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QUANTUM FLUCTUATIONS AND LARGE-DEVIATION PRINCIPLE FOR MICROSCOPIC CURRENTS OF FREE FERMIONS IN DISORDERED MEDIA

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We extend the large-deviation results obtained by N. J. B. Aza and the present authors on atomic-scale conductivity theory of free lattice fermions in disordered media. Disorder is modeled by a random external potential, as in the celebrated Anderson model, and a nearest-neighbor hopping term with random complex-valued amplitudes. In accordance with experimental observations, via the large-deviation formalism, our previous paper showed in this case that quantum uncertainty of microscopic electric current densities around their (classical) macroscopic value is suppressed, exponentially fast with respect to the volume of the region of the lattice where an external electric field is applied. Here, the quantum fluctuations of linear response currents are shown to exist in the thermodynamic limit, and we mathematically prove that they are related to the rate function of the large-deviation principle associated with current densities. We also demonstrate that, in general, they do not vanish (in the thermodynamic limit), and the quantum uncertainty around the macroscopic current density disappears exponentially fast with an exponential rate proportional to the squared deviation of the current from its macroscopic value and the inverse current fluctuation, with respect to growing space (volume) scales.

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1. Introduction

Surprisingly (in view of [Ferry 2012]), experimental measurements [Weber et al. 2012] of electric resistance of silicon nanowires doped with phosphorus demonstrate that the macroscopic laws for charge transport are already accurate at length scales larger than a few nanometers, even at very low temperatures (4.2 K). As a consequence, microscopic (quantum) effects on charge transport can very rapidly disappear with respect to growing space scales. Understanding the breakdown of the classical (macroscopic) conductivity theory at microscopic scales is an important technological issue, because of the growing need for smaller electronic components.

From a mathematical perspective, the convergence of the expectations of microscopic current densities with respect to growing space scales is proved in [Bru et al. 2016; Bru and de Siqueira Pedra 2015a], but no information about the suppression of quantum uncertainty was obtained in the macroscopic limit. In [Aza et al. 2019], in accordance with experimental observations, it was proved for noninteracting lattice fermions with disorder that quantum uncertainty of microscopic electric current densities around their (classical) macroscopic value is suppressed exponentially fast with respect to the volume of the region of the lattice where an external electric field is applied. This is proved in [Aza et al. 2019] via the large-deviation formalism [Deuschel and Stroock 1989; Dembo and Zeitouni 1998], which has been used in quantum statistical mechanics since the 1980s [Aza et al. 2017, Section 7]. Given a fixed electromagnetic field $\mathcal{E}$, we derive in particular in [Aza et al. 2019] the (good) rate function $\mathrm{I}^{(\mathcal{E})}$ associated with microscopic (linear response) current densities[1] $x_L^{(\mathcal{E})} \in \mathbb{R}$, $L \in \mathbb{R}_0^+$, meaning in this case that, in a cubic box of volume L^d (d-dimensional lattice), for any $a, b \in \mathbb{R}$,

$$\mathrm{Prob}\big[x_L^{(\mathcal{E})} \in [a, b]\big] \sim \mathrm{e}^{-L^d \inf_{x \in [a,b]} \mathrm{I}^{(\mathcal{E})}(x)}, \quad \text{as } L \to \infty, \tag{1}$$

with $\mathrm{I}^{(\mathcal{E})} \geq 0$ and $\mathrm{I}^{(\mathcal{E})}(x) = 0$ if and only if x is the macroscopic (linear response) current density, $x^{(\mathcal{E})}$.

In this paper, we complement these studies by rigorously showing two new properties of charge transport of quasifree fermions in disordered media:

(a) The quantum fluctuations of linear response currents exist in the thermodynamic limit and are meanwhile explicitly related to the rate function $\mathrm{I}^{(\mathcal{E})}$, as expected.

(b) In general, the quantum fluctuations of currents do not vanish in the thermodynamic limit and the quantum uncertainty around the macroscopic current density disappears exponentially fast with an exponential rate proportional to $(x - x^{(\mathcal{E})})^2$ and the inverse current fluctuation, with respect to growing space (volume) scales.

Properties (a) and (b) refer to Theorems 3.1 and 3.3, which are the main results of this paper.

Our results show that the experimental measure of the rate function $\mathrm{I}^{(\mathcal{E})}$ (see (1)) leads to an experimental estimate on the corresponding quantum fluctuations. Conversely, an experimental estimate on these quantum fluctuations gives the behavior of the corresponding rate function $\mathrm{I}^{(\mathcal{E})}$ around the macroscopic current density $x^{(\mathcal{E})}$. This fact is certainly not restricted to fermionic currents.

[1] In some direction of $\mathbb{R}^d$.

Note that the existence of quantum fluctuations and associated mathematical structures has been extensively studied for quantum many-body systems. This refers, for instance, to the construction of so-called algebra of normal fluctuations for transport phenomena, which are related to quantum central limit theorems (see, e.g., [Bru et al. 2014; 2016; Goderis et al. 1989a; 1989b; 1989c; 1990a; 1990b; 1991], as well as [Verbeure 2011, Chapter 6]). The explicit relation (a) we derive between quantum fluctuations and the large-deviation formalism in quantum statistical mechanics [Aza et al. 2017, Section 7] is, however, a new general observation on quantum many-body systems.

We use the mathematical framework of [Aza et al. 2019; Bru and de Siqueira Pedra 2015a; 2017a] to study fermions on the lattice. For simplicity we take a cubic lattice $\mathbb{Z}^d$, even if other types of lattices can be considered with very similar methods. Disorder within the conductive material, due to impurities, crystal lattice defects, etc., is modeled by (i) a random external potential, as in the celebrated Anderson model, and (ii) a nearest-neighbor hopping term with random complex-valued amplitudes. In particular, random (electromagnetic) vector potentials can also be implemented. The celebrated tight-binding Anderson model is one particular example of the general case considered here.

In order to prove Property (a), i.e., Theorem 3.1, we use the large-deviation formalism and follow the argument lines of [Aza et al. 2019, Section 4] to show [Aza et al. 2019, Theorem 3.1] via the Akcoglu–Krengel ergodic theorem [Aza et al. 2019, Theorem 4.17], for one has to control the thermodynamical limit of (finite-volume) generating functions that are random. We perform, in particular, the same box decomposition of these random functions, which can be justified with the help of the Bogoliubov-type inequality [Aza et al. 2019, Lemma 4.2] and the "locality" (or space decay) of both the quasifree dynamics and space correlations of KMS states, which is a consequence of Combes–Thomas estimates [Aza et al. 2019, Appendix A] (see [Aza et al. 2019, Section 4.3]). In this paper, we only give the new arguments that are necessary to prove Property (a), like the existence of the thermodynamic limit of quantum fluctuations of currents and the continuity of the second derivative of the generating function. In particular, as in the proof of [Aza et al. 2019, Corollary 4.20], we use the (Arzelà–)Ascoli theorem [Rudin 1991, Theorem A5], which requires uniform bounds on the third-order derivatives of finite-volume generating functions. This proof is much more computational than the one of [Aza et al. 2019, Proposition 4.9], which only controls the first and second derivatives of the same function. Note that derivatives of the logarithm of the expectations of an exponential, like the generating function we consider here, are generally related to so-called "truncated" or "connected" correlations. We demonstrate that it is the case for the third-order derivative we refer to above, allowing the reader to follow the computation of that derivative in a systematic way. Considering the third-order case, the algorithm to compute the derivatives of the generating functions at any order becomes apparent, showing that the generating function is in fact *smooth*. We give below further remarks on that.

In order to prove Property (b), i.e., Theorem 3.3 (Theorem 3.1 being proved), we rewrite the second derivative of the generating function, which is the thermodynamic limit of the quantum fluctuations of currents (Theorem 3.1(i)), as a trace of some explicit positive operator in the one-particle Hilbert space. This quantity can be estimated from below by the Hilbert–Schmidt norm of a kind of current observable in the one-particle Hilbert space. Various computations and estimates then imply Theorem 3.3.

As discussed in [Aza et al. 2019], observe the existence of vast mathematical literature on charged transport properties of fermions in disordered media, see for instance [Schulz-Baldes and Bellissard 1998; Bellissard et al. 1994; Bouclet et al. 2005; Klein et al. 2007; Klein and Müller 2008; 2015; Dombrowski and Germinet 2008; Prodan 2013; Brynildsen and Cornean 2013]. However, it is not the purpose of this introduction to go into the details of the history of this specific research field. For a (nonexhaustive) historical perspective on linear conductivity (Ohm's law), see, e.g., [Bru and de Siqueira Pedra 2015b] or our previous papers [Bru et al. 2014; 2015a; 2015b; 2016; Bru and de Siqueira Pedra 2015a; 2016; 2017a].

To conclude, this paper is organized as follows:

- In Section 2, we describe the mathematical framework, which is the one from [Aza et al. 2019; Bru and de Siqueira Pedra 2015a; 2017a]. It refers to quasifree fermions on the lattice in disordered media. Although all of the problem can be formulated, in a mathematically equivalent way, in the one-particle (or Hilbert space) setting [Aza et al. 2019, Appendix C.3], since the underlying physical system is a many-body one, it is conceptually more appropriate to state our results within the algebraic formulation for lattice fermion systems, as in [Aza et al. 2019; Bru and de Siqueira Pedra 2015a; 2017a]. Short complementary discussions on response of quasifree fermion systems to electric fields can be found in [Aza et al. 2019, Appendix C].
- In Section 3, the main results are stated. In particular, Property (a) described above refers to Section 3.1, while Property (b) is explained in Section 3.2.
- Section 4 gathers all technical proofs. In particular, Sections 4.1–4.2 give preliminary definitions and observations, while Sections 4.3 and 4.4 refer to the proofs of Theorems 3.1(i) and 3.3, respectively.

Notation 1.1. A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$. The Banach space of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The scalar product of any Hilbert space $\mathcal{X}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. We use the convention $\mathbb{R}^+ \doteq \{x \in \mathbb{R} : x > 0\}$, while $\mathbb{R}_0^+ \doteq \mathbb{R}^+ \cup \{0\}$. For any random variable X, $\mathbb{E}[X]$ denotes its expectation and $\mathrm{Var}[X]$ its variance.

2. Setup of the problem

We use the mathematical framework of [Aza et al. 2019; Bru and de Siqueira Pedra 2015a; 2017a] in order to study fermions on the lattice.

2.1. ***Random tight-binding model .*** We consider conducting fermions in a cubic crystal represented by the d-dimensional cubic lattice $\mathbb{Z}^d$ ($d \in \mathbb{N}$). The corresponding one-particle Hilbert space is thus $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d; \mathbb{C})$. Its canonical orthonormal basis is denoted by $\{\mathfrak{e}_x\}_{x\in\mathbb{Z}^d}$, where $\mathfrak{e}_x(y) \doteq \delta_{x,y}$ for all $x, y \in \mathbb{Z}^d$. ($\delta_{x,y}$ is the Kronecker delta).

Disorder in the crystal is modeled via a probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$, defined as follows: Using the sets

$$\mathbb{D} \doteq \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{and} \quad \mathfrak{b} \doteq \left\{\{x, x'\} \subseteq \mathbb{Z}^d : |x - x'| = 1\right\},$$

we define

$$\Omega \doteq [-1, 1]^{\mathbb{Z}^d} \times \mathbb{D}^{\mathfrak{b}} \quad \text{and} \quad \mathfrak{A}_\Omega \doteq \left(\otimes_{x\in\mathbb{Z}^d}\mathfrak{A}_x^{(1)}\right) \otimes \left(\otimes_{\mathbf{x}\in\mathfrak{b}}\mathfrak{A}_{\mathbf{x}}^{(2)}\right),$$

where $\mathfrak{A}_x^{(1)}$, $x \in \mathbb{Z}^d$, and $\mathfrak{A}_{\mathbf{x}}^{(2)}$, $\mathbf{x} \in \mathfrak{b}$, are the Borel σ-algebras of, respectively, the interval $[-1, 1]$ and the unit disc $\mathbb{D}$, both with respect to their usual metric topology. The distribution $\mathfrak{a}_\Omega$ is an *ergodic* probability measure on the measurable space $(\Omega, \mathfrak{A}_\Omega)$. See [Aza et al. 2019] for more details. Below, $\mathbb{E}[\,\cdot\,]$ and $\mathrm{Var}[\,\cdot\,]$ always refer to expectations and variances associated with $\mathfrak{a}_\Omega$.

Given $\vartheta \in \mathbb{R}_0^+$ and $\omega = (\omega_1, \omega_2) \in \Omega$, we define a bounded self-adjoint operator $\Delta_{\omega,\vartheta} \in \mathcal{B}(\mathfrak{h})$ encoding the hopping amplitudes of a single particle in the lattice:

$$[\Delta_{\omega,\vartheta}(\psi)](x) \doteq 2d\psi(x) - \sum_{j=1}^{d} \big((1+\vartheta\overline{\omega_2(\{x, x-e_j\})})\psi(x-e_j) + \psi(x+e_j)(1+\vartheta\omega_2(\{x, x+e_j\}))\big) \quad (2)$$

for any $x \in \mathbb{Z}^d$ and $\psi \in \mathfrak{h}$, where $\{e_k\}_{k=1}^d$ is the canonical basis of $\mathbb{R}^d$. If $\vartheta = 0$, $\Delta_{\omega,0}$ is (up to a minus sign) the usual d-dimensional discrete Laplacian. Random (electromagnetic) vector potentials can also be implemented in our model, since ω_2 takes values in the unit disc $\mathbb{D} \subseteq \mathbb{C}$. Then, the random tight-binding model is the one-particle Hamiltonian defined by

$$h^{(\omega)} \doteq \Delta_{\omega,\vartheta} + \lambda\omega_1, \quad \omega = (\omega_1, \omega_2) \in \Omega, \quad \lambda, \vartheta \in \mathbb{R}_0^+, \quad (3)$$

where the function $\omega_1 : \mathbb{Z}^d \to [-1, 1]$ is identified with the corresponding (self-adjoint) multiplication operator. The celebrated tight-binding Anderson model corresponds to the special case $\vartheta = 0$.

2.2. *C*-algebraic setting* . We denote by $\mathcal{U}$ the universal unital C^*-algebra generated by elements $\{a(\psi)\}_{\psi\in\mathfrak{h}}$ satisfying the canonical anticommutation relations (CAR): For all $\psi, \varphi \in \mathfrak{h}$,

$$a(\psi)a(\varphi) = -a(\varphi)a(\psi), \quad a(\psi)a(\varphi)^* + a(\varphi)^*a(\psi) = \langle\psi, \varphi\rangle_{\mathfrak{h}}\mathbf{1}. \quad (4)$$

As is usual, $a(\psi)$ and $a(\psi)^*$ refer to, respectively, annihilation and creation operators in the fermionic Fock space representation.

For all $\omega \in \Omega$ and $\lambda, \vartheta \in \mathbb{R}_0^+$, a dynamics on the C^*-algebra $\mathcal{U}$ is defined by the unique strongly continuous group $\tau^{(\omega)} \doteq (\tau_t^{(\omega)})_{t\in\mathbb{R}}$ of (Bogoliubov) $*$-automorphisms of $\mathcal{U}$ satisfying

$$\tau_t^{(\omega)}(a(\psi)) = a(\mathrm{e}^{ith^{(\omega)}}\psi), \quad t \in \mathbb{R}, \psi \in \mathfrak{h}. \quad (5)$$

See (3), as well as [Bratteli and Robinson 1997, Theorem 5.2.5], for more details on Bogoliubov automorphisms.

For any realization $\omega \in \Omega$ and disorder strengths $\lambda, \vartheta \in \mathbb{R}_0^+$, the thermal equilibrium state of the system at inverse temperature $\beta \in \mathbb{R}^+$ (i.e., $\beta > 0$) is by definition the unique $(\tau^{(\omega)}, \beta)$-KMS state $\varrho^{(\omega)}$, see [Bratteli and Robinson 1997, Example 5.3.2.] or [Pillet 2006, Theorem 5.9]. It is well known that such a state is stationary with respect to the dynamics $\tau^{(\omega)}$, that is,

$$\varrho^{(\omega)} \circ \tau_t^{(\omega)} = \varrho^{(\omega)}, \quad \omega \in \Omega, t \in \mathbb{R}.$$

The state $\varrho^{(\omega)}$ is also gauge-invariant, quasifree, and satisfies

$$\varrho^{(\omega)}(a^*(\varphi)a(\psi)) = \left\langle \psi, \frac{1}{1+\mathrm{e}^{\beta h^{(\omega)}}}\varphi \right\rangle_{\mathfrak{h}}, \quad \varphi, \psi \in \mathfrak{h}. \quad (6)$$

The gauge-invariant quasifree state with two-point correlation functions given by (6) for $\beta = 0$ is the tracial state (or chaotic state), denoted by $\mathrm{tr} \in \mathcal{U}^*$.

Recall that gauge-invariant quasifree states are positive linear functionals $\rho \in \mathcal{U}^*$ such that $\rho(\mathfrak{1}) = 1$ and, for all $N_1, N_2 \in \mathbb{N}$ and $\psi_1, \ldots, \psi_{N_1+N_2} \in \mathfrak{h}$,

$$\rho(a^*(\psi_1)\cdots a^*(\psi_{N_1})a(\psi_{N_1+N_2})\cdots a(\psi_{N_1+1})) = 0 \tag{7}$$

if $N_1 \neq N_2$, while in the case $N_1 = N_2 \equiv N$,

$$\rho(a^*(\psi_1)\cdots a^*(\psi_N)a(\psi_{2N})\cdots a(\psi_{N+1})) = \det\bigl[\rho(a^*(\psi_k)a(\psi_{N+l}))\bigr]_{k,l=1}^{N}. \tag{8}$$

See, e.g., [Araki 1970/71, Definition 3.1], which refers to a more general notion of quasifree states. The gauge-invariant property corresponds to (7) whereas [Araki 1970/71, Definition 3.1, Condition (3.1)] only imposes the quasifree state to be even, which is a strictly weaker property than being gauge-invariant.

2.3. *Linear response current density*. (i) Paramagnetic currents: Fix $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$. For any oriented edge $(x, y) \in (\mathbb{Z}^d)^2$, we define the paramagnetic[2] current observable by

$$I_{(x,y)}^{(\omega)} \doteq -2\Im\mathrm{m}\bigl(\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}\, a(\mathfrak{e}_x)^* a(\mathfrak{e}_y)\bigr), \tag{9}$$

where, as is usual, the real and imaginary parts of any element $A \in \mathcal{U}$ are respectively defined by

$$\Re\mathrm{e}(A) \doteq \frac{1}{2}(A + A^*) \quad \text{and} \quad \Im\mathrm{m}(A) \doteq \frac{1}{2i}(A - A^*). \tag{10}$$

The self-adjoint elements $I_{(x,y)}^{(\omega)} \in \mathcal{U}$ are seen as current observables, because they satisfy a discrete continuity equation, as explained in [Aza et al. 2019, Appendix C]. This "second-quantized" definition of a current observable and the usual one in the one-particle setting, as in [Schulz-Baldes and Bellissard 1998; Bouclet et al. 2005; Klein et al. 2007], are perfectly equivalent, in the case of noninteracting fermions. See for instance [Aza et al. 2019, Appendix C.3].

(ii) Conductivity: As is usual, $[A, B] \doteq AB - BA \in \mathcal{U}$ denotes the commutator between the elements $A \in \mathcal{U}$ and $B \in \mathcal{U}$. For any finite subset $\Lambda \subsetneq \mathbb{Z}^d$, we define the space-averaged transport coefficient observable $\mathcal{C}_\Lambda^{(\omega)} \in C^1(\mathbb{R}; \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d))$, with respect to the canonical basis $\{e_q\}_{q=1}^d$ of $\mathbb{R}^d$, by the corresponding matrix entries

$$\bigl\{\mathcal{C}_\Lambda^{(\omega)}(t)\bigr\}_{k,q} \doteq \frac{1}{|\Lambda|} \sum_{\substack{x,y\in\Lambda \\ x+e_k,\, y+e_q\in\Lambda}} \int_0^t i[\tau_{-\alpha}^{(\omega)}(I_{(y+e_q,y)}^{(\omega)}), I_{(x+e_k,x)}^{(\omega)}]\, d\alpha + \frac{2\delta_{k,q}}{|\Lambda|} \sum_{x\in\Lambda} \Re\mathrm{e}(\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle a(\mathfrak{e}_{x+e_k})^* a(\mathfrak{e}_x)) \tag{11}$$

[2]Diamagnetic currents correspond to the ballistic movement of charged particles driven by electric fields. Their presence leads to the progressive appearance of paramagnetic currents which are responsible for heat production. For more details, see [Bru and de Siqueira Pedra 2015a; Bru et al. 2015b; Bru and de Siqueira Pedra 2016] as well as [Aza et al. 2019, Appendix C] on linear response currents.

for any $\omega \in \Omega$, $t \in \mathbb{R}$, $\lambda, \vartheta \in \mathbb{R}_0^+$, and $k, q \in \{1, \dots, d\}$. It is the conductivity observable matrix associated with the lattice region Λ and time t. See [Aza et al. 2019, Appendix C]. In fact, the first term in the right-hand side of (11) corresponds to the paramagnetic coefficient, whereas the second one is the diamagnetic component. For more details, see [Bru and de Siqueira Pedra 2016, Theorem 3.7].

(iii) Linear response current density: Fix a direction $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and a (time-dependent) continuous, compactly supported, electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, i.e., the external electric field is a continuous function $t \mapsto \mathcal{E}(t) \in \mathbb{R}^d$ of time $t \in \mathbb{R}$, with compact support. Then, as it is explained in [Aza et al. 2019, Appendix C] as well as in [Bru and de Siqueira Pedra 2015a; 2016] [3], the space-averaged linear response current observable in the lattice region Λ and at time $t = 0$ in the direction $\vec{w}$ is equal to

$$\mathbb{I}_{\Lambda}^{(\omega,\mathcal{E})} \doteq \sum_{k,q=1}^{d} w_k \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \big\{\mathcal{C}_{\Lambda}^{(\omega)}(-\alpha)\big\}_{k,q} \, d\alpha. \tag{12}$$

By [Bru et al. 2016; Bru and de Siqueira Pedra 2015a], the macroscopic (linear response) current density produced by electric fields $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ at time $t = 0$ in the direction $\vec{w}$ is consequently equal to

$$x^{(\mathcal{E})} \doteq \lim_{L\to\infty} \mathbb{E}\big[\varrho^{(\cdot)}\big(\mathbb{I}_{\Lambda_L}^{(\cdot,\mathcal{E})}\big)\big] \in \mathbb{R}, \tag{13}$$

where $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d$ for any $L \in \mathbb{R}_0^+$. In order to obtain the current density at any time $t \in \mathbb{R}$ in the direction $\vec{w}$, it suffices to replace $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ in the last two equations with

$$\mathcal{E}_t(\alpha) \doteq \mathcal{E}(\alpha + t), \qquad \alpha \in \mathbb{R}. \tag{14}$$

For a short summary on response of quasifree fermion systems to electric fields, see [Aza et al. 2019, Appendix C].

2.4. ***Large deviations for microscopic current densities.*** Fix again a direction $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and a time-dependent electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$. Recall that $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d$ for any $L \in \mathbb{R}_0^+$. From [Bru et al. 2016; Bru and de Siqueira Pedra 2015a] combined with [Aza et al. 2019, Corollary 3.2], it follows that the distributions[4] of the microscopic current density observables $(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})_{L\in\mathbb{R}^+}$, in the state $\varrho^{(\omega)}$, weak* converge, for $\omega \in \Omega$ almost surely, to the delta distribution at the macroscopic value $x^{(\mathcal{E})}$, well-defined by (13). By [Aza et al. 2019, Corollary 3.5], the quantum uncertainty around the macroscopic value disappears *exponentially fast*, as $L \to \infty$.

To arrive at that conclusion, we use in [Aza et al. 2019] the large-deviation formalism for the microscopic (linear response) current density in the state $\varrho^{(\omega)}$. More precisely, we prove in [Aza et al. 2019, Corollary 3.2] that, almost surely[5] (or with probability one in Ω), for any Borel subset $\mathcal{G}$ of $\mathbb{R}$ with interior

[3] Strictly speaking, these papers use smooth electric fields, but the extension to the continuous case is straightforward.

[4] Here, as in [Aza et al. 2019], the distribution associated to a selfadjoint element A of a unital C^*-algebra $\mathfrak{A}$ and to a state on this algebra is the probability measure on the spectrum of A representing the restriction of the state to the unital C^*-subalgebra of $\mathfrak{A}$ generated by A. Recall that this measure exists and is unique, by the Riesz–Markov representation theorem.

[5] The measurable subset $\tilde{\Omega} \subseteq \Omega$ of full measure of [Aza et al. 2019, Corollary 3.2] does not depend on $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$.

and closure respectively denoted by $\mathcal{G}^\circ$ and $\bar{\mathcal{G}}$,

$$-\inf_{x\in\mathcal{G}^\circ} \mathrm{I}^{(\mathcal{E})}(x) \leq \liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)}\big(\mathbf{1}\big[\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})} \in \mathcal{G}\big]\big) \leq \limsup_{L\to\infty} \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)}\big(\mathbf{1}\big[\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})} \in \mathcal{G}\big]\big) \leq -\inf_{x\in\bar{\mathcal{G}}} \mathrm{I}^{(\mathcal{E})}(x).$$

By an abuse of notation[6], we applied above the (discontinuous) characteristic function $\mathbf{1}[x \in \mathcal{G}]$ to $\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}$. Here, by [Aza et al. 2019, Theorems 3.1, 3.4, and Corollary 3.2], the so-called good[7] rate function $\mathrm{I}^{(\mathcal{E})}$ is a deterministic, positive, lower-semicontinuous, convex function defined by

$$\mathrm{I}^{(\mathcal{E})}(x) \doteq \sup_{s\in\mathbb{R}}\big\{sx - \mathrm{J}^{(s\mathcal{E})}\big\} \geq 0, \qquad x \in \mathbb{R}, \tag{15}$$

where

$$\mathrm{J}^{(\mathcal{E})} \doteq \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\big[\ln \varrho^{(\cdot)}\big(\mathrm{e}^{|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\cdot,\mathcal{E})}}\big)\big] \in \mathbb{R} \tag{16}$$

for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. By [Aza et al. 2019, Theorem 3.4], $\mathrm{I}^{(\mathcal{E})}$ restricted to the interior of its domain is continuous and, as clearly expected, the rate function $\mathrm{I}^{(\mathcal{E})}$ vanishes on the macroscopic (linear response) current density $x^{(\mathcal{E})}$, i.e., $\mathrm{I}^{(\mathcal{E})}(x^{(\mathcal{E})}) = 0$, whereas $\mathrm{I}^{(\mathcal{E})}(x) > 0$ for all $x \neq x^{(\mathcal{E})}$.

For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, note that (15) means that $\mathrm{I}^{(\mathcal{E})}$ is the Legendre–Fenchel transform of the generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself, which is a well-defined, continuously differentiable, convex function, by [Aza et al. 2019, Theorem 3.1]. Moreover, by [Aza et al. 2019, Corollary 4.20 and Equation (54)], for any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the macroscopic current density defined by (13) can be expressed in terms of the generating function

$$x^{(\mathcal{E})} = \partial_s \mathrm{J}^{(s\mathcal{E})}\big|_{s=0}. \tag{17}$$

3. Main results

In order to provide a rather complete study of conductivity at the atomic scale for free-fermions in a lattice, we analyse here the rate function defined by (15) in much more detail than in [Aza et al. 2019]. See [Aza et al. 2019, Corollary 3.2]. We focus on the behavior of the rate function near the macroscopic value of the current density (see (17)), because it establishes a very interesting connection between exponential suppression of quantum uncertainties at the atomic scale and the concept of *quantum fluctuations*, in the case of currents.

3.1. ***Quantum fluctuations of linear response currents and rate function*** **.** For any inverse temperature $\beta \in \mathbb{R}^+$, disorder strengths $\vartheta, \lambda \in \mathbb{R}_0^+$, disorder realization $\omega \in \Omega$, direction $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and time-dependent electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, the quantum fluctuations of linear response currents in

[6]In fact, the object $\varrho^{(\omega)}\big(\mathbf{1}\big[\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})} \in \mathcal{G}\big]\big)$ can be easily given a precise mathematical sense by using the (up to unitary equivalence) unique cyclic representation of the C^*-algebra $\mathcal{U}$ associated to the state $\varrho^{(\omega)}$, noting that the bicommutant of a $*$-algebra in any representation is a von Neumann algebra, and thus admits a measurable calculus.

[7]It means, in this context, that $\{x \in \mathbb{R} : \mathrm{I}^{(\mathcal{E})}(x) \leq m\}$ is compact for any $m \geq 0$.

cubic boxes are defined to be

$$F_L^{(\omega,\mathcal{E})} \doteq |\Lambda_L| \Big(\varrho^{(\omega)}\big((\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})^2 \big) - \varrho^{(\omega)}\big(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}\big)^2 \Big) \geq 0, \quad L \in \mathbb{R}_0^+, \tag{18}$$

with $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d$ and $\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}(t)$ being the space-averaged linear response current defined by (12). Observe that

$$|\Lambda_L| \, \varrho^{(\omega)}\big(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}(t)\big), \quad L \in \mathbb{R}_0^+,$$

is the (total) current linear response (in the direction $\vec{w}$) to the electric field; and, consequently,

$$F_L^{(\omega,\mathcal{E})} = \frac{1}{|\Lambda_L|} \big(\varrho^{(\omega)}\big((|\Lambda_L| \, \mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})^2 \big) - \varrho^{(\omega)}\big(|\Lambda_L| \, \mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})} \big)^2 \big), \quad L \in \mathbb{R}_0^+, \tag{19}$$

are naturally seen as (normal) quantum fluctuations of the (total) linear response current. Note that these quantum fluctuations are not quite the same current fluctuations of [Bru et al. 2014; 2016], which correspond only to the paramagnetic component of the current, whereas $(F_L^{(\omega,\mathcal{E})})$ also includes the diamagnetic one, and thus refers to the total current.

Recall that $x^{(\mathcal{E})}$ is the macroscopic (linear response) current density defined by (13), and $\mathrm{I}^{(\mathcal{E})}$ (see (15)) is the (good) rate function associated with the large deviation principle of the sequence $\{\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}\}_{L \in \mathbb{R}^+}$ of microscopic current densities, in the KMS state $\varrho^{(\omega)}$ and with speed $|\Lambda_L|$. See, e.g., [Aza et al. 2019, Theorems 3.1, 3.4, and Corollary 3.2]. We are now in a position to connect the quantum fluctuations of (linear) currents with the generating and rate functions associated with the large-deviation principle for microscopic current densities.

Theorem 3.1 (Quantum fluctuations and rate function). *There is a measurable subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the following properties hold true*:

(i) *The generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ defined by (16) belongs to $C^\infty(\mathbb{R}; \mathbb{R})$ and satisfies*

$$\partial_s^2 \, \mathrm{J}^{(s\mathcal{E})}\big|_{s=0} = \lim_{L \to \infty} \mathbb{E}\big[F_L^{(\cdot,\mathcal{E})}\big] = \lim_{L \to \infty} F_L^{(\omega,\mathcal{E})} \geq 0. \tag{20}$$

(ii) *The rate function $I^{(\mathcal{E})}$ satisfies the asymptotics*

$$I^{(\mathcal{E})}(x) = \frac{1}{2\partial_s^2 \, \mathrm{J}^{(s\mathcal{E})}\, |_{s=0}} \big(x - x^{(\mathcal{E})}\big)^2 + o\big((x - x^{(\mathcal{E})})^2\big),$$

provided that $\partial_s^2 \, \mathrm{J}^{(s\mathcal{E})}\, |_{s=0} \neq 0$.

Proof. Fix all parameters of the theorem. By Corollary 4.2, the generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ belongs to $C^2(\mathbb{R}; \mathbb{R})$ and satisfies (20). As explained after Corollary 4.2, under the assumptions of Theorem 3.1, one can straightforwardly extend our arguments to prove that the generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ defined by (16) is infinitely differentiable. Assertion (i) thus holds true. It remains to prove Assertion (ii): Since the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is convex and belongs (at least) to $C^1(\mathbb{R}; \mathbb{R})$ (see, e.g., Assertion (i) or [Aza et al. 2019, Theorem 3.1]), all finite solutions $s(x) \in \mathbb{R}$ to the variational problem (15) for $x \in \mathbb{R}$, i.e.,

$$\mathrm{I}^{(\mathcal{E})}(x) = s(x)x - \mathrm{J}^{(s(x)\mathcal{E})}, \tag{21}$$

satisfy

$$x = f(s(x)), \tag{22}$$

with f being the real-valued function defined by

$$f(s) \doteq \partial_s \mathrm{J}^{(s\mathcal{E})}, \quad s \in \mathbb{R}. \tag{23}$$

Assume now that $\partial_s^2 \mathrm{J}^{(s\mathcal{E})}|_{s=0} \neq 0$, which is equivalent in this case to

$$\partial_s f(0) = \partial_s^2 \mathrm{J}^{(s\mathcal{E})}\big|_{s=0} > 0, \tag{24}$$

by positivity of fluctuations (see (i)). Since, by Corollary 4.2, the mapping $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself belongs (at least) to $C^2(\mathbb{R};\mathbb{R})$, by the inverse function theorem combined with (21)–(24) and (17), there is an open interval

$$\mathcal{I} \subseteq \{f(s) : s \in \mathbb{R} \text{ such that } \partial_s f(s) > 0\} \subseteq \mathbb{R}$$

containing $x^{(\mathcal{E})} = f(0)$ and a C^1-function $x \mapsto s(x)$ from $\mathcal{I}$ to $\mathbb{R}$ such that (21)–(23) hold true. In particular,

$$\partial_s f(s(x)) = \partial_s^2 \mathrm{J}^{(s\mathcal{E})}\big|_{s=s(x)} > 0, \quad x \in \mathcal{I}. \tag{25}$$

Clearly,

$$\partial_x s(x) = \frac{1}{\partial_s f(s(x))}, \quad x \in \mathcal{I}. \tag{26}$$

We thus infer from (21)–(23) and (26), together with (i), that

$$\partial_x \mathrm{I}^{(\mathcal{E})}(x) = s(x), \quad x \in \mathcal{I}.$$

Consequently, $\partial_x \mathrm{I}^{(\mathcal{E})}$ is differentiable on $\mathcal{I}$ with derivative given by

$$\partial_x^2 \mathrm{I}^{(\mathcal{E})}(x) = \partial_x s(x), \quad x \in \mathcal{I}.$$

As a consequence, $\mathrm{I}^{(\mathcal{E})}$ is twice differentiable on $\mathcal{I} \supseteq \{x^{(\mathcal{E})}\}$ and, using the Taylor theorem at the point $x^{(\mathcal{E})}$, one obtains that

$$\mathrm{I}^{(\mathcal{E})}(x) = s(x^{(\mathcal{E})})(x - x^{(\mathcal{E})}) + \frac{1}{2}\partial_x s(x^{(\mathcal{E})})(x - x^{(\mathcal{E})})^2 + o\big((x - x^{(\mathcal{E})})^2\big), \tag{27}$$

provided (24) holds true. Since, by (17), (23), and (26), $s(x^{(\mathcal{E})}) = 0$ and

$$\partial_x s(x^{(\mathcal{E})}) = \frac{1}{\partial_s f(0)} = \frac{1}{\partial_s^2 \mathrm{J}^{(s\mathcal{E})}}\Big|_{s=0},$$

one thus deduces (ii) from (27). □

This theorem is a very interesting observation on the physics of fermionic systems, because it shows that the experimental measure of the rate function of currents around the expected value leads to an experimental estimate on the corresponding quantum fluctuations. Conversely, by Theorem 3.1, an experimental estimate on these quantum fluctuations gives the behavior of the corresponding rate function around the expected value. This phenomenon is certainly not restricted to fermionic currents, and this is a new observation on transport properties of quantum many-body systems, to our knowledge.

Remark 3.2 (Extension of Theorem 3.1). The proof of Theorem 3.1 can be generalized to very general kinetic terms (i.e., it does not really depend on the special choice $\Delta_{\omega,\vartheta}$), provided the pivotal Combes–Thomas estimate holds true for the one-particle Hamiltonian. Note, however, that this would require a new, more complicated, definition of currents, which results from the commutator of the density operator at fixed lattice site with the kinetic term (cf. continuity equations on the CAR algebra [Bru and de Siqueira Pedra 2016, Equations (38)–(39)]). We did not implement this generalization here, because we think that, conceptually, the gain is too small as compared to the drawbacks concerning notations, definitions, and technical proofs. Instead, we aim at obtaining an extension of Theorem 3.1 to weakly interacting fermionic systems by using new constructive methods based on Grassmann–Berezin integrals, Brydges–Kennedy expansions, etc.

3.2. ***Nonvanishing quantum fluctuations of linear response currents*.** By Theorem 3.1, the behavior of the rate function within a neighborhood of the macroscopic current densities is directly related to the quantum fluctuations of the linear response current, provided these fluctuations do not vanish in the thermodynamic limit, i.e., if $\partial_s^2 \mathrm{J}^{(s\mathcal{E})}|_{s=0} \neq 0$ (see Theorem 3.1(i)). We do not expect this situation to appear in presence of disorder. We discuss this issue in Section 4.4, where we give sufficient conditions ensuring nonvanishing quantum fluctuations of linear response currents in the thermodynamic limit. This study leads to the following theorem:

Theorem 3.3 (Sufficient conditions for nonzero quantum fluctuations). *Take $\vartheta, \lambda \in \mathbb{R}_0^+$, $T, \beta \in \mathbb{R}^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ with support in $[-T, 0]$, and $\vec{w} \doteq (w_1, \dots, w_d) \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. Assume that the random variables $\{\omega_1(z)\}_{z\in\mathbb{Z}^d}$ are independently and identically distributed (i.i.d.). Then, for sufficiently small T and ϑ,*

$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})}\big|_{s=0} \geq \frac{\lambda^2 \Upsilon^{(\mathcal{E},\vec{w})}}{\left(1+\mathrm{e}^{\beta(2d(2+\vartheta)+\lambda)}\right)^2} \operatorname{Var}[(\,\cdot\,)_1(0)],$$

with

$$\Upsilon^{(\mathcal{E},\vec{w})} \doteq \left(\int_{-\infty}^{0} \langle w, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d} \alpha^2 \, d\alpha\right)^2 + \frac{1}{2}\sum_{k=1}^{d}\left(w_k \int_{-\infty}^{0} (\mathcal{E}(\alpha))_k \alpha^2 \, d\alpha\right)^2.$$

In particular, $\partial_s^2 \mathrm{J}^{(s\mathcal{E})}|_{s=0} \neq 0$ whenever $\Upsilon^{(\mathcal{E},\vec{w})} > 0$, $\omega_1(0)$ is not almost surely constant (and thus, $\operatorname{Var}[(\,\cdot\,)_1(0)] > 0$, by Chebyshev's inequality), and T, ϑ are sufficiently small.

Proof. This is a direct consequence of (66) and (68) in Section 4. □

By Theorems 3.1 and 3.3, we thus demonstrate that, in general, the quantum fluctuations of linear response currents do not vanish in the thermodynamic limit, and the quantum uncertainty around the macroscopic current density $x^{(\mathcal{E})}$ disappears exponentially fast, as the volume of the cubic box Λ_L grows, with a rate proportional to the squared deviation of the current from $x^{(\mathcal{E})}$ and the inverse current fluctuation. In particular, by combining Theorem 3.1(i) with Theorem 3.3 we can obtain an *explicit* upper bound on the rate function $\mathrm{I}^{(\mathcal{E})}$ around $x^{(\mathcal{E})}$.

The fact that the random variables $\{\omega_1(z)\}_{z\in\mathbb{Z}^d}$ are independently and identically distributed (i.i.d.) in Theorem 3.3 is not essential here: For any $\omega\in\Omega$, let $w^{(\omega)}\doteq(w_1^{(\omega)},\ldots,w_d^{(\omega)})\in\mathbb{R}^d$ be the random vector defined by

$$w_k^{(\omega)}\doteq(2\omega_1(0)-\omega_1(e_k)-\omega_1(-e_k))w_k,\quad k\in\{1,\ldots,d\},$$

with $\{e_k\}_{k=1}^d$ being the canonical basis of $\mathbb{R}^d$. By (64), (66), and (67), it suffices that

$$\mathbb{E}\left[\left|\int_{-\infty}^{0}\langle w^{(\cdot)},\mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\,d\alpha\right|^2\right]=\mathrm{Var}\left[\int_{-\infty}^{0}\langle w^{(\cdot)},\mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\,d\alpha\right]>0$$

in order to ensure nonvanishing quantum fluctuations of linear response currents in the thermodynamic limit, i.e., $\partial_s^2\,\mathrm{J}^{(s\mathcal{E})}\,|_{s=0}\neq 0$.

Theorem 3.3 can be applied to the celebrated tight-binding Anderson model, which corresponds to the special case $\vartheta=0$. This is why we focus on this important example in this theorem. The remaining case of larger parameters $\vartheta, T\in\mathbb{R}_0^+$ can certainly be studied, even if this is not done here.

4. Technical proofs

4.1. ***Quasifree fermions in subregions of the lattice*.** Let $\mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d)\subseteq 2^{\mathbb{Z}^d}$ be the set of all nonempty *finite* subsets of $\mathbb{Z}^d$. Like in [Aza et al. 2019, Section 2.1], we need the sets

$$\mathfrak{Z}\doteq\{\mathcal{Z}\subseteq 2^{\mathbb{Z}^d}:(\forall Z_1,Z_2\in\mathcal{Z})\ Z_1\neq Z_2\Rightarrow Z_1\cap Z_2=\varnothing\},$$
$$\mathfrak{Z}_{\mathrm{f}}\doteq\mathfrak{Z}\cap\{\mathcal{Z}\subseteq\mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d):|\mathcal{Z}|<\infty\}.$$

This kind of decomposition over collections of disjoint subsets of the lattice is important to prove Theorem 3.1(i).

Recall that $\mathfrak{h}\doteq\ell^2(\mathbb{Z}^d;\mathbb{C})$, and $\mathcal{B}(\mathfrak{h})$ is the Banach space of all bounded linear operators acting on $\mathfrak{h}$. One can restrict the quasifree dynamics defined by (5) to collections $\mathcal{Z}\in\mathfrak{Z}$ of disjoint subsets of the lattice by using the orthogonal projections P_Λ, $\Lambda\subseteq\mathbb{Z}^d$, defined on the Hilbert space $\mathfrak{h}$ by

$$[P_\Lambda(\psi)](x)\doteq\begin{cases}\psi(x) & \text{if } x\in\Lambda,\\ 0 & \text{else,}\end{cases}\tag{28}$$

for any $\psi\in\mathfrak{h}$. Then, the one-particle Hamiltonian within $\mathcal{Z}\in\mathfrak{Z}$ is, by definition, equal to

$$h_{\mathcal{Z}}^{(\omega)}\doteq\sum_{Z\in\mathcal{Z}}P_Z h^{(\omega)}P_Z\in\mathcal{B}(\mathfrak{h}),\tag{29}$$

where $h^{(\omega)}\in\mathcal{B}(\mathfrak{h})$ is the random tight-binding model defined by (3) for any $\omega\in\Omega$ and $\lambda,\vartheta\in\mathbb{R}_0^+$. For any $\mathcal{Z}\in\mathfrak{Z}$, it leads to the unitary group $\{\mathrm{e}^{ith_{\mathcal{Z}}^{(\omega)}}\}_{t\in\mathbb{R}}$ acting on the Hilbert space $\mathfrak{h}$.

Similar to (5), for any $\mathcal{Z}\in\mathfrak{Z}$, we consequently define the strongly continuous group $\tau^{(\omega,\mathcal{Z})}\doteq\{\tau_t^{(\omega,\mathcal{Z})}\}_{t\in\mathbb{R}}$ of Bogoliubov $*$-automorphisms of $\mathcal{U}$ by

$$\tau_t^{(\omega,\mathcal{Z})}(a(\psi))=a(\mathrm{e}^{ith_{\mathcal{Z}}^{(\omega)}}\psi),\quad t\in\mathbb{R},\psi\in\mathfrak{h}.$$

This corresponds to replace $h^{(\omega)}$ in (5) with $h_{\mathcal{Z}}^{(\omega)}$. Similarly, for any $\mathcal{Z}\in\mathfrak{Z}$, we define the quasifree state $\varrho_{\mathcal{Z}}{}^{(\omega)}$ by replacing $h^{(\omega)}$ in (6) with the one-particle Hamiltonian $h_{\mathcal{Z}}{}^{(\omega)}$ within $\mathcal{Z}$.

If $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$, then both $\tau^{(\omega,\mathcal{Z})}$ and $\varrho_{\mathcal{Z}}^{(\omega)}$ can be written in terms of bilinear elements[8], defined as follows: The bilinear element associated with an operator in $C \in \mathcal{B}(\mathfrak{h})$ whose range, $\mathrm{ran}(C)$, is finite dimensional is defined by

$$\langle \mathrm{A}, C\,\mathrm{A}\rangle \doteq \sum_{i,j\in I} \langle \psi_i, C\psi_j\rangle_{\mathfrak{h}}\, a(\psi_i)^* a(\psi_j), \tag{30}$$

where $\{\psi_i\}_{i\in I}$ is any orthonormal basis[9] of a finite dimensional subspace

$$\mathcal{H} \supseteq \mathrm{ran}(C) \cup \mathrm{ran}(C^*)$$

of the Hilbert space $\mathfrak{h}$. See [Aza et al. 2019, Definition 4.3]. For any $\omega \in \Omega$ and $\lambda, \vartheta \in \mathbb{R}_0^+$, the range of $h_{\mathcal{Z}}^{(\omega)} \in \mathcal{B}(\mathfrak{h})$ is finite dimensional whenever $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$ and one checks that, for any time $t \in \mathbb{R}$, inverse temperature $\beta \in \mathbb{R}^+$, finite collections $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$ and elements $B \in \mathcal{U}$,

$$\tau_t^{(\omega,\mathcal{Z})}(B) = \mathrm{e}^{it\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle}\, B\, \mathrm{e}^{-it\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle} \quad \text{and} \quad \varrho_{\mathcal{Z}}^{(\omega)}(B) = \frac{\mathrm{tr}\left(B\, \mathrm{e}^{-\beta\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle}\right)}{\mathrm{tr}\left(\mathrm{e}^{-\beta\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle}\right)},$$

where $\mathrm{tr} \in \mathcal{U}^*$ is the tracial state, i.e., the gauge-invariant quasifree state with two-point correlation functions given by (6) for $\beta = 0$. See [Aza et al. 2019, Equations (27)–(28)]. The dynamics corresponds in this case to the usual dynamics written in the Heisenberg picture of quantum mechanics, while the above quasifree state is the Gibbs state at inverse temperature $\beta \in \mathbb{R}^+$, both associated with the Hamiltonian $\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle \in \mathcal{U}$ for $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$.

In order to define the thermodynamic limit, we use the cubic boxes $\Lambda_\ell \doteq \{\mathbb{Z} \cap [-\ell, \ell]\}^d$ for $\ell \in \mathbb{R}_0^+$. Then, as $\ell \to \infty$, for any $t \in \mathbb{R}$, $\tau_t^{(\omega,\{\Lambda_\ell\})}$ converges strongly to $\tau_t^{(\omega)} \equiv \tau_t^{(\omega,\{\mathbb{Z}^d\})}$, while $\varrho_{\{\Lambda_\ell\}}^{(\omega)}$ converges in the weak* topology to $\varrho^{(\omega)} \equiv \varrho_{\{\mathbb{Z}^d\}}^{(\omega)}$. For an explicit proof of these well-known facts, see for instance [Ratsimanetrimanana 2019, Propositions 3.2.9 and 3.2.13].

4.2. ***Current observables in subregions of the lattice .*** Fix once and for all $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. By [Aza et al. 2019, Equation (29)], for any $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$, and $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}$, the linear response current observable is, by definition, equal to

$$\begin{aligned}\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq \sum_{k,q=1}^{d} w_k \sum_{Z\in\mathcal{Z}} \sum_{\substack{x,y\in Z\\ x+e_k,\, y+e_q\in Z}} \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q\, d\alpha \int_0^{-\alpha} ds\, i[\tau_{-s}^{(\omega,\mathcal{Z}^{(\tau)})}(I_{(y+e_q,y)}^{(\omega)}), I_{(x+e_k,x)}^{(\omega)}] \\ +2\sum_{k=1}^{d} w_k \sum_{Z\in\mathcal{Z}} \sum_{x,x+e_k\in Z} \left(\int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q\, d\alpha\right) \Re\mathrm{e}(\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle a(\mathfrak{e}_{x+e_k})^* a(\mathfrak{e}_x)),\end{aligned} \tag{31}$$

with $\{e_k\}_{k=1}^d$ being the canonical basis of $\mathbb{R}^d$. Recall that $\Re\mathrm{e}(A) \in \mathcal{U}$ is the real part of $A \in \mathcal{U}$, see (10). Note from (11)–(12) that

$$\mathfrak{K}_{\{\Lambda\},\{\mathbb{Z}^d\}}^{(\omega,\mathcal{E})} = |\Lambda|\, \mathbb{I}_{\Lambda}^{(\omega,\mathcal{E})}, \qquad \Lambda \in \mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d), \tag{32}$$

are linear response current observables within finite subsets of the lattice.

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[8] This refers to the well-known second-quantization of one-particle Hamiltonians in the Fock space representation.

[9] $\langle \mathrm{A}, C\,\mathrm{A}\rangle$ does not depend on the particular choice of $\mathcal{H}$ and its orthonormal basis.

The above current observables can obviously be rewritten as bilinear elements (30) associated with one-particle operators acting on the Hilbert space $\mathfrak{h}$. In order to give an explicit expression of these operators, we first define, for any $x \in \mathbb{Z}^d$, the shift operator $s_x \in \mathcal{B}(\mathfrak{h})$ by

$$(s_x \psi)(y) \doteq \psi(x+y), \quad y \in \mathbb{Z}^d, \psi \in \mathfrak{h}. \tag{33}$$

Note that $s_x^* = s_{-x} = s_x^{-1}$ for any $x \in \mathbb{Z}^d$. Then, for every $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$, the single-hopping operators are

$$S_{x,y}^{(\omega)} \doteq \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} \, P_{\{x\}} s_{x-y} P_{\{y\}}, \quad x, y \in \mathbb{Z}^d, \tag{34}$$

where $P_{\{u\}}$ is the orthogonal projection defined by (28) for $\Lambda = \{u\}$ and $u \in \mathbb{Z}^d$. Observe that

$$\langle \mathrm{A}, S_{x,y}^{(\omega)} \mathrm{A} \rangle = \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} \, a(\mathfrak{e}_x)^* a(\mathfrak{e}_y), \quad x, y \in \mathbb{Z}^d.$$

Similarly, by the identity

$$\Im\mathrm{m}\{\langle \mathrm{A}, C\, \mathrm{A} \rangle\} = \langle \mathrm{A}, \Im\mathrm{m}\{C\}\, \mathrm{A} \rangle$$

for any $C \in \mathcal{B}(\mathfrak{h})$ whose range is finite dimensional, the paramagnetic current observables defined by (9) equals

$$I_{(x,y)}^{(\omega)} = -2\langle \mathrm{A}, \Im\mathrm{m}\{S_{x,y}^{(\omega)}\}\, \mathrm{A} \rangle, \quad x, y \in \mathbb{Z}^d,$$

for each $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$. For any $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}$, and $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$, the current observable (31) can then be rewritten as

$$\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} = \langle \mathrm{A}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathrm{A} \rangle = \sum_{x,y \in \mathbb{Z}^d} \langle \mathfrak{e}_x, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_y \rangle_{\mathfrak{h}} \, a(\mathfrak{e}_x)^* a(\mathfrak{e}_y), \tag{35}$$

where $K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \in \mathcal{B}(\mathfrak{h})$ is the operator acting on the one-particle Hilbert space $\mathfrak{h}$ defined by

$$\begin{aligned} K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq 4 \sum_{k,q=1}^{d} w_k \sum_{Z \in \mathcal{Z}} \sum_{\substack{x,y \in Z \\ x+e_k, y+e_q \in Z}} & \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \, d\alpha \\ & \times \int_0^{-\alpha} ds\, i\left[\mathrm{e}^{-ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} \Im\mathrm{m}\{S_{y+e_q,y}^{(\omega)}\}\, \mathrm{e}^{ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}}, \Im\mathrm{m}\{S_{x+e_k,x}^{(\omega)}\}\right] \\ & + 2\sum_{k=1}^{d} w_k \sum_{Z \in \mathcal{Z}} \sum_{x, x+e_k \in Z} \left(\int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \, d\alpha\right) \Re\mathrm{e}\{S_{x+e_k,x}^{(\omega)}\}. \end{aligned} \tag{36}$$

Note that the range of this bounded and self-adjoint operator is finite dimensional whenever $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$.

4.3. ***Differentiability class of generating functions .*** The aim of this section is to prove Theorem 3.1(i), in particular that the generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ defined by (16) belongs to $C^2(\mathbb{R}; \mathbb{R})$. By [Aza et al. 2019, Theorem 3.1], we already know that it is a well-defined, continuously differentiable, convex function. So, one has to prove here that the second derivative of the generating function exists and is continuous. To arrive at this assertion, we follow the argument lines of [Aza et al. 2019, Section 4], showing [Aza et al.

2019, Theorem 3.1] via the control of the thermodynamic limit of finite-volume generating functions that are random.

Fix once and for all $\beta \in \mathbb{R}^+$, $\lambda, \vartheta \in \mathbb{R}_0^+$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\omega \in \Omega$, and three finite collections $\mathcal{Z}, \mathcal{Z}^{(\varrho)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_{\mathrm{f}}$, we define the finite-volume generating function

$$\mathrm{J}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} \doteq g^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} - g^{(\omega,0)}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}, \tag{37}$$

where

$$g^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} \doteq \frac{1}{|\cup\mathcal{Z}|} \ln \operatorname{tr}\big(\exp(-\beta\langle \mathrm{A}, h^{(\omega)}_{\mathcal{Z}^{(\varrho)}}\mathrm{A}\rangle)\exp(\mathfrak{K}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}})\big). \tag{38}$$

Recall that the tracial state $\operatorname{tr} \in \mathcal{U}^*$ is the gauge-invariant quasifree state with two-point correlation function given by (6) for $\beta = 0$, while $h^{(\omega)}_{\mathcal{Z}^{(\varrho)}}$ is the one-particle Hamiltonian defined by (29). See also (30) and (31). Compare (37)–(38) with the equalities

$$\begin{aligned}\mathrm{J}^{(\mathcal{E})} &\doteq \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\mathbb{E}\big[\ln \varrho^{(\cdot)}\big(\mathrm{e}^{|\Lambda_L|\mathbb{I}^{(\cdot,\mathcal{E})}_{\Lambda_L}}\big)\big] \\ &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\ln \varrho^{(\omega)}\big(\mathrm{e}^{|\Lambda_L|\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big) = \lim_{L\to\infty}\lim_{L_\varrho\to\infty}\lim_{L_\tau\to\infty} \mathrm{J}^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}},\end{aligned} \tag{39}$$

where the random variable ω is in a measurable subset of full measure [10], by [Aza et al. 2019, Theorem 3.1 and Equation (45)]. Recall that $\Lambda_\ell \doteq \{\mathbb{Z} \cap [-\ell, \ell]\}^d$ for $\ell \in \mathbb{R}_0^+$. See again (16) for the definition of the generating function. In fact, by [Aza et al. 2019, Proposition 4.10], the above local generating functions can be approximately decomposed into boxes of fixed volume, and we use the Akcoglu–Krengel (superadditive) ergodic theorem [Aza et al. 2019, Theorem 4.17] to deduce, via [Aza et al. 2019, Proposition 4.8], the existence of the generating functions as the thermodynamic limit of finite-volume generating functions, as given in (39).

In order to prove that the generating function is continuously differentiable, one uses in [Aza et al. 2019, Corollary 4.20] the (Arzelà–)Ascoli theorem [Rudin 1991, Theorem A5]. This approach requires uniform bounds on the first and second derivatives of the finite-volume generating functions

$$s \mapsto \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}, \quad \mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d),\ \omega \in \Omega,\ \mathcal{Z}, \mathcal{Z}^{(\varrho)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_{\mathrm{f}}. \tag{40}$$

This is done in [Aza et al. 2019, Proposition 4.9], which establishes the following: Fixing $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\beta_1, s_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$, one has

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1] \\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}} \Big\{\big|\partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\big| + \big|\partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\big|\Big\} < \infty. \tag{41}$$

In order to get in the same way the existence and continuity of the second derivative of the generating function, we need to control the *third*-order derivative of the same finite-volume generating functions (40).

[10] The measurable subset $\tilde{\Omega} \subseteq \Omega$ of full measure of [Aza et al. 2019, Theorem 3.1] does not depend on $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$.

Equation (41) is proved by using the CAR (4) and the Combes–Thomas estimate [Aza et al. 2019, Appendix A], in particular the bound

$$\sup_{\lambda\in\mathbb{R}_0^+}\ \sup_{\mathcal{Z}\in\mathfrak{Z}}\ \sup_{\omega\in\Omega}\left|\langle\mathfrak{e}_x, \mathrm{e}^{ith_{\mathcal{Z}}^{(\omega)}}\mathfrak{e}_y\rangle_{\mathfrak{h}}\right| \leq 36\,\mathrm{e}^{|t\eta|-2\mu_\eta|x-y|}, \qquad x, y \in \mathbb{Z}^d,\ \vartheta\in\mathbb{R}_0^+,\ t\in\mathbb{R}, \tag{42}$$

(see [Aza et al. 2019, Equation (7)]), where

$$\mu_\eta \doteq \mu\min\left\{\frac{1}{2}, \frac{\eta}{8d(1+\vartheta)\,\mathrm{e}^{\mu}}\right\}, \tag{43}$$

the parameters $\eta, \mu \in \mathbb{R}^+$ being two arbitrarily fixed (strictly positive) constants. For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\beta_1, s_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$, the Combes–Thomas estimate leads also to the uniform estimates

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1]\\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}}\ \sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\left|\left\langle\mathfrak{e}_y, \frac{1}{1+\mathrm{e}^{-\frac{s}{2}K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}}\,\mathrm{e}^{\beta h_{\mathcal{Z}^{(\varrho)}}^{(\omega)}}\,\mathrm{e}^{-\frac{s}{2}K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}}}\mathfrak{e}_x\right\rangle_{\mathfrak{h}}\right| < \infty \tag{44}$$

(see the end of the proof of [Aza et al. 2019, Proposition 4.9]), as well as

$$\sup_{\vartheta\in[0,\vartheta_1]}\ \sup_{\lambda\in\mathbb{R}_0^+}\ \sup_{\mathcal{Z},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}\ \sup_{\omega\in\Omega}\frac{1}{|\cup\mathcal{Z}|}\sum_{x,y\in\mathbb{Z}^d}\left|\langle\mathfrak{e}_x, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_y\rangle_{\mathfrak{h}}\right| < \infty \tag{45}$$

and

$$\sup_{\vartheta\in[0,\vartheta_1]}\ \sup_{\lambda\in\mathbb{R}_0^+}\ \sup_{\mathcal{Z},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}\ \sup_{\omega\in\Omega}\left|\langle\mathfrak{e}_x, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_y\rangle_{\mathfrak{h}}\right| \leq C_{x,y}^{(\mathcal{E},\vartheta_1)} < \infty \tag{46}$$

for $x, y \in \mathbb{Z}^d$, where $C_{x,y}^{(\mathcal{E},\vartheta_1)} \in \mathbb{R}^+$ are constants satisfying

$$\sup_{x,y\in\mathbb{Z}^d} C_{x,y}^{(\mathcal{E},\vartheta_1)} < \infty \quad\text{and}\quad \sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} C_{x,y}^{(\mathcal{E},\vartheta_1)} < \infty. \tag{47}$$

Recall that $K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \in \mathcal{B}(\mathfrak{h})$ is the operator defining linear response current observables, by (35)–(36).

In order to give a uniform estimate on the third-order derivative of the finite-volume generating functions (40), similar to the proof of (41), we use again the Combes–Thomas estimate, which yields (44)–(47). This proof bears, however, on more complex computations than the one of (41), which only controls the first and second derivatives of the same function.

Proposition 4.1 (Uniform boundedness of third derivatives). *Fix an electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and the parameters $\beta_1, s_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$. Then,*

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1]\\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}}\left|\partial_s^3\,\mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})}\right| < \infty.$$

Proof. For any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and $\mathcal{Z}, \mathcal{Z}^{(\varrho)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_{\mathrm{f}}$, a straightforward computation yields that

$$\begin{aligned}\partial_s^3 \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})} &= \frac{1}{|\cup\mathcal{Z}|} \varpi_s^T \big(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}; \mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}; \mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\big) \\ &= \frac{1}{|\cup\mathcal{Z}|} \Big(\varpi_s\big(\big(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\big)^3\big) - 3\varpi_s\big(\big(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\big)^2\big)\varpi_s\big(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\big) + 2\varpi_s\big(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\big)^3\Big),\end{aligned} \tag{48}$$

where ϖ_s is the (unique) gauge-invariant quasifree state satisfying

$$\varpi_s(a^*(\varphi)a(\psi)) = \left\langle \psi, \frac{1}{1+\mathrm{e}^{-\frac{s}{2}K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}} \mathrm{e}^{\beta h_{\mathcal{Z}^{(\varrho)}}^{(\omega)}} \mathrm{e}^{-\frac{s}{2}K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}}} \varphi \right\rangle_{\mathfrak{h}}, \qquad \varphi, \psi \in \mathfrak{h}. \tag{49}$$

In the first equality of (48), $\varpi_s^T(\cdot\,;\,\cdot\,;\,\cdot)$ denotes the so-called "truncated" or "connected" correlation function of third order, associated with the state ϖ_s. Recall that, for all $A_1, A_2, A_3 \in \mathcal{U}$, this function is defined by

$$\begin{aligned}\varpi_s^T(A_1; A_2; A_3) \doteq \varpi_s(A_1A_2A_3) - \varpi_s(A_1)\varpi_s(A_2A_3) - \varpi_s(A_2)\varpi_s(A_1A_3) \\ - \varpi_s(A_3)\varpi_s(A_1A_2) + 2\varpi_s(A_1)\varpi_s(A_2)\varpi_s(A_3).\end{aligned}$$

(This is similar to [Aza et al. 2019, Proof of Proposition 4.9, until Equation (48)].) Recall that $\{\mathfrak{e}_x\}_{x\in\mathbb{Z}^d}$ is the canonical orthonormal basis of $\mathfrak{h}$, which is defined by $\mathfrak{e}_x(y) \doteq \delta_{x,y}$ for all $x, y \in \mathbb{Z}^d$. By linearity and continuity in each argument of $\varpi_s^T(\cdot\,;\,\cdot\,;\,\cdot)$, one has

$$\begin{aligned}\partial_s^3 \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})} = \frac{1}{|\cup\mathcal{Z}|} \sum_{\substack{x_i, y_i \in \mathbb{Z}^d \\ i\in\{1,2,3\}}} \langle \mathfrak{e}_{x_1}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_{y_1}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x_2}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_{y_2}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x_3}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_{y_3}\rangle_{\mathfrak{h}} \\ \times \varpi_s^T(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1}); a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2}); a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3})).\end{aligned}$$

Note that, by (8) and the fact that ϖ_s is a gauge-invariant quasifree state,

$$\begin{aligned}&\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3})) \\ &= \det\begin{pmatrix} \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1})) & \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_2})) & \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3})) \\ -\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2})) & \varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2})) & \varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_3})) \\ -\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_3})) & -\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3})) & \varpi_s(a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3})) \end{pmatrix} = \sum_{g\in\mathcal{G}_3} \xi_s^g(x_1, y_1, x_2, y_2, x_3, y_3)\end{aligned}$$

(use, for instance, [Bru and de Siqueira Pedra 2017b, Lemma 3.1] to get the above determinant), where

$$\begin{aligned}\mathcal{G}_3 \doteq \{\{(1,1),(2,2),(3,3)\},\{(1,1),(2,3),(3,2)\},\{(1,2),(2,1),(3,3)\}\} \\ \cup\{\{(1,2),(2,3),(3,1)\},\{(1,3),(2,1),(3,2)\},\{(1,3),(2,2),(3,1)\}\}\end{aligned}$$

is a set of oriented graphs with vertex set {1, 2, 3} and

$$\begin{aligned}
\xi_s^{\{(1,1),(2,2),(3,3)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2}))\varpi_s(a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3})),\\
\xi_s^{\{(1,1),(2,3),(3,2)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3})),\\
\xi_s^{\{(1,2),(2,1),(3,3)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_2}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))\varpi_s(a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3})),\\
\xi_s^{\{(1,2),(2,3),(3,1)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq -\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_2}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_3})),\\
\xi_s^{\{(1,3),(2,1),(3,2)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3})),\\
\xi_s^{\{(1,3),(2,2),(3,1)\}}(x_1,y_1,x_2,y_2,x_3,y_3) &\doteq \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_3})).
\end{aligned}$$

By elementary computations, one sees that taking connected correlations corresponds here, as is usual, to only keep the terms associated with connected graphs. That is,

$$\begin{aligned}
&\varpi_s^T(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1}); a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_2}); a^*(\mathfrak{e}_{x_3})a(\mathfrak{e}_{y_3}))\\
&\qquad = \varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3}))\\
&\qquad\qquad -\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_2}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_3})).
\end{aligned}$$

Hence,

$$\partial_s^3 \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})} = \boldsymbol{K}_1 - \boldsymbol{K}_2, \tag{50}$$

where

$$\begin{aligned}
\boldsymbol{K}_1 \doteq \frac{1}{|\cup\mathcal{Z}|}\sum_{\substack{x_i,y_i\in\mathbb{Z}^d\\ i\in\{1,2,3\}}} &\langle \mathfrak{e}_{x_1}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_1}\rangle_{\mathfrak{h}}\,\langle \mathfrak{e}_{x_2}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_2}\rangle_{\mathfrak{h}}\,\langle \mathfrak{e}_{x_3}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_3}\rangle_{\mathfrak{h}}\\
&\times\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3}))
\end{aligned}\tag{51}$$

and

$$\begin{aligned}
\boldsymbol{K}_2 \doteq \frac{1}{|\cup\mathcal{Z}|}\sum_{\substack{x_i,y_i\in\mathbb{Z}^d\\ i\in\{1,2,3\}}} &\langle \mathfrak{e}_{x_1}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_1}\rangle_{\mathfrak{h}}\,\langle \mathfrak{e}_{x_2}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_2}\rangle_{\mathfrak{h}}\,\langle \mathfrak{e}_{x_3}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_3}\rangle_{\mathfrak{h}}\\
&\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_2}))\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_3}))\varpi_s(a^*(\mathfrak{e}_{x_2})a(\mathfrak{e}_{y_3})).
\end{aligned}\tag{52}$$

Applying the triangle inequality, we now obtain that

$$\begin{aligned}
|\boldsymbol{K}_1| \leq \frac{1}{|\cup\mathcal{Z}|}\sum_{\substack{x_i,y_i\in\mathbb{Z}^d\\ i\in\{1,2,3\}}} &|\langle \mathfrak{e}_{x_1}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_1}\rangle_{\mathfrak{h}}|\,|\langle \mathfrak{e}_{x_2}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_2}\rangle_{\mathfrak{h}}|\,|\langle \mathfrak{e}_{x_3}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_3}\rangle_{\mathfrak{h}}|\\
&|\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))|\,|\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))|\,|\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3}))|\\
\leq \sup_{x_3,y_3\in\mathbb{Z}^d}|\langle \mathfrak{e}_{x_3}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_3}\rangle_{\mathfrak{h}}| &\sup_{x_2\in\mathbb{Z}^d}\sum_{y_2\in\mathbb{Z}^d}|\langle \mathfrak{e}_{x_2}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_2}\rangle_{\mathfrak{h}}|\frac{1}{|\cup\mathcal{Z}|}\sum_{x_1,y_1\in\mathbb{Z}^d}|\langle \mathfrak{e}_{x_1}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\mathfrak{e}_{y_1}\rangle_{\mathfrak{h}}|\\
\sup_{x_1\in\mathbb{Z}^d}\sum_{y_3\in\mathbb{Z}^d}|\varpi_s(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_3}))| &\sup_{y_1\in\mathbb{Z}^d}\sum_{x_2\in\mathbb{Z}^d}|\varpi_s(a(\mathfrak{e}_{y_1})a^*(\mathfrak{e}_{x_2}))|\sup_{y_2\in\mathbb{Z}^d}\sum_{x_3\in\mathbb{Z}^d}|\varpi_s(a(\mathfrak{e}_{y_2})a^*(\mathfrak{e}_{x_3}))|.
\end{aligned}$$

We can finally use (44)–(47) and (49) to arrive from the last upper bound at

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1]\\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}} |\boldsymbol{K}_1| < \infty.$$

The absolute value $|\boldsymbol{K}_2|$ of the other term of $\partial_s^3 \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}$ (see (50)–(52)) can be bounded exactly in the same way. By the triangle inequality applied to (50), this concludes the proof. □

We can now sharpen the result given in [Aza et al. 2019, Corollary 4.20], stating that the mapping $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ defined by (16) is continuously differentiable with

$$\partial_s \mathrm{J}^{(s\mathcal{E})} = \lim_{L\to\infty} \frac{\varrho^{(\omega)}\big(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}\, \mathrm{e}^{s|\Lambda_L|\,\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)}{\varrho^{(\omega)}\big(\mathrm{e}^{s|\Lambda_L|\,\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)}.$$

Thanks to (41) and Proposition 4.1, we now obtain the following assertion:

Corollary 4.2 (Differentiability of generating functions). *There is a measurable subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the mapping $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself belongs to $C^2(\mathbb{R};\mathbb{R})$ and*

$$\partial_s \mathrm{J}^{(s\mathcal{E})}\,|_{s=0} = x^{(\mathcal{E})} \doteq \lim_{L\to\infty} \mathbb{E}\big[\varrho^{(\cdot)}\big(\mathbb{I}^{(\cdot,\mathcal{E})}_{\Lambda_L}\big)\big] = \lim_{L\to\infty} \varrho^{(\omega)}\big(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}\big),$$
$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})}\,|_{s=0} = \lim_{L\to\infty} \mathbb{E}\big[\boldsymbol{F}^{(\cdot,\mathcal{E})}_L\big] = \lim_{L\to\infty} \boldsymbol{F}^{(\omega,\mathcal{E})}_L \geq 0,$$

where $\boldsymbol{F}^{(\omega,\mathcal{E})}_L$ is the quantum fluctuation of the linear response current defined by (18) for any $L \in \mathbb{R}_0^+$. See also (13) for the definition of the macroscopic current density $x^{(\mathcal{E})}$.

Proof. [Aza et al. 2019, Corollary 4.19] states, among other things, the existence of a measurable set $\tilde{\Omega}$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and $s \in \mathbb{R}$,

$$\mathrm{J}^{(s\mathcal{E})} = \lim_{L_\tau\geq L_\varrho\geq L\to\infty} \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}}. \tag{53}$$

Fix from now on all parameters $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. By combining (41) and Proposition 4.1 with the mean value theorem and the (Arzelà–)Ascoli theorem [Rudin 1991, Theorem A5], there are three sequences

$$\{L_\tau^{(n)}\}_{n\in\mathbb{N}},\ \{L_\varrho^{(n)}\}_{n\in\mathbb{N}},\ \{L^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_0^+, \tag{54}$$

with $L_\tau^{(n)} \geq L_\varrho^{(n)} \geq L^{(n)}$, such that, as $n \to \infty$, the mappings

$$s \mapsto \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}}, \quad s \mapsto \partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}}, \quad \text{and} \quad s \mapsto \partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}}$$

from $\mathbb{R}$ to itself converge uniformly for s in any compact subset of $\mathbb{R}$. So, the mapping $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is a C^2-function with

$$\partial_s \mathrm{J}^{(s\mathcal{E})} = \lim_{L_\tau \geq L_\varrho \geq L \to \infty} \partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} = \lim_{L\to\infty} \left(\frac{\varrho^{(\omega)}\big(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L} \, \mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)}{\varrho^{(\omega)}\big(\mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)} \right)$$

and

$$\begin{aligned}\partial_s^2 \mathrm{J}^{(s\mathcal{E})} &= \lim_{L_\tau \geq L_\varrho \geq L \to \infty} \partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} \\ &= \lim_{L\to\infty} |\Lambda_L| \left(\frac{\varrho^{(\omega)}\big((\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L})^2 \, \mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)\varrho^{(\omega)}\big(\mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big) - \big(\varrho^{(\omega)}\big(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L} \, \mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)\big)^2}{\big(\varrho^{(\omega)}\big(\mathrm{e}^{s|\Lambda_L| \mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\big)\big)^2} \right).\end{aligned}$$

See (32). Note that the above limits for the first- and second-order derivatives do not need to be taken only along subsequences, by the (Arzelà–)Ascoli theorem [Rudin 1991, Theorem A5] and (53). In particular, for $s = 0$,

$$\partial_s \mathrm{J}^{(s\mathcal{E})}|_{s=0} = \lim_{L\to\infty} \mathbb{E}\big[\varrho^{(\cdot)}\big(\mathbb{I}^{(\cdot,\mathcal{E})}_{\Lambda_L}\big)\big] = \lim_{L\to\infty} \varrho^{(\omega)}\big(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}\big) \tag{55}$$

and

$$\begin{aligned}\partial_s^2 \mathrm{J}^{(s\mathcal{E})}|_{s=0} &= \lim_{L\to\infty} |\Lambda_L| \mathbb{E}\big[\varrho^{(\cdot)}\big((\mathbb{I}^{(\cdot,\mathcal{E})}_{\Lambda_L})^2\big) - \big(\varrho^{(\cdot)}(\mathbb{I}^{(\cdot,\mathcal{E})}_{\Lambda_L})\big)^2\big] \\ &= \lim_{L\to\infty} |\Lambda_L| \big(\varrho^{(\omega)}\big((\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L})^2\big) - \big(\varrho^{(\omega)}(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L})\big)^2\big).\end{aligned} \tag{56}$$

By (18)–(19) and (56), $\partial_s^2 \mathrm{J}^{(s\mathcal{E})}|_{s=0}$ is the thermodynamic limit of the quantum fluctuations of linear response currents. □

From the proof of Proposition 4.1, it is apparent that the n-th derivative $\partial_s^n \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}$, $n \in \mathbb{N}$, has the following structure:

$$\begin{aligned}\partial_s^n \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} &= \frac{1}{|\cup\mathcal{Z}|} \sum_{k=1}^{n} \sum_{x_k,y_k\in\mathbb{Z}^d} \langle \mathfrak{e}_{x_1}, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}} \mathfrak{e}_{y_1}\rangle_{\mathfrak{h}} \cdots \langle \mathfrak{e}_{x_k}, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}} \mathfrak{e}_{y_k}\rangle_{\mathfrak{h}} \\ &\qquad \times \varpi_s^T(a^*(\mathfrak{e}_{x_1})a(\mathfrak{e}_{y_1}); \cdots ; a^*(\mathfrak{e}_{x_n})a(\mathfrak{e}_{x_n})) \\ &= \frac{1}{|\cup\mathcal{Z}|} \sum_{g\in\mathcal{G}_n^c} \sum_{k=1}^{n} \sum_{x_k,y_k\in\mathbb{Z}^d} \operatorname{sign}(g) \langle \mathfrak{e}_{x_1}, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}} \mathfrak{e}_{y_1}\rangle_{\mathfrak{h}} \cdots \langle \mathfrak{e}_{x_k}, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}} \mathfrak{e}_{y_k}\rangle_{\mathfrak{h}} \\ &\qquad \times \prod_{l\in g} k_s(l; x_1, y_1, \ldots, x_n, y_n),\end{aligned}$$

where $\mathcal{G}_n^c$ is the set of all connected oriented graphs g such that, for each vertex $v \in \{1, \ldots, n\}$ of $g \in \mathcal{G}_n^c$, there is exactly one line of the form $(v, \tilde{v}_1) \in g$ and exactly one line of the form $(\tilde{v}_2, v) \in g$, for some $\tilde{v}_1, \tilde{v}_2 \in \{1, \ldots, n\}$. The constants $k_s(l; x_1, y_1, \ldots, x_n, y_n)$, $l \in \{1, \ldots, n\}^2$, $x_1, y_1, \ldots, x_n, y_n \in \mathbb{Z}^d$, are defined by

$$k_s((i, j); x_1, y_1, \ldots, x_n, y_n) \doteq \begin{cases} \varpi_s(a^*(\mathfrak{e}_{x_i})a(\mathfrak{e}_{y_j})) & \text{if } i \leq j, \\ \varpi_s(a(\mathfrak{e}_{y_j})a^*(\mathfrak{e}_{x_i})) & \text{if } i > j. \end{cases}$$

The quantity $\operatorname{sign}(g) \in \{-1, 1\}$ is a sign only depending on the graph $g \in \mathcal{G}_n^c$. By using this expression, exactly as in the special case $n = 3$, for any fixed $n \in \mathbb{N}$ and electric field $\mathcal{E}$, one can bound the n-th derivative $\partial_s^n \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}$ uniformly. This implies that the generating function $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ defined by (16) is a *smooth* function of $s \in \mathbb{R}$, by the (Arzelà–)Ascoli theorem [Rudin 1991, Theorem A5] used as in the proof of Corollary 4.2. We refrain from working out the full arguments to prove this claim since absolutely no new conceptual ingredient would appear in this generalization.

4.4. ***Nonvanishing second derivative of generating functions at the origin .*** We discuss necessary conditions for

$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})} |_{s=0} \neq 0, \tag{57}$$

which is a condition appearing in Theorem 3.1(ii). In other words, the aim of this section is to prove Theorem 3.3. To this end, it is convenient to write this quantity by means of the one-particle Hilbert space $\mathfrak{h}$.

Lemma 4.3 (Quantum fluctuations on the one-particle Hilbert space). *For all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,*

$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})} |_{s=0} = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\Big[\mathrm{Tr}_{\mathfrak{h}}\Big(K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{-\beta h^{(\cdot)}}} K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{\beta h^{(\cdot)}}}\Big)\Big],$$

with $\mathrm{Tr}_{\mathfrak{h}}$ being the trace on $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d; \mathbb{C})$.

Proof. Fix all parameters of the lemma. Using (32) and (35) together with the quasifree property of $\varrho^{(\omega)}$, one obtains from (56) that

$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})} |_{s=0} = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{x,y,u,v\in\mathbb{Z}^d} \langle \mathfrak{e}_x, K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_u, K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_v\rangle_{\mathfrak{h}} \times \varrho^{(\omega)}(a(\mathfrak{e}_y)a(\mathfrak{e}_u)^*)\varrho^{(\omega)}(a(\mathfrak{e}_x)^*a(\mathfrak{e}_v)),$$

because of the identity

$$\rho(a(\mathfrak{e}_x)^*a(\mathfrak{e}_y)a(\mathfrak{e}_u)^*a(\mathfrak{e}_v)) = \rho(a(\mathfrak{e}_x)^*a(\mathfrak{e}_y))\rho(a(\mathfrak{e}_u)^*a(\mathfrak{e}_v)) + \rho(a(\mathfrak{e}_y)a(\mathfrak{e}_u)^*)\rho(a(\mathfrak{e}_x)^*a(\mathfrak{e}_v))$$

for any $x, y, u, v \in \mathbb{Z}^d$ and quasifree state ρ on $\mathcal{U}$, see (4) and (8). By (6) and straightforward computations, the assertion follows. □

Therefore, (57) holds true if

$$\lim_{L\to\infty} \left\{\frac{1}{|\Lambda_L|} \left|\mathrm{Tr}_{\mathfrak{h}}\Big(K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{-\beta h^{(\omega)}}} K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{\beta h^{(\omega)}}}\Big)\right|\right\} \geq \varepsilon > 0$$

for some strictly positive constant $\varepsilon \in \mathbb{R}^+$. To verify this bound, we start with an elementary observation:

Lemma 4.4 (Quantum fluctuations and the Hilbert–Schmidt norm of $K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}$). *For all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,*

$$\mathrm{Tr}_{\mathfrak{h}}\Big(K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{-\beta h^{(\omega)}}} K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \frac{1}{1+\mathrm{e}^{\beta h^{(\omega)}}}\Big) \geq \frac{1}{(1+\mathrm{e}^{\beta(2d(2+\vartheta)+\lambda)})^2} \mathrm{Tr}_{\mathfrak{h}}\big((K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}})^* K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\big).$$

Proof. Fix all parameters of the lemma. By the functional calculus, $(1+\mathrm{e}^{\pm\beta h^{(\omega)}})^{-1}$ are positive operators satisfying

$$\frac{1}{1+\mathrm{e}^{\pm\beta h^{(\omega)}}} \geq \frac{1}{1+\mathrm{e}^{\beta \sup_{\omega\in\Omega}\|h^{(\omega)}\|_{\mathcal{B}(\mathfrak{h})}}}\mathbf{1}_{\mathfrak{h}}\ ,$$

while, for any $\omega=(\omega_1,\omega_2)\in\Omega$ and $\lambda,\vartheta\in\mathbb{R}_0^+$,

$$\|h^{(\omega)}\|_{\mathcal{B}(\mathfrak{h})} \leq \|\Delta_{\omega,\vartheta}\|_{\mathcal{B}(\mathfrak{h})}+\lambda\|\omega_1\|_{\mathcal{B}(\mathfrak{h})} \leq 2d(2+\vartheta)+\lambda, \tag{58}$$

see (2)–(3). Since $K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}$ is a self-adjoint operator (see (36) or (59) below), it thus suffices to use the cyclicity of the trace to prove the lemma. □

Recall that $K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}$ is defined by (36), that is in this case,

$$K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \doteq \sum_{k,q=1}^{d} w_k \int_{-\infty}^{0}\{\mathcal{E}(\alpha)\}_q\left(\delta_{k,q}\boldsymbol{M}_k^{(L,\omega)}+\int_0^{-\alpha}\boldsymbol{N}^{(L,\omega)}_{\gamma,q,k}\,d\gamma\right)d\alpha, \tag{59}$$

where, for any $k,q\in\{1,\dots,d\}$, $\gamma\in\mathbb{R}$, $\vartheta,\lambda\in\mathbb{R}_0^+$, $\omega\in\Omega$, and $L\in\mathbb{R}^+$,

$$\boldsymbol{M}_k^{(L,\omega)} \doteq \sum_{x,x+e_k\in\Lambda_L} 2\Re\mathrm{e}\{S^{(\omega)}_{x+e_k,x}\}, \tag{60}$$

$$\boldsymbol{N}^{(L,\omega)}_{\gamma,q,k} \doteq \sum_{\substack{x,y\in\Lambda_L\\ x+e_k,y+e_q\in\Lambda_L}} 4i\left[\mathrm{e}^{-i\gamma h^{(\omega)}}\,\Im\mathrm{m}\{S^{(\omega)}_{y+e_q,y}\}\,\mathrm{e}^{i\gamma h^{(\omega)}},\Im\mathrm{m}\{S^{(\omega)}_{x+e_k,x}\}\right], \tag{61}$$

with $S^{(\omega)}_{x,y}$ being the single-hopping operators defined by (33)–(34) for any $x,y\in\mathbb{Z}^d$.

The square of the Hilbert–Schmidt norm of $K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}$ is obviously equal to

$$\mathrm{Tr}_{\mathfrak{h}}\left((K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}})^* K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\right)=\sum_{z\in\mathbb{Z}^d}\left\|K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\mathfrak{e}_z\right\|^2_{\mathfrak{h}},$$

and, consequently, we derive an explicit expression for the vectors

$$K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\mathfrak{e}_z\in\mathfrak{h},\qquad z\in\mathbb{Z}^d.$$

This can be directly obtained from (59) together with the following assertion:

Lemma 4.5 (Explicit computations of $\boldsymbol{M}_k^{(L,\omega)}$ and $\boldsymbol{N}^{(L,\omega)}_{\gamma,q,k}$ in the canonical basis). *For all* $k,q\in\{1,\dots,d\}$, $\gamma\in\mathbb{R}$, $\vartheta,\lambda\in\mathbb{R}_0^+$, $\omega\in\Omega$, $\gamma\in\mathbb{R}$, $L\geq 2$, *and* $z\in\Lambda_{L/2}$,

$$\boldsymbol{M}_k^{(L,\omega)}\mathfrak{e}_z=\langle\mathfrak{e}_{z-e_k},\Delta_{\omega,\vartheta}\,\mathfrak{e}_z\rangle_{\mathfrak{h}}\,\mathfrak{e}_{z-e_k}+\langle\mathfrak{e}_{z+e_k},\Delta_{\omega,\vartheta}\,\mathfrak{e}_z\rangle_{\mathfrak{h}}\,\mathfrak{e}_{z+e_k},$$

and, in the limit $L\to\infty$,

$$\boldsymbol{N}^{(L,\omega)}_{\gamma,q,k}\mathfrak{e}_z=\sum_{x,y\in\mathbb{Z}^d}\zeta_{x,y,z}\mathfrak{e}_x+\boldsymbol{R}^{(L,\omega)}_{\gamma,q,k}\mathfrak{e}_z,\qquad \sum_{x,y\in\mathbb{Z}^d}|\zeta_{x,y,z}|^2<\infty,$$

with $\boldsymbol{R}^{(L,\omega)}_{\gamma,q,k}\in\mathcal{B}(\mathfrak{h})$ *satisfying*

$$\lim_{L\to\infty}\left\|\boldsymbol{R}^{(L,\omega)}_{\gamma,q,k}\right\|_{\mathcal{B}(\mathfrak{h})}=0, \tag{62}$$

uniformly with respect to $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and* ϑ, γ *in compact subsets of* $\mathbb{R}_0^+$ *and* $\mathbb{R}$, *respectively, and where, for any* $x, y, z \in \mathbb{Z}^d$,

$$\begin{aligned}\zeta_{x,y,z} \doteq\ & i(1+\vartheta\omega_2(\{x-e_k,x\}))(1+\vartheta\omega_2(\{y,y+e_q\}))\langle \mathfrak{e}_{x-e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}} \\ & -i(1+\vartheta\omega_2(\{x-e_k,x\}))(1+\vartheta\overline{\omega_2(\{y+e_q,y\})})\langle \mathfrak{e}_{x-e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}} \\ & -i(1+\vartheta\overline{\omega_2(\{x+e_k,x\})})(1+\vartheta\omega_2(\{y,y+e_q\}))\langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}} \\ & +i(1+\vartheta\overline{\omega_2(\{x+e_k,x\})})(1+\vartheta\overline{\omega_2(\{y+e_q,y\})})\langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}} \\ & -i(1+\vartheta\omega_2(\{y,y+e_q\}))(1+\vartheta\omega_2(\{z,z+e_k\}))\langle \mathfrak{e}_{y}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{z+e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \\ & +i(1+\vartheta\omega_2(\{y,y+e_q\}))(1+\vartheta\overline{\omega_2(\{z,z-e_k\})})\langle \mathfrak{e}_{y}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{z-e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \\ & +i(1+\vartheta\overline{\omega_2(\{y+e_q,y\})})(1+\vartheta\omega_2(\{z,z+e_k\}))\langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{z+e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y}\rangle_{\mathfrak{h}} \\ & -i(1+\vartheta\overline{\omega_2(\{y+e_q,y\})})(1+\vartheta\overline{\omega_2(\{z,z-e_k\})})\langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{z-e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y}\rangle_{\mathfrak{h}}.\end{aligned}$$

Proof. Fix in all the proof $k, q \in \{1, \ldots, d\}$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\gamma \in \mathbb{R}$, $L \geq 2$, and $z \in \Lambda_{L/2}$. Since, by (33)–(34), for any $x, y \in \mathbb{Z}^d$, $2\Re\mathrm{e}\{S_{x,y}^{(\omega)}\} = \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} P_{\{y\}}s_{y-x}P_{\{x\}} + \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} P_{\{x\}}s_{x-y}P_{\{y\}}$, we deduce from (60) together with (28) and (33) that

$$\begin{aligned}\boldsymbol{M}_k^{(L,\omega)}\mathfrak{e}_z &= \sum_{x,x+e_k\in\Lambda_L}(\delta_{z,x+e_k}\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\,\mathfrak{e}_x + \delta_{z,x}\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}}\,\mathfrak{e}_{x+e_k}) \\ &= \mathbf{1}[z\in\Lambda_L]\mathbf{1}[(z-e_k)\in\Lambda_L]\langle \mathfrak{e}_{z-e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_z\rangle_{\mathfrak{h}}\,\mathfrak{e}_{z-e_k} \\ &\qquad + \mathbf{1}[z\in\Lambda_L]\mathbf{1}[(z+e_k)\in\Lambda_L]\langle \mathfrak{e}_{z+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_z\rangle_{\mathfrak{h}}\mathfrak{e}_{z+e_k}.\end{aligned}$$

If $z \in \Lambda_{L/2} \subseteq \Lambda_L$ and $L \geq 2$, then, obviously, $z, (z-e_k), (z+e_k) \in \Lambda_L$, and the last equality yields the first assertion.

By (33)–(34), for any $x, y \in \mathbb{Z}^d$, $2\Im\mathrm{m}\{S_{x,y}^{(\omega)}\} = i(\langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} P_{\{y\}}s_{y-x}P_{\{x\}} - \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} P_{\{x\}}s_{x-y}P_{\{y\}})$, and we compute that, for any $x, y \in \mathbb{Z}^d$,

$$\begin{aligned}&4i\big[\mathrm{e}^{-i\gamma h^{(\omega)}}\Im\mathrm{m}\{S_{y+e_q,y}^{(\omega)}\}\mathrm{e}^{i\gamma h^{(\omega)}}, \Im\mathrm{m}\{S_{x+e_k,x}^{(\omega)}\}\big] \\ &= i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}\, s_{e_k}P_{\{x\}}\mathrm{e}^{-i\gamma h^{(\omega)}}s_{e_q}P_{\{y\}}\mathrm{e}^{i\gamma h^{(\omega)}} \\ &\quad -i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y}, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\, s_{e_k}P_{\{x\}}\mathrm{e}^{-i\gamma h^{(\omega)}}s_{-e_q}P_{\{y+e_q\}}\mathrm{e}^{i\gamma h^{(\omega)}} \\ &\quad -i\langle \mathfrak{e}_{x}, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}\, s_{-e_k}P_{\{x+e_k\}}\mathrm{e}^{-i\gamma h^{(\omega)}}s_{e_q}P_{\{y\}}\mathrm{e}^{i\gamma h^{(\omega)}} \\ &\quad +i\langle \mathfrak{e}_{x}, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{y}, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\, s_{-e_k}P_{\{x+e_k\}}\mathrm{e}^{-i\gamma h^{(\omega)}}s_{-e_q}P_{\{y+e_q\}}\mathrm{e}^{i\gamma h^{(\omega)}} \\ &\quad -i\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}}\,\mathrm{e}^{-i\gamma h^{(\omega)}}s_{e_q}P_{\{y\}}\mathrm{e}^{i\gamma h^{(\omega)}}s_{e_k}P_{\{x\}} \\ &\quad +i\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\,\mathrm{e}^{-i\gamma h^{(\omega)}}s_{e_q}P_{\{y\}}\mathrm{e}^{i\gamma h^{(\omega)}}s_{-e_k}P_{\{x+e_k\}} \\ &\quad +i\langle \mathfrak{e}_{y}, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}}\,\mathrm{e}^{-i\gamma h^{(\omega)}}s_{-e_q}P_{\{y+e_q\}}\mathrm{e}^{i\gamma h^{(\omega)}}s_{e_k}P_{\{x\}} \\ &\quad -i\langle \mathfrak{e}_{y}, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}}\langle \mathfrak{e}_{x}, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\,\mathrm{e}^{-i\gamma h^{(\omega)}}s_{-e_q}P_{\{y+e_q\}}\mathrm{e}^{i\gamma h^{(\omega)}}s_{-e_k}P_{\{x+e_k\}}.\end{aligned}$$

Using this last equality together with (33)–(34) and (61), we thus get that

$$\begin{aligned}
&\boldsymbol{N}_{\gamma,q,k}^{(L,\omega)}\mathfrak{e}_z \\
&= \sum_{\substack{x,y\in\Lambda_L\\ x+e_k,y+e_q\in\Lambda_L}} \Big\{ i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\, \mathfrak{e}_{x+e_k} \\
&\quad - i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\, \mathfrak{e}_{x+e_k} \\
&\quad - i\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\, \mathfrak{e}_x \\
&\quad + i\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\, \mathfrak{e}_x \\
&\quad - i\delta_{x,z}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q} \\
&\quad + i\delta_{x+e_k,z}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_x\rangle_{\mathfrak{h}}\, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q} \\
&\quad + i\delta_{x,z}\langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}}\, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_y \\
&\quad - i\delta_{x+e_k,z}\langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_x\rangle_{\mathfrak{h}}\, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_y \Big\}.
\end{aligned}$$

By using (2) and (42)–(43) together with

$$\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\mu_\eta(|x-z|+|y-z|)} \le \mathrm{e}^{-\mu_\eta|x-y|} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\mu_\eta(|x-z|+|y-z|)} \le \mathrm{e}^{-\mu_\eta|x-y|} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\mu_\eta|z|},$$

which are simple consequences of the Cauchy–Schwarz and triangle inequalities, all the above summands are absolutely summable, uniformly with respect to $L\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, and ϑ, γ in compact subsets of $\mathbb{R}_0^+$ and $\mathbb{R}$, respectively. For instance, for any (characteristic) functions $f, g : \mathbb{Z}^d \to \{0, 1\}$, one estimates

$$\begin{aligned}
&\sum_{x,y\in\mathbb{Z}^d} f(x)^2 g(y)^2 \big|\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\big| \|\mathfrak{e}_{x+e_k}\|_{\mathfrak{h}} \\
&\quad \le 36^2(1+\vartheta)^2\, \mathrm{e}^{2|\gamma\eta|} \sum_{x,y\in\mathbb{Z}^d} f(x)^2 g(y)^2\, \mathrm{e}^{-2\mu_\eta(|x-e_q-y|+|z-y|)} \\
&\quad \le 36^2(1+\vartheta)^2\, \mathrm{e}^{2|\gamma\eta|} \Big(\sum_{u\in\mathbb{Z}^d} g(u+z)^2\, \mathrm{e}^{-2\mu_\eta|u|}\Big)^{1/2} \\
&\qquad\qquad \times \sum_{x\in\mathbb{Z}^d} f(x)^2\, \mathrm{e}^{-\mu_\eta|x-e_q-z|} \Big(\sum_{y\in\mathbb{Z}^d} g(y+x-e_q)^2\, \mathrm{e}^{-2\mu_\eta|y|}\Big)^{1/2} < \infty.
\end{aligned}$$

(Recall that $\mu_\eta > 0$, by (43).) In fact, by the same arguments combined with

$$\|C\|_{\mathcal{B}(\mathfrak{h})} \le \sup_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} |\langle \mathfrak{e}_x, C\mathfrak{e}_z\rangle_{\mathfrak{h}}|, \qquad C\in\mathcal{B}(\mathfrak{h}),$$

(see [Aza et al. 2019, Lemma 4.1]), the absolutely summable sum

$$\mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_w = \sum_{u\in\mathbb{Z}^d} \mathfrak{e}_u \langle \mathfrak{e}_u, \mathrm{e}^{-i\gamma h^{(\omega)}}\mathfrak{e}_w\rangle_{\mathfrak{h}}, \qquad w\in\mathbb{Z}^d, \tag{63}$$

(see (42)–(43)) and Lebesgue's dominated convergence theorem, in the limit $L\to\infty$ and for any $z\in\Lambda_{L/2}$, there is an operator $\boldsymbol{R}_{\gamma,q,k}^{(L,\omega)} \in \mathcal{B}(\mathfrak{h})$ with vanishing operator norm as $L\to\infty$, uniformly with respect to

$\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and ϑ, γ in compact subsets of $\mathbb{R}_0^+$ and $\mathbb{R}$, respectively, such that

$$\boldsymbol{N}_{\gamma,q,k}^{(L,\omega)} \mathfrak{e}_z = \big(\boldsymbol{N}_{\gamma,q,k}^{(\infty,\omega)} + \boldsymbol{R}_{\gamma,q,k}^{(L,\omega)}\big)\mathfrak{e}_z,$$

where

$$\begin{aligned}
\boldsymbol{N}_{\gamma,q,k}^{(\infty,\omega)} \mathfrak{e}_z \doteq \sum_{x,y\in\mathbb{Z}^d} \Big\{ & i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}}, \langle \mathfrak{e}_x, \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} \mathfrak{e}_{x+e_k} \\
& - i\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} \mathfrak{e}_{x+e_k} \\
& - i\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} \mathfrak{e}_x \\
& + i\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} \mathfrak{e}_x \\
& - i\delta_{x,z}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_{y+e_q} \\
& + i\delta_{x+e_k,z}\langle \mathfrak{e}_{y+e_q}, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_x\rangle_{\mathfrak{h}} \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_{y+e_q} \\
& + i\delta_{x,z}\langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_y \\
& - i\delta_{x+e_k,z}\langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_{y+e_q}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_{x+e_k}\rangle_{\mathfrak{h}} \langle \mathfrak{e}_{y+e_q}, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_x\rangle_{\mathfrak{h}} \mathrm{e}^{-i\gamma h^{(\omega)}} \mathfrak{e}_y \Big\}.
\end{aligned}$$

It suffices now to use again (2) and (63) together with elementary manipulations in each sum of $\boldsymbol{N}_{\gamma,q,k}^{(\infty,\omega)}$ in order to arrive at the second assertion. □

We are now in a position to show (57), at least for $|\gamma|, \vartheta \ll 1$, as a consequence of the next two lemmata:

Lemma 4.6 (Asymptotics for $\vartheta \ll 1$). *For all $k, q \in \{1, \ldots, d\}$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\gamma \in \mathbb{R}$, and $z \in \mathbb{Z}^d$,*

$$\sum_{y\in\mathbb{Z}^d} \zeta_{z,y,z} = 2\Im\mathrm{m}\langle (s_{e_k} - s_{-e_k})\mathfrak{e}_z, \mathrm{e}^{-i\gamma h^{(\omega)}} (s_{e_q} - s_{-e_q})\, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} + \mathcal{O}(\vartheta), \quad \text{as } \vartheta \to 0,$$

uniformly with respect to $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and γ in compact subsets of $\mathbb{R}$. Note that ϑ is not necessarily 0 *in the definition of $h^{(\omega)}$.*

Proof. By Lemma 4.5 at $\vartheta = 0$, one directly computes that, for any $k, q \in \{1, \ldots, d\}$, $\lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\gamma \in \mathbb{R}$, $z \in \mathbb{Z}^d$, and $\vartheta = 0$,

$$\sum_{y\in\mathbb{Z}^d} \zeta_{z,y,z} = \sum_{y\in\mathbb{Z}^d} 2\Im\mathrm{m}\langle \mathfrak{e}_{z+e_k} - \mathfrak{e}_{z-e_k}, \mathrm{e}^{-i\gamma h^{(\omega)}} (\mathfrak{e}_{y+e_q} - \mathfrak{e}_{y-e_q})\rangle_{\mathfrak{h}} \langle \mathfrak{e}_y, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}}\,.$$

If $\vartheta \neq 0$, then one performs the same kind of computation in order to (trivially) deduce the assertion, by (33), Lemma 4.5, and (42)–(43). □

Lemma 4.7 (Asymptotics for $|\gamma| \ll 1$). *For all $k, q \in \{1, \ldots, d\}$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\gamma \in \mathbb{R}$, and $z \in \mathbb{Z}^d$,*

$$\begin{aligned}
&2\Im\mathrm{m}\langle (s_{e_k} - s_{-e_k})\mathfrak{e}_z, \mathrm{e}^{-i\gamma h^{(\omega)}} (s_{e_q} - s_{-e_q})\, \mathrm{e}^{i\gamma h^{(\omega)}} \mathfrak{e}_z\rangle_{\mathfrak{h}} \\
&\qquad = 2\gamma\lambda\delta_{k,q}\{2\omega_1(z) - \omega_1(z+e_k) - \omega_1(z-e_k)\} + \mathcal{O}(\gamma^2),
\end{aligned}$$

as $|\gamma| \to 0$, uniformly with respect to $\omega \in \Omega$ and ϑ, λ in compact subsets of $\mathbb{R}_0^+$.

Proof. By (58), for any $\gamma \in \mathbb{R}$,

$$\mathrm{e}^{i\gamma h^{(\omega)}} = \mathbf{1}_{\mathfrak{h}} + \sum_{n\in\mathbb{N}} \frac{(i\gamma h^{(\omega)})^n}{n!} = \mathbf{1}_{\mathfrak{h}} + i\gamma h^{(\omega)} + \mathcal{O}(\gamma^2), \quad \text{as } |\gamma| \to 0,$$

in the Banach space $\mathcal{B}(\mathfrak{h})$, uniformly with respect to $\omega \in \Omega$ and ϑ, λ in compact subsets of $\mathbb{R}_0^+$. The assertion then follows by direct computations using (2)–(3), (33), and the last equality. □

Lemma 4.8 (Lower bounds on the Hilbert–Schmidt norm of $K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}$). *Take $\vartheta, \lambda, T \in \mathbb{R}_0^+$, $T \in \mathbb{R}^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ with support in $[-T, 0]$, and $\vec{w} \doteq (w_1, \ldots, w_d) \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. If T, ϑ are sufficiently small, then*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\Big[\mathrm{Tr}_{\mathfrak{h}}\Big(\big(K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\big)^* K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\Big)\Big] \geq \frac{\lambda^2}{2} \mathrm{Var}\left[\int_{-\infty}^0 \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d} \alpha^2 \, d\alpha\right] + \mathcal{O}(\vartheta^2) + \mathcal{O}(T^4),$$

uniformly with respect to λ in compact subsets of $\mathbb{R}_0^+$, where $w^{(\cdot)} \doteq (w_1^{(\cdot)}, \ldots, w_d^{(\cdot)}) \in \mathbb{R}^d$ is the random vector defined by

$$w_k^{(\omega)} \doteq (2\omega_1(0) - \omega_1(e_k) - \omega_1(-e_k)) w_k, \quad k \in \{1, \ldots, d\}, \omega \in \Omega. \tag{64}$$

Proof. Fix all parameters of the lemma. Take any $L \geq 2$. Note that

$$\mathrm{Tr}_{\mathfrak{h}}\Big(\big(K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\big)^* K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\Big) \geq \sum_{z\in\Lambda_{L/2}} \big\| K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_z \big\|_{\mathfrak{h}}^2 \geq \sum_{z\in\Lambda_{L/2}} \big|\langle \mathfrak{e}_z, K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_z\rangle_{\mathfrak{h}}\big|^2. \tag{65}$$

By using (59)–(61) and Lemma 4.5, for any $z \in \Lambda_{L/2}$, we have that

$$\begin{aligned}\langle \mathfrak{e}_z, K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_z\rangle_{\mathfrak{h}} = \sum_{k,q=1}^d w_k \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q \int_0^{-\alpha} \sum_{y\in\mathbb{Z}^d} \zeta_{z,y,z} \, d\gamma \, d\alpha \\ + \sum_{k,q=1}^d w_k \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q \int_0^{-\alpha} \langle \mathfrak{e}_z, \boldsymbol{R}^{(L,\omega)}_{\gamma,q,k} \mathfrak{e}_z\rangle_{\mathfrak{h}} \, d\gamma \, d\alpha,\end{aligned}$$

with $\boldsymbol{R}^{(L,\omega)}_{\gamma,q,k} \in \mathcal{B}(\mathfrak{h})$ satisfying (62). Note that $\zeta_{z,y,z}$ is γ-dependent, and its explicit expression is found in Lemma 4.5. If T, ϑ are sufficiently small then, by Lemmata 4.6–4.7, we deduce that, for any $z \in \Lambda_{L/2}$,

$$\begin{aligned}\langle \mathfrak{e}_z, K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}} \mathfrak{e}_z\rangle_{\mathfrak{h}} = \lambda \sum_{k=1}^d w_k \int_{-\infty}^0 \{2\omega_1(z) - \omega_1(z+e_k) - \omega_1(z-e_k)\}\{\mathcal{E}(\alpha)\}_k \alpha^2 \, d\alpha \\ + \mathcal{O}(\vartheta) + \mathcal{O}(T^2) + \sum_{k,q=1}^d w_k \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q \int_0^{-\alpha} \langle \mathfrak{e}_z, \boldsymbol{R}^{(L,\omega)}_{\gamma,q,k} \mathfrak{e}_z\rangle_{\mathfrak{h}} \, d\gamma \, d\alpha,\end{aligned}$$

uniformly with respect to $\omega \in \Omega$ and λ in compact subsets of $\mathbb{R}_0^+$. By the translation invariance of the distribution $\mathfrak{a}_\Omega$ (see [Aza et al. 2019, Equations (1)–(2)]) and (62), it follows that

$$\lim_{L\to\infty} \mathbb{E}\Big[\big|\langle \mathfrak{e}_z, K^{(\cdot,\mathcal{E})}_{\{\Lambda_L\},\{\mathbb{Z}^d\}}\mathfrak{e}_z\rangle_{\mathfrak{h}}\big|^2\Big] = \lambda^2 \mathbb{E}\bigg[\bigg|\int_{-\infty}^{0} \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\bigg|^2\bigg] + \mathcal{O}(\vartheta^2) + \mathcal{O}(T^4)$$
$$= \lambda^2 \operatorname{Var}\Big[\int_{-\infty}^{0} \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\Big] + \mathcal{O}(\vartheta^2) + \mathcal{O}(T^4),$$

uniformly with respect to λ in compact subsets of $\mathbb{R}_0^+$. Thanks to (65), the assertion then follows. Note that

$$\mathbb{E}\bigg[\int_{-\infty}^{0} \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha^2\bigg] = 0. \qquad \square$$

By combining Lemmata 4.4, 4.8, and 4.3, we directly obtain that, for any $\vartheta, \lambda, T \in \mathbb{R}_0^+$, $T, \beta \in \mathbb{R}^+$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ with support in $[-T, 0]$, and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,

$$\partial_s^2 \mathrm{J}^{(s\mathcal{E})}\big|_{s=0} \geq \frac{1}{2\big(1+\mathrm{e}^{\beta(2d(2+\vartheta)+\lambda)}\big)^2}\bigg(\lambda^2 \operatorname{Var}\bigg[\int_{-\infty}^{0} \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\bigg] + \mathcal{O}(\vartheta^2) + \mathcal{O}(T^4)\bigg), \tag{66}$$

provided that T, ϑ are sufficiently small. In particular, if

$$\operatorname{Var}\bigg[\int_{-\infty}^{0} \langle w^{(\cdot)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\bigg] > 0, \tag{67}$$

then $\partial_s^2 \mathrm{J}^{(s\mathcal{E})}\big|_{s=0} > 0$. This last condition is easily satisfied: Because the variance of the sum (or the difference) of uncorrelated random variables is the sum of their variances, if the random variables $\omega_1(0), \omega_1(e_1), \omega_1(-e_1), \ldots, \omega_1(e_d), \omega_1(-e_d)$ are independently and identically distributed (i.i.d.), then

$$\mathbb{E}\bigg[\bigg|\int_{-\infty}^{0} \langle w^{(\omega)}, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\bigg|^2\bigg] = 2\operatorname{Var}[(\,\cdot\,)_1(0)] \times \bigg(2\bigg(\int_{-\infty}^{0} \langle w, \mathcal{E}(\alpha)\rangle_{\mathbb{R}^d}\alpha^2\, d\alpha\bigg)^2 + \sum_{k=1}^{d}\bigg(w_k \int_{-\infty}^{0} (\mathcal{E}(\alpha))_k \alpha^2\, d\alpha\bigg)^2\bigg), \tag{68}$$

which is strictly positive as soon as $\mathcal{E} \neq 0$ and $\omega_1(0)$ is not almost surely constant, by Chebyshev's inequality.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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