \le C_F\frac{R_\varepsilon}{r_\varepsilon}\|f\|_{\dot{H}^s} = C_F\varepsilon^{-(m+M)}\|f\|_{\dot{H}^s}. \tag{A-85}$$

Finally, if we define $\mathbb{P}_{3+4}f \overset{\text{def}}{=} (\mathbb{P}_3 + \mathbb{P}_4)f = (I_d - \mathbb{P}_2)f$ (as $\operatorname{div} f = 0$), then

$$\|\mathbb{P}_{3+4}f\|_{\dot{H}^s} \le C_F(1 + |\nu-\nu'|\varepsilon R_\varepsilon^2)\|f\|_{\dot{H}^s}. \tag{A-86}$$

Remark 44. In the case $\nu = \nu'$ everything is simpler: the eigenvalues have simple explicit expressions: $-\nu|\xi|^2$ (double, μ and μ_0 coincide), $-\nu|\xi|^2 \pm (i|\xi|_F)/(\varepsilon F|\xi|)$, the eigenvectors do not depend on ε and are mutually orthogonal (so that $\mathbb{P}_i$ are of norm 1) and this basis exactly corresponds to the QG/osc decomposition (for divergence-free vector fields): $\mathcal{P} = \mathbb{P}_{3+4}$ and $\mathcal{Q} = \mathbb{P}_2$ so that the quasigeostrophic part only depends on W_2 and the oscillating part only depends on $W_{3,4}$. Finally the operator Γ reduces to a simple anisotropic Laplace operator. We refer to [Charve 2006, Appendix B; 2016; 2018a] for more details.

Remark 45. We emphasize that the leading term in μ is the Fourier symbol of the quasigeostrophic operator Γ. Moreover, the dispersion is related to the term $(i|\xi|_F)/(\varepsilon F|\xi|)$, and when $F=1$ this term reduces to the constant $\frac{i}{\varepsilon}$. This is why dispersion does not occur in the case $F=1$ (we refer to [Chemin 1997; Charve 2018b] for a study of the asymptotics in the special case $F=1$).

A.3.2. *Dispersion, Strichartz estimates.* Combining Proposition 3 from [Charve 2018a] (covering the range $p\geq 4$) with the convolution arguments from the appendix of [Charve 2004] allows us to cover the full range $p\geq 1$ and obtain the following Strichartz estimates satisfied by the last two projections of the solution of system(A-79):

Proposition 46. *Assume that f satisfies* (A-79) *on* $[0,T[$, *where* $\operatorname{div} f_0=0$ *and the frequencies of f_0 and F are localized in $\mathcal{C}_{r_\varepsilon,R_\varepsilon}$. Then there exists a constant $C=C_{F,p,\nu,\nu'}>0$ such that, for $i\in\{3,4\}$ and $p\geq 1$, we have*

$$\|\mathbb{P}_i f\|_{L^p_T L^\infty}\leq CK(\varepsilon)\left(\|f_0\|_{L^2}+\int_0^T\|F_{\text{ext}}(\tau)\|_{L^2}\,d\tau\right).$$

where

$$K(\varepsilon)=\begin{cases}\varepsilon^{\frac14}\dfrac{R_\varepsilon^4}{r_\varepsilon^{\frac52+\frac2p}}\left[\dfrac{4}{\nu_0}\left(\dfrac1p-\dfrac14\right)\right]^{\frac1p-\frac14}=\varepsilon^{\frac14-(4M+(\frac52+\frac2p)m)}\left[\dfrac{4}{\nu_0}\left(\dfrac1p-\dfrac14\right)\right]^{\frac1p-\frac14} & \text{if } p\in[1,4],\\ \varepsilon^{\frac1p}\dfrac{R_\varepsilon^{\frac52+\frac6p}}{r_\varepsilon^{2+\frac4p}}=\varepsilon^{\frac1p-((\frac52+\frac6p)M+(2+\frac4p)m)} & \text{if } p\geq 4.\end{cases}$$

Unfortunately these estimates would be completely useless in our case: we need more flexibility than only L^p-L^∞-estimates, and in the case $\nu\neq\nu'$ we need to take into account the second term G^l as done in [Charve 2004]. We begin with the case $\nu=\nu'$, where we have to deal with the fact that we obtain Strichartz estimates on W_ε, which is not frequency-localized (we improve the method from [Charve 2006, Appendix B]). Then we deal with the case $\nu\neq\nu'$.

A.3.3. *Strichartz estimates in the case $\nu=\nu'$.* The main result of this section is stated as follows:

Proposition 47. *There exists a constant $C_F>0$ such that for any $d\in\mathbb{R}$, $r>4$, $q\geq 1$, and*

$$\theta\in\left]0,\frac{\frac12-\frac1r}{1-\frac4r}\right[\cap]0,1],\qquad p\in\left[1,\frac{4}{\theta\left(1-\frac4r\right)}\right],$$

if f solves (A-79) *for initial data f_0 and external force F_{ext} both with zero divergence and potential vorticity, then* (c_0 *refers to the smaller constant appearing in the Littlewood–Paley decomposition, usually* $c_0=\frac34$).

$$\||D|^d f\|_{\tilde{L}^p_t\dot{B}^0_{r,q}}\leq C_F\frac{C_{p,\theta,r}}{\nu^{\frac1p-\frac\theta4(1-\frac4r)}}\varepsilon^{\frac\theta4(1-\frac4r)}\times\left(\|f_0\|_{\dot{B}^\sigma_{2,q}}+\int_0^t\|F_{\text{ext}}(\tau)\|_{\dot{B}^\sigma_{2,q}}\,d\tau\right),\tag{A-87}$$

where

$$\sigma = d + \tfrac{3}{2} - \tfrac{3}{r} - \tfrac{2}{p} + \tfrac{\theta}{2}\big(1 - \tfrac{4}{r}\big), \quad C_{p,\theta,r} = \big[\tfrac{2}{c_0^2}\big(\tfrac{1}{p} - \tfrac{\theta}{4}(1 - \tfrac{4}{r})\big)\big]^{\frac{1}{p} - \frac{\theta}{4}(1-\frac{4}{r})} \frac{2^{\frac{1}{2}(1-\frac{2}{r}-2\theta(1-\frac{4}{r}))}}{1 - 2^{-\frac{1}{2}(1-\frac{2}{r}-2\theta(1-\frac{4}{r}))}}.$$

Remark 48. It is interesting to compare our Strichartz estimates with the ones from [Iwabuchi et al. 2017; Scrobogna 2017] (see Proposition 5). In our estimates we use the range $r > 4$, whereas in Proposition 5 they consider the case $r \in]2,4[$ and they use it for r close to 3. Our index p is mostly equal to 2 but we can reach $p = 1$ (which is useful when there are derivatives), whereas in [Iwabuchi et al. 2017], $p > 1/\big(1 - \frac{2}{r}\big) > 2$.

Proof of Proposition 47. Let us first assume that $F_{\text{ext}} = 0$. As $\nu = \nu'$, the fact that f_0 is divergence-free and with zero potential vorticity implies that

$$f_0 = \mathbb{P} f_0 = \mathcal{P}\mathbb{P} f_0 = \mathbb{P}_{3+4}\mathbb{P} f_0 = \mathbb{P}_{3+4} f_0,$$

so that we only consider the last two eigenvalues (we recall the eigenvectors are orthogonal). The idea is to push further the Strichartz estimates without the frequency truncation we obtained in [Charve 2006]: we will once more use a simple nonstationary phase argument (see for example the works of Chemin, Desjardins, Gallagher and Grenier [Chemin et al. 2000; 2002; 2006]). As outlined previously, in this special case there is no need to truncate in frequency through the operator $\mathcal{P}_{r_\varepsilon, R_\varepsilon}$ but within the computations we will truncate considering the vertical Littlewood–Paley decomposition ($\dot{\Delta}_k^v u = \varphi(2^{-j} D_3)u$):

$$\|\dot{\Delta}_j f\|_{L_t^p L_x^r} = \|\dot{\Delta}_j f\|_{L^p L^r} = \sum_{k=-\infty}^{j+1} \|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^p L^r}.$$

Now we will use the methods leading to the general Strichartz estimates (previously used when frequencies are truncated on some $\mathcal{C}_{r,R}$) as in our case $r = c_0 2^k$ and $R = C_0 2^j$. We recall that φ is the truncation function involved in the Littlewood–Paley decomposition, we denote by φ_1 another smooth truncation function, with support in a slightly larger annulus than φ and equal to 1 on $\operatorname{supp}\varphi$, and by $\mathcal{B}$ the set

$$\mathcal{B} \overset{\text{def}}{=} \{\psi \in \mathcal{C}_0^\infty(\mathbb{R}_+, \mathbb{R}^3) : \|\psi\|_{L^{\bar{p}}(\mathbb{R}_+, L^{\bar{r}}(\mathbb{R}^3))} \le 1\}.$$

Then following the same classical steps as in [Charve 2006] we get that (we choose for simplicity to write it only for the third eigenvalue) for any $\beta \ge 1$

$$\begin{aligned}
&\|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^p L^r} \\
&\quad = \sup_{\psi \in \mathcal{B}} \int_0^\infty \dot{\Delta}_k^v \dot{\Delta}_j f(t,x)\psi(t,x)\,dx\,dt \\
&\quad = C \sup_{\psi \in \mathcal{B}} \int_0^\infty \int_{\mathbb{R}^3} e^{-\nu t|\xi|^2 + i\frac{t}{\varepsilon}\frac{|\xi|_F}{F|\xi|}} \widehat{\dot{\Delta}_j f_0}(\xi)\varphi_1(2^{-j}\xi)\varphi(2^{-k}|\xi_3|)\hat{\psi}(t,\xi)\,d\xi\,dt \\
&\quad \le C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_j f_0\|_{L^2} \bigg[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^3} K\Big(\nu(t+s), \frac{|t-s|}{\varepsilon}, x\Big) \cdot (\psi(t) * \bar{\psi}(s))(x)\,dx\,ds\,dt\bigg]^{\frac{1}{2}}, \\
&\quad \le C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_j f_0\|_{L^2} \bigg[\int_0^\infty \int_0^\infty \Big\|K\Big(\nu(t+s), \frac{|t-s|}{\varepsilon}, \cdot\Big)\Big\|_{L^{\bar{\beta}}} \|\psi(t) * \bar{\psi}(s)\|_{L^\beta}\,ds\,dt\bigg]^{\frac{1}{2}},
\end{aligned} \tag{A-88}$$

with K defined as follows (we refer to [Charve 2006] for details):

$$K(\sigma,\tau,x)=\int_{A_{j,k}} e^{ix\cdot\xi-\sigma|\xi|^2+i\tau\frac{|\xi|_F}{F|\xi|}}\varphi_1(2^{-j}|\xi|)^2\varphi(2^{-k}|\xi_3|)^2\,d\xi,$$

where

$$A_{j,k}\overset{\text{def}}{=}\{\xi\in\mathbb{R}^3: c_0 2^j\le|\xi|\le C_0 2^j \text{ and } c_0 2^k\le|\xi_3|\le C_0 2^k\}. \tag{A-89}$$

Interpolating the following estimates (we refer to [Charve 2006, Section B.2] for more details), and using as in [Charve 2018a, Section 3.2] that for all $a,b>0$ and $\theta\in[0,1]$ we have $\min(a,b)\le a^{1-\theta}b^\theta$,

$$\begin{aligned}\|K(\sigma,\tau,\cdot)\|_{L^\infty}&\le C_F e^{-c_0^2\sigma 2^{2j}}2^{3j}\min\Big(2^{k-j},\frac{1}{\tau^{\frac12}2^{k-j}}\Big),\\ \|K(\sigma,\tau,\cdot)\|_{L^2}&\le C_F e^{-\frac{c_0^2}{2}\sigma 2^{2j}}2^{\frac{3j}{2}}2^{\frac{k-j}{2}},\end{aligned}$$

we get for any $r\in[2,\infty]$, $\frac1r=\frac{1-\alpha}{\infty}+\frac{\alpha}{2}=\frac{\alpha}{2}$, and $\theta\in[0,1]$

$$\begin{aligned}\|K(\sigma,\tau,\cdot)\|_{L^r}&\le C_F e^{-\frac{c_0^2}{2}\sigma 2^{2j}}\bigg(2^{3j}\frac{2^{(k-j)(1-2\theta)}}{\tau^{\frac{\theta}{2}}}\bigg)^{1-\frac2r}\big(2^{\frac{3j}{2}}2^{\frac{k-j}{2}}\big)^{\frac2r}\\ &\le C_F e^{-\frac{c_0^2}{2}\sigma 2^{2j}}2^{3j(1-\frac1r)}\frac{2^{(k-j)[1-\frac1r-2\theta(1-\frac2r)]}}{\tau^{\frac{\theta}{2}(1-\frac2r)}}.\end{aligned} \tag{A-90}$$

Now we can go back to (A-88), by the Cauchy–Schwarz inequality, fixing $\beta\ge1$ so that

$$\|\psi(t)*\bar\psi(s)\|_{L^\beta}\le\|\psi(t)\|_{L^{\bar r}}\|\psi(s)\|_{L^{\bar r}},$$

that is, choosing $\bar\beta=\frac{\beta}{\beta-1}=\frac r2$ (which implies that $r\ge4$), and using (A-90), we obtain that

$$\begin{aligned}\|\dot\Delta_k^v\dot\Delta_j f\|_{L^pL^r}\le C_F\sup_{\psi\in\mathcal{B}}\|\dot\Delta_j f_0\|_{L^2}\varepsilon^{\frac{\theta}{4}(1-\frac4r)}2^{\frac{3j}{2}(1-\frac2r)}2^{\frac{k-j}{2}(1-\frac2r-2\theta(1-\frac4r))}\\ \times\bigg[\int_0^\infty\int_0^\infty\frac{h(t)h(s)}{|t-s|^{\frac{\theta}{2}(1-\frac4r)}}\,ds\,dt\bigg]^{\frac12},\end{aligned} \tag{A-91}$$

with

$$h(t)=e^{-\frac{c_0^2}{2}\nu t2^{2j}}\|\psi(t)\|_{L^{\bar r}}.$$

Next we will use the Hardy–Littlewood–Sobolev estimates, which we recall in $\mathbb{R}$ for the convenience of the reader (we refer to [Hardy and Littlewood 1930; Sobolev 1938; Lieb 1983]):

Proposition 49. *There exists a constant $C>0$ such that for any function $h_i\in L^{q_i}(\mathbb{R})$ ($q_i>1$ for $i=1,2$) and any $\alpha>0$, with $\frac{1}{q_1}+\frac{1}{q_2}+\alpha=2$, we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{h_1(t)h_2(s)}{|t-s|^\alpha}\,dt\,ds\le C\|h_1\|_{L^{q_1}}\|h_2\|_{L^{q_2}}.$$

Choosing $h_1 = h_2 = h\mathbf{1}_{\mathbb{R}_+}$, $\alpha = \frac{\theta}{2}\left(1-\frac{4}{r}\right) > 0$, and $\frac{1}{q} = 1 - \frac{\theta}{4}\left(1-\frac{4}{r}\right)$, we get that

$$\int_0^\infty \int_0^\infty \frac{h(t)h(s)}{|t-s|^{\frac{\theta}{2}(1-\frac{4}{r})}}\,ds\,dt \le C\|h\|_{L^q}^2 \le C(\|e^{-\frac{c_0^2}{2}\nu 2^{2j}t}\|_{L^m}\|\psi\|_{L^{\bar p}L^{\bar r}})^2$$
$$\le C\left(\frac{1}{\nu^{\frac{1}{m}}}\left[\frac{2}{mc^2}\right]^{\frac{1}{m}} 2^{-\frac{2j}{m}}\|\psi\|_{L^{\bar p}L^{\bar r}}\right)^2 \tag{A-92}$$

for $m \in [1,\infty]$ chosen so that $\frac{1}{m} + \frac{1}{\bar p} = \frac{1}{q}$, that is,

$$\frac{1}{m} = \frac{1}{p} - \frac{\theta}{4}\left(1-\frac{4}{r}\right).$$

Remark 50. Note that this implies the condition $p \le \frac{4}{\theta}\left(1/\left(1-\frac{4}{r}\right)\right)$.

Combining with (A-91), we can write that

$$\|\dot\Delta_k^\nu \dot\Delta_j f\|_{L^pL^r} \le C_F\|\dot\Delta_j f_0\|_{L^2}\varepsilon^{\frac{\theta}{4}(1-\frac{4}{r})}2^{j(\frac{3}{2}-\frac{3}{r}-\frac{2}{p}+\frac{\theta}{2}(1-\frac{4}{r}))}$$
$$\times \frac{2^{\frac{k-j}{2}(1-\frac{2}{r}-2\theta(1-\frac{4}{r}))}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{4}{r})}}\left[\frac{2}{c^2}\left(\frac{1}{p}-\frac{\theta}{4}\left(1-\frac{4}{r}\right)\right)\right]^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{4}{r})}. \tag{A-93}$$

It is possible to sum this for $k \le j+1$ if and only if

$$1 - \frac{2}{r} - 2\theta\left(1-\frac{4}{r}\right) > 0,$$

that is, as $r > 4$, when

$$\theta < \frac{1-\frac{2}{r}}{2\left(1-\frac{4}{r}\right)}.$$

Summing over k we obtain that for all j,

$$\|\dot\Delta_j f\|_{L^pL^r} \le C_F\frac{C_{p,\theta,r}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{4}{r})}}\varepsilon^{\frac{\theta}{4}(1-\frac{4}{r})}2^{j(\frac{3}{2}-\frac{3}{r}-\frac{2}{p}+\frac{\theta}{2}(1-\frac{4}{r}))}\|\dot\Delta_j f_0\|_{L^2}, \tag{A-94}$$

which leads to the desired result in the homogeneous case. The inhomogeneous case (i.e., when $F_{\text{ext}} \neq 0$) easily follows thanks to the Duhamel formula. □

A.3.4. *Strichartz estimates in the case $\nu \neq \nu'$.*

Proposition 51. *There exists a constant $C_{F,\omega} > 0$ (where $\omega = \frac{\max(\nu,\nu')}{\nu_0}$) such that for any $d \in \mathbb{R}$, $r > 4$, and $p < 4/\left(1-\frac{4}{r}\right)$, if f solves* (A-79) *for initial data f_0 and external force $F_{\text{ext}} = F^b + F^l$, all three of them with zero divergence and potential vorticity, then for $i = 3, 4$*

$$\||D|^d\mathbb{P}_i\mathcal{P}_{r_\varepsilon,R_\varepsilon}f\|_{\widetilde L_t^p\dot B_{r,q}^0} \le C_{F,\omega}\frac{D_{p,r}}{\nu_0^{\frac{1}{p}-\frac{1}{4}(1-\frac{4}{r})}}\varepsilon^{\frac{1}{4}(1-\frac{4}{r})}\frac{R_\varepsilon^{4-\frac{9}{r}}}{r_\varepsilon^{\frac{5}{2}+\frac{2}{p}-\frac{6}{r}}}$$
$$\times\left(\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}f_0\|_{\dot B_{2,q}^d} + \|\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^b\|_{L^1\dot B_{2,q}^d} + \frac{1}{\nu_0^{\frac{1}{2}}r_\varepsilon}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^l\|_{L^2\dot B_{2,q}^d}\right), \tag{A-95}$$

where $D_{p,r} = \max(b_{p,r}, d_{p,r})$ *with*

$$b_{p,r} = \big(\tfrac{2}{\nu c^2}\big)^{\frac{1}{p}-\frac{1}{4}(1-\frac{4}{r})}\big(\tfrac{1}{p}-\tfrac{1}{4}\big(1-\tfrac{4}{r}\big)\big)^{\frac{1}{p}-\frac{1}{4}(1-\frac{4}{r})},$$

$$d_{p,r} = 2^{\frac{1}{p}}\big(\tfrac{8}{c^2 p}\big)^{\frac{1}{p}-\frac{1}{4}(1-\frac{4}{r})}\bigg(\int_0^\infty \frac{e^{-x}}{x^{\frac{p}{4}(1-\frac{4}{r})}}\,dx\bigg)^{\frac{1}{p}}.$$

Remark 52. We could prove like in the previous section some refined estimate with $\theta \in]0,1]$ $\big($allowing $p \le 4/\big(\theta\big(1-\frac{4}{r}\big)\big)\big)$ but we will only need the case $\theta = 1$ and p close to 2 in this article.

Proof of Proposition 51. Let us first assume that $F_{\text{ext}} = 0$. With the same notation as in the previous section, we get that (see previous section, as well as [Charve and Ngo 2011; Charve 2018a] for details)

$$\begin{aligned}
&\|\mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f\|_{\widetilde{L}^p(\mathbb{R}_+,L^r(\mathbb{R}^3))}\\
&\quad = \sup_{\psi\in\mathcal{B}} \int_0^\infty\int_{\mathbb{R}^3} \mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f(t,x)\psi(t,x)\,dx\,dt\\
&\quad = \sup_{\psi\in\mathcal{B}} \int_0^\infty\int_{\mathbb{R}^3} e^{-t\tau(\xi)|\xi|^2+it\frac{|\xi|_F}{\varepsilon F|\xi|}+\varepsilon t E(\xi,\varepsilon)}\widehat{\mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f_0}(t,\xi)\chi\bigg(\frac{|\xi|}{2R_\varepsilon}\bigg)\bigg(1-\chi\bigg(\frac{2|\xi_3|}{r_\varepsilon}\bigg)\bigg)\hat{\psi}(t,\xi)\,d\xi\,dt\\
&\quad \le C\sup_{\psi\in\mathcal{B}} \|\mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f_0\|_{L^2}\bigg[\int_0^\infty\int_0^\infty \|L(t,s,\varepsilon,\cdot)\|_{L^{r/2}}\|\psi(t)*\bar{\psi}(s)\|_{L^{r/(r-2)}}\,ds\,dt\bigg]^{\frac{1}{2}},
\end{aligned}\tag{A-96}$$

where

$$L(t,s,\varepsilon,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi-(t+s)\tau(\xi)|\xi|^2+i(t-s)\frac{|\xi|_F}{\varepsilon F|\xi|}+\varepsilon t E(\xi,\varepsilon)+\varepsilon s\overline{E}(\xi,\varepsilon)}\chi\bigg(\frac{|\xi|}{2R_\varepsilon}\bigg)^2\bigg(1-\chi\bigg(\frac{2|\xi_3|}{r_\varepsilon}\bigg)\bigg)^2\,d\xi.$$

Like before, to obtain the $L^{\frac{r}{2}}$-norm, we will interpolate between L^2 and L^∞. It is easy to obtain

$$\|L(s,t,\varepsilon,\cdot)\|_{L^2} \le C_F R_\varepsilon^{\frac{3}{2}} e^{-c^2\frac{\nu_0}{4}(t+s)r_\varepsilon^2},$$

and we refer to [Charve and Ngo 2011; Charve 2018a] where we proved that (there we were working with local-in-time solutions, and we dropped the exponential)

$$\|L(s,t,\varepsilon,\cdot)\|_{L^\infty} \le C_{F,\omega}\frac{R_\varepsilon^3}{r_\varepsilon^2}\min\bigg(1,\frac{R_\varepsilon^3}{r_\varepsilon^2}\bigg(\frac{\varepsilon}{|t-s|}\bigg)^{\frac{1}{2}}\bigg)e^{-c^2\frac{\nu_0}{4}(t+s)r_\varepsilon^2},$$

so that we obtain for any $\beta \ge 2$

$$\|L(s,t,\varepsilon,\cdot)\|_{L^\beta} \le C_{F,\omega} e^{-c^2\frac{\nu_0}{4}(t+s)r_\varepsilon^2}\frac{R_\varepsilon^{6-\frac{9}{\beta}}}{r_\varepsilon^{4-\frac{8}{\beta}}}\bigg(\frac{\varepsilon}{|t-s|}\bigg)^{\frac{1}{2}(1-\frac{2}{\beta})}.$$

Thanks to (A-85), and doing the same as previously, we end up with $\big(\beta = \frac{r}{2}\big)$

$$\|\mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f\|_{L^pL^r} = C_{F,\omega}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon} f_0\|_{L^2}\frac{R_\varepsilon^{4-\frac{9}{r}}}{r_\varepsilon^{3-\frac{8}{r}}}\varepsilon^{\frac{1}{4}(1-\frac{4}{r})}\sup_{\psi\in\mathcal{B}}\bigg[\int_0^\infty\int_0^\infty \frac{g(t)g(s)}{|t-s|^{\frac{1}{2}(1-\frac{4}{r})}}\,ds\,dt\bigg]^{\frac{1}{2}},\tag{A-97}$$

with

$$g(t)=e^{-\frac{c^2}{2}\nu t r_\varepsilon^2}\|\psi(t)\|_{L^{\bar r}}.$$

Using once more Proposition 49, we end up with

$$\|\mathbb{P}_i\mathcal{P}_{r_\varepsilon,R_\varepsilon}f\|_{L^pL^r}\le C_F\frac{b_{p,r}}{\nu_0^{\frac1p-\frac14(1-\frac4r)}}\varepsilon^{\frac14(1-\frac4r)}\frac{R_\varepsilon^{4-\frac9r}}{r_\varepsilon^{\frac52+\frac2p-\frac6r}}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}f_0\|_{L^2}. \tag{A-98}$$

Then it is easy to deduce the nonhomogeneous case with F^b only. Let us now focus on the other external force term; we extend the method from [Charve 2004]. If we denote by $S(t)f_0$ the solution of system (A-79) with $F_{\text{ext}}=0$, we have by the Duhamel formula

$$\begin{aligned}
&\left\|\int_0^t S(t-t')\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_iF^l(t')\,dt'\right\|_{L_t^pL^r}\\
&\quad=\sup_{\psi\in\mathcal{B}}\int_0^\infty\int_{\mathbb{R}^3}\widehat{\mathbb{P}_i\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^l}(t',\xi)\\
&\qquad\times\int_{t'}^\infty e^{-(t-t')\tau(\xi)|\xi|^2+i(t-t')\frac{|\xi|_F}{\varepsilon F|\xi|}+\varepsilon tE(\xi,\varepsilon)}\chi\left(\frac{|\xi|}{2R_\varepsilon}\right)\left(1-\chi\left(\frac{2|\xi_3|}{r_\varepsilon}\right)\right)\hat\psi(t,\xi)\,dt\,d\xi\,dt'\\
&\quad\le C\sup_{\psi\in\mathcal{B}}\|\mathbb{P}_i\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^l\|_{L^2L^2}\left[\int_0^\infty\int_{t'}^\infty\int_{t'}^\infty\|L(t-t',s-t',\varepsilon,\cdot)\|_{L^{r/2}}\|\psi(t)*\bar\psi(s)\|_{L^{r/(r-2)}}\,ds\,dt\right]^{\frac12}\\
&\quad\le C_{F,\omega}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^l\|_{L^2L^2}\frac{R_\varepsilon^{4-\frac9r}}{r_\varepsilon^{3-\frac8r}}\varepsilon^{\frac14(1-\frac4r)}\\
&\qquad\times\sup_{\psi\in\mathcal{B}}\left[\int_0^\infty\int_0^\infty\int_0^\infty\mathbf{1}_{\{t'\le\min(t,s)\}}\frac{e^{-c^2\frac{\nu_0}{4}(t+s-2t')r_\varepsilon^2}}{|t-s|^{\frac12(1-\frac4r)}}\|\psi(t)\|_{L^{\bar r}}\|\psi(s)\|_{L^{\bar r}}\,ds\,dt\,dt'\right]^{\frac12}.
\end{aligned}\tag{A-99}$$

Computing the integral in t',

$$\int_0^{\min(s,t)}e^{c^2\frac{\nu_0}{2}t'r_\varepsilon^2}\,dt'\le\frac{2}{\nu_0r_\varepsilon^2}e^{c^2\frac{\nu_0}{2}\min(t,s)r_\varepsilon^2},$$

and using the fact that $|t-s|=s+t-2\min(s,t)$, we get

$$\begin{aligned}
&\left\|\int_0^t S(t-t')\mathcal{P}_{r_\varepsilon,R_\varepsilon}\mathbb{P}_iF^l(t')\,dt'\right\|_{L_t^pL^r}\\
&\quad\le C_{F,\omega}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon}F^l\|_{L^2L^2}\frac{R_\varepsilon^{4-\frac9r}}{r_\varepsilon^{4-\frac8r}}\varepsilon^{\frac14(1-\frac4r)}\\
&\qquad\qquad\times\sup_{\psi\in\mathcal{B}}\left[\int_0^\infty\int_0^\infty\frac{e^{-c^2\frac{\nu_0}{4}|t-s|r_\varepsilon^2}}{|t-s|^{\frac12(1-\frac4r)}}\|\psi(t)\|_{L^{\bar r}}\|\psi(s)\|_{L^{\bar r}}\,ds\,dt\right]^{\frac12}.
\end{aligned}\tag{A-100}$$

Then setting

$$k(\tau)=e^{-c^2\frac{\nu_0}{4}|\tau|r_\varepsilon^2}|\tau|^{-\frac12(1-\frac4r)},$$

we just have to estimate a convolution

$$\int_0^\infty \int_0^\infty k(t-s)\|\psi(t)\|_{L^{\bar{r}}}\|\psi(s)\|_{L^{\bar{r}}}\, ds\, dt \le \|k\|_{L^{p/2}}\|\psi\|^2_{L^{\bar{p}}L^{\bar{r}}}, \tag{A-101}$$

provided that $p \ge 2$ and $\frac{p}{4}\left(1-\frac{4}{r}\right) < 1$ so that $k \in L^{\frac{p}{2}}$, whose norm is featured in the constant $d_{p,r}$ and we have

$$\begin{aligned}\|\mathbb{P}_i \mathcal{P}_{r_\varepsilon,R_\varepsilon} f\|_{L^pL^r} \le C_{F,\omega}\frac{D_{p,r}}{\nu_0^{\frac{1}{p}-\frac{1}{4}(1-\frac{4}{r})}}\varepsilon^{\frac{1}{4}(1-\frac{4}{r})}\frac{R_\varepsilon^{4-\frac{9}{r}}}{r_\varepsilon^{\frac{5}{2}+\frac{2}{p}-\frac{6}{r}}}\\ \times\left(\|\mathcal{P}_{r_\varepsilon,R_\varepsilon} f_0\|_{L^2} + \|\mathcal{P}_{r_\varepsilon,R_\varepsilon} F^b\|_{L^1L^2} + \frac{1}{\nu_0^{\frac{1}{2}} r_\varepsilon}\|\mathcal{P}_{r_\varepsilon,R_\varepsilon} F^l\|_{L^2L^2}\right).\end{aligned} \tag{A-102}$$

Finally, to obtain the announced estimates, we just have to apply this estimate to $\dot{\Delta}_j|D|^d f$. $\square$

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