PURE and APPLIED ANALYSIS

PAA

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msp

vol. 2 no. 3 2020

PURE and APPLIED
ANALYSIS
Vol. 2, No. 3, 2020
https://doi.org/10.2140/paa.2020.2.603

L^p ESTIMATES FOR BAOUENDI–GRUSHIN OPERATORS

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We prove L^p estimates for the Baouendi–Grushin operator $\Delta_x + |x|^\alpha \Delta_y$ in $L^p(\mathbb{R}^{N+M})$, $1 < p < \infty$, where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. When $p = 2$ more general weights belonging to the reverse Hölder class $B_2(\mathbb{R}^N)$ are allowed.

1. Introduction

We prove L^p estimates for the Baouendi–Grushin operator $L = \Delta_x + |x|^\alpha \Delta_y$ in $L^p(\mathbb{R}^{N+M})$, $1 < p < \infty$, where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$; more specifically, we prove the L^p boundedness of the operators $D_{x_i x_j} L^{-1}$, $|x|^\alpha D_{y_i y_j} L^{-1}$ and $|x|^{\frac{\alpha}{2}} D_{x y_i} L^{-1}$. We use these results to characterize the domain of the operator L, denoted by $D_p(L)$, where the solution of the equation $\lambda u - Lu = f$ exists and is unique for any $f \in L^p(\mathbb{R}^{N+M})$ and $\lambda > 0$. In an equivalent way, we describe the domain under which L generates an (analytic and symmetric) semigroup in $L^p(\mathbb{R}^{N+M})$.

If α is an even integer, the operator is hypoelliptic but is not a sublaplacian in the sense of [Folland 1975] and our estimates seem to be known only for $\alpha = 2$ and even $N \geq 2$; see [Koch et al. 2015]. When α is an unrestricted positive real number, many results are known on local regularity of the equation $Lu = f$; see for example [Franchi and Serapioni 1987; Franchi et al. 1994; Franchi and Lanconelli 1984; Garofalo and Vassilev 2007] for the unique continuation property. We refer to [Robinson and Sikora 2008] for heat kernel estimates even in a more general context. However, we are not aware of global regularity results for the second derivatives of u, with the exception of [Wang 2003], where global Hölder regularity is proved for every $\alpha > 0$ and of [Kim 1999], where L^p estimates are proved, when $\alpha = 1$ and $N = 1$, in the half-plane $x > 0$ for the inhomogeneous problem $Lu = f$, $u(0, y) = g(y)$. When $g = 0$ the estimates in [Kim 1999] reduce to ours: even though our results are valid in the whole space, they can be rephrased when $N = 1$ in the half-space $x > 0$ for Dirichlet or Neumann boundary conditions by considering odd and even functions with respect to x, respectively.

We prove L^p estimates through an interpolation theorem in the absence of kernels in homogeneous spaces due to Z. Shen [2005, Theorem 3.1], see also [Auscher and Martell 2007, Theorem 3.14], and weighted mean value inequalities for subsolutions of the elliptic equation $Lu = 0$, with respect to the balls associated with the subelliptic distance defined by the operator, proved in [Franchi and Serapioni 1987; Chanillo and Wheeden 1986]. Some of these results can probably be generalized to the case when $|x|^\alpha$ is replaced by a weight function $\phi(x)$ belonging to the reverse Hölder class $B_p(\mathbb{R}^N)$. This is the

MSC2010: 35H20, 35J70, 47F05.
Keywords: Baouendi–Grushin operators, degenerate elliptic equations, subelliptic equations, L^p estimates.

case when $p = 2$, where the result is not obtained via integration by parts but using maximal inequalities due to P. Auscher and B. Ben Ali [2007] for Schrödinger operators with B_2 potentials. However, local estimates for subsolutions seem to be known only in special cases and they are crucial in our approach when $p \neq 2$. Another restriction comes from the estimates of the mixed derivatives, that is, for the operator $|x|^{\frac{\alpha}{2}} D_{xy_i} L^{-1}$, where our proof relies on scaling. In order to unify our approach and to improve the readability, we consider only the $|x|^\alpha$ case in our L^p estimates. Perturbation arguments from this model case allow us to treat different powers near 0 and ∞ or power-like behavior, but we prefer not to deal with these variants here.

The paper is organized as follows. In Section 2 we define the operator in $L^2(\mathbb{R}^{N+M})$ through a form and prove L^2 estimates via partial Fourier transform and maximal regularity results on Schrödinger operators. In Section 3 we briefly recall the subelliptic distance associated with L and the main geometrical objects needed in L^p estimates. The latter are proved in Section 4, where a separate subsection deals with mixed derivatives.

Notation. We use L^p for $L^p(\mathbb{R}^{N+M})$ and C_c^∞ for $C_c^\infty(\mathbb{R}^{N+M})$. L_c^∞ stands for the space of all bounded measurable functions on $\mathbb{R}^{N+M}$ having compact support. $\mathcal{S}$ is the Schwartz space and $\mathcal{S}'$ the space of tempered distributions. We also write $B(r) := \{x \in \mathbb{R}^N : |x| < r\}$, $B(x_0, r) = x_0 + B(r)$.

2. L^2 estimates

Let $\phi : \mathbb{R}^N \to [0, +\infty[$ be a nonnegative continuous function and set

$$\mathbb{R}^N \supseteq \Omega_N = \{x \in \mathbb{R}^N : \phi(x) > 0\}, \qquad \Omega = \Omega_N \times \mathbb{R}^M.$$

Let L be the operator defined on smooth functions by

$$L = \Delta_x + \phi(x)\Delta_y,$$

where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. Setting

$$a = \left(\begin{array}{c|c} I_N & 0 \\ \hline 0 & \phi I_M \end{array}\right) = (a_{ij})$$

or

$$a_{ij}(x, y) = \begin{cases} 1 & \text{if } i = j \le N, \\ \phi(x) & \text{if } N+1 \le i = j \le N+M \end{cases} \tag{1}$$

and 0 elsewhere, we can write

$$L = \operatorname{div}(a\,\nabla)$$

and, therefore, L is formally self-adjoint with respect to the Lebesgue measure.

Remark 2.1. L is nondegenerate in the x-direction but degenerates in the y-direction outside Ω. Accordingly, $\nabla_x u$ will denote the distributional gradient (with respect to x) of u in the whole space $\mathbb{R}^{N+M}$ and $\nabla_y u$ only its distributional gradient (with respect to y) in Ω.

We give a formal definition of L through a symmetric form:

Definition 2.2. Consider the sesquilinear form $\mathfrak{a}$ in L^2 defined by

$$\mathfrak{a}(u, v) := \int_{\mathbb{R}^{N+M}} [\langle \nabla_x u, \nabla_x \bar{v}\rangle + \phi(x)\langle \nabla_y u, \nabla_y \bar{v}\rangle]\, dx\, dy,$$
$$D(\mathfrak{a}) := \{u \in L^2 : u \in H^1_{\mathrm{loc}}(\Omega),\ \nabla_x u,\ \phi^{\frac{1}{2}}\nabla_y u \in L^2\}.$$

According to the remark above, we require that the weak gradient $\nabla_y u$ exists only in Ω.

We summarize in the following lemma the main properties of a. Note that, due to the assumptions on ϕ, $\mathfrak{a}$ is locally uniformly elliptic on Ω.

Lemma 2.3. *The form $\mathfrak{a}$ is densely defined, nonnegative, symmetric and closed in L^2 and the following properties hold*:

(i) *If Q is an orthogonal matrix in $\mathbb{R}^M$, $y_0 \in \mathbb{R}^M$ and $I_{Q+y_0}u(x, y) = u(x, Qy + y_0)$, then for every $u, v \in D(\mathfrak{a})$, one has $I_{Q+y_0}u, I_{Q+y_0}v \in D(\mathfrak{a})$ and*

$$\mathfrak{a}(I_{Q+y_0}u, I_{Q+y_0}v) = \mathfrak{a}(u, v).$$

(ii) *If ϕ is homogeneous of degree $\alpha \geq 0$, i.e., $\phi(sx) = s^\alpha \phi(x)$ for $x \in \mathbb{R}^N$, $s > 0$, then defining the dilation*

$$I_s u(x, y) = u(sx, s^{\frac{2+\alpha}{2}} y),$$

for every $u, v \in D(\mathfrak{a})$ one has $I_s u, I_s v \in D(\mathfrak{a})$ and

$$\mathfrak{a}(I_s u, I_s v) = s^{2-N-\frac{2+\alpha}{2}M}\mathfrak{a}(u, v).$$

Proof. Clearly, due to the positivity of ϕ, $\mathfrak{a}$ is a nonnegative symmetric form in L^2. The closedness of the form follows easily since $\mathfrak{a}$ is locally uniformly elliptic in Ω. The proofs of (i) and (ii) follow by a straightforward computation. □

Let $-L$ be the operator associated with $\mathfrak{a}$, that is,

$$D(L) := \left\{u \in D(\mathfrak{a}) : \text{there exists } v \in L^2 \text{ such that } \mathfrak{a}(u, w) = \int_{\mathbb{R}^{N+M}} v\bar{w}\, dx\, dy \text{ for all } w \in D(\mathfrak{a})\right\}, \qquad -Lu := v. \tag{2}$$

The basic properties of L are listed below.

Proposition 2.4. *The operator $-L$ defined in (2) is nonnegative and self-adjoint. Moreover*:

(i) $C_c^\infty \hookrightarrow D(L) \hookrightarrow \{u \in L^2 \cap W^{2,2}_{\mathrm{loc}}(\Omega)\ ;\ Lu \in L^2\}$ *and for every $u \in C_c^\infty$*

$$Lu = \Delta_x u + \phi(x)\Delta_y u.$$

(ii) *L generates a contractive analytic semigroup $\{e^{zL} : z \in \mathbb{C}_+\}$ in L^2.*

(iii) *The semigroup $\{e^{tL} : t > 0\}$ is submarkovian; i.e., it is positive and L^∞-contractive.*

(iv) *If Q is an orthogonal matrix in $\mathbb{R}^M$ and $y_0 \in \mathbb{R}^M$, then*

$$L = I^{-1}_{Q+y_0} L I_{Q+y_0}, \quad I_{Q+y_0}u(x, y) = u(x, Qy + y_0).$$

(v) *If ϕ is homogeneous of degree α, then*

$$s^2 L = I_s^{-1} L I_s, \quad I_s u(x,y) = u(sx, s^{\frac{2+\alpha}{2}} y), \quad s > 0.$$

Proof. Part (i) is clear by construction and from interior elliptic regularity (see, however, the proof of Theorem 2.9 for justifying the integration by parts). The generation property of L follows by standard results, see [Ouhabaz 2005, Chapter 1, Section 4]; the positivity of e^{tL} as well its L^∞-contractivity is a consequence of the Beurling–Deny criteria satisfied by the form $\mathfrak{a}$, see [Ouhabaz 2005, Corollary 2.18], and note that $\mathfrak{a}$ is real; that is, $\mathfrak{a}(u,v) \in \mathbb{R}$ whenever u, v are real functions. Concerning (iv) and (v), let $u \in D(L)$, $v \in D(\mathfrak{a})$ and $s > 0$. Then

$$\begin{aligned}\mathfrak{a}(I_s u, v) &= s^{2-N-\frac{2+\alpha}{2}M} \mathfrak{a}(u, I_{s^{-1}} v)\\ &= -s^{2-N-\frac{2+\alpha}{2}M} \int_{\mathbb{R}^{N+M}} (Lu) I_{s^{-1}} \bar{v}\, dx\, dy = -s^2 \int_{\mathbb{R}^{N+M}} (I_s L u) \bar{v}\, dx\, dy;\end{aligned}$$

hence $I_s u \in D(L)$ and $L I_s u = s^2 I_s L u$. The proof for I_{Q+y_0} is similar. □

The following proposition shows that C_c^∞ is dense in $D(L)$ with respect to the graph norm.

Proposition 2.5. *C_c^∞ is a core for the operator $(L, D(L))$ and the form $\mathfrak{a}$.*

Proof. Since $I - L$ is invertible we have to show that $(I-L)(C_c^\infty)$ is dense in L^2 or, equivalently, that $(I-L)(C_c^\infty)^\perp = \{0\}$. To this aim let $v \in L^2$ such that

$$\int_{\mathbb{R}^{N+M}} (I-L) u\, \bar{v}\, dx\, dy = 0 \quad \text{for all } u \in C_c^\infty.$$

Taking the partial Fourier transform with respect to the y-variable and applying the Fubini and Plancherel theorems we get

$$\int_{\mathbb{R}^{N+M}} [\hat{u}(x,\xi) - \Delta_x \hat{u}(x,\xi) + \phi(x)|\xi|^2 \hat{u}(x,\xi)]\, \bar{\hat{v}}(x,\xi)\, dx\, d\xi = 0 \quad \text{for all } u \in C_c^\infty.$$

Choosing $u = A(x)B(y) \in C_c^\infty$ we have $\hat{u}(x,\xi) = A(x)\widehat{B}(\xi)$ and

$$\int_{\mathbb{R}^{N+M}} [A(x) - \Delta_x A(x) + \phi(x)|\xi|^2 A(x)]\, \widehat{B}(\xi) \bar{\hat{v}}(x,\xi)\, dx\, d\xi = 0. \tag{3}$$

Fix $\xi_0 \in \mathbb{R}^M$, $r > 0$ and let

$$w(\xi) = \frac{1}{|B(\xi_0, r)|} \chi_{B(\xi_0,r)} \in L^2(\mathbb{R}^M).$$

Let $(B_n)_n \in C_c^\infty(\mathbb{R}^M)$ be a sequence of test functions such that $B_n \to \check{w}$ in $L^2(\mathbb{R}^M)$; then $\widehat{B}_n \to w$ in $L^2(\mathbb{R}^M)$ and taking the limit as $n \to \infty$ in (3) with $\widehat{B}$ replaced by $\widehat{B}_n$ we obtain

$$\frac{1}{|B(\xi_0, r)|} \int_{B(\xi_0,r)} d\xi \int_{\mathbb{R}^N} [A(x) - \Delta_x A(x) + \phi(x)|\xi|^2 A(x)]\, \bar{\hat{v}}(x,\xi)\, dx = 0.$$

Letting $r \to 0$ and using the Lebesgue differentiation theorem, we have for a.e. $\xi_0 \in \mathbb{R}^M$

$$\int_{\mathbb{R}^N} [A(x) - \Delta_x A(x) + \phi(x)|\xi_0|^2 A(x)]\, \bar{\hat{v}}(x,\xi_0)\, dx = 0,$$

which, since u was arbitrary, is valid for every $A \in C_c^\infty(\mathbb{R}^N)$. The operator $\Delta_x - \phi(\cdot)|\xi|^2$ is a Schrödinger operator in $L^2(\mathbb{R}^N)$ with nonpositive potential $-\phi|\xi|^2 \in L^2_{\text{loc}}(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is dense in the domain $D(\Delta_x - \phi(\cdot)|\xi|^2)$ with respect to the graph norm; see [Kato 1972]. The last equation then implies $\hat{v}(\cdot, \xi_0) = 0$ for a.e. $\xi_0 \in \mathbb{R}^M$, which proves the required claim.

Since $D(L)$ is dense in $D(L^{1/2}) = D(\mathfrak{a})$, see [Kato 1966, Theorem VI.2.23], the second statement follows from the first. □

In order to prove the main result of this section we recall the definition of B_p-weights. Let $1 < p \le \infty$. Then $\omega \in B_p(\mathbb{R}^N)$, where $B_p(\mathbb{R}^N)$ is the class of the reverse Hölder weights of order p, if $\omega \in L^p_{\text{loc}}$, $\omega > 0$ a.e. and there exists a positive constant C such that the inequality

$$\left(\frac{1}{|B|}\int_B \omega^p\right)^{\frac{1}{p}} \le \frac{C}{|B|}\int_B \omega \tag{4}$$

holds for every ball B. If $p = \infty$, the left-hand side of the inequality above has to be replaced by the essential supremum of ω on B. The smallest positive constant C such that (4) holds is the B_p constant of ω. We recall that powers $|x|^\alpha$ belong to $B_\infty(\mathbb{R}^N)$ whenever $\alpha \ge 0$. This is easily seen first considering balls of radius 1 (and large centers) and then scaling.

Remark 2.6. In the proof of the following result we need the maximal L^2 inequalities for Schrödinger operators $\Delta - V$, $0 \le V \in B_2(\mathbb{R}^N)$, shown in [Auscher and Ben Ali 2007, Theorem 1.1, Corollary 1.3]. They say that the operator $V(\Delta - V)^{-1}$ is bounded in $L^2(\mathbb{R}^N)$; moreover, the norm of $V(\Delta - V)^{-1}$ is bounded by a constant which depends only on N and the B_2 constant of V. This last fact is not explicitly stated in [Auscher and Ben Ali 2007], even though it follows from the proofs, but can be found in [Carbonaro et al. 2008, Theorem 3.6] in the more general setting of parabolic Schrödinger operators $D_t - \Delta + V$. In fact the norm of $V(\Delta - V)^{-1}$ depends on C in (4) and the constant in the Harnack inequality for the Laplacian in $\mathbb{R}^N$. See also [Shen 1995, Theorem 0.3] where, however, $N \ge 3$ and $V \in B_q$ for some $q \ge \frac{N}{2}$.

Theorem 2.7. *Assume that $\phi : \mathbb{R}^N \to [0, +\infty[$ belongs to $B_2(\mathbb{R}^N)$. Then for every $1 \le i, j \le N$, $1 \le h, k \le M$, one has*

$$\|D_{x_i x_j} u\|_2 + \|\phi D_{y_h y_k} u\|_2 \le C\|Lu\|_2, \quad u \in D(L).$$

Moreover

$$\|\nabla_x u\|_2 + \|\phi^{\frac{1}{2}} \nabla_y u\|_2 \le C(\|Lu\|_2 + \|u\|_2), \quad u \in D(L).$$

Proof. By Proposition 2.5 we may assume that $u \in C_c^\infty$. Consider the partial Fourier transform with respect to the y-variable. Let $v(x, \xi) = \hat{u}(x, \xi)$. Then, setting $Lu = f$, we have

$$\Delta_x v(x, \xi) - \phi(x)|\xi|^2 v(x, \xi) = \hat{f}(x, \xi) \in L^2.$$

Observe now that, for every fixed $\xi \in \mathbb{R}^M$, $\Delta_x - \phi(\cdot)|\xi|^2$ is a Schrödinger operator in $\mathbb{R}^N$ with potential $\phi|\xi|^2$. Moreover, since $\phi \in B_2(\mathbb{R}^N)$, it immediately follows that $\phi|\xi|^2$ satisfies the reverse Hölder condition with the same constant as ϕ. By Remark 2.6 above, we have

$$|\xi|^4 \int_{\mathbb{R}^N} \phi(x)^2 |v(x,\xi)|^2 \, dx \le C \int_{\mathbb{R}^N} |\hat{f}(x,\xi)|^2 \, dx,$$

with a constant C not depending on ξ. Integrating the last inequality over $\mathbb{R}^M$, we get

$$\int_{\mathbb{R}^{N+M}} |\xi|^4 \phi(x)^2 |v(x,\xi)|^2 \, dx \, d\xi \le C \int_{\mathbb{R}^{N+M}} |\hat{f}(x,\xi)|^2 \, dx \, d\xi.$$

Since $|\cdot|^2 v(x,\cdot) = \widehat{\Delta_y u}(x,\cdot)$ we get, using Fubini's theorem and the Plancherel theorem in $\mathbb{R}^M$,

$$\int_{\mathbb{R}^{N+M}} \phi(x)^2 |\Delta_y u(x,y)|^2 \, dx \, dy \le C \int_{\mathbb{R}^{N+M}} |f(x,y)|^2 \, dx \, dy,$$

which reads as $\|\phi \Delta_y u\|_2 \le C \|Lu\|_2$; by difference we also get $\|\Delta_x u\|_2 \le C\|Lu\|_2$.

The Calderón–Zygmund theorem applied separately to each variable implies

$$\|D_{x_i x_j} u\|^2_{L^2(\mathbb{R}^N)} \le C(N) \|\Delta_x u\|^2_{L^2(\mathbb{R}^N)}, \quad 1 \le i, j \le N,$$
$$\|D_{y_h y_k} u\|^2_{L^2(\mathbb{R}^M)} \le C(M) \|\Delta_y u\|^2_{L^2(\mathbb{R}^M)}, \quad 1 \le h, k \le M.$$

Integrating the previous inequalities (with the last one multiplied by $\phi(x)^2$) over $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, we get the first claim.

Concerning the gradient estimates, it is enough to observe that, by interpolation, for every $\epsilon > 0$,

$$\|\nabla_x\|_{L^2(\mathbb{R}^N)} \le \epsilon \sum_{i,j=1}^{N} \|D_{x_i x_j} u\|_{L^2(\mathbb{R}^N)} + \frac{C}{\epsilon} \|u\|_{L^2(\mathbb{R}^N)}.$$

The estimates for the first-order derivatives with respect to x immediately follow after integration over $\mathbb{R}^M$ and by using the first part of the theorem. For the gradient with respect to y, we start, analogously, from

$$\|\nabla_y\|_{L^2(\mathbb{R}^M)} \le \epsilon \sum_{h,k=1}^{M} \|D_{y_h y_k} u\|_{L^2(\mathbb{R}^M)} + \frac{C}{\epsilon} \|u\|_{L^2(\mathbb{R}^M)}.$$

Choosing $\epsilon = \phi(x)^{1/2}$, the claim follows after the integration over $\mathbb{R}^N$ and by using the first part of the theorem. □

Remark 2.8. If $\phi \in B_N(\mathbb{R}^N)$ and $N \ge 3$, it can be proved along the same lines that

$$\|\phi^{\frac{1}{2}} D_{x_i y_h} u\|_2 \le C \|Lu\|_2, \quad u \in D(L).$$

In fact, the partial Fourier transform of $D_{x_i y_h} u$ is $-i\xi_h D_{x_i} \hat{u}(x,\xi)$ and $\phi|\xi|^2$ satisfies the $B_N(\mathbb{R}^N)$ reverse Hölder condition with the same constant as ϕ. By [Shen 1995, Theorem 0.8]

$$|\xi|^2 \int_{\mathbb{R}^N} \phi(x) |\nabla_x v(x,\xi)|^2 \, dx \le C \int_{\mathbb{R}^N} |\hat{f}(x,\xi)|^2 \, dx, \tag{5}$$

with a constant C not depending on ξ. Integrating over $\mathbb{R}^M$ and using Plancherel's theorem, we get

$$\int_{\mathbb{R}^{N+M}} \phi(y) |D_{x_i y_h} u(x,y)|^2 \, dx \, dy \le C \int_{\mathbb{R}^{N+M}} |f(x,y)|^2 \, dx \, dy.$$

The above result probably holds also for $N = 1, 2$ since the maximal inequality (5) is discussed in [Auscher and Ben Ali 2007] (see the comments after Corollary 1.5); the authors say that their methods give the result for all N but a detailed proof is not given. Since we shall not use this remark in what follows we omit further investigation.

In the following theorem we characterize the domain of the operator L.

Theorem 2.9. *If $\phi \in B_2(\mathbb{R}^N)$ then the domain of the operator L defined in (2) satisfies*

$$D(L) = \{u \in L^2 : \nabla_x u,\ D_{x_i x_j} u,\ \phi^{\frac{1}{2}} \nabla_y u,\ \phi D_{y_h y_k} u \in L^2\}. \tag{6}$$

Proof. Let $\widetilde{D}(L)$ be the set defined in the right-hand side of equality (6). Theorem 2.7 then implies $D(L) \subseteq \widetilde{D}(L)$. To prove the equality it is then enough to prove that the operator $(L, \widetilde{D}(L))$ is dissipative since in this case $I - L : \widetilde{D}(L) \to L^2$ is an injective extension of the resolvent operator $I - L : D(L) \to L^2$ and so both operators must coincide. Let $u \in \widetilde{D}(L)$; then, by the definition, for every compact set $\omega \Subset \Omega$, we have $u, D^2 u \in L^2(\omega)$; hence $u \in H^2_{\text{loc}}(\Omega)$. Moreover a section argument, see for example [Ziemer 1989, Theorem 2.1.4], shows that for a.e. $x \in \Omega_N$, we have $u(x, \cdot) \in H^2(\mathbb{R}^M)$ and

$$\int_{\mathbb{R}^M} u\, \Delta_y u\, dy = -\int_{\mathbb{R}^M} |\nabla_y u|^2\, dy \quad \text{for a.e. } x \in \Omega_N.$$

Then multiplying by ϕ, integrating in x and using Fubini's theorem we get

$$\int_{\mathbb{R}^{N+M}} \phi(x)\, u\, \Delta_y u\, dx\, dy = -\int_{\mathbb{R}^{N+M}} \phi(x) |\nabla_y u|^2\, dx\, dy.$$

Analogous reasoning applied to the y-sections shows that

$$\int_{\mathbb{R}^{N+M}} u\, \Delta_x u\, dx\, dy = -\int_{\mathbb{R}^{N+M}} |\nabla_x u|^2\, dx\, dy.$$

The last two inequalities imply

$$\int_{\mathbb{R}^{N+M}} u\, Lu\, dx\, dy = -\int_{\mathbb{R}^{N+M}} (|\nabla_y u|^2 + \phi(x) |\nabla_y u|^2)\, dx\, dy \le 0,$$

which, since $u \in \widetilde{D}(L)$ was arbitrary, implies the dissipativity of $(L, \widetilde{D}(L))$. □

The next proposition provides regularity properties of the solution of the resolvent equation with respect to the y-variables.

Proposition 2.10. *Let $u \in D(L)$ be such that $u - Lu = f \in C_c^\infty$. Then for every multiindex α one has $D_y^\alpha u \in D(L)$ and*

$$D_y^\alpha u - L D_y^\alpha u = D_y^\alpha f.$$

Proof. Let $u \in D(L)$ be such that $u - Lu = f \in C_c^\infty$. Then

$$\int_{\mathbb{R}^{N+M}} (uv + \langle \nabla_x u, \nabla_x v\rangle + \phi(x) \langle \nabla_y u, \nabla_y v\rangle)\, dx\, dy = \int_{\mathbb{R}^{N+M}} f v\, dx\, dy \quad \text{for every } v \in D(\mathfrak{a}). \tag{7}$$

For $h \in \mathbb{R}^M$ let $D_h g(z) := (g(x, y+h) - g(x, y))$ and let us take, in the last equation, $v = D_{-h} D_h u \in D(\mathfrak{a})$. Then, since $D_{-h} = D_h^*$, one has

$$\int_{\mathbb{R}^{N+M}} (|D_h u|^2 + |D_h \nabla_x u|^2 \phi(x) + |D_h \nabla_y u|^2)\, dx\, dy = \int_{\mathbb{R}^{N+M}} D_h f\, D_h u\, dx\, dy$$
$$\le \|D_h f\|_2 \|D_h u\|_2 \le \tfrac{1}{2}(\|D_h f\|_2^2 + \|D_h u\|_2^2). \quad (8)$$

In particular for every $\omega \Subset \Omega$ there exists some positive constant $C = C(\omega)$ such that

$$\|D_h \nabla u\|_{L^2(\omega)} \le C|h| \|\nabla f\|_{L^2(\omega)}$$

for sufficiently small h; this proves that ∇u is weakly differentiable in ω in the y-variable and that $D_{y_i} u \in H^1_{\mathrm{loc}}(\Omega)$. Moreover, if $e_1, \dots, e_M$ is the standard basis of $\mathbb{R}^M$, $t \ne 0$, and $h = te_i$, then dividing both sides of (8) by t and taking the limit for $t \to 0$ we obtain

$$\frac{1}{2} \int_{\mathbb{R}^{N+M}} (|D_{y_i} u|^2 + |D_{y_i} \nabla_x u|^2 \phi(x) + |D_{y_i} \nabla_y u|^2)\, dx\, dy \le \int_{\mathbb{R}^{N+M}} |D_{y_i} f|^2\, dx\, dy,$$

which proves that $D_{y_i} u \in D(\mathfrak{a})$. Let us fix now $v \in C_c^\infty$; using (7) with v replaced by $D_{-te_i} v$ we get

$$\begin{aligned}
\int_{\mathbb{R}^{N+M}} D_{te_i} f\, v\, dx\, dy &= \int_{\mathbb{R}^{N+M}} f\, D_{-te_i} v\, dx\, dy \\
&= \int_{\mathbb{R}^{N+M}} \big(u D_{-te_i} v + \langle \nabla_x u, \nabla_x D_{-te_i} v\rangle + \phi(x)\langle \nabla_y u, \nabla_y D_{-te_i} v\rangle\big)\, dx\, dy \\
&= \int_{\mathbb{R}^{N+M}} \big(D_{te_i} u v + \langle D_{te_i} \nabla_x u, \nabla_x v\rangle + \phi(x)\langle D_{te_i} \nabla_y u, \nabla_y v\rangle\big)\, dx\, dy.
\end{aligned}$$

Dividing both sides of the last equation by t and taking the limit for $t \to 0$ we obtain

$$\int_{\mathbb{R}^{N+M}} D_{y_i} f\, v\, dx\, dy = \int_{\mathbb{R}^{N+M}} \big(D_{y_i} u v + \langle D_{y_i} \nabla_x u, \nabla_x v\rangle + \phi(x)\langle D_{y_i} \nabla_y u, \nabla_y v\rangle\big)\, dx\, dy.$$

Since by Proposition 2.5 C_c^∞ is a core for $\mathfrak{a}$ and since v is arbitrary in the last equation, we have that $D_{y_i} u \in D(L)$ and $D_{y_i} u - L(D_{y_i} u) = D_{y_i} f$, which is the required claim for $|\alpha| = 1$. An inductive argument easily proves the claim for any multiindex α. Moreover, since $D_y^\alpha u = (I - L)^{-1} D_y^\alpha f$, for some $C = C(\alpha) > 0$ we have

$$\|D_y^\alpha u\|_2 \le C \|D_y^\alpha f\|_2.$$ □

We end this section by proving a version of Kato's inequality adapted to L, which will be used for proving L^p-estimates.

Proposition 2.11. *Let $u \in D(L)$ and let us define*

$$\mathrm{sign}(u) = \begin{cases} 0 & \text{if } u(x) = 0, \\ u(x)/|u(x)| & \text{if } u(x) \ne 0. \end{cases}$$

Then $|u|$ satisfies the distributional inequality

$$-\mathfrak{a}(|u|, \varphi) \ge \int_{\mathbb{R}^{N+M}} \mathrm{sign}(u)\, Lu\, \phi\, dx\, dy \quad \text{for any } 0 \le \varphi \in C_c^\infty.$$

Proof. We suppose first that $u \in C_c^\infty$. If

$$u_\epsilon(x) = \sqrt{|u|^2 + \epsilon^2}$$

then $u_\epsilon \geq |u|$ and

$$u_\epsilon(a\,\nabla u_\epsilon) = u(a\,\nabla u) \tag{9}$$

(here a is the matrix defined in (1)). Thus (9) implies

$$\begin{aligned} |\nabla_x u_\epsilon| &\leq |u|\,|u_\epsilon|^{-1}|\nabla_x u| \leq |\nabla_x u|, \\ \phi(x)|\nabla_y u_\epsilon| &\leq |u|\,|u_\epsilon|^{-1}\phi(x)|\nabla_y u| \leq \phi(x)|\nabla_y u|. \end{aligned} \tag{10}$$

Taking the divergence of (9) we obtain

$$u_\epsilon L u_\epsilon + |\nabla_x u_\epsilon|^2 + \phi(x)|\nabla_y u_\epsilon|^2 = uLu + |\nabla_x u|^2 + \phi(x)|\nabla_y u|^2,$$

so by (10)

$$Lu_\epsilon \geq \frac{u}{u_\epsilon} Lu. \tag{11}$$

Integrating by parts the right-hand side of (11), it follows that

$$-\mathfrak{a}(u_\epsilon, \varphi) \geq \int_{\mathbb{R}^{N+M}} \frac{u}{u_\epsilon}\, Lu\, \varphi\, dx\, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty.$$

Letting $\epsilon \to 0$ we get

$$-\mathfrak{a}(|u|, \varphi) \geq \int_{\mathbb{R}^{N+M}} \operatorname{sign}(u)\, Lu\, \varphi\, dx\, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty.$$

Let now $u \in D(L)$ and let $u_n \in C_c^\infty$ be such that $u_n \to u$ in $D(L)$. Up to a subsequence, if necessary, we can also suppose that $u_n \to u$ almost everywhere. Since also $u_n \to u$ in $D(\mathfrak{a})$ by the last inequalities

$$-\mathfrak{a}(|u_n|, \varphi) \geq \int_{\mathbb{R}^{N+M}} \operatorname{sign}(u_n)\, Lu_n\, \varphi\, dx\, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty;$$

the claim follows letting $n \to \infty$. □

3. The distance d associated with L

Let $\alpha > 0$ and let

$$L = \Delta_x + |x|^\alpha \Delta_y$$

be the self-adjoint operator defined in Section 2 with $\phi(x) = |x|^\alpha$. In this section we introduce a natural metric d on $\mathbb{R}^{N+M}$ associated with L and which makes the triple $(\mathbb{R}^{N+M}, d, \mathcal{L})$, consisting of $\mathbb{R}^{N+M}$ equipped with the distance d and the Lebesgue measure $\mathcal{L}$, a homogeneous space in the sense of [Coifman and Weiss 1971; 1977].

Definition 3.1. Let $\gamma : [0, T] \to \mathbb{R}^{N+M}$ be an absolutely continuous curve. We say that γ is a subunit curve if for a.e. $t \in [0, T]$ one has

$$\langle \dot\gamma(t), \xi\rangle^2 \leq |\xi_x|^2 + |x|^\alpha |\xi_y|^2 \quad \text{for every } \xi = (\xi_x, \xi_y) \in \mathbb{R}^{N+M}.$$

For every $z_1, z_2 \in \mathbb{R}^{N+M}$ we define

$$\begin{aligned} d(z_1, z_2) &= \inf\{T \in \mathbb{R}^+ : \text{ there exists a subunit curve } \gamma : [0, T] \to \mathbb{R}^{N+M},\ \gamma(0) = z_1,\ \gamma(T) = z_2\} \\ &= \sup\{\psi(z_2) - \psi(z_1) : \psi \in W^{1,\infty}(\mathbb{R}^{N+M}),\ |\nabla_x \psi|^2 + |x|^\alpha |\nabla_y \psi|^2 \le 1\}. \end{aligned} \tag{12}$$

We remark that d is a well-defined distance and that any pair of points $z_1, z_2 \in \mathbb{R}^{N+M}$ can be joined by a subunit curve; see [Franchi and Serapioni 1987, Section 2, Example 3.6] and [Franchi and Lanconelli 1984, Definition 2.4]. A proof of the equality in (12) can be found in [Jerison and Sánchez-Calle 1987, Proposition 3.1].

For $z_0 \in \mathbb{R}^{N+M}$, $r > 0$, we write

$$S(z_0, r) := \{z \in \mathbb{R}^{N+M} : d(z_0, z) < r\}$$

to denote the balls of $\mathbb{R}^{N+M}$ with respect to the metric d. In the next proposition we clarify the structure of the metric and define an equivalent system of balls which are explicit and easier to work with. For $z_0 = (x_0, y_0) \in \mathbb{R}^{N+M}$, $r > 0$, let us define the cylindrical set

$$Q(z_0, r) := B(x_0, r) \times B(y_0, r(x_0)), \qquad r(x_0) := r(r + |x_0|)^{\frac{\alpha}{2}}. \tag{13}$$

Proposition 3.2. *There exist two positive constants $C_1, C_2 > 0$ such that the distance function d satisfies for every $z_1 = (x_1, y_1)$, $z_1 = (x_2, y_2) \in \mathbb{R}^{N+M}$*

$$C_1 F(z_1, z_2) \le d(z_1, z_2) \le C_2 F(z_1, z_2),$$

where

$$F(z_1, z_2) = |x_1 - x_2| + \left(\frac{|y_1 - y_2|}{(|x_1| + |x_2|)^{\frac{\alpha}{2}}} \wedge |y_1 - y_2|^{\frac{2}{2+\alpha}} \right).$$

In particular

$$|S(z_0, r)| \simeq \begin{cases} r^{N+M(1+\frac{\alpha}{2})} & \text{if } r \ge |x_0|, \\ r^{N+M} |x_0|^{M\frac{\alpha}{2}} & \text{if } r \le |x_0|, \end{cases}$$

and the metric balls satisfy the doubling property

$$|S(z_0, sr)| \le C s^{N+M(1+\frac{\alpha}{2})} |S(z_0, r)| \quad \text{for every } z_0 \in \mathbb{R}^{N+M},\ s \ge 1.$$

Furthermore there exists a constant $c > 1$ such that for every $z_0 = (x_0, y_0) \in \mathbb{R}^{N+M}$, $r > 0$,

$$Q(z_0, c^{-1} r) \subseteq S(z_0, r) \subseteq Q(z_0, cr).$$

In particular $|S(z_0, r)| \simeq r^{N+M}(r + |x_0|)^{M\alpha/2}$ and $(\mathbb{R}^{N+M}, d, \mathcal{L})$ is a metric space of homogeneous type.

Proof. The first part of the statement is proved in [Robinson and Sikora 2008, Proposition 5.1, Corollary 5.2] $\big($take in that paper $\delta_1 = \delta_1' = 0$, $\delta_2 = \delta_2' = \frac{\alpha}{2}$, $D = D' = N + M\big(1 + \frac{\alpha}{2}\big)\big)$. A proof of the second part can be found in [Franchi and Serapioni 1987, Proposition 2.7, Example 3.6] and [Franchi et al. 1994, Proposition 1]. □

4. L^p estimates

Let $1 < p < \infty$. In this section we assume that $\phi(x) = |x|^\alpha$, with $\alpha > 0$, and consider therefore the operator

$$L = \Delta_x + |x|^\alpha \Delta_y$$

in L^p with $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. Property (iii) of Proposition 2.4 shows that the symmetric semigroup $(e^{tL})_{t\geq 0}$ generated by L in L^2 is submarkovian. Then by a standard result, see for example [Ouhabaz 2005, Chapter 3], it induces a consistent family of strongly continuous semigroups on L^p for any $1 < p < \infty$, still denoted by $(e^{tL})_{t\geq 0}$. Moreover $(e^{tL})_{t\geq 0}$ extends to a contractive holomorphic semigroup on a sector; see [Ouhabaz 2005, Theorem 3.13].

Definition 4.1. For any $p \in (1, \infty)$ we define the sectorial operator $(L, D_p(L))$ as the generator of the extrapolated semigroup $(e^{tL})_{t\geq 0}$ in L^p. We also write $D_2(L) = D(L)$.

$D_p(L) \cap D(L)$ is dense in L^p; in fact, if $f \in C_c^\infty$, then $e^{tL} f \in D_p(L) \cap D(L)$ and converges to f in L^p. Then $D_p(L) \cap D(L)$, being a dense invariant set, is by construction a core for $(L, D_p(L))$.

Theorem 2.7 holds in the specific situation since $|x|^\alpha \in B_\infty(\mathbb{R}^N)$ and we prove that those estimates extend to $1 < p < \infty$.

We recall that $|x|^\beta$ belongs to $A_t(\mathbb{R}^N)$, the class of Muckenhoupt weights of order $t \geq 1$, whenever $0 \leq \beta < N(t-1)$. This means that

$$\left(\frac{1}{|B|}\int_B |x|^\beta \, dx\right)\left(\frac{1}{|B|}\int_B |x|^{\beta(1-t')} \, dx\right)^{t-1} \leq C$$

for any ball (or cube) B of $\mathbb{R}^N$; see for example [Duoandikoetxea 2001, Chapther 7.3]. However, we need Muckenhoupt weights in $(\mathbb{R}^{N+M}, d, \mathcal{L})$ with respect to the metric defined in Section 3. Since $|x|^\beta$ is independent of y and since the balls S in this space are equivalent to the cylinders $Q(z, r)$ defined in (13), which are products of balls in R^N and $\mathbb{R}^M$ respectively, one easily verifies that

$$\left(\frac{1}{|S|}\int_S |x|^\beta \, dx\, dy\right)\left(\frac{1}{|S|}\int_S |x|^{\beta(1-t')} \, dx\, dy\right)^{t-1} \leq C$$

for every ball S (or cylinder) in $(\mathbb{R}^{N+M}, d, \mathcal{L})$.

A theory on these classes of weights in homogeneous spaces is presented for example in [Strömberg and Torchinsky 1989, Chapter I], to which we refer for the proofs of the results needed in what follows. In particular, we recall that Muckenhoupt weights induce doubling measures. The following well-known consequence of the definition is crucial in our approach.

Lemma 4.2. *If $\phi(x, y) = |x|^\beta$, $t \geq 1$, and $\beta < N(t-1)$, there exists $c > 0$ such that the inequality*

$$\left(\frac{1}{|Q|}\int_Q g\right)^t \leq \frac{c}{\phi(Q)}\int_Q g^t \phi \tag{14}$$

holds for all nonnegative functions g and all cylinders Q in $(\mathbb{R}^{N+M}, d, \mathcal{L})$. Here

$$\phi(Q) = \int_Q \phi.$$

Proof. By Hölder's inequality one has

$$\left(\frac{1}{|Q|}\int_Q g\right)^t = \left(\frac{1}{|Q|}\int_Q g\phi^{\frac{1}{t}}\phi^{-\frac{1}{t}}\right)^t \le \left(\frac{1}{|Q|}\int_Q g^t\phi\right)\left(\frac{1}{|Q|}\int_Q \phi^{1-t'}\right)^{t-1}$$

and the claim follows from the A_t property of $|x|^\beta$ in $(\mathbb{R}^{N+M}, d, \mathcal{L})$. □

The A_t property of $\phi = |x|^\alpha$, combined with mean value inequalities for Baouendi–Grushin operators, allows us to characterize the domain of the operator. We prove the following result.

Theorem 4.3. *For every $1 \le i, j \le N$, $1 \le h, k \le M$, the operators $|x|^\alpha D_{y_h y_k}(I-L)^{-1}$, $D_{x_i x_j}(I-L)^{-1}$, originally defined in L^2, extend to bounded operators in L^p.*

The main tool is the following result due to Shen [2005, Theorem 3.1], which can be considered as a version of the Calderón–Zygmund theorem in the absence of kernels. The original proof, where Euclidean balls are used, can be modified to work also for our space $(\mathbb{R}^{N+M}, d, \mathcal{L})$. Indeed an improved version of Shen's result in more general homogeneous spaces, which covers the cases of our interest, can be found in [Auscher and Martell 2007, Theorem 3.14 and Section V].

Theorem 4.4. *Let $1 \le p_0 < q_0 \le \infty$. Suppose that T is a sublinear bounded operator on L^{p_0}. Suppose moreover that there exist $\alpha_2 > \alpha_1 > 1$, $C > 0$, such that*

$$\left(\frac{1}{|Q|}\int_Q |Tf|^{q_0}\right)^{\frac{1}{q_0}} \le C\left(\frac{1}{|\alpha_1 Q|}\int_{\alpha_1 Q} |Tf|^{p_0}\right)^{\frac{1}{p_0}}$$

for all cylinders Q and for all $f \in C_c^\infty$, with support in $\mathbb{R}^{N+M} \setminus \alpha_2 Q$. Then, for $p_0 \le p < q_0$, there exists a positive constant C_p such that for all $f \in C_c^\infty$

$$\|Tf\|_p \le C_p \|f\|_p.$$

We briefly describe our strategy of proof of Theorem 4.3. We first prove the a priori estimates for $p \ge 2$ by applying the above theorem to the operator $T = |x|^\alpha D_{y_h y_k}(I-L)^{-1}$, with $p_0 = 2$, arbitrary $q_0 > 2$ and $\alpha_1 = 3$, $\alpha_2 = 4$. Therefore we have to prove that, if Q is a cylinder and $f \in C_c^\infty$ has support in $\mathbb{R}^{N+M} \setminus 4Q$, then $u = (I-L)^{-1} f$ satisfies

$$\left(\frac{1}{|Q|}\int_Q ||x|^\alpha D_{y_h y_k} u|^{q_0}\right)^{\frac{1}{q_0}} \le C\left(\frac{1}{|3Q|}\int_{3Q} ||x|^\alpha D_{y_h y_k} u|^2\right)^{\frac{1}{2}}$$

for some positive C independent of f. Observe that u satisfies in $4Q$ the equation

$$u - Lu = u - \Delta_x u - |x|^\alpha \Delta_y u = 0.$$

Moreover, by Proposition 2.10, the operator L commutes with the second-order derivatives with respect to y and $v = D_{y_h y_k} u$ satisfies the same equation in $4Q$.

To get the a priori estimates in the case $1 < p \le 2$, we apply Shen's theorem to the adjoint operator T^*.

As a first step we recall a mean value inequality for subsolutions of L, that is, for functions v satisfying the inequality $Lv \ge 0$ in Q, in a weak sense. This means that $\mathfrak{a}(u, \varphi) \le 0$ for any $0 \le \varphi \in C_c^\infty(Q)$.

Lemma 4.5 (see [Franchi and Serapioni 1987, Theorem 5.7]). *There exists a positive constant C such that, if v is a local subsolution of L in $4Q$, then*

$$\sup_Q |v| \le C\left(\frac{1}{|3Q|}\int_{3Q} v^2\right)^{\frac{1}{2}}.$$

The previous mean value inequality remains true also for $0 < r < \infty$. It follows from the self-improvement of the right-hand side in weak reverse Hölder estimates; see for example [Chanillo and Wheeden 1986, Theorem 4.1]. We give a simple self-contained proof which is a simplification of [Bernicot et al. 2016, Theorem B.1] for the sup-norm. Note that the case $r > 2$ follows from Hölder's inequality.

Lemma 4.6. *For every $0 < r < \infty$, there exists a positive constant C_r such that, if v is local subsolution of L in $4Q$, then*

$$\sup_Q |v| \le C_r\left(\frac{1}{|3Q|}\int_{3Q} |v|^r\right)^{\frac{1}{r}}.$$

Proof. Let $r < 2$ and $\mathcal{Q}$ be the collection of all cylinders Q' contained in Q. For $\epsilon \in (0, 1)$ let

$$C_r(\epsilon) := \sup_{Q' \in \mathcal{Q}} \frac{\sup_{Q'} |v|}{\left(\frac{1}{|3Q'|}\int_{3Q'} |v|^r\right)^{\frac{1}{r}} + \epsilon} \le \epsilon^{-1} \sup_Q |v|.$$

Let us fix $Q' \in \mathcal{Q}$ and let Q'' be a cylinder centered at some point of Q' and such that $9Q'' \subseteq 3Q'$. We assume that the radii $r(Q'')$, $r(Q')$ of Q'' and Q' satisfy $c^{-1}r(Q') \le r(Q'') \le cr(Q')$ for some fixed constant $c \in (0, 1)$. Applying Lemma 4.5 in Q'' we get

$$\begin{aligned}
\sup_{Q''} |v| &\le C_2\left(\frac{1}{|3Q''|}\int_{3Q''} v^2\right)^{\frac{1}{2}} \le C_2\big(\sup_{3Q''} |v|\big)^{1-\frac{r}{2}}\left(\frac{1}{|3Q''|}\int_{3Q''} |v|^r\right)^{\frac{1}{2}} \\
&\le C_2 C_r(\epsilon)^{1-\frac{r}{2}}\left[\left(\frac{1}{|9Q''|}\int_{9Q''} |v|^r\right)^{\frac{1}{r}} + \epsilon\right]^{1-\frac{r}{2}}\left(\frac{1}{|3Q''|}\int_{3Q''} |v|^r\right)^{\frac{1}{2}} \\
&\le C' C_2 C_r(\epsilon)^{1-\frac{r}{2}}\left[\left(\frac{1}{|9Q''|}\int_{9Q''} |v|^r\right)^{\frac{1}{r}} + \epsilon\right] \le C'' C_2 C_r(\epsilon)^{1-\frac{r}{2}}\left[\left(\frac{1}{|3Q'|}\int_{3Q'} |v|^r\right)^{\frac{1}{r}} + \epsilon\right].
\end{aligned}$$

Since Q'' is arbitrary

$$\sup_{Q'} |v| \le C'' C_2 C_r(\epsilon)^{1-\frac{r}{2}}\left[\left(\frac{1}{|3Q'|}\int_{3Q'} |v|^r\right)^{\frac{1}{r}} + \epsilon\right].$$

Taking the supremum over $Q' \in \mathcal{Q}$ we get $C_r(\epsilon) \le (C''C_2)^{2/r}$ and the thesis follows letting $\epsilon \to 0$. □

Now we prove that Lemma 4.6 holds if we replace the Lebesgue measure with $|x|^\beta\, dx$, $\beta > 0$.

Lemma 4.7. *Fix $0<s<\infty$ and v as in Lemma 4.5. Then*

$$\sup_Q |v| \le \left(\frac{C}{\phi(3Q)}\int_{3Q}\phi|v|^s\right)^{\frac{1}{s}},$$

where C depends only on s, p and the B_p constant of $\phi(x)=|x|^\beta$ and

$$\phi(3Q)=\int_{3Q}\phi.$$

Proof. Let $0<s<\infty$ and Q be a cylinder of $\mathbb{R}^{N+M}$. We fix t as in Lemma 4.2. By using Lemma 4.6 with $r=\frac{s}{t}$ and (14) we obtain

$$\sup_Q |v| \le C\left(\frac{1}{|3Q|}\int_{3Q}|v|^{\frac{s}{t}}\right)^{\frac{t}{s}} \le C\left(\frac{1}{\phi(3Q)}\int_{3Q}\phi|v|^s\right)^{\frac{1}{s}}. \qquad \square$$

By combining the estimate in Lemma 4.7 and the B_p property we deduce the following.

Corollary 4.8. *Let $0<s<\infty$, $1<p<\infty$ and v as in Lemma 4.5. Then*

$$\left(\frac{1}{|Q|}\int_Q (|x|^\beta|v|^s)^p\right)^{\frac{1}{p}} \le \frac{C}{|3Q|}\int_{3Q}|x|^\beta|v|^s,$$

where C depends only on s, p and the B_p constant of $|x|^\beta$.

Proof. Using the B_p property of $\phi=|x|^\beta$ and Lemma 4.7 we obtain

$$\left(\frac{1}{|Q|}\int_Q(\phi|v|^s)^p\right)^{\frac{1}{p}} \le \left(\frac{1}{|Q|}\int_Q\phi^p\right)^{\frac{1}{p}}\sup_Q|v|^s \le C\left(\frac{1}{|Q|}\int_Q\phi\right)\sup_Q|v|^s \le \frac{C}{|3Q|}\int_{3Q}\phi|v|^s. \qquad \square$$

We can now prove our main result.

Proof of Theorem 4.3. We first consider the operators $|x|^\alpha D_{y_hy_k}(I-L)^{-1}$.

Let us preliminarily treat the case $p\ge 2$. Let us fix $q_0>2$ and let Q be a cylinder in $\mathbb{R}^{N+M}$ and $f\in C_c^\infty$ a smooth function with support in $\mathbb{R}^{N+M}\setminus 4Q$. We set

$$T=|x|^\alpha D_{y_hy_k}(I-L)^{-1}, \quad u=(I-L)^{-1}f, \quad v=D_{y_hy_k}(I-L)^{-1}f.$$

By Theorem 2.7, T is bounded on L^2. Since $f=0$ in $4Q$, by Proposition 2.10 $v-Lv=0$ in $4Q$. Combining the last equality with Kato's inequality of Proposition 2.11, we get

$$-\mathfrak{a}(|v|,\varphi)\ge\int \operatorname{sign} v\, Lv\,\varphi=\int|v|\varphi\ge 0 \quad \text{for all } 0\le\varphi\in C_c^\infty(4Q).$$

It follows that $|v|$ is a local subsolution of L. Note that v is a local solution of $v-Lv=0$ but we cannot assert that it is a local subsolution of L, that is, $Lv\ge 0$, since its sign is not given. By Corollary 4.8 with $s=2$ and $\beta=2\alpha$ we have

$$\left(\frac{1}{|Q|}\int_Q(|x|^{2\alpha}|v|^2)^q\right)^{\frac{1}{q}} \le \frac{C}{|3Q|}\int_{3Q}|x|^{2\alpha}|v|^2, \quad 1<q<\infty,$$

or, equivalently,

$$\left(\frac{1}{|Q|}\int_Q (|x|^\alpha |v|)^{2q}\right)^{\frac{1}{2q}} \le \left(\frac{C}{|3Q|}\int_{3Q} (|x|^\alpha |v|)^2\right)^{\frac{1}{2}}, \quad 1 < q < \infty.$$

It follows that for $2 \le q_0 < \infty$

$$\left(\frac{1}{|Q|}\int_Q |Tf|^{q_0}\right)^{\frac{1}{q_0}} = \left(\frac{1}{|Q|}\int_Q (|x|^\alpha |D_{y_h y_k} u|)^{q_0}\right)^{\frac{1}{q_0}}$$
$$\le \left(\frac{C}{|3Q|}\int_{3Q} (|x|^\alpha |D_{y_h y_k} u|)^2\right)^{\frac{1}{2}} = \left(\frac{C}{|3Q|}\int_{3Q} |Tf|^2\right)^{\frac{1}{2}}.$$

By Theorem 4.4, T extends to a bounded operator in L^p for every $2 \le p < q_0$. Since we can choose q_0 arbitrarily, the case $2 \le p < \infty$ follows.

To treat the case $p < 2$ we consider the adjoint operator

$$T^* = D_{y_h y_k}(I - L)^{-1}|x|^\alpha,$$

which is bounded in L^2. By duality, the boundedness of T in L^p for every $1 < p \le 2$ is equivalent to that of T^* in L^p for every $p \ge 2$. As before we fix $q_0 > 2$ and we prove that T^* satisfies Shen's assumption for every $2 \le q_0 < \infty$. Let Q be a cylinder in $\mathbb{R}^{N+M}$ and $f \in C_c^\infty$ with support in $\mathbb{R}^{N+M} \setminus 4Q$; set

$$u = (I - L)^{-1}(|x|^\alpha f), \qquad v = D_{y_h y_k} u.$$

Then v satisfies $v - Lv = 0$ in $4Q$. By arguing as above, $|v|$ is a local subsolution of L; hence Lemma 4.5 yields a positive constant C such that

$$\sup_Q |v| \le C\left(\frac{1}{|3Q|}\int_{3Q} v^2\right)^{\frac{1}{2}}.$$

It follows that

$$\left(\frac{1}{|Q|}\int_Q |T^* f|^{q_0}\right)^{\frac{1}{q_0}} = \left(\frac{1}{|Q|}\int_Q |D_{y_h y_k} u|^{q_0}\right)^{\frac{1}{q_0}}$$
$$\le \sup_Q |v| \le C\left(\frac{1}{|3Q|}\int_{3Q} v^2\right)^{\frac{1}{2}} = \left(\frac{C}{|3Q|}\int_{3Q} |T^* f|^2\right)^{\frac{1}{2}}$$

and the proof is complete by Theorem 4.4 applied to T^*.

By difference the operator $\Delta_x (I - L)^{-1}$ is bounded on L^p and, integrating with respect to y the classical Calderón–Zygmund estimates in the x-variables we deduce the L^p boundedness of $D_{x_i x_j}(I - L)^{-1}$. □

Remark 4.9. Theorem 4.3 holds for every $0 \le \phi \in B_2(\mathbb{R}^N)$ when $1 < p \le 2$. In fact, the properties of the powers $|x|^\alpha$ stated in Lemma 4.7 and Corollary 4.8 have been used in the above proof only when $p > 2$.

We can now give a partial description of the domain $D_p(L)$, which will be strengthened in Theorem 4.21 after proving L^p-estimates for mixed derivatives.

Proposition 4.10. *Let $p \in (1, \infty)$. Then one has*

$$D_p(L) = \{u \in L^p : \nabla_x u,\ D_{xx}u,\ |x|^{\frac{\alpha}{2}}\nabla_y u,\ |x|^\alpha D_{yy}u \in L^p\}. \tag{15}$$

Moreover

$$\|D_{x_ix_j}u\|_p + \||x|^\alpha D_{y_hy_k}u\|_p \le C\|Lu\|_p, \quad u \in D_p(L). \tag{16}$$

Proof. Let $1 < p < \infty$ and let $\widetilde{D}_p(L)$ be the set defined in the right-hand side of equality (15).

Let us preliminarily prove that $D_p(L) \subseteq \widetilde{D}_p(L)$.

Theorem 4.3 and the consistency of the resolvent operators in L^2 and in L^p imply that

$$\|D_{x_ix_j}u\|_p + \||x|^\alpha D_{y_hy_k}u\|_p \le C\|(I-L)u\|_p \tag{17}$$

for any $u \in (I-L)^{-1}(C_c^\infty)$ which is dense in $D_p(L)$ with respect to the graph norm. This implies that (17) extends to $D_p(L)$, proving that u has pure second-order distributional derivatives which satisfy $D_{x_ix_j}, |x|^\alpha D_{y_hy_k} \in L^p$ and that

$$\|D_{x_ix_j}u\|_p + \||x|^\alpha D_{y_hy_k}u\|_p \le C\|(I-L)u\|_p \le C(\|u\|_p + \|Lu\|_p), \quad u \in D_p(L). \tag{18}$$

As in Theorem 2.7, an interpolation argument shows that $\nabla_x u,\ |x|^{\frac{\alpha}{2}}\nabla_y u \in L^p(\mathbb{R}^{N+M})$; i.e., $u \in \widetilde{D}_p(L)$.

To get homogeneous estimates, we use Proposition 2.4(v), and apply (18) to $u(x, y) = v(sx, s^{(2+\alpha)/2}y)$, $s > 0$, thus obtaining

$$\|D_{x_ix_j}u\|_p + \||x|^\alpha D_{y_hy_k}u\|_p \le C(\|Lv\|_p + s^{-2}\|v\|_p).$$

Letting $s \to \infty$ we obtain (16).

To prove that $\widetilde{D}_p(L) = D(L)$, we proceed as in the proof of Theorem 2.9 and show that the operator $(L, \widetilde{D}_p(L))$ is dissipative. Let $u \in \widetilde{D}_p(L)$; then the same sectional argument of (6) shows that for a.e. $x \in \Omega_N$, we have $u(x, \cdot) \in W^{2,p}(\mathbb{R}^M)$ and from [Metafune and Spina 2008]

$$\int_{\mathbb{R}^M} u|u|^{p-2}\Delta_y u\, dy = -(p-1)\int_{\mathbb{R}^M} |\nabla_y u|^2|u|^{p-2}\, dy \quad \text{for a.e. } x \in \Omega_N.$$

Then multiplying by $|x|^\alpha$, integrating in x and using Fubini's theorem we get

$$\int_{\mathbb{R}^{N+M}} |x|^\alpha u|u|^{p-2}\Delta_y u\, dx\, dy = -(p-1)\int_{\mathbb{R}^{N+M}} |x|^\alpha|\nabla_y u|^2|u|^{p-2}\, dx\, dy.$$

Analogously

$$\int_{\mathbb{R}^{N+M}} u|u|^{p-2}\Delta_x u\, dx\, dy = -(p-1)\int_{\mathbb{R}^{N+M}} |\nabla_x u|^2|u|^{p-2}\, dx\, dy.$$

The last two inequalities imply

$$\int_{\mathbb{R}^{N+M}} u|u|^{p-2}\, Lu\, dx\, dy = -(p-1)\int_{\mathbb{R}^{N+M}} (|\nabla_x u|^2 + |x|^\alpha|\nabla_y u|^2)|u|^{p-2}\, dx\, dy \le 0,$$

which, since $u \in \widetilde{D}_p(L)$ is arbitrary, implies the dissipativity of $(L, \widetilde{D}_p(L))$. □

Remark 4.11. The previous theorem implies, in particular, the equivalence between the graph norm $\|u\|_p + \|Lu\|_p$ and the norm

$$\|u\|_p + \|\nabla_x u\|_p + \||x|^{\frac{\alpha}{2}} \nabla_y u\|_p + \|D_{xx} u\|_p + \||x|^{\alpha} D_{yy} u\|_p,$$

since this last clearly dominates $\|Lu\|_p$.

The following proposition shows that C_c^∞ also is a core for $(L, D_p(L))$.

Proposition 4.12. *For any $p \in (1, \infty)$, C_c^∞ is a core for the operator $(L, D_p(L))$.*

Proof. Let $u \in D_p(L)$; we preliminarily approximate u with functions in $D_p(L)$ having compact support in $\mathbb{R}^{N+M}$. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a smooth function such that $\chi_{B_1} \le \eta \le \chi_{B_2}$ and, for every $n \in \mathbb{N}$, $x \in \mathbb{R}^N$, define $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. Set $u_n = \eta_n u$. Now u_n has, by construction, compact support in x and, using the characterization in (15), one can easily recognize that $u_n \in D_p(L)$. Lebesgue's theorem immediately implies that u_n, $|x|^{\alpha/2}\nabla_y u_n$, $|x|^\alpha D_{yy} u_n$ tend to u, $|x|^{\alpha/2}\nabla_y u$, $|x|^\alpha D_{yy} u$ in L^p, respectively. Concerning the x-gradient, we have

$$\begin{aligned}\|\nabla_x(\eta_n u) - \nabla_x(u)\|_p^p &\le \int_{\mathbb{R}^{N+M}} |\eta_n - 1|^p |\nabla_x u|^p \, dx\, dy + \int_{\mathbb{R}^{N+M}} |\nabla_x \eta_n|^p |u|^p \, dx\, dy \\ &\le \int_{\mathbb{R}^{N+M}} |\eta_n - 1|^p |\nabla u|^p \, dx\, dy + C n^{-p} \int_{\{n \le |x| \le 2n\}} |u|^p \, dx\, dy,\end{aligned}$$

which tends to 0 by dominated convergence. Similarly $D_{xx} u_n$ tends to $D_{xx} u$ in L^p. By Proposition 4.10 and Remark 4.11, this proves that u_n tends to u in $D_p(L)$. Using a similar argument with η replaced by an analogous cut-off function $\eta \in C_c^\infty(\mathbb{R}^M)$, we can approximate u with functions in $D_p(L)$ having compact support also in the y-variable.

Let us suppose first that $p < 2$ and let $u \in D_p(L)$ (the case $p = 2$ is already proved in Proposition 2.5). From the first part of the proof, we can suppose u has compact support. Let $\eta \in C_c^\infty$ such that $\eta = 1$ on the support of u and, using Proposition 2.5, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of C_c^∞ functions such that $u_n \to u$ in $D_2(L)$ as $n \to \infty$. This implies

$$\|L(\eta u_n - u)\|_p + \|\eta u_n - u\|_p \le C[\|L(\eta u_n - u)\|_2 + \|\eta u_n - u\|_2] \to 0 \quad \text{as } n \to \infty,$$

where C depends on the measure of the support of u. This proves the claim for $p < 2$.

The proof of the case $p > 2$ can be carried out by slightly adapting the arguments used in the proof of Proposition 2.5. We equivalently show that $(I - L)(C_c^\infty)$ is dense in L^p and to this aim let $v \in L^{p'}(\mathbb{R}^{N+M})$ such that

$$\int_{\mathbb{R}^{N+M}} (I - L)u\, \bar{v}\, dx\, dy = 0 \quad \text{for all } u \in C_c^\infty.$$

Since $1 < p' < 2$, the partial Fourier transform of $v(x, \cdot) \in L^{p'}(\mathbb{R}^M)$, with respect to the y-variable, exists as a function in $L^p(\mathbb{R}^M)$ for a.e. $x \in \mathbb{R}^N$. Therefore, taking in the last equality the Fourier transform with respect to the y-variable and applying the Fubini and Plancherel theorems, we get

$$\int_{\mathbb{R}^{N+M}} [\hat{u}(x, \xi) - \Delta_x \hat{u}(x, \xi) + |x|^\alpha |\xi|^2 \hat{u}(x, \xi)]\, \bar{\hat{v}}(x, \xi)\, dx\, d\xi = 0 \quad \text{for all } u \in C_c^\infty.$$

By [Semenov 1977, Theorem 1.1], since the potential is nonnegative and in L^p_{loc}, we know $C_c^\infty(\mathbb{R}^N)$ is a core for $\Delta_x - \phi(\,\cdot\,)|\xi|^2$ in L^p; proceeding then as in the proof of Proposition 2.5 we conclude that $\hat{v}(\,\cdot\,,\xi) = 0$ for a.e. $\xi \in \mathbb{R}^M$ and the proof follows. □

Mixed derivatives. By using classical covering results and Rellich inequalities we obtain here L^p estimates for the mixed second-order derivatives. To simplify the notation we write x for any of the variables x_i, $i = 1, \dots, N$, and y for any of the variables y_h, $h = 1, \dots, M$.

Theorem 4.13. *For every $u \in D_p(L)$*

$$\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \le C\|Lu\|_p.$$

We need a Rellich-type inequality.

Lemma 4.14. *Let $p \neq N, \frac{N}{2}$. There exist a positive constant C such that for $u \in C_c^\infty$ satisfying $u(0,y) = 0$, $\nabla_x u(0,y) = 0$ for all $y \in \mathbb{R}^M$, we have*

$$\left\| \frac{u}{|x|^2} \right\|_p \le C\|Lu\|_p.$$

Proof. Let $u \in C_c^\infty$ such that $u(0,y) = 0$, $\nabla_x u(0,y) = 0$ so that $\|u/|x|^2\|_p < \infty$. Then by [Metafune et al. 2019, Theorem 4.2; 2015, Theorem 3.3]

$$\int_{\mathbb{R}^N} \left| \frac{u}{|x|^2} \right|^p dx \le C \int_{\mathbb{R}^N} |\Delta_x u|^p \, dx.$$

Integrating the previous inequality over $\mathbb{R}^M$ and using Theorem 4.3,

$$\left\| \frac{u}{|x|^2} \right\|_p \le C\|\Delta_x u\|_p \le C(\|Lu\|_p + \||x|^\alpha \Delta_y u\|_p) \le C\|Lu\|_p. \qquad \square$$

Remark 4.15. The above Rellich inequality uses Theorem 4.3 to replace Δ_x with the operator L. However, even its version in dimension 1 (that is, for D_{xx} rather than L) is not obvious and probably cannot be obtained by integration by parts; see, e.g., [Metafune et al. 2019], where it is shown that Rellich inequalities can be proved for the Laplacian in $L^p(\mathbb{R}^N)$ when $p < \frac{N}{2}$, a condition which is never verified in dimension 1.

We first prove mixed derivatives estimates assuming that both u and $\nabla_x u$ vanish for $x = 0$. We need the following covering result.

Proposition 4.16 [Cupini and Fornaro 2004, Proposition 6.1]. *For every $0 \le k < \frac{1}{2}$ there exists a natural number $\zeta = \zeta(N,k)$ with the following property. Given $\mathcal{F} = \{x + B(\rho(x))\}_{x\in\mathbb{R}^N}$, where $\rho : \mathbb{R}^N \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant k, there exist a countable subcovering $\{x_n + B(\rho(x_n))\}_{n\in\mathbb{N}}$ of $\mathbb{R}^N$ such that at most ζ among the double balls $\{x_n + B(2\rho(x_n))\}_{n\in\mathbb{N}}$ overlap.*

The following lemma is an immediate consequence of the classical Calderón–Zygmund inequalities for the Laplacian.

Lemma 4.17. *Let $1 < p < \infty$. Then for every $a \in \mathbb{R}$, $u \in C_c^\infty$, one has*

$$\|D_{xx}u\|_p + \|a^2 D_{yy}u\|_p + \|aD_{xy}u\|_p \le C\|\Delta_x u + a^2\Delta_y u\|_p.$$

Proof. It is sufficient to apply the Calderón–Zygmund inequalities to $v(x, y) = u(x, ay)$. □

Lemma 4.18. *Let $p \neq \frac{N}{2}$, N. One has*

$$\|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \le C\|Lu\|_p$$

for every $u \in C_c^\infty$ such that $u(0, y) = 0$, $\nabla_x u(0, y) = 0$ for all $y \in \mathbb{R}^M$.

Proof. We fix $x_0 \in \mathbb{R}^N$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B(0, 1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(0, 2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta((x - x_0)/\rho)$, where $\rho = \frac{1}{4}|x_0|$. We apply Lemma 4.17 to the function $\vartheta_\rho u$ and obtain

$$\|D_{xx}(\vartheta_\rho u)u\|_p + \||x_0|^\alpha D_{yy}(\vartheta_\rho u)\|_p + \||x_0|^{\frac{\alpha}{2}} D_{xy}(\vartheta_\rho u)\|_p \le C\|\Delta_x(\vartheta_\rho u) + |x_0|^\alpha \Delta_y(\vartheta_\rho u)\|_p.$$

By the classical interpolation inequalities for the gradient we get, for every $\eta > 0$,

$$\begin{aligned}&\|D_{xx}u\|_{L^p(B(x_0,\rho))} + \||x_0|^\alpha D_{yy}u\|_{L^p(B(x_0,\rho))} + \||x_0|^{\frac{\alpha}{2}} D_{xy}u\|_{L^p(B(x_0,\rho))}\\ &\le C\left(\|\Delta_x u + |x_0|^\alpha \Delta_y u\|_{L^p(B(x_0,2\rho))} + \frac{1}{\rho}\|\nabla_x u\|_{L^p(B(x_0,2\rho))} + \frac{1}{\rho^2}\|u\|_{L^p(B(x_0,2\rho))} + \frac{|x_0|^{\frac{\alpha}{2}}}{\rho}\|\nabla_y u\|_{L^p(B(x_0,2\rho))}\right)\\ &\le C\left(\|\Delta_x u + |x_0|^\alpha \Delta_y u\|_{L^p(B(x_0,2\rho))} + \eta\|\Delta_x u\|_{L^p(B(x_0,2\rho))} + \eta\||x_0|^\alpha \Delta_y u\|_{L^p(B(x_0,2\rho))} + \frac{1}{\eta\rho^2}\|u\|_{L^p(B(x_0,2\rho))}\right).\end{aligned}$$

Since

$$\rho = \tfrac{1}{4}|x_0|, \quad \tfrac{1}{2}|x_0| \le |x| \le \tfrac{3}{2}|x_0|, \quad x \in B(x_0, 2\rho),$$

we get

$$\begin{aligned}&\|D_{xx}u\|_{L^p(B(x_0,\rho))} + \||x|^\alpha D_{yy}u\|_{L^p(B(x_0,\rho))} + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_{L^p(B(x_0,\rho))}\\ &\quad\le C\Bigg(\|\Delta_x u + |x|^\alpha \Delta_y u\|_{L^p(B(x_0,2\rho))} + \eta\|\Delta_x u\|_{L^p(B(x_0,2\rho))}\\ &\qquad\qquad + \eta\||x|^\alpha \Delta_y u\|_{L^p(B(x_0,2\rho))} + \frac{1}{\eta}\left\|\frac{u}{|x|^2}\right\|_{L^p(B(x_0,2\rho))}\Bigg).\end{aligned} \tag{19}$$

Let $\{B(x_n, \rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 4.16 such that at most ζ among the double balls $\{B(x_n, 2\rho(x_n))\}$ overlap. Writing (19) with x_n instead of x_0 and summing over n it follows that

$$\begin{aligned}&\|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p\\ &\qquad\le C\left(\|\Delta_x u + |x|^\alpha \Delta_y u\|_p + \eta\|\Delta_x u\|_p + \eta\||x|^\alpha \Delta_y u\|_p + \frac{1}{\eta}\left\|\frac{u}{|x|^2}\right\|_p\right).\end{aligned}$$

By choosing η small enough we get

$$\|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \le C\left(\|\Delta_x u + |x|^\alpha \Delta_y u\|_p + \left\|\frac{u}{|x|^2}\right\|_p\right).$$

and the claim follows from Lemma 4.14. □

Next we prove mixed derivatives estimates assuming that either u or $\nabla_x u$ vanishes for $x=0$.

Lemma 4.19. *If $p \neq \frac{2N}{2-\alpha}, \frac{N}{2}, N$, then*

$$\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \leq C\|Lu\|_p$$

for every $u \in C_c^\infty$ such that $u(0,y)=0$ or $u_x(0,y)=0$ for all $y \in \mathbb{R}^M$.

Proof. Let $u \in C_c^\infty$ such that $u(0,y)=0$ and let $v(x,y)=\frac{1}{\lambda}u(\lambda x, y)$. Then $v(0,y)=0$ and $\nabla_x v(0,y) = \nabla_x u(0,y)$. This implies that $w=u-v$ satisfies $w(0,x)=w_x(0,x)=0$. Moreover

$$\||x|^{\frac{\alpha}{2}} D_{xy}v\|_p = \lambda^{-\frac{\alpha}{2}-\frac{N}{p}} \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p$$

and, applying Theorem 4.3,

$$\begin{aligned}\|Lv\|_p &\leq \|v_{xx}\|_p + \||x|^\alpha \Delta_y v\|_p \\ &= \lambda^{1-\frac{N}{p}}\|u_{xx}\|_p + \lambda^{-\alpha-1-\frac{N}{p}}\||x|^\alpha \Delta_y u\|_p \leq C(\lambda)\|Lu\|_p.\end{aligned}$$

Applying Lemma 4.18 to w we then have

$$\begin{aligned}\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p &\leq \||x|^{\frac{\alpha}{2}} D_{xy}w\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}v\|_p \leq C(\|Lw\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}v\|_p) \\ &\leq C(\|Lu\|_p + \|Lv\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}v\|_p) \leq C(\lambda)\|Lu\|_p + C\||x|^{\frac{\alpha}{2}} D_{xy}v\|_p \\ &= C(\lambda)\|Lu\|_p + C\lambda^{-\frac{\alpha}{2}-\frac{N}{p}}\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p.\end{aligned}$$

The claim then follows by choosing λ large enough such that $C\lambda^{-\alpha/2-N/p} \leq \frac{1}{2}$.

Assume now $u_x(0,y)=0$ and let $v(x,y)=u(\lambda x, y)$. Then $u(0,y)=v(0,y)$ and $v_x(0,y) = \lambda u_x(0,y)=0$. Moreover

$$\||x|^{\frac{\alpha}{2}} D_{xy}v\|_p = \lambda^{1-\frac{\alpha}{2}-\frac{N}{p}} \||x|^{\frac{\alpha}{2}} D^2_{xy}u\|_p.$$

It follows that $w=u-v$ satisfies $w(0,x)=w_x(0,x)=0$. Hence an analogous argument yields

$$\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \leq C(\lambda)\|Lu\|_p + C\lambda^{1-\frac{\alpha}{2}-\frac{N}{p}}\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p.$$

Choosing λ large enough or small enough according to $1-\frac{\alpha}{2}-\frac{N}{p}>0$ or $1-\frac{\alpha}{2}-\frac{N}{p}<0$ we get the claim for $1-\frac{\alpha}{2}-\frac{N}{p} \neq 0$ or, equivalently, $p \neq \frac{2N}{2-\alpha}$. □

Proof of Theorem 4.13. Let us suppose, preliminarily, $p \neq \frac{2N}{2-\alpha}$, $p \neq \frac{N}{2}$, $p \neq N$ and let $u \in C_c^\infty$. We introduce the operators

$$Pu(x,y) = \frac{u(x,y)+u(-x,y)}{2}, \qquad Qu(x,y) = \frac{u(x,y)-u(-x,y)}{2}.$$

Observe that

$$u = Pu + Qu, \quad Qu(0,y)=0, \quad \nabla_x(Pu)(0,y)=0, \quad y \in \mathbb{R}^M,$$

P and Q commute with the second-order derivatives and $\|P(Lu)\|_p + \|Q(Lu)\|_p$ is equivalent to $\|Lu\|_p$. Moreover

$$L(Pu) = P(Lu), \quad L(Qu) = Q(Lu).$$

We can therefore apply the results in Lemma 4.19 to Pu and Qu. For the mixed second-order derivatives we get

$$\||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \le \||x|^{\frac{\alpha}{2}} P(D_{xy}u)\|_p + \||x|^{\frac{\alpha}{2}} Q(D_{xy}u)\|_p = \||x|^{\frac{\alpha}{2}} D_{xy}(Pu)\|_p + \||x|^{\frac{\alpha}{2}} Q_{xy}(Qu)\|_p$$
$$\le C(\|L(Pu)\|_p + \|L(Qu)\|_p) = C(\|P(Lu)\|_p + \|Q(Lu)\|_p) \le C\|Lu\|_p.$$

By density the proof extends to $u \in D_p(L)$.

Suppose now $p = \frac{2N}{2-\alpha}$. Observe that, by the previous part of the proof, the operator $|x|^{\alpha/2} D_{xy}(I-L)^{-1}$ is bounded in L^p for some $p_1 < \frac{2N}{2-\alpha} < p_2$, with $p_1,\ p_2 \neq \frac{N}{2}, N$. The Riesz–Thorin interpolation theorem then yields the boundedness of $|x|^{\alpha/2} D_{xy}(I-L)^{-1}$ also for $p = \frac{2N}{2-\alpha}$; the same scaling argument used in the proof of Proposition 4.10 then proves the required claim. We can argue similarly for $p = N$, $p = \frac{N}{2}$. □

As a corollary we improve gradient estimates near $x = 0$ showing that $|x|^{\alpha/2-1}\nabla_y u \in L^p,\ u \in D_p(L)$, when $\left(\frac{\alpha}{2}-1\right)p + N > 0$. This last condition is necessary for the above integrability, since otherwise the weight $|x|^{\alpha/2-1}$ is not locally p-summable near $x = 0$. We also recall that $|x|^{\alpha/2}\nabla_y u \in L^p$ by Proposition 4.10.

Corollary 4.20. *Let $\left(\frac{\alpha}{2}-1\right)p + N > 0$. Then for every $u \in D_p(L)$*

$$\||x|^{\frac{\alpha}{2}-1}\nabla_y u\|_p \le C\|Lu\|_p.$$

Proof. By density we may assume that $u \in C_c^\infty$. By the Hardy inequality, see for example [Metafune et al. 2015, Proposition 8.1],

$$\int_{\mathbb{R}^N} |x|^{\left(\frac{\alpha}{2}-1\right)p} |\nabla_y u|^p \, dx \le \left(\frac{p}{\left(\frac{\alpha}{2}-1\right)p + N}\right)^p \int_{\mathbb{R}^N} |x|^{\frac{\alpha}{2}p} |D_{xy}u|^p \, dx.$$

Integrating over $\mathbb{R}^M$ and using Theorem 4.13, the claim follows. □

We can now strengthen Proposition 4.10.

Theorem 4.21. *Let $p \in (1, \infty)$. Then one has*

$$D_p(L) = \{u \in L^p : \nabla_x u,\ D_{xx}u \in L^p,\ |x|^{\frac{\alpha}{2}}\nabla_y u,\ |x|^{\frac{\alpha}{2}} D_{xy}u,\ |x|^\alpha D_{yy}u \in L^p\}.$$

In particular the graph norm $\|u\|_p + \|Lu\|_p$ is equivalent to

$$\|u\|_p + \|\nabla_x u\|_p + \||x|^{\frac{\alpha}{2}}\nabla_y u\|_p + \|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p.$$

Moreover, if $\left(\frac{\alpha}{2}-1\right)p + N > 0$, then also $|x|^{\frac{\alpha}{2}-1}\nabla_y u \in L^p$.

Proof. The proof follows from Proposition 4.10, Theorem 4.13 and Corollary 4.20. □

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Received 27 Nov 2019. Revised 24 Jan 2020. Accepted 15 Mar 2020.

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