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RATIO TESTS FOR CONVERGENCE OF SERIES

Ralph Palmer Agnew

1. Introduction. The following theorem was proved and used by Jehlke [2] to obtain elegant improvements of the classic tests of Gauss and Weierstrass for convergence of series of real and of complex terms.

THEOREM 1. *If the terms of two series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are such that*

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(1 + c_n) \qquad (n = 0, 1, \cdots), \tag{1}$$

where $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent, then the two series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are both convergent or both divergent.*

It is the main object of this note to prove that Theorem 1 is a best possible theorem in that no hypothesis weaker than the hypothesis that $\sum_{n=0}^{\infty} |c_n| < \infty$ is sufficient to imply the conclusion of the theorem. The final result, Theorem 4, is obtained from two preliminary theorems, Theorems 2 and 3, which seem to have independent interest.

2. Preliminary theorems. We first establish the following result.

THEOREM 2. *Let* $c_n \neq -1$, $n = 0, 1, 2, \cdots$. *In order that the sequence* $\{c_n\}$ *be such that* $\sum_{n=0}^{\infty} b_n$ *converges whenever* (1) *holds and* $\sum_{n=0}^{\infty} a_n$ *converges, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n| < \infty . \tag{2}$$

Proof. To prove Theorem 2, let (1) hold. Then

$$\frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n}(1 + c_n) \qquad (n = 0, 1, 2, \cdots), \tag{3}$$

and hence

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$$\frac{b_n}{a_n} = \frac{b_0}{a_0}(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \qquad (n = 1, 2, \cdots). \tag{4}$$

Let

$$p_n = \frac{b_0}{a_0}(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \qquad (n = 1, 2, \cdots). \tag{5}$$

Then $b_n = p_n a_n$. But by a well-known theorem of Hadamard [1], $\Sigma_{n=0}^{\infty} p_n a_n$ converges whenever $\Sigma_{n=0}^{\infty} a_n$ converges if and only if $\Sigma_{n=0}^{\infty} |p_{n+1} - p_n| < \infty$. But (5) implies that

$$p_{n+1} - p_n = \frac{b_0}{a_0}(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n, \tag{6}$$

and the conclusion of Theorem 2 follows.

THEOREM 3. *Let $c_n \neq -1$, $n = 0, 1, 2, \cdots$. In order that the sequence $\{c_n\}$ be such that $\Sigma_{n=0}^{\infty} a_n$ converges whenever* (1) *holds and $\Sigma_{n=0}^{\infty} b_n$ converges, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} \frac{c_n}{1 + c_n} \right| < \infty. \tag{7}$$

Proof. Theorem 3 may be proved by revising the proof of Theorem 2 to use the relations

$$\frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} \frac{1}{1 + c_n} \qquad (n = 0, 1, 2, \cdots) \tag{8}$$

instead of (1) or, which amounts to the same thing, replacing $1 + c_k$ by $1/(1 + c'_k)$ in (2) and then removing the primes.

3. Theorem. Our main result is the following.

THEOREM 4. *Let $c_n \neq -1$, $n = 0, 1, 2, \cdots$. In order that this sequence be such that the two series $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever* (1) *holds, it is necessary and sufficient that $\Sigma_{n=0}^{\infty} |c_n| < \infty$.*

Proof. To prove necessity, suppose $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds. Then, by Theorems 2 and 3, both (2) and (7) hold. Denoting the nth terms of the series in (2) and (7) by u_n and v_n, we see that, as $n \longrightarrow \infty$, we have $u_n \longrightarrow 0$ and $v_n \longrightarrow 0$ and hence

$$u_n v_n = \frac{c_n^2}{1 + c_n} \longrightarrow 0 \,. \tag{9}$$

This implies that $c_n \longrightarrow 0$ and hence that $|1/(1 + c_n)| > 1/2$ for n sufficiently great. This and (7) imply that

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} c_n \right| < \infty \,. \tag{10}$$

If we let $x_n = |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})|$, then (2) and (10) imply that

$$\sum_{n=1}^{\infty} (x_n + x_n^{-1}) \; |c_n| < \infty \,. \tag{11}$$

But the mere fact that $x_n > 0$ implies that $(x_n + x_n^{-1}) \geq 2$, and it follows that $\Sigma_{n=0}^{\infty} |c_n| < \infty$. This proves necessity. To prove sufficiency, suppose that $\Sigma_{n=0}^{\infty} |c_n| < \infty$. Then the infinite product $\prod(1 + c_k)$ converges to a number not zero, and this means that each of $(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})$ and $[(1 + c_0)(1 + c_1) \cdots (1 + c_n)]^{-1}$ converges to a number not zero. This and $\Sigma_{n=0}^{\infty} |c_n| < \infty$ imply (2) and (7). Therefore Theorems 2 and 3 imply that $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ are both convergent or both divergent. This completes the proof of Theorem 4.

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TOPOLOGIES FOR FUNCTION SPACES

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1. Introduction. Let Z^Y denote the class of continuous functions (or "mappings," or "maps")

$$f: Y \to Z \tag{1.1}$$

of a topological space Y into another Z. A great variety of topologies t may be introduced into Z^Y making it into a topological space $Z^Y(t)$. The topologies we deal with in this paper can be classified by using the notion of "continuous convergence" of directed sets (generalized sequences) f_μ in Z^Y as follows: with no reference to any topology Z^Y, we can say f_μ *converges continuously* (Frink [1]; Kuratowski [2]) to f (f_μ and f are elements of Z^Y) if

$$f_\mu(y_\nu) \to f(y) \tag{1.2}$$

whenever $y_\nu \to y$ in Y. (We use the "$\to$" for convergence as in (1.2), as well as for indicating the domain-range relation as in (1.1). The context prevents confusion.) We can classify the topologies t for Z^Y according as to whether

(1.3) convergence in $Z^Y(t)$ implies continuous convergence

or

(1.4) continuous convergence implies convergence in $Z^Y(t)$.

Certainly there are other topologies possible in Z^Y, but we do not discuss these. There may be a topology t satisfying both (1.3) and (1.4), but if so it is unique; see (5.6).

An apparently different approach to the same classification is suggested by homotopy theory. Beside Y and Z, consider a third space X. For a function g defined on $X \times Y$ with values in Z, we can define $g^*(x)$ mapping X into Z^Y by setting $g^*(x)(y) = g(x,y)$. Then a topology t for Z^Y may be such that, for any X,

(1.5) if g is continuous, then g^* is continuous,

or

(1.6) if g^* is continuous, then g is continuous.

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It is proved ((2.4), (2.5)) that (1.5) is equivalent to (1.3) and (1.6) is equivalent to (1.4). We call the former class of topologies *proper*, and the latter *admissible*.

The following questions about this class of topologies in Z^Y are considered in this paper: What are the relations (in the sense of the conventional partial ordering of topologies) of the proper topologies to the admissible topologies? What can be said about the order-type of the proper topologies? of the admissible topologies?

We write $s \leq t$ if s and t are topologies such that a set open in $Z^Y(s)$ is open in $Z^Y(t)$. Then (a) if $s \leq t$ and t is proper, s is proper; (b) if $s \leq t$ and s is admissible, so also is t; (c) if s is proper and t is admissible, then $s \leq t$; (d) there is at most one proper admissible topology, and such a topology is both the greatest proper and least admissible topology.

The proper topologies form a principal ideal in the lattice of all topologies for Z^Y; thus there is always a greatest proper topology. The admissible topologies are much more disorganized. We state some findings for the special case in which Z is the real line. (e) When Y is not locally compact, but is completely regular, there is no least admissible topology and (hence) no proper admissible topology; (f) if Y is a metric space, not locally compact, then there always exists a pair of admissible topologies none of whose common lower bounds are admissible.

When Y is locally compact, there does exist a proper admissible topology, as is well known, which we call the k-topology (see below (4.3)). We ask: To what extent do any of these properties of the k-topology persist when Y is not locally compact? It is always proper, but sometimes not the greatest of the proper topologies even if Y is completely regular. Admissibility does not often persist (See (c), above).

We consider a special class of topologies, the *set-open topologies*, whose definition is patterned after that of the k-topology except that arbitrary families $\{A\}$ of sets are admitted. We determine fairly complete criteria as to whether a given one is proper or admissible. The k-topology is always the greatest proper set-open topology, when Z is metric, and also the g. l. b. of all admissible set-open topologies.

A subclass of the set-open topologies are the *σ-topologies* defined in terms of coverings (just as the k-topology is definable in terms of the covering by open sets with compact closure when Y is compact). These topologies are admissible, and for any pair there is a common lower bound.

Considering that the space F of closed subsets of Y can be regarded as a function space, we felt it appropriate to point out that the usual Hausdorff topology, even when Y is a compactum, is not proper, and that k-topology is not a Hausdorff topology.

One interesting by-product of our investigation of admissible topologies is that it enables us to answer in the negative, surprisingly enough, the following simple question about topological products:

Let Y be a space, let s, t be two topologies for a set X, and let u be the greatest lower bound of the topologies s and t. Then is the (product) topology of

$$X(u) \times Y$$

the greatest lower bound of the topologies of $X(s) \times Y$, $X(t) \times Y$?

Finally, we determine a necessary and sufficient condition that, when Y is a locally compact regular space, X is a set, and a topology t has been given to $X \times Y$, a topology s can be found for X such that t is the product topology of

$$X(s) \times Y.$$

2. **Admissible topologies and proper topologies.** By a space Y we shall mean a set Y in which certain subsets, including Y and the empty set, are designated as open, and which have the property that their finite intersections and arbitrary unions are also open; no separation axioms are assumed. A *basis* for a space Y is a collection σ of open sets such that any open set in Y can be represented as the union of sets of σ; a *subbasis* for the space Y is a collection of open sets which, together with their finite intersections, form a basis. *Compactness* in this paper shall always be the bicompactness of Alexandroff-Hopf [1]; a space with the property that every infinite subset has a limit point is called *Fréchet compact.* A space Y is *locally compact* if every point lies in an open set having compact closure. Y is *completely regular* if, given any $y \in Y$ and open U with $y \in U$, there exists a continuous real-valued function f satisfying $f(y) = 1$ and $f(x) = 0$ for $x \notin U$. If Y_1 and Y_2 are two spaces, the topological product $Y_1 \times Y_2$ is the space whose points are all collections of ordered pairs (y_1, y_2), $y_1 \in Y_1$, $y_2 \in Y_2$, and in which a basis consists of all sets of form $(U_1 \times U_2)$, U_i being open in Y_i, $i = 1,2$.

If Y and Z are two spaces, the symbol "$f: Y \to Z$" will always denote a *continuous* mapping of Y into Z; the totality of all such continuous maps will be written Z^Y. Various topologies can be introduced into the set Z^Y; a set Z^Y with a topology is called a function space. In this section, we shall single out two important types of topologies in Z^Y, and give elementary consequences of the definitions.

(2.1) DEFINITION. Let Z and Y be two given spaces. A topology t in Z^Y is called *admissible* if the mapping $\omega(y,f) = f(y)$ of $Y \times Z^Y$ into Z is continuous in y and f.

The set Z^Y with the topology t will be denoted by $Z^Y(t)$, but when no ambiguity is involved regarding the topology t under discussion, the (t) will be omitted. The mapping ω will be called the evaluation mapping.

We now make the following observation. Let X, Y, Z, be three spaces and g a mapping of $X \times Y$ into Z. Setting $g^*(x)(y) = g(x,y) = f_x(y)$ and varying x, we can evidently regard g^* as a mapping of X into Z^Y. Conversely, if we have a mapping g^* of X into Z^Y, we can write $g(x,y) = g^*(x)(y) = f_x(y)$ and regard g as a mapping of $X \times Y$ into Z. Two maps g and g^* related as just described will be called *associated.*

We can now show the intimate relationship between mappings of cartesian products and admissible topologies in function spaces: the continuity of any g^* implies the continuity of the associated g.

(2.2) THEOREM. *Let Z and Y be two given spaces. A topology t in Z is admissible if and only if*

(2.21) *for every space X, $g^*: X \to Z^Y(t)$ implies $g: X \times Y \to Z$, where g is the associated mapping.*

Proof. Assume t is admissible, and $g^*: X \to Z^Y(t)$. Define

$$h: Y \times X \to Y \times Z^Y(t) \text{ by } h(x,y) = (g^*(x), y).$$

If ω is the evaluation map, we have $\omega h: Y \times X \to Z$, and it is not hard to see that ωh is the mapping associated with g^*. Hence, (2.21) holds.

Assume now (2.21) holds. In particular, select $X = Z^Y(t)$ and the identity map $I^*: Z^Y(t) \to Z^Y(t)$; by (2.21) this means the associated map of $Y \times Z^Y(t)$ into Z is continuous, and this associated mapping is precisely the evaluation mapping.

The other important class of topologies in Z^Y is given in the following definition.

(2.3) DEFINITION. Let Z and Y be two spaces. A topology t in Z^Y is *proper* if for every space X, $g: X \times Y \to Z$ implies $g^*: X \to Z^Y(t)$, where g^* is the associated map.

An extremely useful equivalent formulation of the notion "proper" can be given which is based on directed sets and continuous convergence. We therefore insert an explanatory paragraph (cf. Birkhoff [2]).

A *directed system* Δ is a partially ordered system with the property that for any $\mu, \mu' \in \Delta$, there exists a $\mu'' \in \Delta$ with $\mu'' \geq \mu$, $\mu'' \geq \mu'$. Every directed system Δ gives rise to a directed *space* Δ' by addition of one ideal point ∞ satisfying $\infty \geq \mu$ for all $\mu \in \Delta$. The topology in $\Delta' = \Delta \cup \{\infty\}$ is obtained by defining all μ to be open sets, and neighborhoods of ∞ to be all sets of form $\{\mu; \mu \geq \mu'$ for some $\mu'\}$, $\mu' \in \Delta$. If $\Gamma = \{\nu\}$ is another directed system, the set

$$\Delta \times \Gamma = (\mu, \nu)$$

of all pairs is also a directed system if we define $(\mu,\nu) \geq (\mu', \nu')$ whenever both $\mu \geq \mu'$ and $\nu \geq \nu'$. A (Δ-) *directed set in a space* Y is a function on a directed system Δ with values in Y, and is denoted by $\{y_\mu\}_{\mu\epsilon\Delta}$ or more briefly by $\{y_\mu\}$; a directed set $\{y_\mu\}$ converges to y (in symbols, $y_\mu \rightarrow y$) if for every neighborhood V of y there exists a $\mu' \epsilon \Delta$ with $y_\mu \epsilon V$ for all $\mu \geq \mu'$. Furthermore, we have $g: Y \rightarrow Z$ if and only if for every directed set $\{y_\mu\}$, $y_\mu \rightarrow y$ implies $g(y_\mu) \rightarrow g(y)$ (see Tukey [1, p.28]). Let $\{f_\mu)$ be a directed set in the set Z^Y; f_μ *converges continuously* to $f \epsilon Z^Y$ if for every y and every neighborhood W of $f(y)$ there is a μ' and a neighborhood V of y such that for $\mu \geq \mu'$ we have $f(V) \subset W$. (This definition is equivalent to that made in Section 1.) Notice that the idea of continuous convergence does *not* require any topology in Z^Y.

With these preliminaries, we prove the following result.

(2.4) THEOREM. *Let Z and Y be two spaces. A topology t in Z^Y is proper if and only if for every directed system Δ and every Δ-directed set $\{f_\mu\}$ in $Z^Y(t)$, the continuous convergence of f_μ to f implies $f_\mu \rightarrow f$ according to t.*

Proof. Suppose, first, t is proper and let f_μ converge continuously to f, $\{f_\mu\}$ being directed by Δ. Let Δ' be the corresponding directed space. Then define $g(\mu,y) = f_\mu(y)$, $g(\infty,y) = f(y)$. Now we have $g: \Delta' \times Y \rightarrow Z$, by the definition of continuous convergence. Hence $f_\mu = g^*(\mu) \rightarrow g^*(\infty) = f$ as desired.

Now suppose continuous convergence always implies convergence, and suppose we have $g: X \times Y \rightarrow Z$. Suppose $x_\mu \rightarrow x$ in X. It is easy to see that $g^*(x_\mu)$ converges continuously to $g^*(x)$ since g is continuous. Thus $g^*(x_\mu) \rightarrow g^*(x)$ in Z. This proves that we have $g^*: X \rightarrow Z^Y$. Consequently t is proper.

We remark that if every continuously convergent *sequence* in $Z^Y(t)$ converges, the topology need not necessarily be proper.

A rather parallel criterion for admissibility can also be stated. We formulate it now but leave the proof, which resembles that of (2.4), to the reader.

(2.5) THEOREM. *Let Z and Y be two spaces. A topology t in Z^Y is admissible if and only if for every directed system Δ and Δ-directed set $\{f_\mu\}$ the convergence $f_\mu \rightarrow f$ in $Z(t)$ implies the continuous convergence of f_μ to f.*

Kuratowski [11] has shown that the idea of continuous convergence can be used to introduce a convergence (in Kuratowski's case [10], L^*-convergence) in Z provided also Y and Z are L^*-spaces. The convergence obtained is both admissible and proper, in a suitable sense (see Kuratowski [11]). There is not always a corresponding *topology* in Z^Y associated with this convergence, but the poor showing of *topologies* in this connection (see (6.01)) seems to commend this step beyond the class of topological spaces, as Kuratowski points out.

3. Comparison of topologies. Since we are going to be concerned with various topologies for Z^Y it is natural to recall that there is a useful partial ordering for all the topologies on a fixed fundamental set E. For references, see Birkhoff [5, p.173].

To define this partial ordering, it is useful to take the attitude that a topology t on a set E *is* (rather than merely *determines*) the class of those sets which are open in the topology. Thus a topology is a subset of 2^E, the class of all subsets of E. Hence for two topologies t and u on the same set E the set-theoretic statement of inclusion, "$t \subset u$," is meaningful and leads to the following definition.

(3.1) DEFINITION. If t and u are two topologies for a set E, we shall write $t \leq u$ or $u \geq t$ or *t is smaller than u* or *u is greater than t* when every set open in t is open in u, that is, when $t \subset u$.

Notice that the statement "t is smaller than u" is not comparable with the statement "u is not greater than t," since the former is not intended to exclude the possibility: $t = u$. If $t \leq u$ we shall sometimes call u an *expansion* of t and t a *contraction* of u. This partial ordering is easily seen to have the property that $t \leq u$ if and only if the identity mapping

$$E(u) \to E(t) \tag{3.2}$$

is continuous.

Since the class $\tau(E)$ of all topologies is a subset of 2^E, and since the relation "$\leq$" defined above is just that which is inherited from the natural partial ordering (by inclusion) (see Birkhoff [5]) in 2^E, we have the following result.

(3.3) THEOREM. *The relation "$\leq$" in the class $\tau(E)$ of topologies on E is a partial ordering.*

$\tau(E)$ is not a sublattice of 2^E because, while $t \cap u$ is always a topology, $t \cup u$ is not always a topology. This does not exclude the possibility that $\tau(E)$ be nevertheless a lattice (see Birkhoff [5, p. 19]).

(3.4) THEOREM (Birkhoff [4]). *$\tau(E)$ is a lattice; that is, for two topologies t and u there is a least upper bound $t \vee u$ and a greatest lower bound $t \wedge u$. In fact, every subset T of $\tau(E)$ has a least upper bound (briefly: "join")*

$$\bigvee_{t \in T} t$$

and a greatest lower bound (briefly: "meet")

$$\bigwedge_{t \in T} t \quad ;$$

that is, $\tau(E)$ is a complete lattice with greatest and least members.

The greatest lower bound of a class T of topologies has the open sets

$$\bigcap_{t \in T} t$$

which are open in all. On the other hand, if we take as a *basis* the class of sets

$$\bigcup_{t \in T} t$$

we obtain a topology which is easily seen to be the least one including all the members of T. The discrete topology is the greatest of all topologies and the *trivial* topology (only E and the void set are open) is least.

The familiar classes of topologies (for example, Hausdorff, completely regular, normal) are not all well behaved with reference to expansion and contraction (see Hewitt [9]). Since there will emerge some trouble with admissible topologies under the operation "meet," it is only fair to show that things do not go smoothly with every one of the familiar classes of topologies. The following theorem is intended only for orientation. The statement that a property T is preserved under "meeting of two" means that if t and u have property T then so does $t \wedge u$, and so on for the other terms to be used.

(3.5) THEOREM. *In the lattice of topologies on a set E,*

(3.51) *The Riesz* [T_1], *Hausdorff* [T_2], *and Urysohn* (*see Hewitt* [9]) *separation properties are each preserved under arbitrary expansions* (*Hewitt* [9]), *and hence under joining*:

(3.52) *Although not preserved under arbitrary expansion* (*Hewitt* [9]), *regularity and complete regularity are preserved under joining of two*;

(3.53) *Riesz separation is preserved under meeting* (*Birkhoff* [4]);

(3.54) *Hausdorff and Urysohn separation, regularity, complete regularity, normality, complete normality, and metrizability are not generally preserved under meeting of two.*

Proof. Statements (3.51) through (3.53) may be found in the references or easily proved. We content ourselves by supplying an example supporting (3.54).

Let E be any denumerable infinite set, and let x_1, x_2 be a pair of distinct elements of E. Consider the topology t_1 in which any set is open if it either excludes x_1 or has a finite complement. This (compact) space has all the properties mentioned in (3.54). By interchanging the roles of x_1 and x_2 we obtain another topology t_2. Since $t_1 \wedge t_2$ is a non-Hausdorff Riesz space, all the properties in (3.24) also fail since each guarantees Hausdorff separation when points are closed sets. This completes (3.5).

The result (3.54) just obtained entitles one to consider that perhaps the comparison of topologies based on (3.1) is not the most satisfactory one possible. However, no other generally applicable definition of ordering seems to have been proposed anywhere.

We now apply these ideas to topologies on a function space Z^Y.

(3.6) THEOREM. *Let t and u be topologies on* Z^Y. *If t is admissible and* $u \geq t$ *then u is admissible. If u is proper and* $t \leq u$ *then t is proper.*

These facts follow at once from the definition and property (3.2). In particular, admissibility is preserved under joining, and properness is preserved under meeting.

We shall see in (5.1) and (5.2) that the proper topologies form a principal ideal (t_m) in the lattice of all topologies; that is, there exists a topology t_m such that t is proper if and only if $t \leq t_m$. In particular, they constitute a sublattice. For admissible topologies, there sometimes exists a topology u such that $u \leq t$ precisely for the admissible topologies, but sometimes (see (6.3)) not even $t \wedge u$ is admissible when t and u are.

The general position of the admissible topologies with respect to the proper ones is this:

(3.7) THEOREM. *If t is proper and u is admissible then* $t \leq u$.

Proof. Since u is admissible, the mapping

$$\omega: Z^Y(u) \times Y \to Z$$

is continuous. From the definition of "proper," we obtain

$$\omega^+: Z^Y(u) \to Z^Y(t).$$

From (3.2) we conclude that $t \leq u$.

See also (6.01) below.

4. Examples of function spaces. In this section we shall give examples of function spaces, some having a proper topology, and some an admissible topology; we also investigate in some detail a method for introducing topologies in the set Z^Y. Notice that the discrete topology in the set Z^Y is always an (the greatest) admissible topology, and the trivial topology (3.4) in Z^Y is always a (the smallest) proper topology. We proceed to less trivial methods for introducing a topology.

(4.01) DEFINITION. Let A and B be subsets of the spaces Y and Z respectively. The symbol (A,B) denotes the set of all $f \in Z^Y$ satisfying $f(A) \subset B$.

We utilize this notation to define a class of topologies in Z^Y: the σ-topologies. Let σ be an arbitrary covering of Y by *open* sets; we keep σ fixed throughout this discussion. Introduce a topology in Z^Y as follows. Let F be any closed set in Y contained in some member of σ, and V an open set in Z. The class of all sets of form (F,V) is taken as a subbasis in Z^Y.

(4.02) DEFINITION. The topology in Z^Y thus determined by σ is called the *σ-topology.*

(4.1) THEOREM. *Let Y be regular, Z arbitrary. Then for any σ, the σ-topology in Z^Y is always admissible.*

Proof. We are to show that we have $\omega: Y \times Z^Y \to Z$. Let $f \in Z^Y$, $y \in Y$, and let W be a neighborhood of $f(y)$ in Z. Since Y is regular, we can find an open V containing y with closure V^- in $f^{-1}(W)$ and also in some member of σ; then $\omega(V \times (V^-, W)) \subset W$ and ω is continuous, as was to be shown.

The following fact about σ-topologies is to be compared with (6.3) below.

(4.11) THEOREM. *Let σ_1 and σ_2 be open coverings of Y. If σ_1 is a refinement of σ_2 then the σ_1-topology is less than or equal to the σ_2-topology. If Y is regular, the meet of two σ-topologies is also admissible.*

Proof. The first assertion is obvious. It implies the second as follows. Let σ be a common refinement of σ_1 and σ_2. When Y is regular, the σ-topology is admissible, and since

$$\sigma\text{-topology} \leq \sigma_1\text{-topology} \wedge \sigma_2\text{-topology},$$

the latter is admissible. There is no reason why the latter should be a σ-topology, of course.

Variants of the σ-topologies can be found by varying the allowable sets in Y, that is, by permitting open, or arbitrary, subsets of members of σ to be used in the definition of the subbasis. Although these variants of σ-topologies are also always admissible (when Y is regular) there is a reason for preferring the σ-topologies. To see this, we first remark that the existence of a proper admissible topology in Z^Y is a desirable property. For example, it is easily seen that with such a topology, the homotopy of two maps $Y \to Z$ is equivalent with their being joined by an arc in the functional space. Now, when Y is regular, it is easy to see that the σ-topology is always less than or equal any of its variants, so that the former is "nearer" to the proper topologies than any of the latter. For this reason, the σ-topologies appear better suited to our work.

We shall now introduce a class of topologies including the class of σ-topologies. Let Y and Z be as before and let a family $\{A\}$ of subsets of Y be given. Taking the family of sets (A, W) (see (4.01)), where W is open in Z and A belongs to $\{A\}$, as a subbase in Z^Y we obtain a topology.

(4.2) DEFINITION. The topology described above is called the $\{A\}$-open topology. Any such topology will be called an S-topology, or *set-open* topology. The space Z^Y with the $\{A\}$-open topology will be written $Z^Y\ (S : \{A\})$.

One reason for still limiting W to *open* subsets of Z in (A, W) is that in this way we can be sure that if we consider only the class of constant functions, it will be homeomorphic to Z.

The next result shows how a large class of proper topologies may be obtained. (Recall that σ-topologies provide a way of obtaining admissible topologies.)

(4.21) THEOREM. *Let Z and Y be two spaces. If all the sets in $\{A\}$ are compact, then the $\{A\}$-open topology in Z^Y is proper.*

Proof. Suppose we have $g: X \times Y \to Z$; to prove g^* continuous, it is enough to show that given any subbasic open set $(A, V) \ni g^*(x_0)$, there is a neighborhood U of x_0 with $g^*(U) \subset (A, V)$. Using the continuity of g and the definition of g^*, we see that from $x_0 \times A$ being contained in the open set $g^{-1}(V)$ we have to conclude that $U \times A \subset g^{-1}(V)$ for some neighborhood U of x_0. To do this, for each (x_0, y) in $x_0 \times A$ we find a set W_y open in $X \times Y$ with $(x_0, y) \in W_y \subset g^{-1}(V)$; this gives a covering of $x_0 \times A$, and the compactness of $x_0 \times A$ allows us to extract a finite covering. The intersection of the projections of the sets of this finite covering on X gives an open U containing x_0 and clearly $U \times A \subset g^{-1}(V)$.

On the basis of this theorem, an important special case of the set-open topologies is singled out: the case where $\{A\}$ is the collection of *all* the compact subsets of Y (see Arens [2], Fox [7]). We will call this special case, for ready reference, the *k-topology*. For separation properties of the k-topology, see Arens [2]. For example, if Z is a Hausdorff space, the k-topology is a Hausdorff topology. It is evident that the k-topology is the greatest set-open proper topology based on compact sets.

The proof of the properness of a set-open topology contains essentially the following question: What conditions on the sets $\{A\}$ insure that, for every X, an open V in $X \times Y$ containing $x_0 \times A$ also contains an "open tube" $U \times A$ (U a neighborhood of x_0 in X)? With this observation, we are ready to approach the problem: Which of the set-open topologies are proper? Our procedure enables us to answer a more inclusive question: What conditions on the class $\{A\}$ follow from the assumption that the $\{A\}$-open topology is $\leq$ every admissible topology? (See (3.7).) A sufficient condition has been given in (4.2); we have several necessary conditions, but have not found both necessary and sufficient conditions, except in isolated instances.

We first treat the special case of real-valued functions.

(4.3) THEOREM. *Let Y be a completely regular space, and E_1 the Euclidean line. If the $\{A\}$-open topology in E_1^Y is $\leq$ every admissible topology, then the sets of $\{A\}$ must all have compact closure.*

Proof. Let B be any set of $\{A\}$ and σ: $\{V\}$ an arbitrary covering of the closure B^- of B. We are to show that we can extract a finite covering of B^-.

Let f be the constant function 0 in E_1^Y. Then $f \in (B, W)$, where W is the complement of 1 in E_1. Now form the σ-topology based on the covering of Y by the sets $\{V\}$ together with the complement of B^-. By (4.11) this topology is admissible, and by hypothesis there exists a neighborhood

$$U = (C_1, C_2, \cdots, C_n; W_1, W_2, \cdots, W_n)$$

in Z^Y(σ-topology) such that $f \in U \subset (B, W)$. Let C denote the closed union of $C_1, \cdots, C_n$. If C does not contain B^-, there is a point b in B^- which is not in C, and which hence has a neighborhood V not meeting C; since $b \in B^-$, it follows that V contains some point b' in B not in C. Construct a continuous real-valued function r with $r(b') = 1$ and $r(y) = 0$ for $y \notin V$. It is clear that $r \in U$ since it coincides with f on C, but evidently $r \notin (B, W)$. Hence, B^- *is* contained in C. Let $V_1, \cdots, V_n$ be sets of the covering σ containing the closed sets $C_1, \cdots, C_n$ respectively; then B^- is contained in the union of the former. Hence, B^- is compact, as was to be shown.

It is evident that a similar theorem holds for mappings of a completely regular space Y into any space Z that contains at least one non-degenerate arc. Thus an application of the special case (4.3) yields the same conclusion in many more general cases.

(4.31) THEOREM. *Let Y be a completely regular space, and Z a space containing a non-degenerate arc. A necessary and sufficient condition that a set-open topology based on closed sets be $\leq$ every admissible topology is that it be a proper topology.*

Proof. The necessity stems from (4.3) and (4.21). The sufficiency arises from (3.7).

(4.311) COROLLARY. *Let Y be a completely regular space, and Z a space containing a non-degenerate arc. A set-open topology based on closed sets is proper if and only if all the sets are compact.*

The following concept is useful in the further investigation of $\{A\}$ when the $\{A\}$-open topology is proper. Let B be a subset of Y. A point y_0 of Y is *inessential to B* if, for every $f: Y \to E_1$ there exists a y in B, $y \neq y_0$, such that $f(y) = f(y_0)$. Note that in a metric space Y no point is inessential to any B. In a completely regular space, y_0 is essential (that is, not inessential) to B if and only if y_0 is a G_δ-set relative to B.

(4.4) THEOREM. *Let Y be an arbitrary space, Z a space containing a non-degenerate arc. If the $\{A\}$-open topology in Z^Y is $\leq$ every admissible topology, then each set A of $\{A\}$ must contain all points of its closure which are essential to A.*

Proof. One may regard E_1 as embedded in Z. Let A belong to $\{A\}$, and let $Y \in A^- - A$. Suppose y_0 is essential to A. Then there exists an $f: Y \to E_1$ with

$f(y) \neq f(y_0)$ for every y in A. Let us take $f(y_0) = 0$.

Select $X = E_1$ and define $g: E_1 \times Y \to E_1$ by the condition $g(x,y) = x + f(y)$. We can now define an admissible topology t in Z^Y as follows: (a) the set

$$G = \{g^*(x);\ x \in E_1\}$$

is open and homeomorphic to E_1; (b) all other elements are isolated. This topology is clearly admissible, due to the continuity of g. By hypothesis there must exist a set U open in $Z^Y(t)$ with $f \in U \subset (A, W)$; and due to the definition of t, this U is (or at least contains) the image of some interval $(-e, e)$, $e > 0$. Now, since y_0 is a limit point of A, and f is continuous, there must exist a y_0 in A with $-e < f(y_0) < e$. Construct $f_1 = g^*[-f(y_1)]$. This f_1 belongs to U, but it does not belong to (A, W) since $f_1(y_1) = -f(y_1) + f(y_1) = 0$. This is a contradiction, and shows that y_0 is inessential to A. This proves (4.4).

We give an example to show that the sets $\{A\}$ on which a proper S-topology is based need not be closed. Let Y be any uncountable set in which all points are declared open sets except one, y_0, whose neighborhoods are defined as the complements of finite sets excluding y_0. Introduce a set-open topology t into $E_1{}^Y$ based on the non-closed set $A = Y - y_0$. Note that $g \in (Y - y_0, W)$ if and only if $g \in (Y, W)$, since otherwise g would assume a value at y_0 different from all its other values, and y_0 would be a G_δ. Thus this topology is the same as that based on $A_1 = Y$, which is proper, by (4.3), since A_1 is compact.

With the aid of this Theorem (4.4), one can refine the results of (4.31) and (4.311). We state the result but leave the proof to the reader.

(4.41) THEOREM. *Let Y be a completely regular space in which every point is a G_δ, and Z any space that contains a non-degenerate arc. A set-open topology in Z^Y is proper if and only if it is based on sets that are all compact. A necessary and sufficient condition that a set-open topology in Z^Y be proper is that it be $\leq$ every admissible topology.*

If a simple condition be satisfied by Z, we lose no proper set-open topologies by limiting ourselves to $\{A\}$-open topologies where every A is compact. This is shown in the next theorem.

(4.5) THEOREM. *Let Y be a completely regular space, Z a metric space containing a non-degenerate arc. If an $\{A\}$-open topology in Z^Y is $\leq$ every admissible topology, then it is equivalent to the set-open topology based on the compact sets $\{A^-\}$.*

Proof. We have, by (4.3), that all the sets A^- are compact; on the basis of (4.4) every A contains all points of A^- that are essential to A. Let A_0 be the set of points of A^- inessential to A; then $A \cup A_0 = A^-$. The theorem will be proved when we show that, for the subbasic open sets, we have $(A \cup A_0, W) = (A, W)$. The inclusion $(A \cup A_0, W) \subset (A, W)$ is evident. To prove also $(A, W) \subset (A \cup A_0, W)$ we need only show that $f \in (A, W)$ implies $f(a_0) \in W$ for every $a_0 \in A_0$. To this end denote by d the metric in Z, and set $F(y) = d(f(y), f(a_0))$; then we have

$$F : Y \to E_1 ,$$

and since $a_0 \in A_0$, this means there must be a point $a \in A$ with $F(a) = F(a_0) = 0$; this in turn implies that $f(a_0) = f(a) \in W$, finishing the proof as indicated. (All that is really needed for this theorem is that (a) Z be completely regular, (b) every point of Z be a G_δ, and (c) Z contain a non-degenerate arc.)

If we specialize Z instead of Y, we get a more complete converse to (4.2). Let us take Z to be the *Sierpinski space* consisting of two points, which we call 0 and 1, with the empty set, the entire space, and the point 0 as the only open sets.

(4.6) THEOREM. *Let Y be an arbitrary space, S the Sierpinski space. A necessary and sufficient condition that an $\{A\}$-open topology in S^Y be $\leq$ every admissible topology is that all the sets of $\{A\}$ be compact.*

Proof. The sufficiency follows from (4.2). We need only prove the necessity. Let B be an arbitrary set of the collection $\{A\}$ and $\{V_\beta\}$ an arbitrary covering of B. We shall reduce $\{V_\beta\}$ to a finite covering. Let V be the union of all the V_β.

Let $k \in S^Y$ be the function which is 0 precisely on V. Introduce an admissible topology t in S^Y as follows: (a) All elements of S^Y except k are isolated, (b) the neighborhoods of k are of form $(V_1 \cup V_2 \cup \cdots \cup V_\beta, 0)$ where the V_i are open sets with $V_i \subset V_{\beta_i}$. In fact, t is admissible, as is not hard to verify. The hypothesis then gives us a neighborhood in t with

$$k \in (V_1 \cup V_2 \cup \cdots \cup V_n , 0) \subset (B, 0).$$

Selecting $V_{\beta_i} \supset V_i$, we form their union G. This set covers B; for otherwise, if g is the function vanishing precisely on G we have $g \in (V_1 \cup \cdots \cup V_n , 0)$ and $g \notin (B, 0)$, a contradiction. Hence B is compact, proving (4.6).

We now turn to the admissible case, and seek conditions under which a set-open topology is admissible. For convenience we make another definition. A family of sets $\{A\}$ is a *regular* family in Y if, given any y in Y and neighborhood U of y, there exists an A in $\{A\}$ contained in U and containing y in its interior.

This concept permits the following statement.

(4.7) THEOREM. *Let Z and Y be arbitrary spaces. A set-open topology in Z^Y based on a regular family of sets is always admissible.*

Proof. We are to show that we have $\omega: Y \times Z^Y \to Z$. Let $f \in Z^Y$, $y \in Y$ and W a neighborhood of $f(y)$; then $f^{-1}(W)$ is open in Y, and $y \in f^{-1}(W)$; by regularity of the family $\{A\}$ we can find an A with $y \in \operatorname{int} A$, $A \subset f^{-1}(W)$. It is clear that $\omega[\operatorname{int} A \times (A_1, W)] \subset W$, and so ω is continuous.

This theorem has two interesting consequences, the first of which has been known for a long time; see Fox [7] and Arens [2].

(4.71) THEOREM. *Let Y be a regular locally compact space, and Z arbitrary. Then the k-topology* (see (4.2)) *in Z^Y is admissible and proper.*

Proof. The totality of all compact subsets of Y, since Y is locally compact and regular, forms a regular family.

A corollary of (4.71) particularly useful in discussions of homotopies is the following.

(4.72) COROLLARY (Fox [7]). *Let Y be regular, and let X and Y both satisfy the first axiom of countability. Then $g: X \times Y \to Z$ is equivalent with*

$$g^*: X \to Z^Y(k),$$

for any Z.

Proof. One half of the result comes from (4.2); we prove $g^*: X \to Z^Y(k)$ implies $g: X \times Y \to Z$. Note that (4.71) implies g is continuous on all sets of form $A \times Y$ where A is a compact subset of X. In particular, g is sequentially continuous, and with our hypothesis this implies that g is continuous.

5. The proper topologies. The situation of the proper topologies in the class of all topologies in a class Z^Y is a particularly simple one: with the partial ordering of (3.1), they form an ideal with a (smallest and a) greatest element. We establish first the completeness of the class of proper topologies.

(5.1) LEMMA. *Let Z and Y be arbitrary spaces; let $\{t_\alpha\}$ an arbitrary collection of proper topologies in Z^Y. Then $\wedge_\alpha t_\alpha$ and $\vee_\alpha t_\alpha$ are also proper topologies in Z^Y.*

Proof. That $\wedge_\alpha t_\alpha$ is a proper topology is immediate from (3.6). To prove the remaining part, suppose we have $g: X \times Y \to Z$; we are to show that we have $g^*: X \to Z^Y(\vee_\alpha t_\alpha)$. Select an open set U in $Z^Y(\vee_\alpha t_\alpha)$; since it is sufficient to consider only the subbasic open sets, this selected set can be assumed open in some topology t_β. Since t_β is proper, the inverse image of U under g^* is open in X. Hence g^* is continuous, as was to be shown.

Since an application of (5.1) gives a greatest and a least proper topology, we may reformulate (5.1) in the following way.

(5.2) THEOREM. *Let Z and Y be arbitrary spaces. With the partial ordering of* (3.1), *the proper topologies in Z^Y form an ideal with a greatest element.*

The least proper topology is, of course, the trivial topology. The (unique) greatest proper topology t_m can be characterized as follows: let $\{t_\alpha\}$ be the collection of all the proper topologies in Z^Y; then $t_m = \vee_\alpha t_\alpha$. We have been unable to characterize this greatest proper topology more directly in terms of the topological structures of Z and Y. So far as its properties are concerned, since $t_m \geq k$-topology, all properties of the k-topology invariant under expansion (such as Hausdorff separation, disconnection) are inherited by t_m. A method for obtaining the greatest proper topology that sometimes works will be given in Section 6; we merely remark here that a proper admissible topology in Z^Y is the greatest proper topology.

The problem initiating this paper was to determine the status of the k-topology. From (4.71) and the above remark, we note that if Y is locally compact, the k-topology is in fact the greatest proper topology in Z^Y. We ask then if the k-topology has any distinguished role in the hierarchy of proper topologies for Z^Y. Theorems (5.4) and (5.5), below, will give a reason why the k-topology is a convenient topology to be used in function spaces. However, it is *not* distinguished by being always the greatest proper topology in Z^Y. We now present an example.

(5.3) THEOREM. *Let Z be the unit interval* $[0,1]$ *in* E_1. *Then there exists a completely regular space Y such that the k-topology in Z^Y is not the greatest proper topology.*

Proof. Let Y be the set of all ordered pairs of positive integers, and one additional element which we will call ∞. The topology in Y is obtained by taking each pair (i, j) as an isolated point, and the neighborhoods of ∞ to be all sets obtained as follows: if N, J_{N+1}, $J_{N+2}, \cdots$ is any collection of integers, the set

$$V = \{(i, j);\ i > N \text{ and } j > J_i\}$$

is a neighborhood of ∞. We remark that, in this space Y, all the compact sets are finite sets; see Arens [3, p. 234] .

Define a function $f_n\colon Y \to Z$, $f_n(i, j) = 0$ or 1 according as $i \neq n$ or $i = n$, and $f_n(\infty) = 0$. Expand the topology of $Z^Y(k)$ by declaring the set $B = \{f_1, f_2, \cdots\}$ *closed*, thus obtaining a topology k^+ in Z . We note first that $k^+ > k$ since $f_n \to 0$ in the k-topology, but not in the k^+-topology.

The theorem will be proved when we show that k^+ is a proper topology in Z^Y. To this end, let g_μ converge continuously to g (see (2.4)); we are to prove $g_\mu \to g$. Our proof breaks into two cases.

Case 1: $g \neq 0$. Since $Z^Y(k)$ is a Hausdorff space (see (4.3)), if $g \neq 0$, we can find a k-neighborhood U of 0 that excludes the sequence B, because $f_n \to 0$ implies that 0 is the only limit point of B in $Z^Y(k)$. Since the topologies of $Z^Y(k^+)$ and $Z^Y(k)$ coincide at all points $g \neq 0$, and k is a proper topology, this means g_μ is ultimately in U, and so converges to g in k and in k^+.

Case 2: $g = 0$. Let U^+ be a neighborhood of 0 in k^+. Then $U^+ = U - B$ (U open in k-topology), and g_μ is again ultimately in U. Let us assume that nevertheless $g_\mu \not\to 0$ in k^+. Then we must have $g_\mu \in B$ cofinally; that is, given

μ, there exists a $\mu' \geq \mu$ with $g_{\mu'} \in B$. The $g_{\mu'}$ still converge continuously to 0. Hence there is a neighborhood V of ∞ (say the one described in the first paragraph) and a μ_0 such that for $\mu' \geq \mu_0$, we have $g_{\mu'} = 0$ on V. But since $g_{\mu'} \in B$, we have $g_{\mu'} = f_{n(\mu')}$; and since $g_{\mu'} = f_{n(\mu')} = 0$ on V for $\mu' \geq \mu_0$, we would have $1 \leq n(\mu') \leq N$ for $\mu' \geq \mu_0$. Since the $f_{n(\mu')}$ converge continuously to 0, they converge also in the k-topology (see (2.4)). Hence there exists a μ_1 such that $\mu \geq \mu_1$ implies $f_{n(\mu)}(i, 1) = 0$ for $i = 1, 2, \cdots, N$; this means that for $\mu' \geq \mu_0$, $\mu' \geq \mu_1$, we have $g_{\mu'} = 0 = f_{n(\mu')}$. Thus the $g_{\mu'}$ are finally all 0, contradicting that they all lie on B. Hence, $g_\mu \to 0$ in k^+. This concludes the proof of the fact that k^+ is proper.

If we do not require Y to be a completely regular space, then a construction of a proper topology greater than the k-topology becomes simpler. We append such an example for later use.

(5.31) LEMMA. *Let Y be a completely regular space, satisfying the first axiom of countability, and Z the unit interval in E_1. Let $Y^\sim$ be an expansion of the topology of Y in such a way that the sets Z^Y and $Z^{Y^\sim}$ are the same. Then for any space X, $g: X \times Y^\sim \to Z$* implies *$g: X \times Y \to Z$.*

Proof. We first establish the following results.

(5.311) *A point y in $Y^\sim$ has a basis of neighborhoods of the form $V - D^\sim$, where V is open in Y, $D^\sim$ is closed in $Y^\sim$, and $V \cap D^\sim$ has no interior in Y.*

To see this, note that any neighborhood of y in $Y^\sim$ has the form $Y^\sim - E^\sim$, where $E^\sim$ is closed in $Y^\sim$. Suppose now that the interior I of $E^\sim$ in Y has y as limit point. Pick a basis $V_1, V_2, \cdots$ of y in Y such that

$$V_1 \supset V_2^-, V_2 \supset V_3^-, \cdots,$$

in such a manner that $I \cap (V_n - V_{n+1}^-) \neq 0$. Define $f_n: Y \to Z$ by

$$f_n(y) = \begin{cases} 1 \text{ at some point of } I \cap (V_n - V_{n+1}^-) \\ 0 \text{ for } y \notin I \cap (V_n - V_{n+1}^-). \end{cases}$$

Then $f = \Sigma_n f_n$ is a continuous on Y except at y. In $Y^\sim$, however, it is continuous even at y because $f = 0$ on $Y^\sim - E^\sim$. The real-valued continuous functions being the same, this can only happen if I does *not* have y as a limit point in Y. Hence, we can pick V so that $V \cap E^\sim$ has no interior in Y, as was to be shown.

(5.312) $g: X \times Y^\sim \to Z$ *implies* $g: X \times Y \to Z$.

With the notations of (5.311), $g: X \times Y^\sim \to Z$ implies that there exists a neighborhood U of x_0 and a $V - D^\sim$ containing y_0 such that

$$|g(x, y) - g(x_0, y_0)| < \epsilon$$

for $(x, y) \in U \times (V - D^{\sim})$. Now, each point of $V \cap D^{\sim}$ is a limit point of $V - D^{\sim}$; so if $y \in (V \cap D^{\sim})$, there exists a sequence $\{y_\mu\}$ with $y_\mu \in (V - D^{\sim})$ and $y_\mu \to y$ in Y. Since $g(x, \cdot)$ is still continuous on Y, we have $|g(x,y) - g(x_0, y_0)| \leq E$ and therefore $g: X \times Y \to Z$, as desired.

(5.32) THEOREM. *Let Y be a completely regular space, satisfying the first axiom of countability, and let Z be the unit interval of E_1. Let $Y^{\sim}$ be an expansion of the topology of Y that introduces no new continuous real-valued functions. Let $Z^{Y^{\sim}}(k^{\sim})$ be the functional space with the k-topology, and let $Z^{Y^{\sim}}(k)$ be the same functional space with the k-topology of Z^Y. Then $k > k^{\sim}$, and k is also proper. Hence, the k-topology in $Z^{Y^{\sim}}$ is not the greatest proper topology.*

Proof. It is not hard to verify that in fact $k > k^{\sim}$. To see that k is also proper, note that $g: X \times Y^{\sim} \to Z$ implies $g: X \times Y \to Z$, which yields

$$g^*: X \to Z^Y(k) = Z^{Y^{\sim}}(k),$$

as was to be shown.

An example of such a space is exhibited in (6.21) below.

The position of the k-topology in the proper topologies for Z^Y can now be somewhat clarified; and a reason for its utility will appear from the following sequence of theorems.

(5.4) THEOREM. *Let Y be a completely regular space, and Z an arbitrary space. Then the k-topology is the greatest of the proper set-open topologies based on closed sets.*

(5.5) THEOREM. *Let Y be a completely regular space, Z a metric space containing a non-degenerate arc. Then the k-topology is the greatest of the proper set-open topologies.*

The proofs of these theorems are immediate from (4.311) and (4.5).

(5.51) COROLLARY. *Let Y be a completely regular space, and Z a metric space containing a non-degenerate arc. If the k-topology is not the greatest proper topology, then the greatest proper topology is not a set-open topology.*

From consideration of the results in (3.6) it is clear that, if a proper admissible topology exists, there must be a delicate balancing of open sets. In general, there is no such topology (see (6)). But we now show there can never be more than one, if any.

(5.6) THEOREM. *Let Z and Y be arbitrary spaces. If t is a proper admissible topology in Z^Y, then t is unique. That is, there is no other proper admissible topology except t.*

Proof. Let s be another proper admissible topology. Since s is proper and t is admissible, by (3.6), we have $s \leq t$. Reversing the roles of s and t, we get also $t \leq s$. Hence, $t = s$.

Some of the considerations of this section can be applied to more general situations in which topologies for a set X are considered, there being no function space in the picture. The following generalization of (5.1) is related to a construction of Choquet's [6, p. 85].

(5.7) LEMMA. *Let L be a "notion of convergence," that is, a rule which assigns, to some directed sets in X, a point. Let a topology t for X be called "L-proper" if whenever L assigns x to $\{x_\mu\}$ then $x_\mu \to x$ in X. Then, if T is any family of L-proper topologies, the topology*

$$t^- = \bigvee_{t \in T} t$$

is also L-proper.

Proof. Let L assign x to $\{x_\mu\}$. Let a neighborhood U of x in $X(t^-)$ be given. There exist $t_1, \cdots, t_n$ in T such that $U = U_1 \cap \cdots \cap U_n$, where U_i is open in $X(t_i)$. Hence there is some μ_i such that the relation $\mu \geq \mu_i$ implies $x_i \in U_i$. Select $\mu_0 \geq \mu_1, \cdots, \mu_n$. For $\mu \geq \mu_0$, we have $x_\mu \in U$. Thus t^- is proper.

To obtain (5.1), let L be the notion of continuous convergence, and let T be the class of all L-proper topologies.

One may also define what is meant by an "L-admissible" topology. A topology t for X is *L-admissible* if whenever $x_\mu \to x$ in X, then L assigns x to $\{x_\mu\}$. The next Section shows that the L-admissible topologies do not always have the property dual to that established for the L-proper topologies in (5.7), not even when T is limited to finite sets.

6. Admissible topologies.

The σ-topologies provide many examples of admissible topologies. We shall now see that proper admissible topologies are scarce, and that the hierarchy of admissible topologies rarely forms a lattice.

(6.01) THEOREM. *A proper admissible topology for Z^Y is both the greatest proper topology and the least admissible topology.*

Proof. The proof rests on the fact that any proper topology is smaller than or equal to any given admissible topology (see (3.7)).

One conceivable way of determining whether there is a proper admissible topology is to examine the greatest proper topology itself. It is unique and is admissible if and only if there exists a proper admissible topology. A direct examination of the greatest proper topology seems rather cumbersome. The following partial converse to (5.71) shows that, under fairly general conditions, when Y is not locally compact, there is no proper admissible topology.

(6.1) THEOREM (Fox [7, Theorem 3]). *If Y is separable and metrizable and Z is the real line, and if there exists a proper admissible topology, then Y is locally compact.*

Now (6.1) is, by (6.01), immediately deducible from the following somewhat stronger result.

(6.2) THEOREM (Arens [2, Theorem 3]). *If Y is a completely regular space and Z is the real interval $[0,1]$, and if there exists a least admissible topology for Z^Y, then Y is locally compact.*

In particular, the "separable and metrizable" in (6.1) can be replaced by "completely regular." As to the necessity of complete regularity in (6.2): a locally compact Hausdorff space must be completely regular. However, this justification for assuming Y to be completely regular is not as convincing as is the following example.

(6.21) THEOREM. *There exists a non-locally-compact Hausdorff space Y such that Z^Y can be given a least admissible topology, where Z is the real interval $[0,1]$.*

Proof. Let Y_0 be the real interval $[0,1]$ with the ordinary topology. Let y be the interval $[0,1]$ with the topology generated by the following subbasic open sets (cf. Alexandroff-Hopf [1, p.31]): first, the complement of the set D, where is the set of numbers $1/n$ $(n = 1, 2, \cdots)$; second, the open sets of Y_0. Thus Y differs from Y_0 only at 0, but the mutilation at 0 is enough to make Y an irregular Hausdorff space. Hence it cannot be locally compact. The remaining part of the argument hinges on the fact that a function f: $[0,1] \to Z$ is continuous on Y if and only if it is continuous on Y_0. We leave the proof of this to the reader. (This in itself implies that Y is not completely regular.) Let $\{A\}$ be the class of sets which are compact in Y_0. In Z^Y these determine a set-open topology (see (4.2)) which we shall call the k_0-topology. It is clearly admissible, by (4.7). Let t be any other admissible topology for Z^Y. Suppose $f \in Z^Y$ and let (A, W_1) be a neighborhood of f in $Z^Y(k_0)$. Now the image $f(A)$ is surely compact in Z so that we can find an open set W in Z such that $f(A) \subset W$ and $W^- \subset W_1$. For each y in A there is a neighborhood $V_1(y)$ of y and a neighborhood $U(f, y)$ of f in $Z^Y(t)$ so that $z \in V_1(y)$ and so that the relation $g \in U(f, y)$ implies $g(z) \in W$. Let $V(y)$ be the interior of the closure of $V_1(y)$. It is easy to see that $g(z) \in W_1$ for $z \in V(y)$ and g as before. Since each $V(y)$ is open in Y_0, where A is compact, we can find $y_1, \cdots, y_n$ such that

$$A \subset V(y_1) \cup \cdots \cup V(y_n).$$

Let $U(f) = U(f, y_1) \cap \cdots \cap U(f, y_n)$. It is easy to see that $U(f)$ is contained in (A, W_1) (see Arens [2, p.482]). We infer from this that $t \geq k_0$.

The complete regularity is used in the proof of (6.2) (see Arens [2, p.483]) in constructing an element of Z^Y which distinguishes a closed set from a disjoint point. Hence the proof may be duplicated in any other case where such separation by continuous functions is always possible.

An *application*, rather than extension, of (6.2) shows that in (6.2) the space Z may be taken as any T_0-space containing a non-degenerate arc. Thus we obtain a result when Z is the Sierpinski space $\{0,1\}$. However, in this particular case the proof of (6.2) can be adapted so as to give a still better result: complete regularity is relaxed to regularity. We state the result but leave the proof to the reader.

(6.25) THEOREM. *If Y is a regular space and Z is the Sierpinski space* (see (4.6)), *and if there exists a least admissible topology for Z^Y, then Y is locally compact.*

Returning to the remark just made about having an arc in Z, we wish to show that this requirement cannot be simply omitted. Let us consider an extreme example in which Y is connected but Z is totally disconnected. Then Z^Y consists only of constant functions and can hence be given the topology of Z, which is both proper and admissible, regardless of any other properties of Y.

Theorem (6.2) says that when Y is completely regular, and Z is $[0,1]$ but Y is not locally compact, then there is at least one class $\mathfrak{A}$ of admissible topologies whose greatest lower bound is not admissible. The class $\mathfrak{A}$ which (6.2) exhibits for this purpose is a large one—in fact the largest possible. One might ask whether any *two* admissible topologies have an admissible "meet" topology (greatest lower bound) (see (3.4)), especially since we know of an extensive class of admissible topologies (see (4.11)) for which the meet of two is always admissible. The following theorem shows that the answer is "no" for any metric non-locally compact space Y. We define a *Fréchet-compact* set to be one in which every infinite subset has a limit point.

(6.3) THEOREM. *Let Y be a completely regular Hausdorff space in which each point has a countable basis, and let Z be the real line or the interval $[0,1]$. If the meet of every pair of admissible S-topologies for Z^Y is admissible then Y is locally Fréchet-compact.*

Proof. Suppose Y is not locally Fréchet-compact. One can then find a point y_0 for which one can construct a basis $V_1 \supset V_2 \supset V_3 \supset \cdots$, and a sequence of infinite sets $r_1, r_2, r_3, \cdots$, none of which has a limit point and such that r_n is contained in $V_n - V_{n+1}^-$. Break each r_n into disjoint infinite subsets s_n, t_n. Let r be the union of $r_1, r_2, r_3, \cdots$. A set A will be called an *R-set* if A^- intersects r in but a finite set. Let B_n be any open subset of $V_n - V_{n+1}$ containing all of t_n and no points of s_n, and let A_n be any R-set. Then let $(V_n - B_n) \cup A_n$ be called an *S_n-set* $(n = 1, 2, \cdots)$.

Define an S-topology in Z^Y by taking as a subbase the sets (A, W) where W is open in Z and A is an R-set or an S_n-set for some n. (The notation (A, W), defined earlier, refers to the class of functions sending A into W.) Designate this topology simply by "S".

We first show that S is admissible.

Let f and y be given, as well as a neighborhood W of $f(y)$. We consider two cases.

Case 1: $y = y_0$. There is a neighborhood V_n on which f has all its values in W. For each x in s_n obtain a neighborhood B_x contained in $V_n - \bar{V}_{n+1}$ and avoiding the set t_n. Let B_n be the union of these B_x. Then $(V_n - B_n, W)$ is a neighborhood of f, and for g therein and x in V_{n+1} we surely have $g(x)$ in W.

Case 2: $y = y_n$. We can find a neighborhood A of y which is an R-set and on which f has values only in W. For g in (A, W) and x in A we have $g(x)$ in W. With the completion of this second case we have shown the admissibility.

Replacing each S by a T, each s by a t, and each t by an s, regardless of subscripts, we obtain the definition and admissibility of another topology, T.

(6.31) *The meet* $S \wedge T$ *is not admissible.*

Suppose it were admissible. Let W be the complement of 1 in Z, and let f_0 be a function in Z^Y such that $f_0(y_0) = 0$ and $f_0(y) = 1$ for some y in r_1. Then there is a set U open in both S and T, and a V_p, such that:

(6.32) *The relations* $f \in U$ *and* $y \in V_p$ *together imply* $f(y) \in W$.

Continuing the proof of (6.3) we now deduce the following from (6.32):

(6.33) *If* U *contains an* f *which assumes the value* 1 *on some point of* r_m, *then it contains an* f_1 *which assumes that value on some point of* r_n *for some* n *greater than* m.

Proof. We may suppose that f in U assumes the value 1 on some point of s_m. Now f has a neighborhood $U_T = (A_1, \cdots, A_j; W_1, \cdots, W_j)$, where the latter expression denotes the intersection of $(A_1, W_1), \cdots, (A_j, W_j)$ in T. Some W_i clearly excludes 1, for otherwise the constant function 1 belongs to U, violating (6.32). Let $W_1, \cdots, W_k$ be those that exclude 1, and suppose A_1 is a T_n-set with the lowest value of n. The closures of the finitely many R-sets figuring in U_T clearly cannot cover t_n. Hence there is a point y_1 in t_n with a neighborhood V intersecting none of $A_1, \cdots, A_j$. We construct a continuous real-valued function g with $g(y_1) = 1 - f(y_1)$, vanishing outside V, and having 0 and $1 - f(y_1)$ as its bounds. Let $f_1 = f + g$; this function has the property required by (6.33), but it remains to show that we have $n > m$. If we had $n \leq m$ then s_m would be inside A_1. Now f assumes the value 1 somewhere on s_m. Thus 1 belongs to W_1. This contradicts the earlier finding that W_1 does not contain 1. Hence (6.33) is proved.

To prove (6.3) we observe that from the f_0 and an iterated application of (6.33) we finally obtain an f in U which assumes the value 1 somewhere on V_p. This contradicts (6.32). Hence (6.3) is proved.

As in the case of (6.2), this result (6.3) can be extended automatically to the case in which Z merely possesses a non-degenerate arc, and is a T_0-space. The "Sierpinski space" (see (4.6)) is an example. However, one can do a little better in the case of the Sierpinski space, as follows.

(6.4) THEOREM. *Let Y be a Hausdorff space in which each point has a countable basis, and let Z be the Sierpinski space* $\{0,1\}$ (see (4.6)). *If the meet of every pair of S-admissible topologies in Z^Y is admissible, then Y is locally compact.*

The reader can obtain the proof out of that of (6.3), observing these changes: (a) instead of having no limit points, the r_n have no *complete* limit points (see Alexandroff and Urysohn [14, p.7]); (b) the R-sets may intersect each r_n in a set of power less than that of r_n; (c) the value j need not exceed 1; and (d) the sets B_n may be taken as t_n. In fact, our method of investigating these matters was to consider first the case where Z is the Sierpinski space.

The following observation is of interest. In (6.3) we saw that the meet t of two admissible topologies may not have enough open sets to be itself admissible, unless Y is locally compact. Although there are obviously topologies which are neither admissible nor proper, one might wonder whether any such are accessible through lattice operations from admissible topologies. In other words, if the t above is not admissible is it necessarily proper? A consideration of the proof of (6.3) shows that the meet t of S and T need not be proper: each set of the form (A, W), where A is an R-set, is open in t; but there is no reason why all R sets should be compact, and hence (by (4.41)) why t should be proper.

An observation which sometimes leads to the identification of the greatest proper topology is this: If the meet of two admissible topologies is proper, this meet is the greatest proper topology. An application of this to the reasoning of (6.3) yields a result which should be compared with the earlier example (5.3).

(6.5) THEOREM. *When Y is not locally compact, and Z is the Sierpinski space* $\{0,1\}$, *then the k-topology may be the greatest proper topology.*

Proof. Let Y be the space of pairs of positive integers (i, j) with an added point "∞". Neighborhoods of ∞ shall be V_n $(n = 1, 2, \cdots)$ of points with $j \geq n$, plus ∞ itself. Other points are isolated. Define an S-topology as follows. Let A_n be the set of points (i, j) where $j \geq n+1$ when i is odd and $j \geq n$ when i is even, plus ∞ itself. Let A_∞ be void. A set denoted by B shall be any finite set. The S-topology shall be based on such A or B sets; call it S. The sets of the form $(A_n \cup B, 0)$, where $0 \in \{0,1\}$ is the open point, clearly form a basis. Interchange "odd" and "even" in the above and arrive at another topology T. Both these topologies are admissible, and hence $\geq k$, whence $S \wedge T \geq k$. Let U be open in $S \wedge T$, and contain the function 0. Select an $(A_n \cup B, 0)$ containing 0 and contained in U, the former being open in S. Then there must be an element f in U which has the value 1 at almost all points with $j < n$, and also at almost all points (i, j) with $j = n$ and i odd. Now select another neighborhood $(A_p \cup B_1, 0)$ of f in U, the former being open in T. Then clearly $p > n$. There is then a g in U with value 1 at almost all points with $j < p$ and at almost all points (i, j) with $j = p$ and i even. By induction we arrive at a set K of points, finitely many on each row, such that, if $h(i, j) = 0$ for $j \geq N$ (any N) and for $(i, j) \in K$, then we have $h \in U$. Let $K_1 = K \cup \{\infty\}$. What we have just said is that if $h \in (K_1, 0)$ then $h \in U$. Hence U is open in $Z^Y(k)$. If in the first neighborhood $(A_n \cup B, 0)$ we have $n = \infty$, then we have already a k-open set containing 0 and contained in U. The argument is still simpler for any function which is not 0 identically. Thus we have $S \wedge T \leq k$, or $S \wedge T = k$.

It is probable that this example can be adapted to the case where Z is the real interval $[0,1]$.

7. The space of closed sets. Let S be the Sierpinski space of two points 0 and 1, where $\{0\}$ is open and $\{1\}$ is not open. Consider S^Y for any space Y. Let $f \in S^Y$, and let F be the class of points on which $f(y) = 1$. Then F is evidently a closed set, and clearly every closed set can be obtained in this way. The notation of S has been so chosen that the correspondence $f \leftrightarrow F$ preserves the lattice operations (S obviously is a lattice, and this introduces lattice ordering into S^Y in an obvious way) and the Boolean ring operations (where we use intersection and symmetric difference $(F_1 \cup F_2) - (F_1 \cap F_2)$ in the class $\mathfrak{F}$ of closed sets). We sum up this situation briefly as a theorem.

(7.1) THEOREM. *S^Y and $\mathfrak{F}$ are isomorphic.*

We shall henceforth prefer the symbol "$\mathfrak{F}$", or "$\mathfrak{F}(Y)$", to "S^Y", and shall write the elements as $F, F_1, \cdots$, using F in a dual way when we write

(7.11) $y \in F$ if and only if $F(y) = 1$.

Having decided to regard $\mathfrak{F}$ as a space of continuous functions, we naturally investigate first the interpretation of any kind of convergence in $\mathfrak{F}$ which does not require introduction of any topology in the function space.

(7.2) THEOREM. *Let $\{F_\mu\}$ be a directed set in $\mathfrak{F}$. Then F_μ converges continuously* (see *circa* 2.4) *to F if and only if $F \supset \limsup F_\mu$.*

Before proceeding to the proof, we must explain what the lim sup of a directed set of sets E_μ is. Generalizing Hausdorff's definition (Alexandroff-Hopf [1, p.111]) in an obvious way (see Choquet [6]) we say:

(7.21) $y \in \lim\sup E_\mu$ if and only if for every μ and every neighborhood V of y there is a $\mu' \geq \mu$ such that $E_{\mu'}$ intersects V.

We add the customary companion:

(7.22) $y \in \lim\inf E_\mu$ if and only if for every neighborhood V of y there is a μ such that for every $\mu' \geq \mu$, $E_{\mu'}$ intersects V.

Proof of 7.2. Let $\lim\sup F_\mu$ be denoted by L. Then $L(y) = 0$ precisely if for some μ and some V of y and every $\mu' \geq \mu$ we have $F_{\mu'}(y) = 0$. Now suppose that F_μ converge continuously to F (the definition precedes (2.4)), and suppose $F(y) = 0$. It follows at once that $L(y) = 0$. Hence the continuous convergence implies $L \subset F$ (see (7.11)). Conversely, suppose $F \supset L$. To check continuous convergence we need only consider y such that $F(y) = 0$. Then $L(y) = 0$. Reading the second sentence of this proof, we see that the condition of continuous convergence is satisfied.

Observe that continuous limits are not unique. Everything converges continuously to Y itself, for example.

The condition that a topology for $\mathfrak{Z}$ should be admissible is easily deducible from (2.5) and (7.2).

(7.23) THEOREM. *A topology t for* $\mathfrak{Z}$ *is admissible if and only if for every directed set of closed sets* $\{F_\mu\}$ *which converges to F according to t we have* $F \supset \lim\sup F_\mu$.

We shall now consider the significance of proper topologies.

(7.3) THEOREM. *Let X and Y be spaces, and let* Φ *be a closed subset of* $X \times Y$. *For each x let* F_x *be the closed set of points y for which* $(x, y) \in \Phi$. *Then a necessary and sufficient condition that a topology for* $\mathfrak{Z}$ *be proper is that for every* Φ *the associated mapping*

$$f: X \to \mathfrak{Z}; \; f(x) = F_x,$$

be continuous.

According to (6.25), (6.01), and (4.71), when Y is a regular space we can obtain a topology t with the properties of (7.23) and (7.3) if and only if Y is locally compact, and that topology will be the k-topology. We wish to compare this topology with that introduced by Hausdorff into $\mathfrak{Z}$ when Y is a compact metric space (see Alexandroff-Hopf [1]) and further generalized by Choquet [6, pp. 87-93]. Hausdorff's topology H is surely not the same as the k-topology, for $S^Y(H)$ is a Hausdorff space whereas in $S^Y(k)$ the closed set Y has, as its only neighborhood, $\mathfrak{Z}$ itself. Theorem VI (Alexandroff-Hopf [1, p. 115]) shows that convergence in $S^Y(H)$ fulfills the condition of (7.23), so that H is admissible, and thus $H > k$. As a matter of fact, the void set is omitted in Hausdorff's treatment; but a formal application of definitions shows that it would be isolated, as it is in the k-topology.

Since Hausdorff's topology H makes $\mathfrak{Z}$ compact [1], there is no other Hausdorff topology H' lying between k and H. This supports the conjecture that H is the least admissible Hausdorff topology.

8. Topological products. The techniques of this paper enable us to give an answer to the following question: If X is an arbitrary set and Y a *space*, if T is a topology in the set of all couples (x,y), $x \in X$, $y \in Y$, yielding a topological space P, when can a topology t be introduced in X so that P is the topological product $X(t) \times Y$?

(8.1) THEOREM. *With the notations as above, assume T has the following properties:*

(8.11) *For each fixed x_0, the mapping $f_{x_0}(y) = (x_0, y)$ of Y into P is continuous;*

(8.12) *The mapping $g(x,y) = y$ of P into Y is continuous;*

(8.13) *Given any two points (x_0, y_0), (x_0, y_0^-) of P, and any neighborhood V of (x_0, y_0), there exists a neighborhood W of y_0 and a neighborhood V' of (x_0, y_0^-) such that $(x, y^-) \in V'$ and $y \in W$ imply $(x,y) \in V$.*

Then, if P^Y has a proper and admissible topology, there exists a topology t in X with $X(t) \times Y = P$.

Proof. Since by (8.11), for each x, $f_x(y) \in P^Y$, the map F^*, $F^*(x)(y) = f_x(y)$, is a one-to-one mapping of X into P^Y, and so in all that follows we shall consider $X \subset P^Y$. We now give X the topology t of a subset of the space P^Y. Then we have $F^*: X(t) \to P^Y$; and, due to the admissibility, for the associated map we have $F: X(t) \times Y \to P$. It is evident that F is the identity map.

On the basis of (8.13) we note that defining $h[y, (x, y^-)] = (x, y)$ we get $h: Y \times P \to P$; by the properness we find $h^*: P \to P^Y$, and it is easy to see that h^* maps P into $X \subset P^Y$, so that we have $h^*: P \to X$. Using (8.12), we also have $g: P \to Y$. Defining $H(x,y) = [h^*(x,y), g(x,y)]$, we thus have $H: P \to X(t) \times Y$, and H is the identity map. Hence, from the above we find that $X(t) \times Y$ and P are homeomorphic, and the theorem is proved.

Note that in case Y is locally compact and regular, then from (4.71) the k-topology in P^Y, for any space P, is admissible and proper.

(8.2) THEOREM. *Let Y be a locally compact regular space, and let X be an arbitrary set. A necessary and sufficient condition that a topology T in the set $X \times Y$ be a product topology with one factor the space Y, is that T satisfy* (8.11) *through* (8.13).

Proof. The necessity of (8.11) through (8.13) is immediate from elementary properties of topological products. The sufficiency of (8.11) through (8.13) stems from (4.71) and (8.1).

Another result on the behavior of topological products that is implied by our results will now be given. Let Y be a fixed space, and X a set carrying topologies s and t. In the set of all pairs, $X \times Y$, let S be the product topology of $Y \times X(s)$, T the product topology of $Y \times X(t)$, and R the product topology in $Y \times X(s \wedge t)$.

(8.3) THEOREM. *In the set $X \times Y$, we always have $R \leq S \wedge T$. Furthermore:*

(8.31) *If Y is locally compact and regular, then $R = S \wedge T$;*

(8.32) *If Y is not locally compact, then in general $R \neq S \wedge T$.*

Proof. If W is open in R, and $(y, x) \in W$, then there is a set of the form $U \times V$, U open in Y, V open in $X(s)$ and in $X(t)$, with $(y, x) \in U \times V \subset W$. This means that $U \times V$ is open in S and in T, hence in $S \wedge T$. It follows that W is open in $S \wedge T$, so that $R \leq S \wedge T$.

Ad (8.31). Let P denote the set $X \times Y$ with topology $S \wedge T$. Since Y is locally compact, P^Y has, by (4.71), an admissible and proper topology. We first remark that for the identity map, we have $g: Y \times X(s) \to P$; due to properness, we have for the associated map $g^*: X(s) \to P^Y$. Using the same map, we also obtain $g^*: X(t) \to P^Y$, and so evidently we have $g^*: X(s \wedge t) \to P^Y$. By admissibility we find for the identity map $g: Y \times X(s \wedge t) \to P$. This shows that $S \wedge T \leq R$ and, with what we have already shown, this gives $S \wedge T = R$.

Ad (8.32). Let us take Y to be the space of (6.4) and X to be the set Z^Y of (6.4). Let s and t be two admissible topologies whose meet is *not* admissible. We show that $R \neq S \wedge T$. First we note that the evaluation map ω of $Y \times Z^Y(s \wedge t)$ into Z is not continuous in R since $s \wedge t$ is not admissible. But ω is clearly continuous in $S \wedge T$, since it is continuous in both S and T. This proves that $R \neq S \wedge T$, and also establishes the theorem.

In case both factors are allowed to change, the assertion (8.32) always holds, regardless of whether Y is compact or not.

(8.4) THEOREM. *Let X be an arbitrary set, and let s and t be two topologies in X. If S is the topology of $X(s) \times X(s)$, T the topology of $X(t) \times X(t)$, and R the topology of $X(s \wedge t) \times X(s \wedge t)$, then in general $R \neq S \wedge T$.*

Proof. We take X to be the countable set of (3.54) and s and t the two *compact* Hausdorff topologies mentioned there. We remark that it is trivial to prove that the diagonal $D = \{(x, y);\ x = y\}$ in a space $X(r) \times X(r)$ is closed if and only if r is a Hausdorff topology To prove the theorem, we note that on the basis of this remark, D is closed in S and in T, hence in $S \wedge T$, but that D is not closed in R since $s \wedge t$ is not a Hausdorff topology.

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DISTRIBUTIVE LATTICES WITH A THIRD OPERATION DEFINED

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1. Introduction. If L is the direct union of two distributive lattices, one may define a new operation $*$ between any two elements (a,b) and (c,d) of L by

$$(a,b) * (c,d) = (a \cap c,\ \ b \cup d). \tag{1}$$

This operation $*$ is:

$P1$. Idempotent
$P2$. Commutative
$P3$. Associative
$P4$. Distributive with $*$, $\cup$, $\cap$ in all possible ways.

The main results of this paper are the following.

First (Theorem 16), this is essentially the only way in which an operation with properties $P1$-$P4$ can arise in a distributive lattice. That is, if L is a distributive lattice with a binary operation $*$ having properties $P1$-$P4$, then L is a sublattice of the direct union of two distributive lattices, and the operation $*$ is given by equation (1).

Second (Theorem 9), if

$P5$. L contains an identity element e for the operation $*$,

then L is the entire direct union. Here $P5$ is sufficient but not necessary; a necessary and sufficient condition is given in Theorem 17. In case $*$ is identical with $\cup$ or $\cap$, Theorems 9, 16, and 17 still hold, but give trivial decompositions.

Finally, Section 5 shows that the presence of an operation $*$ is equivalent to the existence of a partial ordering with certain properties, so that our theorems may be restated so as to apply to distributive lattices with an auxiliary partial ordering.

2. Preliminary considerations. Throughout the paper, L is a distributive lattice with an operation $*$ having at least properties $P1$-$P4$. By an isomorphism

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between two such algebraic structures, we shall mean a one-to-one correspondence which preserves the operations $\cup$, $\cap$, $*$; as is customary, in the direct union $A \times B$ of two such algebraic structures, all operations act coordinatewise; for example, $(a,b) * (c,d) = (a * c,\ b * d)$.

For later reference, we collect here several simple consequences of $P1$-$P4$. The proofs consist of repeated applications of the idempotent and other laws, and will be presented briefly and without annotation of the separate steps. These results will be used frequently in later proofs without any explicit reference being made. In these theorems, small Latin letters represent arbitrary elements of L.

THEOREM 1. $x \cap y \leq x * y \leq x \cup y$.

Proof. We have

$$(x \cap y) \cup (x * y) = [(x \cap y) \cup x] * [(x \cap y) \cup y] = x * y;$$

thus $x \cap y \leq x * y$. Similarly for the other inequality.

THEOREM 2. *If* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, *then* $x_1 * y_1 \leq x_2 * y_2$.

Proof. We have

$$(x_1 * y_1) \cap (x_1 * y_2) = x_1 * (y_1 \cap y_2) = x_1 * y_1;$$

thus $x_1 * y_1 \leq x_1 * y_2$. Similarly, $x_1 * y_2 \leq x_2 * y_2$, and the theorem follows.

THEOREM 3. $x * (x \cup y) = x \cup (x * y)$ *and* $x * (x \cap y) = x \cap (x * y)$.

Proof. Clearly,

$$x * (x \cup y) = (x * x) \cup (x * y) = x \cup (x * y).$$

Similarly for the other equation.

THEOREM 4. $x * y = (x \cap y) * (x \cup y)$.

Proof. The result follows from the continued equation,

$$\begin{aligned}(x \cap y) * (x \cup y) &= [(x \cap y) * x] \cup [(x \cap y) * y] \\ &= [(x * y) \cap x] \cup [(x * y) \cap y] \\ &= (x * y) \cap (x \cup y) = x * y.\end{aligned}$$

THEOREM 5. $x * (x * y) = x * y$.

Proof. Clearly,

$$x * (x * y) = (x * x) * y = x * y.$$

THEOREM 6. *If* $x \leq u \leq x * y \leq v \leq y$, *then* $u * v = x * y$.

Proof. Since $x \leq u$ and $x * y \leq v$, Theorem 2 shows that

$$x * y = x * (x * y) \leq u * v.$$

Similarly,

$$u * v \leq (x * y) * y = x * y.$$

3. **The operation $*$ has properties $P1$-$P5$.** In this section, we prove one of the main results of the paper (Theorem 9), using the assumption that L contains an element e which is an identity element for the operation $*$.

THEOREM 7. *If* $a \leq e$ *and* $c \leq e$, *then* $a * c = a \cap c$.

Proof. Since $a * c = (a \cap c) * (a \cup c)$, it is sufficient to consider the case $a \leq c \leq e$ and prove $a * c = a$. But then $a \leq a \leq a * e \leq c \leq e$, and Theorem 6 shows that $a * c = a * e = a$.

THEOREM 8. *If* $b \geq e$ *and* $d \geq e$, *then* $b * d = b \cup d$.

The proof is similar to that of Theorem 7.

THEOREM 9. *If L is a distributive lattice with a binary operation $*$ having properties $P1$-$P5$, then L is isomorphic to the direct union of two distributive lattices A, B each with an operation $*$ having properties $P1$-$P5$; and if (a,b), (c,d) are any two elements of $A \times B$, then* $(a,b) * (c,d) = (a \cap c,\ b \cup d)$.

Proof. Set

$$A = \{a \mid a \leq e\}, \quad B = \{b \mid b \geq e\};$$

then, with the same operations as in L, A and B are distributive lattices each with an operation $*$ having properties $P1$-$P5$.

We prove that the correspondence $(a,b) \rightarrow a * b$ is the required isomorphism from $A \times B$ onto L. It is clearly a single valued correspondence from $A \times B$ into L. It covers L because, for any element x of L, we have $x \cap e \in A$, $x \cup e \in B$ and, by Theorem 4, $(x \cap e) * (x \cup e) = x * e = x$. It is one-to-one because, for any $a \in A$, $b \in B$, we have $e \cap (a * b) = (e \cap a) * (e \cap b) = a * e = a$. Thus if $a * b = c * d$, $c \in A$, $d \in B$, then $a = c$. Similarly, $b = d$.

This correspondence preserves the three operations $\cup$, $\cap$, $*$. For instance,

$$(a,b) \cup (c,d) = (a \cup c,\ b \cup d) \rightarrow (a \cup c) * (b \cup d) = (a \cup c) * [(b \cup d) * (b \cup d)].$$

By Theorem 8, $b \cup d = b * d$; making this replacement in one parenthesis only, and rearranging the factors connected by $*$, we have $[(a \cup c) * b] * [(b \cup d) * d]$. But $b = b \cup c$, $d = a \cup d$, so that we have

$$[(a \cup c) * (b \cup c)] * [(b \cup d) * (a \cup d)]$$
$$= [(a * b) \cup c] * [(b * a) \cup d] = (a * b) \cup (c * d),$$

whence the operation $\cup$ is preserved by our correspondence. Similarly for $\cap$.

For the operation $*$,

$$(a,b) * (c,d) = (a * c,\ b * d) \to (a * c) * (b * d) = (a * b) * (c * d).$$

Thus our correspondence is an isomorphism.

By Theorems 7 and 8, $(a,b) * (c,d) = (a \cap c,\ b \cup d)$. This completes the proof.

REMARK. The element e will be the I in A and the O in B. The lattice A will have an O if and only if L has one; B will have an I if and only if L has one.

4. The operation $*$ has properties $P1$-$P4$. In this section, we prove one of the main results of the paper (Theorem 16). The method employed is to complete L in such a way that $*$ has properties $P1$-$P5$ and then to apply Theorem 9. Several preliminary definitions and theorems will be of use.

DEFINITION 1. We extend the operations $\cup$, $\cap$, $*$ to act on any subsets H, K of L by defining $H \cup K = \{x \cup y \mid x \in H,\ y \in K\}$, and similarly for the other operations.

Notice that $H \cup K$, for example, is a subset of L, and is usually neither the supremum of the elements in the subsets H and K nor the point set union of H and K.

DEFINITION 2. A subset P of L is a *$*$-ideal* if $P * L \subset P$.

For any fixed $a \in L$, the set $a * L$ is a $*$-ideal; it is called the *principal $*$-ideal generated by a.*

THEOREM 10. *An element x of L is in the principal $*$-ideal A generated by a if and only if $a * x = x$.*

The sufficiency is evident. To prove necessity, we note that if $x \in A$, then $x = a * y$; by Theorem 5 it follows that $a * x = a * (a * y) = x$.

DEFINITION 3. A subset H of L is *intervally closed* if $x \in H$, $y \in H$, and $x \leq z \leq y$ imply $z \in H$. The *interval closure* of a set G is the smallest intervally closed subset containing G.

It is easily seen that the interval closure of any set is the collection of elements which lie between two elements of the set.

DEFINITION 4. A subset R of L is *special* if it is

(a) a $*$-ideal,
(b) a sublattice, and
(c) intervally closed.

THEOREM 11. *Each principal $*$-ideal is also a special subset.*

Proof. Let A be the principal $*$-ideal generated by a, and let x, y be any two elements of A. Then $a * (x \cup y) = (a * x) \cup (a * y) = x \cup y$, and $x \cup y \in A$, by Theorem 10. Similarly $x \cap y \in A$, and A is a sublattice of L.

If x, y are any two elements of A, and $x \leq z \leq y$, we must show that z is in A. Since A contains $a \cap x$ and $a \cup y$, there is no loss in generality in supposing that $x \leq a \leq y$. Set $a * (a \cup z) = u$. We prove first that $u = a \cup z$. Since $a * y = y$ and $a \leq u \leq y$, Theorem 6 shows that $u * y = y$, so that

$$(a \cup z) \cap (u * y) = (a \cup z) \cap y = a \cup z. \tag{2}$$

By Theorem 5, $u * (a \cup z) = u$. From the definition of u and Theorem 1, we have $u \leq a \cup z$, so that

$$[(a \cup z) \cap u] * [(a \cup z) \cap y] = u * (a \cup z) = u. \tag{3}$$

But, from the distributive law, the left-hand members of equations (2) and (3) are equal, and $u = a \cup z$.

We now proceed with the proof that $z \in A$. Set

$$v = z \cap (a * z) = z * (a \cap z).$$

Then, by Theorem 1, $a \cap z \leq v \leq z$, and

$$a \cup v = a \cup [z * (a \cap z)] = (a \cup z) * [a \cup (a \cap z)] = (a \cup z) * a = u = a \cup z.$$

But now

$$v = (a \cap z) \cup v = (a \cup v) \cap (z \cup v) = (a \cup z) \cap (z \cup v) = (a \cup z) \cap z = z.$$

That is, $z \cap (a * z) = z$, whence $z \leq a * z$. Similarly, $z \geq a * z$. Thus $z = a * z$ and, by Theorem 10, $z \in A$.

THEOREM 12. *If P is any $*$-ideal, the interval closure of the sublattice generated by P is a special subset of L.*

Proof. Let Q be the sublattice generated by P, and let $\bar{Q}$ be the interval closure of Q. Then evidently $\bar{Q}$ is intervally closed.

$\bar{Q}$ is a sublattice because if $x, y \in \bar{Q}$, there exist elements u_1, u_2, v_1, v_2 of Q such that $u_1 \leq x \leq v_1$, $u_2 \leq y \leq v_2$. Then $u_1 \cup u_2 \leq x \cup y \leq v_1 \cup v_2$ and, since Q

is a lattice, $x \cup y \in \bar{Q}$. Similarly $x \cap y \in \bar{Q}$, so that $\bar{Q}$ is a sublattice of L.

$\bar{Q}$ is a $*$-ideal. Since $*$ distributes over $\cup$, $\cap$, Q is a $*$-ideal, and Theorem 2 then shows that $\bar{Q}$ is a $*$-ideal. This completes the proof.

REMARK. It is evident that any special subset containing P must contain $\bar{Q}$. Thus Theorem 12 gives a construction for the smallest special subset containing (generated by) a given $*$-ideal.

THEOREM 13. *If R, S are special subsets of L, then $R * S$, $R \cup S$, $R \cap S$ are special subsets of L.*

Proof. That each of the sets $R * S$, $R \cup S$, $R \cap S$ is a $*$-ideal is a simple consequence of the distributive laws and Definition 1.

To see that $R * S$ is a special subset, note that $R * S$ is contained in both R and S, since both are $*$-ideals; but clearly $R * S$ contains the point-set intersection of R and S since $*$ is idempotent. Thus $R * S$ is this intersection, which is easily seen to be a special subset of L.

$R \cup S$ is intervally closed because, if $r_1 \cup s_1 \leq x \leq r_2 \cup s_2$, then

$$r_1 \cap r_2 \leq x \cap r_2 \leq r_2$$

and, since R is a special subset, $x \cap r_2 \in R$. Similarly, $x \cap s_2 \in S$. But then

$$(x \cap r_2) \cup (x \cap s_2) = x \cap (r_2 \cup s_2) = x$$

lies in $R \cup S$, and $R \cup S$ is intervally closed.

$R \cup S$ is a sublattice of L because, if $r_1 \cup s_1$, $r_2 \cup s_2$ are any two elements of $R \cup S$, clearly $(r_1 \cup s_1) \cup (r_2 \cup s_2) = (r_1 \cup r_2) \cup (s_1 \cup s_2)$ lies in $R \cup S$. Also, since $r_1 \cap r_2 \leq (r_1 \cup s_1) \cap (r_2 \cup s_2)$ and $s_1 \cap s_2 \leq (r_1 \cup s_1) \cap (r_2 \cup s_2)$, we have

$$(r_1 \cap r_2) \cup (s_1 \cap s_2) \leq (r_1 \cup s_1) \cap (r_2 \cup s_2) \leq (r_1 \cup r_2) \cup (s_1 \cup s_2).$$

But the two extreme elements of this sequence of inequalities lie in $R \cup S$; thus, since $R \cup S$ is intervally closed, the center element also lies in $R \cup S$. This completes the proof that $R \cup S$ is a special subset; dually, $R \cap S$ is a special subset.

DEFINITION 5. $\mathfrak{L} = \{R, S, T, \cdots\}$ is the collection of all special subsets of L with the three operations $\cup$, $\cap$, $*$.

THEOREM 14. *The set $\mathfrak{L}$ with the operations $\cup$, $\cap$ is a distributive lattice and $*$ has properties $P1$-$P5$.*

Proof. Theorem 13 shows that $\mathfrak{L}$ is closed under the operations $\cup$, $\cap$, $*$. To show that $\mathfrak{L}$ is a distributive lattice, we prove

1. $R \cup R = R, \ R \cap R = R$,
2. $R \cup S = S \cup R, \ R \cap S = S \cap R$,
3. $(R \cup S) \cup T = R \cup (S \cup T), \ (R \cap S) \cap T = R \cap (S \cap T)$,
4. $R \cup (R \cap S) = R$,
5. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$.

Numbers 1, 2, and 3 are evident. To prove 4, we note that clearly

$$R \cup (R \cap S) \supset R;$$

we show that $R \cup (R \cap S) \subset R$. If $x = r_1 \cup (r_2 \cap s)$ is any element of $R \cup (R \cap S)$, then $r_1 \leq x \leq r_1 \cup r_2$, and $x \in R$.

To prove 5, we note that clearly $(R \cup S) \cap (R \cup T) \supset R \cup (S \cap T)$; we show that $(R \cup S) \cap (R \cup T) \subset R \cup (S \cap T)$. If $x = (r_1 \cup s) \cap (r_2 \cup t)$ is any element of $(R \cup S) \cap (R \cup T)$, then $(r_1 \cap r_2) \cup (s \cap t) \leq x \leq (r_1 \cup r_2) \cup (s \cap t)$, and

$$x \in R \cup (S \cap T).$$

The proofs that the operation $*$ has properties $P1$-$P4$ are similar to those just given and will be omitted. For $P5$, the lattice L itself is a special subset of L and acts as the identity element for the operation $*$ in $\mathfrak{L}$.

THEOREM 15. *The correspondence* $x \rightarrow$ *the principal* $*$*-ideal generated by* x *is an isomorphism of* L *onto a sublattice of* $\mathfrak{L}$ *which identifies the operations* $*$ *in* L *and the sublattice of* $\mathfrak{L}$.

Proof. By Theorem 10, if x, y generate the same principal $*$-ideal, then $y = x * y = y * x = x$, so that the above correspondence is one-to-one.

To prove that this correspondence is an isomorphism, let

$$x \rightarrow X = x * L, \ \ y \rightarrow Y = y * L; \ \text{ then } x \cup y \rightarrow (x \cup y) * L = Z.$$

Clearly $Z \subset X \cup Y$. Conversely, if $w = (x * u) \cup (y * v)$ is any element of $X \cup Y$, then $(x \cup y) * (u \cap v) \leq w \leq (x \cup y) * (u \cup v)$, and $w \in Z$. The proofs for $\cap$, $*$ are similar, and will be omitted.

Theorems 9, 14, and 15 give immediately our main result:

THEOREM 16. *If* L *is any distributive lattice with an operation* $*$ *having properties* $P1$-$P4$, *then* L *is isomorphic to a sublattice of the direct union of two distributive lattices* A, B, *each with an operation* $*$ *having properties* $P1$-$P5$; *and if* (a,b), (c,d) *are any two elements of* $A \times B$, *then*

$$(a,b) * (c,d) = (a \cap c, \ b \cup d).$$

THEOREM 17. *If* L *is any distributive lattice with an operation* $*$ *having properties* $P1$-$P4$, *then* L *is isomorphic to a direct union in which the operation* $*$

is given by equation (1) *if and only if each pair of elements of L is contained in some principal *-ideal.*

Necessity. If L is a direct union with $*$ given by equation (1), the two arbitrary elements (a,b), (c,d) are contained in the principal $*$-ideal generated by $(a \cup c, b \cap d)$.

Sufficiency. By Theorem 16, L may be considered as a sublattice of a direct union in which $*$ is given by equation (1). Let (x_1,y_1) be any fixed element of L, and define $A = \{x \mid (x,y_1) \in L\}$, $B = \{y \mid (x_1,y) \in L\}$; then $A \times B \subset L$. In fact, if (x,y_1) and (x_1,y) are in the principal $*$-ideal generated by (a,b), then

$$a \geq x \cup x_1, \quad b \leq y \cap y_1,$$

and L contains

$$[(x,y_1) \cap (a,b)] * [(x_1,y) \cup (a,b)] = (x,b) * (a,y) = (x,y).$$

Conversely, $L \subset A \times B$, for if (x,y) is any element of L, and (a,b) generates a principal $*$-ideal containing (x,y) and (x_1,y_1), then L contains

$$\begin{aligned}[(x,y) \cap (a,b)] \cup \{(x_1,y_1) \cap [(x,y) * (x_1,y_1)]\} \\ = (x,b) \cup \{(x_1,y_1) \cap (x \cap x_1, y \cup y_1)\} = (x,y_1).\end{aligned}$$

Similarly, (x_1,y) is in L, and $(x,y) \in A \times B$.

CAUTION. The decomposition of L will be trivial (one of A, B consisting of a single element) if and only if $*$ is identical with $\cup$ or $\cap$.

5. **The ordering equivalent to $*$.** In any distributive lattice L with an operation $*$ having properties $P1$-$P4$, we may define an auxiliary order relation by making $x \succ y$ mean $x * y = y$. It is easily seen that this order relation has the following properties:

- $O1$. $x \succ x$;
- $O2$. $x \succ y$, $y \succ x$ imply $x = y$;
- $O3$. $x \succ y$, $y \succ z$ imply $x \succ z$;
- $O4$. Any two elements x, y of L have a greatest lower bound (namely $x * y$);
- $O5$. The operation of taking the greatest lower bound is distributive with itself and with the two lattice operations in all possible ways.

Conversely, if L is any distributive lattice (with no additional operation $*$ defined) with an auxiliary order relation having properties $O1$-$O5$, then the operation $*$ defined in L by setting $x * y$ equal to the greatest lower bound of x and y has properties $P1$-$P4$. Moreover, the operation $*$ will have property $P5$ if and only if the order relation satisfies:

$O6$. There is a greatest element e in L.

Our results may thus be restated as theorems concerning distributive lattices with an auxiliary order relation.

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CONCERNING HEREDITARILY INDECOMPOSABLE CONTINUA

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1. Introduction. A continuum is indecomposable if it is not the sum of two proper subcontinua. It is hereditarily indecomposable if each of its subcontinua is indecomposable. In [3] Knaster gave an example of a hereditarily indecomposable continuum which was not a point. In this paper we study some properties of the Knaster example and describe some other hereditarily indecomposable continua.

2. Chained hereditarily indecomposable continua are homeomorphic. The hereditarily indecomposable continuum given [3] by Knaster was a plane continuum which was described in terms of covering bands. For each positive number ϵ, it could be covered by an ϵ-chain. Moise used [5] a hereditarily indecomposable continuum to exhibit a continuum which was topologically equivalent to each of its nondegenerate subcontinua. He called it a pseudo-arc and noted that it was similar (if not in fact topologically equivalent) to Knaster's example. It could be chained. Bing used [2] such a continuum as an example of a homogeneous plane continuum. Anderson showed [1] that the plane is the sum of a continuous collection of such continua. Theorem 1 reveals that all of these continua are topologically equivalent.

We follow the definitions used in [2]. In particular, we recall the following. A *chain* $D = [d_1, d_2, \cdots, d_n]$ is a collection of open sets $d_1, d_2, \cdots, d_n$ such that d_i intersects d_j if and only if i is equal to $j-1, j$, or $j+1$. If the links are of diameter less than ϵ, the chain is called an ϵ-*chain*. We do not suppose that the links of a chain are necessarily connected.

If the chain $E = [e_1, e_2, \cdots, e_n]$ is a refinement of the chain $D = [d_1, d_2, \cdots, d_m]$, E is called *crooked* in D provided that if $k - h > 2$ and e_i and e_j are links of E in links d_h and d_k of D, respectively, then there are links e_r and e_s of E in links d_{k-1} and d_{h+1}, respectively, such that either $i > r > s > j$ or $i < r < s < j$.

EXAMPLE 1. *The pseudo-arc.* The following description of a chained hereditarily indecomposable continuum appeared in [2] and is much like one given

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earlier in [5]. In the plane let D_1, D_2, $\cdots$ be a sequence of chains between the distinct points p and q such that for each positive integer i, (a) D_{i+1} is crooked in D_i, (b) no link of D_i has a diameter of more than $1/i$, and (c) the closure of each link of D_{i+1} is a compact subset of a link of D_i. The common part of the sum of the links of D_1, the sum of the links of D_2, $\cdots$ is a pseudo-arc. That it is hereditarily indecomposable is shown in [2] and [5].

A continuum can be chained if for each positive number ϵ, the continuum can be covered by an ϵ-chain. A *composant* of a continuum W is a set H such that for some point p of W, H is the sum of all proper subcontinua of W containing p. We recall that a nondegenerate indecomposable compact continuum has uncountably many composants and no two of its composants intersect each other. The following result holds in a metric space.

THEOREM 1. *The compact nondegenerate hereditarily indecomposable continua M and M' are homeomorphic if each can be chained.*

In fact, if p and q are points of different composants of M while p' and q' are points of different composants of M', there is a homeomorphism carrying M into M', p into p', and q into q'.

Proof. Since M can be chained, there is a sequence C_1, C_2, $\cdots$ such that C_i is a $1/i$-chain covering M, each element of C_i intersects M, and C_{i+1} is a refinement of C_i.

First, we show that there is an integer n_2 so large that C_{n_2} is crooked in $C_1 = [c_{1,1}, c_{1,2}, \cdots, c_{1,t_1}]$. If this were not true there would be elements $c_{1,h}$ and $c_{1,k}$ of C_1 such that $k - h > 2$ and for infinitely many integers m, $C_m = [c_{m,1}, c_{m,2}, \cdots, c_{m,t_m}]$ would have two links $c_{m,i}$ and $c_{m,j}$ in $c_{1,h}$ and $c_{1,k}$ respectively such that if $c_{m,r}$ is in $c_{1,k-1}$ and between $c_{m,i}$ and $c_{m,j}$, then there is not a link of C_m in $c_{1,h+1}$ which is between $c_{m,r}$ and $c_{m,j}$. Denote by W_m the sum of $c_{m,i}$, $c_{m,r}$, and the elements of C_m between them, where we suppose that no element of C_m in $c_{1,k-1}$ is between $c_{m,i}$ and $c_{m,r}$. Let V_m be the sum of $c_{m,r}$, $c_{m,j}$, and the elements of C_m between them.

Let a_1, a_2, $\cdots$ be an increasing sequence of integers such that both W_{a_1}, W_{a_2}, $\cdots$ and V_{a_1}, V_{a_2}, $\cdots$ converge. But the limiting set W of W_{a_1}, W_{a_2}, $\cdots$ is a continuum which intersects $\bar{c}_{1,h}$ but not $\bar{c}_{1,k}$. Also, the limiting set V of V_{a_1}, V_{a_2}, $\cdots$ is a continuum which intersects $\bar{c}_{1,k}$ but not $\bar{c}_{1,h}$. Hence the assumption that there is no integer n_2 such that C_{n_2} is crooked in C_1 has led to the contradiction that the hereditarily indecomposable continuum M contains the decomposable continuum $W + V$.

Hence, there is a subsequence C_{n_1}, C_{n_2}, C_{n_3}, $\cdots$ of C_1, C_2, C_3, $\cdots$ such that $C_{n_{i+1}}$ is crooked in C_{n_i}.

Let p and q be points belonging to different composants of M. Then for each integer j, there is an integer k greater than j such that the subchain of C_{n_k} from p to q has a link that intersects the first link of C_{n_j} and has a link that intersects the last link of C_{n_j}. To see that this is so, let W_i be the sum of the links of the subchain of C_{n_i} from p to q. Since the limiting set of each subsequence W_1, W_2, $\cdots$ is a continuum in M containing $p + q$, each such limiting set is M. Hence, some $W_k (k > j)$ intersects both the first and last links of C_{n_k}, and the subchain of C_{n_k} corresponding to W_k has links intersecting the first and last links of C_{n_j}.

We find from Theorem 4 of [2] that there is a chain E_j such that the first link of E_j contains p, the last contains q, E_j is a consolidation of C_{n_j}, while each link of E_j lies in the sum of two adjacent links of C_{n_j} and is therefore of diameter less than $2/j$. Hence, there is no loss of generality in supposing that each of the chains C_1, C_2, $\cdots$ is from p to q. Therefore, there is a sequence D_1, D_2, $\cdots$ of chains from p to q such that for each positive integer i, (a) D_{i+1} is crooked in D_i, (b) the closure of each link of D_{i+1} is a subset of a link of D_i, (c) no link of D_i has a diameter of more than $1/i$, and (d) M is the common part of the sum of the links of D_1, the sum of the links of D_2, $\cdots$.

Similarly, we find that if p' and q' are points of different composants of M', there is a sequence D_1', D_2', $\cdots$ of chains from p' to q' such that (a) D_{i+1}' is crooked in D_i', (b) the closure of each element of D_{i+1}' is a subset of an element of D_i', (c) no element of D_i' has a diameter of more than $1/i$, and (d) D_i covers M'. Theorem 12 of [2] shows that there is a homeomorphism that carries M into M', p into p', and q into q'.

The preceding theorem shows that M is homogeneous and homeomorphic with each of its nondegenerate subcontinua. It also reveals that the continua studied by Knaster [3], Moise [5], Bing [2], and Anderson [1] are all topologically equivalent and are pseudo-arcs.

QUESTION. It would be interesting to know if each nondegenerate bounded hereditarily indecomposable plane continuum which does not separate the plane is homeomorphic to M. This question would be answered in the affirmative if it were shown that each bounded atriodic plane continuum which does not separate the plane can be chained (see Section 6, below).

3. Most continua are pseudo-arcs. Mazurkiewicz showed [4] that the continua contained in a circle plus its interior which were not hereditarily indecomposable were of the first category. We go even further to show that those which are not pseudo-arcs are of the first category.

The *space of compact continua* in the metric space S is the metric space $C(S)$ whose points are the compact continua of S, where the distance between two elements g_1, g_2 of $C(S)$ is the Hausdorff distance between them in S, namely L.U.B. (distance from x to g_i in S), $x \in g_1 + g_2$; $i = 1,2$. By saying that *most compact continua of S have a certain property*, we mean that there is a dense inner limiting (G_δ) subset W of $C(S)$ such that each element of W has the property (when regarded as a continuum in S). The collection of continua of S with this property is said to be of the second category.

The following theorem holds in a space S which is either a Hilbert space or a Euclidean n-space ($n > 1$).

THEOREM 2. *Most compact continua are pseudo-arcs.*

Proof. Let F_i be the collection of all compact continua f in S such that f can not be covered by a $1/i$-chain. If $f_1, f_2, \cdots$ is a sequence of elements of F_i converging to a compact continuum f_0, f_0 is an element of F_i because if a $1/i$-chain covers f_0, it covers some f_j. Hence, F_i is closed in $C(S)$.

Let G_i be the set of all compact continua K such that K contains a subcontinuum K' which is the sum of two continua K_1 and K_2 such that K_1 contains a point at a distance (in S) of $1/i$ or more from K_2, while K_2 contains a point at a distance of $1/i$ or more from K_1. Then G_i is a closed subset of $C(S)$. Furthermore, the collection of points of S is closed in $C(S)$.

The collection of all pseudo-arcs is dense in $C(S)$. For suppose that g is an element of $C(S)$ and ϵ is a positive number. There is a broken line ab whose distance (in $C(S)$) from g is less than $\epsilon/2$. Let D be an $\epsilon/2$-chain from a to b covering ab such that each element of D is the interior of a sphere. There is a pseudo-arc h containing $a + b$ which is covered by D. The distance from g to h is less than ϵ in $C(S)$.

Each element of $C(S)$ not belonging to ΣF_i can be chained and each element of $C(S)$ not belonging to ΣG_i is hereditarily indecomposable. Let W be the set of all elements g of $C(S)$ such that g is not a point of S, g is not an element of any F_i, and g is not an element of any G_i. By Theorem 1, each element of W is topologically equivalent to M. However, W is a dense inner limiting subset of $C(S)$.

4. Decomposition of a pseudo-arc. Like a simple closed curve, the pseudo-arc is homogeneous; and like an arc it is homeomorphic with each of its nondegenerate subcontinua. We now show that like both a simple closed curve and an arc, it is topologically equivalent to itself under a nontrivial monotone decomposition.

THEOREM 3. *If an upper semi-continuous collection of continua fills a chained compact continuum, the resulting decomposition space is chained.*

Proof. Suppose G is an upper semi-continuous collection of continua filling the chained continuum M and that the resulting decomposition space M' is given a metric. We show that for each positive number ϵ, an ϵ-chain in M' covers M'.

Let δ be a positive number so small that if two elements of G are no farther apart in M than δ, then the points in M' corresponding to them are no farther apart than $\epsilon/5$ in M'. Let $[c(1), c(2), \cdots, c(n)]$ be a δ-chain covering M. Let $n_1 = 1$, $n_2, n_3, \cdots, n_j = n$ be a monotone increasing sequence of integers such that an element of G intersects $c(n_i)$ and $c(n_{i+1})$, but none intersects $c(n_i)$ and $c(n_{i+1}+1)$. Denote by $D(i,j)$ the open subset of M' consisting of those points corresponding to elements of G which are covered by $c(i) + c(i+1) + \cdots + c(j)$. Then

$$D(n_1, n_5),\ D(n_4, n_8),\ D(n_7, n_{11}), \cdots,\ D(n_{3k+1}, n_j),$$

$$j - 5 \leq 3k + 1 < j - 3$$

is an ϵ-chain covering M'.

The following result follows from Theorems 1 and 3.

THEOREM 4. *If M is a pseudo-arc and G is an upper semi-continuous collection of proper subcontinua of M filling M, the resulting decomposition space is topologically equivalent to M.*

5. Other types of hereditarily indecomposable plane continua. The pseudo-arc can be imbedded in the plane. We now show that there are nondegenerate hereditarily indecomposable plane continua which are not topologically equivalent to the pseudo-arc. In fact, there are as many topologically different hereditarily indecomposable plane continua as there are plane continua.

A method which differs from the one below of showing that there are topologically different nondegenerate plane continua is to consider in 3-space the intersection of a plane with a hereditarily indecomposable continuum which separates 3-space. That there are such continua in 3-space will be shown in my paper, *Higher dimensional hereditarily indecomposable continua,* which proves that there are hereditarily indecomposable continua of all dimensions.

EXAMPLE 2. *A hereditarily indecomposable continuum that separates the plane.* A circular chain differs from a chain in that its first and last links intersects each other. In the plane let $D_1, D_2, \cdots$ be a sequence of circular chains such that (a) each element of D_i is the interior of a circle of diameter less than $1/i$, (b) the closure of each element of D_{i+1} lies in an element of D_i, (c) the sum of the elements of D_i is topologically equivalent to the interior of an annulus ring, (d) each complementary domain of the sum of the elements of D_{i+1} contains a complementary domain of the sum of the elements of D_i, and (e) if E_i is a proper subchain of D_i and E_{i+1} is a subchain of D_{i+1} contained in E_i, then E_{i+1} is crooked in E_i. Condition (c) is superfluous since it follows from condition (a). The required continuum M is the intersection of the sum of the elements of D_1, the sum of the elements of $D_2, \cdots$.

Suppose that a chain $D_i = [d_1, d_2, \cdots, d_n]$ satisfying the preceding conditions has been found. To see that there is a chain D_{i+1} satisfying the required conditions, it might be convenient first to consider a chain $D_i' = [d_1', d_2', \cdots, d_{3n}']$ in D_i which follows the following pattern: $d_1', d_{n+1}', d_{2n+1}'$ are subsets of d_1; d_2', d_{n+2}', d_{3n}' are subsets of d_2; $d_3', d_{n+3}', d_{3n-1}'$ are subsets of $d_3; \cdots$; d_n', d_{2n}', d_{2n+1}' are subsets of d_n. Roughly speaking, D_i' goes through D_i twice in one direction and once in the opposite direction. Then the circular chain D_{i+1} satisfying the required conditions is the sum of two chains one of which is crooked in $[d_1', d_2', \cdots, d_{2n+1}']$ and the other of which is crooked in

$$[d'_{2n+1}, d'_{2n+2}, \cdots, d'_{3n}, d'_i].$$

That M separates the plane follows from conditions (c) and (d). We show that it is hereditarily indecomposable by showing that each of its proper subcontinua is indecomposable.

Suppose M' is a proper subcontinuum of M. Let m be an integer so large that some element of D_m does not intersect M'. If $E_k (k \geq m)$ is the collection of elements of D_k which intersect M', then $E_m, E_{m+1}, \cdots$ is a sequence of chains such that E_{k+1} is crooked in E_k. If M' were the sum of two proper subcontinua H and K, there would be a point p of H; a point q of K, and an integer w such that the distances from p to K and from q to H are each greater than $2/w$. Suppose e_i and e_j are elements of E_{w+1} containing p and q respectively. Since E_{w+1} is crooked in E_w, there are elements e_r and e_s in E_{w+1} such that e_r separates e_i from e_s in E_{w+1}, each point of e_s is nearer than $2/w$ to p, and each point of e_r is nearer to q than $2/w$. Then H would not be connected because it has a point in e_s, a point in

e_i, but none in e_r.

QUESTIONS. Theorem 1 shows that if M_1 and M_2 are two nondegenerate chained hereditarily indecomposable continua, they are topologically equivalent and each is homogeneous. Suppose W_1 and W_2 are two continua, each defined as described in Example 2. It would be interesting to know if W_1 would necessarily be homeomorphic to W_2. Also, is any such continuum homogeneous?

EXAMPLE 3. *Another hereditarily indecomposable continuum.* The following example of a hereditarily indecomposable plane continuum is somewhat of a combination of Examples 1 and 2.

If W is a nondegenerate indecomposable plane continuum, it contains a nondegenerate subcontinuum H such that no point of H is accessible from the complement of W. To see that this is true, consider two parallel lines L_1 and L_2, each of which separates W. Let G be an uncountable collection of mutually exclusive subcontinua of W each irreducible from L_1 to L_2, and let K be the sum of L_1, L_2, and the closure of the sum of the elements of G. If D is a complementary domain of K between L_1 and L_2, its closure does not intersect three elements of G. Hence, some element of G is not accessible from any complementary domain of K between L_1 and L_2. If H is a subcontinuum of this element of G which does not intersect $L_1 + L_2$, no point of H is accessible from the complement of W.

If W_0 is a nondegenerate hereditarily indecomposable plane continuum, there is a point p of W_0 such that if W' is a nondegenerate subcontinuum of W_0 containing p, then p is not accessible from the complement of W'. To find such a point, let $W_1, W_2, \cdots$ be a sequence of continua such that W_i is a subcontinuum of W_{i-1}, no point of W_{i+1} is accessible from the complement of W_i, and W_i is of diameter less than $1/i$. Then $W_0 \cdot W_1 \cdot W_2 \cdot \cdots$ is a point p; and if W' is a nondegenerate subcontinuum of W_0 containing p, it contains one of the W_i's. Hence, p is not accessible from the complement of W'.

Let M_1 be a pseudo-arc in the plane and M_2 be a hereditarily indecomposable plane continuum as described in Example 2. Let p be a point of M_1 such that p is not accessible from the complement of any nondegenerate subcontinuum of M_1. By a theorem of R. L. Moore, there is a continuous transformation T of the plane into itself such that $T^{-1}(p) = (M_2$ plus its interior) and the inverse of each other point is a point. We show that $M_3 = (T^{-1}(M_1)$ minus the interior of $M_2)$ is hereditarily indecomposable. That M_2 and M_3 are not homeomorphic follows from the fact that M_2 is irreducible with respect to separating the plane but M_3 is not.

If W is a subcontinuum of M_3 intersecting M_2 and containing a point of $M_3 - M_2$, it contains M_2. For suppose that it does not contain some point of M_2. Then there is an arc α from the exterior of M_2 to M_2 that does not intersect W. But $T(\alpha)$ would be an arc, revealing that p is accessible from the complement of $T(W)$. This is contrary to the definition of p.

Suppose that some subcontinuum M' of M_3 is the sum of two proper subcontinua H and K. Since M_2 is hereditarily indecomposable, M' is not a subset of M_2. Since M_1 is hereditarily indecomposable, we may suppose that $T(H) = T(M')$. But H would equal M' because T is one-one on the exterior of M_2 and H contains M_2 if it intersects it.

A variation of the method used in obtaining Example 3 may be used to get other topologically different hereditarily indecomposable plane continua. Instead of replacing a point of M_1 by a continuum homeomorphic to M_2, we can replace each of several points of M_1 by such a continuum.

THEOREM 5. *There are as many topologically different hereditarily indecomposable bounded plane continua as there are real numbers.*

Proof. Suppose $n_1, n_2, \cdots$ is a monotone increasing sequence of positive integers. The collection of such sequences has the power of the continuum. For each such sequence we describe a hereditarily indecomposable plane continuum such that no two of these continua are topologically equivalent.

The hereditarily indecomposable continuum associated with $n_1, n_2, \cdots$ will have one composant containing exactly n_1 continua each topologically equivalent to M_2 of Example 2, another composant containing exactly n_2 continua each topologically equivalent to M_2, $\cdots$; no other composant contains a continuum topologically equivalent to M_2.

Let M_1 be a pseudo-arc in the plane. Suppose

$$p_{1,1}, p_{1,2}, \cdots, p_{1,n_1}, p_{2,1}, \cdots$$

is a converging sequence of different points of M_1 such that $p_{i,j}$ is not accessible from the complement of any nondegenerate subcontinuum of M_1 containing it, and $p_{i,j}$ belongs to the composant containing $p_{r,s}$ if and only if $i = r$.

Suppose $M_{1,1}, M_{1,2}, \cdots, M_{1,n_1}, M_{2,1}, \cdots$ is a sequence of mutually exclusive continua in the plane all topologically equivalent to M_2 of Example 2 such that the sequence converges to a point and each of the continua lies in the exterior of each of the others. There is a continuous transformation T of the plane into itself such that $T^{-1}(p_{i,j})$ is ($M_{i,j}$ plus its interior), and $T^{-1}(q)$ is a point if q is not a $p_{i,j}$. The argument used in Example 3 shows that $M_3 = (T^{-1}(M_1)$ minus the sum of the interiors of the $M_{i,j}$'s) is a hereditarily indecomposable continuum. Furthermore, one composant of M_3 contains exactly n_1 mutually exclusive subcontinua, each topologically equivalent to M_2, another composant contains exactly n_2 such subcontinua, · · ·, while the other composants of M_3 contain no such continua.

6. **Added in proof.** R. D. Anderson answered the question at the end of Section 2 in the negative by announcing at the February, 1951, meeting of the American Mathematical Society in New York that there are nondegenerate bounded hereditarily indecomposable plane continua other than pseudo-arcs which do not separate the plane.

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GENERALIZATIONS OF HYPERGEODESICS

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1. Introduction. In a recently published paper [1] the author studied families of hypergeodesic curves on a surface in ordinary euclidean space of three dimensions. Here we wish to define a more general class of curves on a surface which will contain all the hypergeodesics as a subclass and at the same time will possess most of the properties of the subclass of hypergeodesics.

The summation convention of tensor analysis with regard to repeated indices will be observed with Greek letter indices taking on the values 1, 2, and Latin letter indices taking on the values 1, 2, 3. The notation of Eisenhart [2] will be used throughout.

2. The differential equation. Consider a surface S in ordinary space of three dimensions represented by the three parametric equations

$$x^i = x^i(u^1, u^2) \qquad (i = 1,2,3)$$

referred to a rectangular cartesian coordinate system. A family of hypergeodesics on S is defined as the set of all solutions $u^\alpha = u^\alpha(s)$ $(\alpha = 1,2)$ of a differential equation [1]

$$K_g = \Omega_{\alpha\beta\gamma} u'^\alpha u'^\beta u'^\gamma \,, \tag{2.1}$$

where K_g is the expression for geodesic curvature of a curve C given by $u^\alpha = u^\alpha(s)$ $(\alpha = 1,2)$ in which the parameter s is arc-length, the primes indicate differentiation with respect to s, and the $\Omega_{\alpha\beta\gamma}$ are the covariant components of a tensor of the third order relative to transformations of the surface coordinates u^1 and u^2. If we use the scalar Ω to abbreviate the right member of (2.1), the equation reads $K_g = \Omega$ where Ω is a polynomial homogeneous of degree three in the parameters u'^1 and u'^2 with coefficients as analytic functions of u^1 and u^2. Division by $(u'^1)^3$ and some further simplification reduces this differential equation to a form stating that the second derivative of u^2 with respect to u^1 is equal to a cubic in the first derivative.

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In order to retain most of the properties of hypergeodesics for our generalizations we replace only the polynomial Ω by a rational function U/V with the same homogeneity property. Thus a curve $C: u^{\alpha} = u^{\alpha}(s)$ $(\alpha = 1,2)$ will be called a *supergeodesic* if it satisfies a differential equation of the form

$$K_g = W \tag{2.2}$$

where the scalar W is the quotient U/V of the two scalars

$$U \equiv U_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} u'^{\alpha_1} u'^{\alpha_2} u'^{\alpha_3} \cdots u'^{\alpha_n} \tag{2.3}$$

and

$$V \equiv V_{\beta_1 \beta_2 \beta_3 \cdots \beta_{n-3}} u'^{\beta_1} u'^{\beta_2} u'^{\beta_3} \cdots u'^{\beta_{n-3}} . \tag{2.4}$$

If we divide equation (2.2) by $(u'^1)^3$ it reduces to

$$\frac{d}{du^1}\left(\frac{du^2}{du^1}\right) = \frac{A_0 + A_1 \dfrac{du^2}{du^1} + A_2\left(\dfrac{du^2}{du^1}\right)^2 + \cdots + A_n\left(\dfrac{du^2}{du^1}\right)^n}{B_0 + B_1 \dfrac{du^2}{du^1} + B_2\left(\dfrac{du^2}{du^1}\right)^2 + \cdots + B_{n-3}\left(\dfrac{du^2}{du^1}\right)^{n-3}} ; \tag{2.5}$$

or, if we divide by $(u'^2)^3$, a similar equation is obtained.

It is easy to see that there exists a unique solution [3, p.106] to (2.5) at any point (u^1, u^2) in any direction (u'^1, u'^2) for which V does not vanish. Hence, (2.2) defines a two-parameter family of supergeodesics on the surface S with the property that at any point of S there is a supergeodesic in every direction except those directions in which $V = 0$. It may be said that $V = 0$ defines $n - 3$ one-parameter families of curves which are never tangent to any supergeodesic of the family defined by (2.2) within a region R on S at each point of which U and V considered as polynomials in u'^{α} are relatively prime. (This excludes points of S where U and V have common factors.)

Now W is a polynomial in u'^{α} if and only if the family defined by (2.2) is a family of hypergeodesics. Otherwise, (2.2) represents a more general family of supergeodesics.

3. Supergeodesic curvature. We define the *supergeodesic curvature* of a curve $u^{\alpha} = u^{\alpha}(s)$ at a point P of the curve as the scalar K_s given by

$$K_s \equiv K_g - W. \tag{3.1}$$

The *supergeodesic curvature vector* we define as the vector whose contravariant components λ^δ are given by

$$\lambda^\delta \equiv (K_g - W)\,\mu^\delta \equiv K_s\mu^\delta \tag{3.2}$$

where $\mu^\delta \equiv \epsilon^{\gamma\delta} g_{\gamma\eta} u'^\eta$ is a unit vector which makes a right angle with the unit vector u'^α.

We see from (3.1) and (2.2) that *a curve C is a supergeodesic of the family if and only if the supergeodesic curvature along the curve is identically zero in s.*

4. **The elements of the cone related to the supergeodesics through a point.** The elements of the cone enveloped by the osculating planes of a family of supergeodesics through a point P of the surface S may be determined by the procedure sketched in Section 4 of [1]. The only change is that the symbol Ω be replaced by W. This replacement is possible since only the homogeneity of Ω was used, and W possesses this same homogeneity property. The direction numbers c^h of the element of the cone corresponding to the supergeodesic in the direction u'^α will have the values

$$c^h = \epsilon^{\alpha\beta} \frac{\partial K_n}{\partial \mu'^\beta} x^h{}_{,\alpha} \qquad (K_n = 0, \quad W \neq 0) \tag{4.1}$$

or

$$c^h = r^\alpha x^h{}_{,\alpha} + X^h \qquad (K_n \neq 0, \quad V \neq 0) \tag{4.2}$$

where

$$r^\alpha \equiv \epsilon^{\beta\alpha} \frac{\partial}{\partial u'^\beta} (W/K_n) \qquad (K_n \neq 0, \quad V \neq 0). \tag{4.3}$$

5. **A geometric interpretation of supergeodesic curvature.** It can be demonstrated that the supergeodesic curvature of a curve $C: u^\alpha = u^\alpha(s)$ is the curvature of the curve C' which is the projection of the curve C upon the tangent plane at the point P, the lines of projection being parallel to that element of the cone determined by the direction u'^α at P. The proof of this property consists of replacing Ω by W in Section 5 of [1]. Of course at points of C for which $K_n = 0$ or $V = 0$ there can be in general no geometric interpretation of this type since the element lies in the tangent plane or does not exist.

6. Supergeodesic torsion of a curve. The torsion at P of the supergeodesic $u^{\alpha} = u^{\alpha}(s)$ of the family having the same direction as a curve C at P (if such a supergeodesic of the family exists) will be called the supergeodesic torsion τ_S of C at the point P. The expression for τ_S as found by the calculations in Section 6 of [1] with Ω replaced by W is

$$(6.1) \qquad \tau_s = \frac{csc^2\phi}{L}\,\epsilon_{\beta\delta}\left\{r^{\beta}\left[(1/L)_{,\alpha} + \frac{r^{\gamma}}{L}\,d_{\alpha\gamma}\right] - \left[(r^{\beta}/L)_{,\alpha} - \frac{1}{L}\,d_{\alpha\gamma}\,g^{\gamma\beta}\right]\right\} u'^{\alpha}u'^{\delta} \qquad (K_n \neq 0, \quad W \neq 0),$$

where ϕ is the angle between the vector c^h as given by (4.2) and the unit tangent vector $x^i_{,\sigma}u'^{\sigma}$, L is the length of the vector c^h, and r^{β} is defined by (4.3).

7. A geometric condition that a supergeodesic be a plane curve. If we find the differential equation for the special related intersector net of the complex of cone elements for the family of supergeodesics under consideration, it will be exactly the same as the differential equation

$$(7.1) \qquad \tau_S = 0,$$

as can be verified by simply replacing Ω by W in Section 7 of [1]. Now a curve of the special related intersector net is a curve for which the elements c^h at each point of the curve corresponding to the direction u'^{α} of the curve form a developable surface. Hence, we may state that *a supergeodesic not in an asymptotic direction is a plane curve if and only if it is a curve of the special intersector net of the complex of cone elements.*

Geometrically speaking the theorem reads: *A supergeodesic not in an asymptotic direction is a plane curve if and only if the one-parameter family of cone elements, which are the elements of contact of the osculating planes of the supergeodesic with the cones, constitutes a developable.*

8. A study of the special related intersector net. If we discard the nonzero multipliers from the left side of the differential equation (7.1) and make use of (4.3), the differential equation reads

$$(8.1) \quad \epsilon_{\beta\delta}\left\{\left[\epsilon^{\gamma\beta}\,\frac{\partial}{\partial u'^{\gamma}}\left(\frac{U}{K_nV}\right)\right]_{,\alpha} + \epsilon^{\delta\gamma}\,\frac{\partial}{\partial u'^{\delta}}\left(\frac{U}{K_nV}\right)d_{\alpha\gamma} + d_{\alpha\gamma}\,g^{\gamma\beta}\right\}u'^{\alpha}u'^{\delta} = 0.$$

After performing the partial and covariant differentiation in (8.1) and clearing the fractions we see that the equation is of the first order and in general (when U and $K_n V$ are relatively prime) of degree $N = 3n - 1$ where n is the degree of homogeneity of U in the parameters u'^{α}.

If U and $K_n V$ have exactly m common factors then the degree N of (8.1) is $3(n-m)-1$. For hypergeodesics, V divides U so that N is 8 in general. However, for union curves, $K_n V$ divides U so that N is 2. Now union curves are specializations of hypergeodesics in that K_n divides Ω [1]. A similar specialized class of supergeodesics containing the class of union curves as a subclass is obtained when K_n divides U.

9. **Pangeodesics.** In the case of the family of pangeodesics [4, pp. 203-204] on the surface, it is observed that the differential equation is of the type (2.5) and hence the pangeodesics constitute an example of a family of supergeodesics. For the pangeodesics n is 6 so that in general N is 17.

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RELATIONS AMONG CERTAIN RANGES OF VECTOR MEASURES

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1. Introduction and definitions. The purpose of the present paper is to prove certain measure theoretical results concerning ranges of measures. One of our results (the closure and convexity result implied by Theorem 4) may be regarded as a generalization of a theorem of Liapounoff [5]. The results obtained here have applications to statistics and the theory of games.

Throughout this paper $\{x\} = X$ denotes an arbitrary space, and $\{S\} \equiv \mathfrak{S}$ denotes a Borel field of subsets of X; that is, $\mathfrak{S}$ is a nonempty family of subsets of X which is closed with respect to the operations of complementation (with respect to X) and countable union. The phrase, *S is measurable*, will be used as synonymous with $S \in \mathfrak{S}$.

A real-valued countably additive set function defined for all measurable sets will be called a *measure*. Thus we admit measures assuming negative or infinite values. A measure cannot, however, assume the value $+\infty$ for one measurable set and $-\infty$ for another such set, since in such a case additivity cannot be defined satisfactorily. A measure is called *finite* if it assumes finite values for all measurable sets. It is called *nonnegative* if it assumes nonnegative values for all such sets.

We say that $f(x)$ is a *measurable function* if it is real-valued, defined for all $x \in X$, and if, moreover, the set f_c of all $x \in X$ for which $f(x) < c$ is measurable for every real number c. A *step function* is a measurable function which assumes only a *finite* number of values.

If n is a positive integer and $\eta_j(x)\,(j = 1, \cdots, n)$ are nonnegative measurable functions satisfying

$$(1) \qquad \eta_1(x) + \cdots + \eta_n(x) = 1 \qquad \text{for every } x \in X,$$

then $\eta(x) = [\eta_1(x), \cdots, \eta_n(x)]$ will be called a *probability n*-vector. The functions $\eta_j(x)$ are called the *components* of this vector. If all the components of

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$\eta(x)$ are step functions, then $\eta(x)$ is called a (probability) *step n-vector.* We shall occasionally denote such vectors by $\eta^0(x)$. If, in particular, all components of $\eta(x)$ assume only the two values zero and one, that is, if for every $x \in X$ one $\eta_j(x)$ is equal to one and all others vanish, then $\eta(x)$ is called a *pure n-vector.* Such vectors will be denoted by $\eta^*(x)$. If the jth component ($j = 1, \cdots, n$) of $\eta^*(x)$ is considered as the characteristic function of a set S_j, then the sets $S_1, \cdots, S_n$ are measurable and disjoint and their union is X. Conversely, if $S_1, \cdots, S_n$ is a decomposition of X into n disjoint measurable sets, and $\eta_j^*(x)$ is the characteristic function of S_j, then $\eta^*(x) = [\eta_1^*(x), \cdots, \eta_n^*(x)]$ is a pure n-vector. We therefore call $\eta^*(x)$ also a *decomposition n-vector* or, more specifically, a decomposition n-vector corresponding to the decomposition $X = S_1 \cup \cdots \cup S_n$.

Let $\mu_k(S)$ $(k = 1, \cdots, p)$ be a finite set of measures, and let $\eta(x)$ be a probability n-vector. We denote by $v(\eta) = v(\eta; \mu_1, \cdots, \mu_p)$ the np dimensional vector (or point in np space),

$$\left[\int_X \eta_1(x)\, d\mu_1(x), \cdots, \quad \int_X \eta_1(x)\, d\mu_p(x), \right.$$

$$\left. \int_X \eta_2(x)\, d\mu_1(x), \cdots, \quad \int_X \eta_p(x)\, d\mu_p(x)\right].$$

The set of all points $v(\eta) = v(\eta; \mu_1, \cdots, \mu_p)$ corresponding to all probability n-vectors $\eta(x)$ is called the *n-range* of $\mu_1, \cdots, \mu_p$ and is further denoted by $V_n(\mu_1, \cdots, \mu_p)$ or, more concisely, by V_n. In the same way we define the *step n-range* of $\mu_1, \cdots, \mu_p$ as the set of all points $v(\eta^0) = v(\eta^0; \mu_1, \cdots, \mu_p)$ corresponding to all step n-vectors $\eta^0(x)$, and denote it by $V_n^0(\mu_1, \cdots, \mu_p)$ or V_n^0. Similarly V_n^* or $V_n^*(\mu_1, \cdots, \mu_p)$ denotes the set of all points

$$v(\eta^*) = v(\eta^*; \mu_1, \cdots, \mu_p)$$

corresponding to all pure n-vectors $\eta^*(x)$ and is called the *decomposition n-range* of $\mu_1, \cdots, \mu_p$. When no confusion is possible we replace n-range in the above terms by range.

It is shown in Section 2 that if $\mu_1, \cdots, \mu_p$ are finite measures then the range $V_n(\mu_1, \cdots, \mu_p)$ is compact and convex and coincides with the step-range $V_n^0(\mu_1, \cdots, \mu_p)$. Actually a stronger result is proved; this states that the points $v(\eta; \mu_1, \cdots, \mu_p)$ for which the components of $\eta(x)$ assume at most 2^{np-p+1}

different values already fill V_n. Applying a theorem of Liapounoff we deduce, in Section 3, the result that if the measures are atomless then the decomposition range $V_n^*(\mu_1, \cdots, \mu_p)$ is identical with $V_n(\mu_1, \cdots, \mu_p)$. This result is extended in Section 4 to arbitrary (not necessarily finite) atomless measures. Applications of these results to statistics and the theory of games are briefly indicated in Section 5.

2. Identity of the step range and the range for finite measures. First, we prove the following result.

THEOREM 1. *If $\mu_1, \cdots, \mu_p$ are finite measures, then for every n, the range $V_n(\mu_1, \cdots, \mu_p)$ is a compact and convex set in Euclidean np dimensional space.**

Proof. Let $A = v(\eta)$ and $A' = v(\eta')$ be any two points of V_n. Then every point of the segment joining them is represented vectorially by $cA + (1-c)A'$, with $0 < c < 1$. But such a point is clearly $v[c\eta + (1-c)\eta']$ and, since $c\eta + (1-c)\eta'$ is a probability n-vector, the point also belongs to the range. Thus V_n is convex.

The proof of compactness is more difficult. We start by establishing a lemma on sequences of measures.

LEMMA 1. *Let $\{B\} = \mathfrak{B}$ be a Borel field of subsets of X generated by countably many sets. Let μ^t $(t = 1, 2, \cdots)$ and μ be measures over $\mathfrak{B}$ satisfying, for all $B \in \mathfrak{B}$,*

$$(2) \qquad 0 \leq \mu^t(B) \leq \mu(B) < \infty \qquad (t = 1, 2, \cdots).$$

Then there exists a measure ν over $\mathfrak{B}$ satisfying

$$(3) \qquad 0 \leq \nu(B) \leq \mu(B) \quad \text{for all } B \in \mathfrak{B},$$

and a sequence of integers t_q $(q = 1, 2, \cdots)$ satisfying

$$(4) \qquad 0 < t_1 < t_2 < \cdots < t_q < t_{q+1} < \cdots$$

such that

$$(5) \qquad \lim_{q=\infty} \mu^{t_q}(B) = \nu(B) \qquad \text{for every} \quad B \in \mathfrak{B}.$$

---

* For the special case when X is a finite-dimensional Euclidean space and all the μ_k are absolutely continuous, this theorem follows from Theorems 3.1 and 3.2a of [6].

The proof of the lemma proceeds as follows. Let $B_0 = X, B_1, \cdots, B_n \cdots$ be a countable basis of $\mathfrak{B}$. Then, according to the well-known diagonal procedure of Cantor, there exists a sequence (4) for which

$$\beta_r = \lim_{q=\infty} \mu^{t_q}(B_r)$$

exists for $r = 0, 1, 2, \cdots$. To prove the existence of the limit in (5) for every B, and the fact that this limit is a measure, it suffices to show that: (a) if $\mu^{t_q}(B)$ tends to a limit as $q \longrightarrow \infty$, then so does $\mu^{t_q}(\bar{B})$ where $\bar{B}$ is the complement of B with respect to X; and (b) if $B^s (s = 1, 2, \cdots)$ are disjoint sets of $\mathfrak{B}$ for which $\lim_{q=\infty} \mu^{t_q}(B^s)$ exists for $s = 1, 2, \cdots$, then we have also

$$\lim_{q=\infty} \mu^{t_q}\left(\bigcup_{s=1}^{\infty} B^s\right) = \sum_{s=1}^{\infty} \lim_{q=\infty} \mu^{t_q}(B^s) . \tag{6}$$

Now (a) follows immediately when we write $\mu^t(\bar{B}) = \mu^t(X) - \mu^t(B)$ and observe that $\mu^{t_q}(X)$ has the limit β_0. To prove (6) it is sufficient to observe that the functions μ^t are countably additive, that by (2) we have $\mu^{t_q}(B^s) \leq \mu(B^s)$, and that $\sum_{s=1}^{\infty} \mu(B^s)$ is a convergent series of nonnegative terms. (This is the standard bounded convergence argument.) Since (2) and (5) obviously imply (3), the proof of Lemma 1 is completed.

Let now $\eta^t(x) (t = 1, 2, \cdots)$ be any sequence of probability n-vectors. The compactness of V_n will be proved if we show that there exist a probability n-vector and a sequence (4) satisfying

$$\lim_{q=\infty} \int_X \eta_j^{t_q}(x)\, d\mu_k(x) = \int_X \eta_j(x)\, d\mu_k(x) \qquad (j = 1, \cdots, n;\ k = 1, \cdots, p) . \tag{7}$$

Denote by $B_{j,\rho}^t (t = 1, 2, \cdots; j = 1, \cdots, n;\ \rho$ rational with $0 \leq \rho \leq 1)$ the set of all x for which $\eta_j^t(x) \leq \rho$, and let $\{B\} = \mathfrak{B} \subset \mathfrak{S}$ be the smallest Borel field containing these sets. Write $|\mu_k|$ for the absolute measure* associated with μ_k. Put $\mu(B) = |\mu_1|(B) + \cdots + |\mu_p|(B)$ for every $B \in \mathfrak{B}$. Then $\mathfrak{B}$, μ, and

*That is, $|\mu_k|(S) = \sup\ [|\mu_k(S')| + |\mu_k(S'')|]$ for all decompositions of S into two disjoint measurable sets S' and S''.

μ^t, defined by

$$\mu^t(B) = \int_B \eta_1^t(x)\ d\mu(x) \qquad (t = 1, 2, \cdots, B \in \mathfrak{B}),$$

satisfy the conditions of Lemma 1. Hence there exists a nonnegative measure ν_1 over $\mathfrak{B}$ and a sequence (4), for which

$$\lim_{q=\infty} \int_B \eta_1^{t_q}(x)\ d\mu(x) = \nu_1(B) \tag{8}$$

for every $B \in \mathfrak{B}$.

Again applying Lemma 1, we can extract from the sequence t_q a further subsequence for which (8) holds with the subscript 1 replaced by 2. Repeating this $n-1$ times, and again denoting, for simplicity of writing, the final subsequence by t_q, we see that there exist nonnegative measures $\nu_1, \cdots, \nu_n$ over $\mathfrak{B}$ and a sequence (4) satisfying

$$\lim_{q=\infty} \int_B \eta_j^{t_q}(x)\ d\mu(x) = \nu_j(B) \qquad (j = 1, \cdots, n) \tag{9}$$

for every $B \in \mathfrak{B}$. Clearly, we have

$$\nu_1(B) + \cdots + \nu_n(B) = \mu(B) \qquad (B \in \mathfrak{B}). \tag{10}$$

By the Radon-Nikodym theorem there exist $\mathfrak{B}$-measurable functions $f_j(x)$ $(j = 1, \cdots, n)$ such that

$$\nu_j(B) = \int_B f_j(x)\ d\mu(x) \qquad (j = 1, \cdots, n) \tag{11}$$

for every $B \in \mathfrak{B}$. Since the ν_j are nonnegative measures, we may assume that the f_j are nonnegative functions; and, because of (10), we may further assume that $f_1(x) + \cdots + f_n(x) = 1$ for every x. The f_j are $\mathfrak{B}$-measurable and are, *a fortiori*, $\mathfrak{S}$-measurable; hence $[f_1(x), \cdots, f_n(x)]$ is a probability n-vector. We denote this vector by $\eta(x)$ and proceed to show that (7) holds with this η and the above constructed sequence (4) satisfying (9).

Let $g_k(x)(k = 1, \cdots, p)$ denote a $\mathfrak{B}$-measurable Radon-Nikodym derivative

$d\mu_k(x)/d\mu(x)$. Then, replacing f_j in (11) by η_j, we have

$$\int_X \eta_j(x)\, d\mu_k(x) = \int_X \eta_j(x)\, g_k(x)\, d\mu(x)$$

$$= \int_X g_k(x)\, d\int \eta_j(x)\, d\mu(x) = \int_X g_k(x)\, d\nu_j(x).$$

Similarly, the left side of (7) may be rewritten as

$$\lim_{q=\infty} \int_X g_k(x)\, d\int \eta_j^{t_q}(x)\, d\mu(x) ,$$

and thus (7) follows from (9). This completes the proof of Theorem 1.

For any compact convex set C in a Euclidean space, we designate as *extreme points of* C, all those points of C which are not *interior* points of any segment lying in C. Our next result is the following.

THEOREM 2. *If the measures* $\mu_1, \cdots, \mu_p$ *are finite, and* $v(\eta)$ *is an extreme point of* V_n, *then the set of* x *for which* $0 < \eta_j(x) < 1$ *for at least one* j $(j = 1, \cdots, n)$ *is a null-set* for each of the measures* $\mu_1, \cdots, \mu_p$. *In particular, all extreme points of* V_n *belong to the decomposition range* V_n^*.

Proof. Let Y denote the set of x defined in the theorem. If Y is not a null-set for μ_{k_0} with $1 \leq k_0 \leq p$, then there exist integers j_0, j_1 with $1 \leq j_0 < j_1 \leq n$, a number $\delta > 0$, and a measurable set $Z \subset Y$, such that

(12) $$\delta < \eta_j(x) < 1 - \delta \quad \text{for } x \in Z \quad \text{and} \quad j = j_0, j_1 ,$$

and

$$\mu_{k_0}(Z) \neq 0 .$$

Let $\zeta = \zeta(x) = [\zeta_1(x), \cdots, \zeta_n(x)]$ be the vector defined as follows:

$$\zeta_{j_0}(x) = -\zeta_{j_1}(x) = \begin{cases} \delta \text{ if } x \in Z \\ 0 \text{ if } x \notin Z \end{cases}$$

---

*A measurable S is a *null-set* for the measure μ if $\mu(S') = 0$ for every measurable $S' \subset S$.

and all other components vanish identically.

Because of (12), $\eta(x) + \theta\zeta(x)$ is a probability n-vector whenever $-1 \leq \theta \leq 1$. Since

$$\int_X [\eta_{j_0}(x) + \zeta_{j_0}(x)]\, d\mu_{k_0}(x) - \int_X [\eta_{j_0}(x) - \zeta_{j_0}(x)]\, d\mu_{k_0}(x)$$

$$= 2\delta\mu_{k_0}(Z) \neq 0,$$

the points $v(\eta + \zeta)$ and $v(\eta - \zeta)$ are different. But clearly as θ increases from -1 to $+1$ the point $v(\eta + \theta\zeta)$ moves from $v(\eta - \zeta)$ to $v(\eta + \zeta)$ along the segment connecting them. Moreover, $v(\eta)$ is the middle point of this segment and thus it cannot be an extreme point of V_n.

If $v(\eta)$ is an extreme point, then Y is a null-set for all μ_k. Therefore, if we put $\eta^*(x) = \eta(x)$ for $x \notin Y$ and, say, $\eta_1^*(x) = 1$ for $x \in Y$, the decomposition vector $\eta^*(x)$ thus defined satisfies $v(\eta^*) = v(\eta)$. This proves the last assertion of Theorem 2.

THEOREM 3. *If the measures $\mu_1, \cdots, \mu_p$ are finite, then the step-range V_n^0 coincides with the range V_n. More precisely, every point of V_n may be represented as $v(\eta^0)$, where η^0 is a step n-vector whose components assume not more than 2^{np-p+1} different values.*

Proof. According to Theorem 1, V_n is a compact convex set in Euclidean np-dimensional space. However, because of the p equations

$$\sum_{j=1}^{n} \int_X \eta_j(x)\, d\mu_k(x) = \mu_k(x) \qquad (k = 1, \cdots, p)$$

V_n lies in an $N = np - p$ dimensional linear subspace. Hence, according to well-known facts on convex bodies, every point P of V_n may be represented vectorially by

$$P = c_1 P_1 + \cdots + c_N P_N + c_{N+1} P_{N+1},$$

where $P_1, \cdots, P_{N+1}$ are extreme points of V_n and $c_1, \cdots, c_{N+1}$ are nonnegative constants whose sum is 1. According to Theorem 2, we have $P_r = v(\eta^{*r})$ with η^{*r} a decomposition n-vector $(r = 1, 2, \cdots, N + 1)$.

Hence, putting $\eta^0 = \Sigma_{r=1}^{N+1} c_r \eta^{*r}$, we have $P = v(\eta^0)$. Clearly, for every x, every component of $\eta^0(x)$ equals $\Sigma_{r \epsilon K} c_r$, where K is a subset of $\{1,2,\cdots,N+1\}$. There being 2^{N+1} such subsets, Theorem 3 is proved.

3. Identity of the range and the decomposition range for finite atomless measures. A measurable set S is called an *atom* of the measure μ if $\mu(S) \neq 0$ and if, moreover, for every measurable $S' \subset S$ we have either $\mu(S') = 0$ or $\mu(S') = \mu(S)$. If the measure $\mu(S)$ has no atoms it is called *atomless*.

For atomless measures we can improve on Theorem 3 by establishing the following result.

THEOREM 4. *If $\mu_1, \cdots, \mu_p$ are finite atomless measures then, for every n, the range V_n and the decomposition range V_n^* are identical.*

According to Theorem 1, the common range is convex and compact.

Proof. In view of Theorem 3 it suffices to prove that, in the present case, $V_n^* = V_n^0$.

For this purpose we shall use the following fact: If $\mu_1, \cdots, \mu_p$ are finite and atomless, then, given $0 \leq c \leq 1$, there exists a measurable set S for which

$$\mu_k(S) = c\mu_k(X) \qquad (k = 1, \cdots, p) . \tag{13}$$

The existence of such a set S follows immediately from a result of Liapounoff [5] (see also [3]) according to which, under the above stated conditions, the set of points $\mu_1(S), \cdots, \mu_p(S)$ in Euclidean p-space corresponding to all measurable S is convex. Indeed, the empty set Λ and X are certainly measurable and $(1 - c)\ \mu_k(\Lambda) + c\mu_k(X) = c\mu_k(X)$ for all k.

To complete the proof of Theorem 4, we use the following lemma.

LEMMA 2. *If $\mu_1, \cdots, \mu_p$ are finite and atomless and $c_1, \cdots, c_n$ are non-negative numbers satisfying $c_1 + \cdots + c_n = 1$, then there exists a decomposition of X into n disjoint measurable sets $S_1, \cdots, S_n$ having the property that*

$$\mu_k(S_j) = c_j\mu_k(X) \qquad (j = 1, \cdots, n;\ k = 1, \cdots, p) . \tag{14}$$

Indeed, according to (13) there exists a measurable S_1 satisfying (14) for $j = 1$. Similarly, there exists a measurable $S_2 \subset X - S_1$ satisfying

$$\mu_k(S_2) = \frac{c_2}{c_2 + \cdots + c_n}\ \mu_k(X - S_1) = c_2\mu_k(X) ,$$

where we interpret

$$\frac{c_2}{c_2 + \cdots + c_n}$$

as zero if $c_2 = \cdots = c_n = 0$. That is, S_2 satisfies (14) for $j = 2$. In the same manner $S_j \subset X - \bigcup_{i=1}^{j-1} S_i$ satisfying (14) may be obtained for $j = 1, \cdots, n-1$. But then

$$\mu_k\left(X - \bigcup_{j=1}^{n-1} S_j\right) = 1 - (c_1 + \cdots + c_{n-1})\,\mu_k(X) = c_n\mu_k(X) ;$$

thus

$$S_n = X - \bigcup_{j=1}^{n-1} S_j$$

satisfies (14) for $j = n$ as required. Hence, Lemma 2 holds.

The proof of Theorem 4 can now easily be completed. Let $\eta^0(x)$ be any step n-vector. Then X can be decomposed into a finite number of disjoint measurable subsets Y_t over each of which all the components of $\eta^0(x)$ are constant. According to Lemma 2, Y_t may be decomposed into n disjoint measurable sets $S_{1,t}, \cdots, S_{n,t}$ such that

$$\mu_k(S_{j,t}) = \int_{Y_t} \eta_j^0(x)\, d\mu_k(x) \qquad (j = 1, \cdots, n;\ k = 1, \cdots, p) . \tag{15}$$

Putting $S_j = \bigcup_t S_{j,t} (j = 1, \cdots, n)$ we have, from (15),

$$\int_X \eta_j^0(x)\, d\mu_k(x) = \mu_k(S_j) \qquad (j = 1, \cdots, n;\ k = 1, \cdots, p) .$$

Thus the point $v(\eta^0; \mu_1, \cdots, \mu_p)$ coincides with $v(\eta^*; \mu_1, \cdots, \mu_p)$, where $\eta_j^*(x) = 1$ if $x \in S_j$ and zero otherwise. In other words, $V_n^0 \subset V_n^*$. Since the converse inclusion is obvious, Theorem 4 is proved.

Remarks. (a) Liapounoff [5] proved that if the conditions of Theorem 4 are satisfied then the set of all points $[\mu_1(S), \cdots, \mu_p(S)]$ in Euclidean p-space corresponding to all measurable S is convex and compact. This result is clearly implied by the convexity and compactness of V_n^*; thus the convexity and compactness of V_n^* may be considered as a generalization of Liapounoff's theorem. If we put $\overline{S} = X - S$, then Liapounoff's result is easily seen to be equivalent to the statement that the set of all points $[\mu_1(S), \cdots, \mu_p(S), \mu_1(\overline{S}), \cdots, \mu_p(\overline{S})]$ in Euclidean $2p$ dimensional space is convex and compact. But this amounts precisely to the assertion that V_n^* is convex and compact for $n = 2$. That this assertion remains valid also for $n > 2$ is precisely the generalization of Liapounoff's result contained in Theorem 4.

We used in our proof the convexity part of Liapounoff's result. This is, however, the easier part (cf. Halmos [3]), and thus our method furnishes also a new proof of Liapounoff's theorem.

(b) The values 0 and 1 are among those which the components of η^0 in Theorem 3 are allowed to assume. Hence, on combining the results of Theorems 3 and 4 we see that, if all but p' of the measures $\mu_1, \cdots, \mu_p$ are atomless, we may replace p by p' in the exponent of 2 in Theorem 3. This estimate is again independent of the number of atoms.

(c) If the measures $\mu_1, \cdots, \mu_p$ in Theorem 4 are not assumed to be atomless, then of course V_n^* need not be convex. It is, however, compact as can easily be seen on decomposing into atomless and purely atomic parts and dealing separately with each (see, for example, [3]).

(d) For some applications the following is of importance: If η is a probability n-vector, then there exists a decomposition n-vector η^* with $v(\eta^*) = v(\eta)$ having the further property that, for every $x \in X$ and $j = 1, \cdots, n$, the vanishing of $\eta_j(x)$ implies that of $\eta_j^*(x)$. This assertion follows easily from Theorem 4. Indeed, X may be decomposed into a finite number of measurable sets Y with the following property: If $\eta_j(x) = 0$ for some $x \in Y$, then $\eta_j(x) = 0$ for all $x \in Y$. Let $j_1, \cdots, j_m$ be those j for which $\eta_j(x) > 0$ when $x \in Y$. We may now define $\eta^*(x)$ for $x \in Y$ by applying Theorem 4 (with X replaced by Y and n by m) to the m-vector formed by these components, and putting $\eta_j(x) = 0$ for all other j and $x \in Y$. Combining these definitions for all sets Y, we obtain an η^* with the required property.

4. Extension to arbitrary atomless measures. The assumption of finiteness in Theorem 4 is unnecessary. Indeed, we shall prove the following result.

THEOREM 5. *If the measures* $\mu_1, \cdots, \mu_p$ *are atomless, then, for every* n, *the range V_n and the decomposition range V_n^* are identical.*

Since the measures are now allowed to assume infinite values, the components of $V(\eta)$ are no longer necessarily finite and one should look upon V_n and V_n^* as imbedded in Euclidean space extended by allowing each coordinate to assume also infinite values.

Before proceeding to the proof we establish the following lemmas.

LEMMA 3. *If μ is a nonnegative atomless measure with $\mu(X) = \infty$, and u is any finite positive number, then there exists a measurable set T with $\mu(T) = u$.*

Proof. Since μ is nonnegative and atomless, there exists a set S with $0 < \mu(S) < \infty$. We first show that $\alpha = \sup \mu$ for all such sets S is infinite. Indeed, assume α finite; then, for every integer m, there exists a measurable S_m with $\mu(S_m) > \alpha - 1/m$. Put $S' = \bigcup_{m=1}^{\infty} S_m$; then $\mu(S') = \alpha$. But $\mu(X - S') = \infty$; hence, there exists a measurable $S'' \subset X - S'$ with $0 < \mu(S'') = b < \infty$. Thus $\alpha < \mu(S' \cup S'')$, contradicting the assumption that α is finite.

Therefore, given u there exists a measurable T' with $u < \mu(T') < \infty$. But then, according to the intermediary values theorem of Sierpinski (see, for example, [2, 52]), or the one dimensional case of Liapounoff's theorem, there exists a measurable $T \subset T'$ with $\mu(T) = u$.

LEMMA 4. *If μ is a nonnegative atomless measure with $\mu(X) = \infty$, and q is any positive integer, then X may be decomposed into q measurable disjoint sets $X_1, \cdots, X_q$ with $\mu(X_1) = \cdots = \mu(X_q) = \infty$.*

Proof. According to Lemma 3, there exist a set T_1 with $\mu(T_1) = 1$, a set $T_2 \subset X - T_1$ with $\mu(T_2) = 1$, a set $T_3 \subset X - (T_1 \cup T_2)$ with $\mu(T_3) = 1$, and so on. Putting $X_i = \bigcup_{n=0}^{\infty} T_{qn+i}$ for $i = 1, \cdots, q - 1$ and $X_q = X - \bigcup_{i=1}^{q-1} X_i$ we obtain the required result.

LEMMA 5. *If $\nu_1, \cdots, \nu_m$ are nonnegative atomless measures with $\nu_1(X) = \cdots \nu_m(X) = \infty$, and q is any positive integer, then X may be decomposed into q measurable sets $X_1, \cdots, X_q$ satisfying $\nu_i(X_1) = \cdots = \nu_i(X_q) = \infty$ for $i = 1, \cdots, m$.*

Proof. For $m = 1$, this lemma reduces to the preceding one. Assume $m > 1$ and the lemma proved for $m - 1$. According to Lemma 4, X is the union of m disjoint measurable sets $Y_1, \cdots, Y_m$ with $\nu_m(Y_1) = \cdots = \nu_m(Y_m) = \infty$. For every $i\,(i = 1, \cdots, m-1)$ let i' denote the smallest integer for which $\nu_i(Y_{i'}) = \infty$. (Since $\nu_i(X) = \infty$ we have $1 \le i' \le m$.) Put $Y' = \bigcup_{i=1}^{m-1} Y_{i'}$ and $Y'' = X - Y'$. Then $\nu_m(Y'') = \infty$ and Y'' is the union of disjoint measurable sets $Y_1'', \cdots, Y_q''$ with $\nu_m(Y_1'') = \cdots = \nu_m(Y_q'') = \infty$. Also $\nu_i(Y'') = \infty$ for $i = 1, \cdots, m-1$ and hence, by the assumption of induction, it can be decomposed into measurable sets $Y_1', \cdots, Y_q'$ with $\nu_i(Y_1') = \cdots = \nu_i(Y_q')$ for $i = 1, \cdots, m-1$. Putting $X_1 = Y_1' \cup Y_1'', \cdots, X_q = Y_q' \cup Y_q''$, we obtain the required decomposition.

LEMMA 6. *Let μ, ν be nonnegative atomless measures with $\mu(X) < \infty$, $\nu(X) = \infty$. Then either X may be decomposed into countably many measurable sets, each having finite ν measure, or there exists a measurable set T with $\mu(T) = 0$, $\nu(T) = \infty$.*

Proof. For every positive integer t consider the measure μ_t defined by

$$\mu_t(S) = \nu(S) - t\mu(S) .$$

According to Hahn (see for example [2, p.18] or [4, p.121]) X may be decomposed into two disjoint measurable sets Y_t and $\overline{Y}_t$ with $\mu_t(S) \le 0$ for every measurable $S \subset Y_t$ and $\mu_t(S) > 0$ for every measurable $S \subset \overline{Y}$. Clearly,

$$\nu(Y_t) \le t\mu(Y_t) \le t\mu(X) < \infty .$$

Put now $Y_t' = Y_1 \cup \cdots \cup Y_t$ and $Z_1 = Y_1'$, $Z_t = Y_t' - Y_{t-1}'$ for $t = 2, 3, \cdots$, and denote by Z_0 the complement of $\bigcup_{t=1}^{\infty} Z_t$. Then $X = \bigcup_{t=0}^{\infty} Z_t$ and $\nu(Z_t) < \infty$ for $t \ge 1$. If $\nu(Z_0) < \infty$ then this is a decomposition of X into countably many sets of finite ν measure. If, on the other hand, $\nu(Z_0) = \infty$ then, by Lemma 3, there exists for every integer u a measurable $T_u \subset Z_0$ with $\nu(T_u) = u$. Moreover, $\mu(T_u) = 0$ since, according to the construction of Z_0, $\mu(S) > 0$ for $S \subset Z_0$ implies $\nu(S) = \infty$. Thus $T = \bigcup_{u=1}^{\infty} T_u$ has the properties required in Lemma 6.

Proof of Theorem 5. Since every measure is the difference between two nonnegative measures, we may assume throughout the proof that the measures $\mu_k\,(k = 1, \cdots, p)$ are nonnegative.

Let η be any probability n-vector. For every $j\,(j = 1, \cdots, n)$ we denote by $Y_{j,0}$ the set of x for which $\eta_j(x) = 0$ and by $Y_{j,t}\,(t = 1, 2, \cdots)$ the set of x for which

$$\frac{1}{t+1} < \eta_j(x) < \frac{1}{t} .$$

We use Y to denote any set of the form

$$\bigcap_{j=1}^{n} Y_{jt_j} \quad \text{with } t_j = 0, 1, 2, \cdots \qquad (j = 1, \cdots, n) .$$

The space X is thus decomposed into countably many sets Y having the following property: There exists a nonempty subset $J = J(Y)$ of $\{1, \cdots, n\}$ and a *positive* $\delta = \delta(Y)$ such that for all $x \in Y$ we have

$$\eta_j(x) > \delta > 0 \quad \text{if } j \in J, \qquad \eta_j(x) = 0 \quad \text{if } j \notin J. \tag{16}$$

Let Y be any such set and consider the subset K' of $\{1, \cdots, p\}$ consisting of all those k for which Y can be decomposed into countably many sets, all having finite μ_k measure. If K' is empty, we call Y final, if not we decompose Y into countably many measurable sets Y' with $\mu_k(Y') < \infty$ for $k \in K'$. Let Y' be any such set and denote by K'' the subset of $\{1, \cdots, p\}$ consisting of all k for which Y' can be decomposed into countably many sets, all having finite μ_k measure. Clearly, $K' \subset K''$. If $K' = K''$ we call Y' final, if not we decompose it into countably many Y'' with $\mu_k(Y'') < \infty$ for $k \in K''$. Again a $K''' \supset K''$ is defined and Y'' is called final if $K'' = K'''$, and so on. After not more than p steps we always end with a final set Z.

We have thus decomposed Y, and hence X, into countably many sets Z having the following property: To every Z there corresponds a decomposition of $\{1, 2, \cdots, p\}$ into two disjoint sets K and K such that $\mu_k(Z) < \infty$ if $k \in K$, while if $k \in \overline{K}$ then Z cannot be decomposed into countably many sets, all having finite μ_k measure. Furthermore, since Z is contained in some Y, (16) holds for all $x \in Z$.

Next, we show how to decompose Z into disjoint measurable sets $Z_1, \cdots, Z_n$ satisfying

$$\mu_k(Z_j) = \int_Z \eta_j(x)\, d\mu_k(x) \qquad (j = 1, \cdots, n;\ k = 1, \cdots, p) . \tag{17}$$

(If $\eta_j(x) = 0$ for all $x \in Z$, the right side of (17) is understood to be 0 even when $\mu_k(Z) = \infty$.)

If K is empty, then the possibility of such a decomposition is assured by Theorem 4.

If K is empty then, by (16), the integral in (17) is infinite if $j \in J$ and is zero otherwise. By Lemma 5 it is possible to decompose Z into sets Z_j $(j \in J)$ with

$$\mu_k(Z_j) = \infty \quad \text{for} \quad k = 1, \cdots, p .$$

Denoting the empty set, for $j \notin J$, by Z_j, we have a decomposition satisfying (17).

Finally, assume both K and $\overline{K}$ nonempty. We define a nonnegative measure μ by $\mu(S) = \Sigma_{k \in K} \mu_k(S)$. Clearly, μ is atomless and $\mu(Z) < \infty$. According to Lemma 6 there exists, for every $k \in \overline{K}$, a measurable $T_k \subset Z$ with $\mu(T_k) = 0$, $\mu_k(T_k) = \infty$. Let T be the union of $T_k (k \in \overline{K})$. Then (see the treatment of the case when K is empty) it is possible to decompose T into disjoint measurable sets $Z_1', \cdots, Z_n'$ so that Z_j' is empty for $j \notin J$, while for all $j \in J$ and $k \in \overline{K}$ we have $\mu_k(Z_j') = \infty$. Since $\mu(T) = 0$ we have, for all j, $\mu_k(Z_j') = 0$ whenever $k \in K$. Let $T' = Z - T$; then it is possible, by Theorem 4, to decompose T' into disjoint measurable sets $T_1', \cdots, T_n'$ such that T_j' is empty for $j \notin J$, while for $j \in J$ and $k \in K$ we have

$$\mu_k(Y_j) = \int_{T'} \eta_j(x)\, d\mu_k(x) = \int_Z \eta_j(x)\, d\mu_k(x) .$$

Putting $Z_j = T_j' \cup Z_j'$ for $j = 1, \cdots, n$, we have a decomposition satisfying (17).

We now define the decomposition n-vector η^* as follows: For $x \in Z$, put $\eta_j^*(x) = 1$ if $x \in Z_j$, and $\eta_j^*(x) = 0$ for all other $x \in Z$. Because of the countable additivity of the measures and the integrals, (17) implies $v(\eta^*) = v(\eta)$ and the proof is completed.

Remarks. (a) The last remark after Theorem 4 applies also here. Indeed, our construction in the proof of Theorem 5 yields a vector having the properties required of η^* in that remark.

(b) In applications usually X can be decomposed into countably many sets of finite μ_k measure $(k = 1, \cdots, p)$. For this special case Theorem 5 is, of course, an immediate consequence of Theorem 4.

4. Application to statistics and the theory of games.* Theorem 4 (together with its extension mentioned in the last remark of the preceding section) has

* A more detailed discussion and other results, including a discussion of the sequential statistical decision problem, are contained in our paper, *Elimination of randomization in certain statistical decision procedures and zero-sum two-person games*, Annals of Mathematical Statistics, 22, No. 1, March, 1951. A brief discussion of these applications was also given in an earlier publication [1] .

immediate applications to the following statistical decision problem: Let $y = \{y_1, \cdots, y_t\}$ be a random vector with t components, where t is a given positive integer. For every point $x = (x_1, \cdots, x_t)$ of the t-dimensional Euclidean space X, let $F(x)$ denote the probability that $y_i < x_i$ for $i = 1, \cdots, t$; that is, $F(x)$ is the distribution function of y. The distribution function $F(x)$ is assumed to be unknown. It is known, however, that $F(x)$ is one of the distribution functions $F_1(x), \cdots, F_m(x)$. An observation x is made on y and according to the observed value x the statistician may adopt any one of n decisions $j (j = 1, \cdots, n)$. Let $W_{i,j}(x)$ denote the loss sustained by the statistician when $F_i(x)$ is the true distribution of y, x is the observed value of y, and the jth decision is adopted. $W_{i,j}(x)$ is assumed to be a finite nonnegative and measurable function of x. If the statistician, on observing the value x, adopts the various decisions with probabilities $\eta_j(x)$, where these are nonnegative measurable functions satisfying (1), then the risk, or expected loss, when $F_i(x)$ is the true distribution function, is given by

$$r_i(\eta) = \sum_{j=1}^{n} \int_X W_{i,j}(x)\ \eta_j(x)\ dF_i(x)\ .$$

The decision function $\eta_j(x)$ is said to be nonrandomized if for every x all but one of the $\eta_j(x)$ vanish. Theorem 4 yields without difficulty the following result: *If the distribution functions $F_i(x) (i = 1, \cdots, m)$ are atomless then, given any decision function $\eta(x)$, there exists a nonrandomized decision function $\eta^*(x)$ such that $r_i(\eta) = r_i(\eta^*) (i = 1, \cdots, m)$.*

Similar application can be made to the theory of games. In fact, the above described statistical decision problem may be interpreted as a zero-sum two-person game as follows: Player 1 has a finite number of pure strategies $i (i = 1, \cdots, m)$, while a pure strategy of Player 2 is a nonrandomized decision function $\eta^*(x)$ (decomposition n-vector). If i is the pure strategy of Player 1 and $\eta^*(x)$ the pure strategy of Player 2, the outcome is defined by

$$R[i, \eta^*(x)] = r_i(\eta^*)\ .$$

A mixed strategy of Player 1 is represented by a vector $\xi = (\xi_1, \cdots, \xi_m)$ with nonnegative components whose sum is one, while a mixed strategy of Player 2 is given by a probability n-vector $\eta(x)$. The expected value of the outcome corresponding to the mixed strategies ξ and $\eta(x)$ is given by

$$R[\xi, \eta(x)] = \sum_{i=1}^{m} \xi_i r_i(\eta) .$$

The above stated result for the statistical decision problem can be restated in game terminology as follows: *If the distribution functions* $F_i(x)(i = 1, \cdots, m)$ *are atomless, then given any mixed strategy* $\eta(x)$ *of Player* 2, *there exists a pure strategy* $\eta^*(x)$ *such that* $R[\xi, \eta^*(x)] = R[\xi, \eta(x)]$ *for all strategies* ξ *of Player* 1.

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SCHLICHT TAYLOR SERIES WHOSE CONVERGENCE ON THE UNIT CIRCLE IS UNIFORM BUT NOT ABSOLUTE

P. ERDÖS, F. HERZOG, AND G. PIRANIAN

1. Summary. That a Taylor series which converges uniformly on the unit circle C need not converge absolutely on C was proved by Hardy [2] (see also Landau [4, p.68]; for a simpler example, see Herzog and Piranian [3, Section 4]). *The present paper exhibits two functions that are schlicht on the closed unit disc, and whose Taylor series converge uniformly but not absolutely on C. Each of the examples satisfies an additional restrictive requirement: the first function has only one singular point on C, and the Taylor series*

$$\sum_{k=0}^{\infty} a_k z^{m_k} \tag{1}$$

of the second function has the property that $\lim(m_{k+1} - m_k) = \infty$.

The condition that (1) represent a schlicht function and converge uniformly but not absolutely on C imposes restrictions on the sequence of exponents $\{m_k\}$. For the condition implies that $\sum_{k=0}^{\infty} m_k |a_k|^2 < \infty$ (see Landau [4, p.65]); since, by Schwarz's inequality, we have

$$\left(\sum_{k=0}^{\infty} |a_k|\right)^2 \leq \sum_{k=0}^{\infty} m_k |a_k|^2 \cdot \sum_{k=0}^{\infty} 1/m_k\,,$$

it follows that

$$\sum_{k=0}^{\infty} 1/m_k = \infty. \tag{2}$$

It remains an open question whether the condition implies a restriction on $\{m_k\}$ which is stronger than (2).

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In the construction of both examples, the basic idea consists of the observation that if

$$h(z) \equiv z + k\omega[1 - (1 - z/\omega)^{1/n}] \tag{3}$$

(where k is a real constant, $|\omega| = 1$, n is a positive integer and the function $(1 - z/\omega)^{1/n}$ is chosen to be positive when $z = \omega/2$), then $h(z)$ maps the unit disc into a region which consists roughly of the unit disc with a tooth of length k protruding at the point $z = \omega$. The tooth can be made arbitrarily narrow by choosing n large enough. If additional terms are joined to the right member of (3), the map of the unit disc by $h(z)$ bristles with teeth; and if the lengths, widths and positions of these teeth are chosen appropriately, the Taylor series of $h(z)$ converges uniformly, but not absolutely, on C. The geometric and analytic motivation for the devices that induce $h(z)$ to satisfy the additional requirements will be obvious from the text.

2. The first example. Let $\{\phi_j\}$ be a decreasing sequence of real numbers $(2\pi > \phi_1, \phi_j \to 0)$, and $\{\delta_j\}$ a sequence of positive numbers such that the discs $|z - e^{i\phi_j}| < \delta_j$ are disjoint. For each index j, Ω_j shall denote the complement, relative to the disc $|z| < 2$, of the union of the disc $|z - e^{i\phi_j}| < \delta_j$ and the line segment $z = re^{i\phi_j}$, $1 < r < 2$. Also, for each index j, ρ_j shall denote a real number subject to the condition

$$1 < \rho_j < 1 + \delta_j/2 \; ; \tag{4}$$

N_j shall denote a positive integer such that, for every ρ_j satisfying (4) and every n_j greater than N_j, the function

$$f_j(z) \equiv 1 - (1 - z/\omega_j)^{1/n_j}$$

$(\omega_j = \rho_j e^{i\phi_j}$, $(1 - z/\omega_j)^{1/n_j}$ positive when $z = \omega_j/2)$ satisfies the inequality $|f_j(z)| < 2^{-j}$ throughout Ω_j.

We now proceed to select the integers n_j in such a way that, in a suitable region about the origin, the series

$$z + \sum_{j=1}^{\infty} \omega_j f_j(z)/j \tag{5}$$

converges uniformly to a function which is endowed with the desired properties. To this end, we define the numbers $a_{m,j}$ by the equations

$$f_j(z) = \sum_{m=1}^{\infty} a_{m,j}(ze^{-i\phi_j})^m ,$$

keeping in mind that at this stage of the discussion the numbers $a_{m,j}$ must be regarded as functions of the still undetermined constants ρ_j and n_j. It should be noted that the $a_{m,j}$ are all positive; that, for each fixed j, they form a decreasing sequence whose first element is $(\rho_j\, n_j)^{-1}$; and that

(6) $$\sum_{m=1}^{\infty} a_{m,j} = f_j(e^{i\phi_j}) = 1 - (1 - 1/\rho_j)^{1/n_j} .$$

Let n_1 be an integer greater than N_1; and let ρ_1 be a real number satisfying condition (4), and near enough to one so that $(1 - 1/\rho_1)^{1/n_1} < 2^{-1}$. Once n_ν and ρ_ν have been chosen for $\nu = 1, 2, \cdots, j-1$, let M_j denote an integer so large that

(7) $$\sum_{\nu=1}^{j-1} \sum_{m>M_j} a_{m,\nu} < 2^{-j} ,$$

and let n_j be greater than N_j and so large that, for all ρ_j satisfying (4),

(8) $$\sum_{m\leq M_j} a_{m,j} < 2^{-j} ;$$

finally, let ρ_j be chosen near enough to one so that

(9) $$(1 - 1/\rho_j)^{1/n_j} < 2^{-j} .$$

Then the series (5) converges uniformly in some closed region whose interior contains all points of the closed unit disc except the point $z = 1$. Its sum $F(z)$ is therefore continuous on the closed disc, and holomorphic at all its points except at $z = 1$. The Taylor series $\sum_{m=1}^{\infty} a_m z^m$ of $F(z)$ does not converge absolutely on C; for

$$a_m = \sum_{\nu=1}^{\infty} (\omega_\nu/\nu)\; a_{m,\nu}\, e^{-i\phi_\nu m} , \qquad m \geq 2,$$

and therefore

$$|a_m| \geq a_{m,j}/j - 2 \sum_{\nu \neq j} a_{m,\nu} \ ;$$

hence it follows from (6)–(9) that

$$\sum_{M_j < m \leq M_{j+1}} |a_m| \geq j^{-1} - O(2^{-j}) \ .$$

That $\Sigma_{m=1}^{\infty} a_m z^m$ converges uniformly on C can be shown directly; but it will also follow from the continuity of $F(z)$ and Fejér's Theorem, once univalence has been established (see Fejér [1] or Landau [4, pp. 65, 66]).

To establish univalence of the function $F(z)$, it is sufficient to note that

$$(d/dz)[-\omega_j (1 - z/\omega_j)^{1/n_j}] = (1/n_j)(1 - z/\omega_j)^{1/n_j - 1},$$

whence the argument of the quantity on the left is $-(1 - 1/n_j) \arg(1 - z/\omega_j)$; since $-\pi/2 < \arg(1 - z/\omega_j) < \pi/2$, the real part of the derivative of $\omega_j f_j(z)$ is positive throughout the open unit disc, and therefore $\Re F'(z) > 1$ when $|z| < 1$. This implies that $|F(z_1) - F(z_2)| \geq |z_1 - z_2|$ for all pairs of points z_1 and z_2 in the open unit disc; and because $F(z)$ is continuous in the closed unit disc, it is schlicht in the closed unit disc.

3. The second example. The schlicht function whose Taylor series has Fabry gaps and converges uniformly but not absolutely on C is obtained from the first example by simple modifications. Let

$$G(z) \equiv z + \sum_{j=1}^{\infty} g_j(z) \ , \tag{10}$$

where

$$g_j(z) \equiv k_j z \{1 - [1 - (z/\omega_j)^{p_j}]^{1/n_j}\} \ ;$$

the symbols ω_j and n_j play the same role as in the first example; k_j is a certain real number; and p_j is an integer, much smaller than n_j. For the sake of intuitive clarity, it should be observed that the value of $g_j(z)$ is $k_j z$ when $(z/\omega_j)^{p_j} = 1$, and that it is small whenever $|z| \leq |\omega_j|$ and $(z/\omega_j)^{p_j}$ is very different from one. A rough idea of the image of C under the mapping by $G(z)$ can be obtained by attaching to C a tooth of length k_1 at each of the points

$$z = \exp[i(\phi_1 + 2\pi h/p_1)], \qquad h = 0, 1, 2, \cdots, p_1 - 1,$$

then adding further sets of teeth as dictated by the parameters k_2, ω_2, p_2, and so forth.

A rigorous proof that the parameters can actually be chosen in such a way that the function $G(z)$ is schlicht and will be schlicht after it has been modified through the introduction of gaps in its Taylor series is based on the study of $\Re\, \psi'(z)$, where

$$\psi(z) \equiv z\{1 - [1 - (z/\omega)^p]^{1/n}\}$$

($|\omega| > 1$, p and n integers, $1 \leq p < n$). If $t = (z/\omega)^p$, then

$$\psi'(z) = 1 + (1 - t)^{1/n-1}[(1 + p/n)\, t - 1] \equiv \Phi(t)\,.$$

We wish to show that

$$\Re\, \psi'(z) > -3p/n\,, \qquad |z| \leq 1. \tag{11}$$

In order to do this we shall prove that

$$\Re\Phi(t) > -3p/n\,, \qquad |t| \leq 1, \quad t \neq 1. \tag{12}$$

Since $\Phi(t)$ is holomorphic for $|t| \leq 1$, $t \neq 1$, it will suffice to show that (12) holds

(a) when t is inside the unit circle (of the t-plane) and sufficiently near the point $t = 1$;

(b) when $|t| = 1$, $t \neq 1$.

Since the coefficients of the powers of t in the power series of $\Phi(t)$ are all real, we may restrict ourselves in (a) and (b) to values of t whose imaginary part is nonnegative; if t has one of these values,

$$0 \geq \arg(1 - t) > -\pi/2\,.$$

(a) Let $t = u + iv$, and consider those values of t for which

$$u^2 + v^2 < 1\,, \qquad \frac{1 + p/2n}{1 + p/n} < u < 1\,, \qquad 0 \leq v < \frac{p}{2n^2(1 + p/n)}\,.$$

We then have

$$0 \leq \arg(1-t)^{1/n-1} < (\pi/2)(1-1/n)$$

and

$$0 \leq \arg[(1+p/n)\,t-1] = \arctan \frac{(1+p/n)v}{(1+p/n)\,u-1} < \frac{p/2n^2}{p/2n} < \frac{\pi}{2n},$$

whence $\Re\,\Phi(t) > 1$.

(b) Let $t = e^{i\theta}$, where $0 < \theta \leq \pi$. A simple computation gives

$$\begin{aligned} (13) \qquad \Re\Phi(t) &= 1 - \left(2 \sin \frac{\theta}{2}\right)^{1/n-1} \\ &\quad \times \left\{\left(1+\frac{p}{n}\right) \sin \frac{(n+1)\theta - \pi}{2n} + \sin \frac{(n-1)\theta+\pi}{2n}\right\} \\ &= 1 - \left(2 \sin \frac{\theta}{2}\right)^{1/n} \cos \frac{\pi-\theta}{2n} \\ &\quad - \frac{p}{n}\left(2 \sin \frac{\theta}{2}\right)^{1/n-1} \sin \frac{(n+1)\theta-\pi}{2n}. \end{aligned}$$

(b_1) If $0 < \theta \leq \pi/(n+1)$ then, from the second expression for $\Re\,\Phi(t)$ in (13), we have

$$\Re\Phi(t) \geq 1 - (2 \sin \theta/2)^{1/n} \cos[(\pi-\theta)/2n] > 0.$$

(b_2) If $\pi/(n+1) < \theta \leq \pi$, then the content of the braces in the first expression for $\Re\,\Phi(t)$ in (13) is less than

$$(1+p/n)(2 \sin \theta/2) \cos [(\pi-\theta)/2n],$$

and hence

$$\Re\Phi(t) > 1 - 2^{1/n}(1+p/n) > -3p/n.$$

This establishes the validity of (12), and therefore that of (11).

Now let

$$\{p_j\} = \{1, 2, 2, 4, 4, 4, 4, 8, 8, 8, \cdots\};$$

$$k_j = 1/p_j;$$

$$\{\phi_j/2\pi\} = \{0, 0, 1/4, 0, 1/16, 2/16, 3/16, 0, 1/64, 2/64, \cdots\}.$$

The choice of the parameters n_j and ρ_j is similar to the analogous procedure in the first example. However, here we restrict ourselves entirely to the closed unit disc and choose as the region Ω_j the complement, relative to $|z| \leq 1$, of the union of certain neighborhoods of the points

$$\exp[i(\phi_j + 2\pi h/p_j)], \qquad h = 0, 1, 2, \cdots, p_j - 1.$$

These neighborhoods are chosen sufficiently small so that if a point z of the closed unit disc fails to lie in Ω_j, it lies in Ω_r whenever $r \neq j$ and $p_r = p_j$. Furthermore, the indices n_j should be greater than p_j and such that

$$\sum_{j=1}^{\infty} 1/n_j < 1/8.$$

In this manner we will again arrive at the result that the series in (10) converges uniformly for $|z| \leq 1$, and that the convergence of the Taylor series for $G(z)$ is not absolute on $|z| = 1$ because, as in the first example,

$$\sum_{M_j < m \leq M_{j+1}} |a_m| \geq k_j - O(2^{-j}) \tag{14}$$

and $\sum_{j=1}^{\infty} k_j = \infty$.

The function $G(z)$ has all the properties that are required of the second example (see Summary), except that it fails to possess Fabry gaps. In order to introduce these, we replace each $g_j(z)$ by a partial sum $s_j(z)$ of its Taylor series. Because the Taylor series of $g_j(z)$ and $g_j'(z)$ converge uniformly in the closed unit disc, it is possible to choose the degrees P_j of the polynomials $s_j(z)$ large enough so that

$$|g_j(z) - s_j(z)| < 2^{-j}$$

when $|z| \leq 1$ (this ensures uniform convergence of the series

$$S(z) \equiv z + \sum_{j=1}^{\infty} s_j(z)$$

on the closed unit disc); so that

$$|g_j'(z) - s_j'(z)| < 1/n_j$$

when $|z| \leq 1$, and in turn

$$\Re S'(z) > 1 - 4 \sum_{j=1}^{\infty} 1/n_j > 1/2$$

(this guarantees univalence of the function $S(z)$ in the closed unit disc); and so that the analogue to (14) holds for the Taylor series of $S(z)$. The function $S(z)$ then has the desired properties.

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ON DEDEKIND'S FUNCTION $\eta(\tau)$

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1. Introduction. A transformation of the form

$$\tau' = \frac{a\tau + b}{c\tau + d}, \tag{1.1}$$

where a, b, c, d are rational integers satisfying

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 1, \tag{1.2}$$

is called a *modular transformation.* Without loss of generality we may assume $c \geq 0$. A function $f(\tau)$, analytic in the upper halfplane $\mathfrak{I}(\tau) > 0$, and satisfying the functional equation

$$f(\tau) = (c\tau + d)^k \, f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{1.3}$$

is called a *modular form of dimension k.* An example of a modular form is the discriminant

$$\Delta(\tau) = \exp\{2\pi i\tau\} \prod_{m=1}^{\omega} (1 - \exp\{2\pi i m\tau\})^{24}, \tag{1.4}$$

which is of dimension -12; that is, it satisfies the equation*

$$\Delta(\tau') = (c\tau + d)^{12} \Delta(\tau). \tag{1.5}$$

An important role in the theory of modular functions is played by the function

$$\eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \prod_{m=1}^{\infty} (1 - \exp\{2\pi i m\tau\}), \tag{1.6}$$

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*Cf. Hurwitz [6]; however, he gives this formula only in homogeneous coordinates.

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which is the 24th root of $\Delta(\tau)$. The transformation formula for this function may be obtained from (1.5) and is conveniently written as:

$$\eta(\tau') = \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{-i(c\tau + d)}\ \eta(\tau)\ . \tag{1.7}$$

Since we have assumed $c \geq 0$ and $\Im(\tau) > 0$, the radicand has a nonnegative real part. By the square root we always mean the principal branch; that is, $\Re(\sqrt{\ }) > 0$. The ϵ appearing in (1.7) is a 24th root of unity. The purpose of the present paper is to determine this ϵ completely.

Investigations concerning this root of unity were carried out first by Dedekind [2] and later by Tannery and Molk [10] and Rademacher [8; 9]. However, they use the theory of log $\eta(\tau)$, which requires much more than is needed for this purpose. Hurwitz discusses only $[\Delta(\tau)]^{1/12}$ and remarks that the transformation formula of $\eta(\tau)$ can be obtained by means of θ-functions. The investigations of Hermite [5] are likewise not sufficient for our purpose, because he discusses only $\eta^3(\tau)$, and therefore a third root of unity remains still undetermined.

In the following, we shall approach the determination of ϵ directly by investigations of the function $\eta(\tau)$, which, by a well-known formula due to Euler, can be written as the following sum:

$$\begin{aligned}\eta(\tau) &= \exp\left\{\frac{\pi i \tau}{12}\right\} \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda} \exp\{\pi i \tau\lambda(3\lambda - 1)\} \\ &= \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda} \exp\left\{3\pi i \tau\left(\lambda - \frac{1}{6}\right)^2\right\}.\end{aligned} \tag{1.8}$$

Our starting point is formula (1.8); our principal tools are a Poisson transformation formula and Gaussian sums.

2. Application of a Poisson formula. We introduce a new variable z with $\Re(z) > 0$ by the substitution*

$$\tau' = \frac{iz}{c} + \frac{a}{c}, \qquad c > 0;\ (a, c) = 1, \tag{2.1}$$

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* This requires $c \neq 0$, but the case $c = 0$ is trivial.

and obtain, from (1.8)

$$\eta\left(\frac{a}{c}+\frac{iz}{c}\right)=\sum_{\lambda=-\infty}^{+\infty}(-1)^{\lambda}\exp\left\{\frac{3\pi i}{c}(a+iz)\left(\lambda-\frac{1}{6}\right)^2\right\} \tag{2.2}$$

$$=\sum_{j \bmod 2c}\exp\pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^2\right\}$$

$$\times\sum_{q=-\infty}^{+\infty}\exp\left\{-\frac{3\pi z}{c}\left(2cq+j-\frac{1}{6}\right)^2\right\}.$$

To the inner sum,

$$F_c(z)=\sum_{q=-\infty}^{+\infty}\exp\left\{-12\pi cz\left(q+\frac{6j-1}{12c}\right)^2\right\},$$

we apply Poisson's formula (cf. [11]),

$$\sum_{m=-\infty}^{+\infty}\exp\{-\pi(m+\alpha)^2t\}=\frac{1}{\sqrt{t}}\sum_{m=-\infty}^{+\infty}\exp\left\{2\pi im\alpha-\frac{\pi m^2}{t}\right\}, \qquad \Re(t)>0,$$

and obtain

$$F_c(z)=\frac{1}{2\sqrt{3cz}}\sum_{q=-\infty}^{+\infty}\exp\left\{2\pi iq\,\frac{6j-1}{12c}-\frac{\pi q^2}{12cz}\right\}.$$

Putting this in (2.2), we get:

$$\eta\left(\frac{a}{c}+\frac{iz}{c}\right)=\frac{1}{\sqrt{3cz}}\sum_{q=-\infty}^{+\infty}\exp\left\{\frac{-\pi q^2}{12cz}\right\}T_q(c), \tag{2.3}$$

where

$$T_q(c)=\frac{1}{2}\sum_{j \bmod 2c}\exp\pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^2+q\,\frac{6j-1}{6c}\right\}$$

$$=\frac{1}{2}\exp\pi i\left\{\frac{a-2q}{12c}\right\}[1+\exp\pi i\{3ac+c-a+q\}]$$

$$\times\sum_{j=1}^{c}\exp\left\{\frac{\pi i}{c}[3aj^2+j(c-a+q)]\right\}.$$

But, a and c being coprime, and thus

$$3ac + c - a \equiv 1 \pmod{2},$$

only the T_q with odd subscripts actually appear so that we have

(2.4) $$T_{2r+1}(c) = \exp \pi i \left\{\frac{a-4r-2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i}{c}\left[3aj^2 + j(c-a+1+2r)\right]\right\}.$$

In order to have a complete square in the exponent we multiply each term of the sum by

$$\exp \pi i \left\{j\,\frac{ad-1}{c}(c-1+2r)\right\} = \exp \pi i \,\{jb(c+1)\}.$$

As we do not wish to change T_{2r+1} by this multiplication, we have to assume that, for c even, b also is even. Using the abbreviation

(2.5) $$\beta = cd + d - 1,$$

we obtain from (2.4):

(2.6) $$T_{2r+1}(c) = \exp \pi i \left\{\frac{a-4r-2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}\left[36j^2 + 12j(cd+d-1+2rd)\right]\right\}$$

$$= \exp \pi i \left\{\frac{a-a\beta^2-2}{12c} - \frac{r}{3c}(ad^2r + ad\beta + 1)\right\}$$

$$\times \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}(6j+\beta+2rd)^2\right\}.$$

In the sum appearing here, j can be taken as running over any full residue system mod c, because $\beta \equiv c \pmod{2}$ and therefore the sum remains unchanged if j is replaced by $j + c$. Consequently, β can be chosen arbitrarily, mod 6, and $T_{2r+1}(c)$ can be simplified by the substitution $r = 3\mu + \nu$. We note that

$$\exp \pi i \left\{\frac{-r}{3c}(ad^2r + ad\beta + 1)\right\}$$

$$= \exp \pi i \left\{\frac{-\mu}{c}(3\mu d + 3\mu bcd + 2d\nu + bc\beta + cd + d)\right.$$

$$\left. - \frac{\nu}{3c}(d\nu + bcd\nu + bc\beta + cd + d)\right\};$$

and considering

$$\exp\{-\pi i\mu(b\beta+d+3\mu bd)\} = \exp\{-\pi i\mu(bcd-b+d)\} = \exp\{\pi i\mu\},$$

we obtain

$$T_{6\mu+2\nu+1}(c) = \exp\pi i\left\{\frac{a-a\beta^2-2}{12c} - \frac{\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right] - \frac{\mu}{c}\left[3\mu d+d(1+2\nu)+c\right]\right\} H_{a,c}(\beta+2\nu d)$$

with the abbreviation

$$(2.7) \qquad H_{a,c}(\beta) = \sum_{j \bmod c} \exp\left\{\frac{\pi i a}{12c}(6j+\beta)^2\right\}, \qquad \beta \equiv c \pmod 2.$$

Looking back to (2.3), we see that the result we have obtained so far may be written as:

$$(2.8)\quad \eta\left(\frac{a}{c}+\frac{iz}{c}\right) = \frac{1}{\sqrt{3cz}}\exp\pi i\left\{\frac{a-a\beta^2-2}{12c}\right\} \times \sum_{\nu=0}^{2}\exp\left\{\frac{-\pi i\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right]\right\} U_\nu(z)\, H_{a,c}(\beta+2d\nu),$$

with

$$U_\nu(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\mu^2-\frac{d}{c}\mu(2\nu+1)\right]-\frac{3\pi}{cz}\left(\mu+\frac{2\nu+1}{6}\right)^2\right\}.$$

These expressions are easy to sum, since, according to (1.8), we have

$$U_0(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\left(\mu^2+\frac{\mu}{3}\right)\right]-\frac{3\pi}{cz}\left(\mu+\frac{1}{6}\right)^2\right\}$$

$$= \exp\left\{\frac{\pi id}{12c}\right\}\eta\left(-\frac{d}{c}+\frac{i}{cz}\right);$$

and, replacing μ by $-\mu-1$, we see that

$$U_1(z) = -U_1(z), \quad \text{or} \quad U_1(z) = 0 \, ,$$

$$U_2(z) = -\exp\left\{\pi i \, \frac{2d}{c}\right\} U_0(z) \, .$$

Now, by the meaning of z in (2.1), we get

$$-\frac{d}{c} + \frac{i}{cz} = \frac{-d\tau' + b}{c\tau' - a} = \tau \, ,$$

and have therefore:

$$\eta(\tau') = \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{a(1-\beta^2)-2+d}{12c} \right\} \times \left[H_{a,c}(\beta) - \exp\left\{ \frac{-2\pi i}{3} (d+2bd+b\beta) \right\} \times H_{a,c}(\beta + 4d) \right] \sqrt{-i(c\tau + d)} \; \eta(\tau) \, . \tag{2.9}$$

Comparing this with (1.7), we see that we have obtained so far:

$$\begin{aligned} \epsilon &= \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{a(1-\beta^2)-2+d}{12c} \right\} \times \left[H_{a,c}(\beta) - \exp\left\{ \frac{-2\pi i}{3} (d+2bd+b\beta) \right\} H_{a,c}(\beta + 4d) \right] \\ &= \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{bd(1-c^2)-cd}{12} + \frac{(1-d)(b+ad)}{6} \right\} \times \left[H_{a,c}(\beta) - \exp\left\{ \frac{-2\pi i}{3} (d+2bd+b\beta) \right\} H_{a,c}(\beta + 4d) \right] \end{aligned} \tag{2.91}$$

and it remains to be shown that this is a root of unity.

3. Reduction to Gaussian sums. The sums $H_{a,c}(\beta)$ which appear in (2.91) are defined in (2.7) only for $\beta \equiv c \pmod 2$. In this section, however, it will be more convenient to consider the more general sums*

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*We have used the letters h and k instead of a and c in order to indicate that the investigations of this section are independent from our previous results.

(3.1) $$H_{h,k}(\gamma) = \frac{1}{2} \sum_{j \bmod 2k} \exp\left\{\frac{\pi i h}{12k}(6j+\gamma)^2\right\},$$

with no restriction on γ. These sums can be expressed in terms of Gaussian sums

(3.2) $$G(h,k) = \sum_{j \bmod k} \exp\left\{\frac{2\pi i h}{k} j^2\right\}.$$

Comparing the definitions (3.1) and (3.2) one finds immediately that:

$$H_{h,k}(0) + H_{h,k}(1) + H_{h,k}(2) + H_{h,k}(3) + H_{h,k}(4) + H_{h,k}(5) = \frac{1}{4} G(h, 24k),$$

$$H_{h,k}(0) + H_{h,k}(2) + H_{h,k}(4) = \frac{1}{2} G(h, 6k),$$

$$H_{h,k}(0) + H_{h,k}(3) = \frac{1}{4} G(3h, 8k).$$

If we consider that

$$H_{h,k}(-\gamma) = H_{h,k}(\gamma) = H_{h,k}(\gamma + 6n),$$

we get the following relations:

(3.31) $$H_{h,k}(0) = \frac{1}{2} G(3h, 2k),$$

(3.32) $$H_{h,k}(3) = \frac{1}{4} G(3h, 8k) - \frac{1}{2} G(3h, 2k),$$

(3.33) $$H_{h,k}(2) = \frac{1}{4} G(h, 6k) - \frac{1}{4} G(3h, 2k),$$

(3.34) $$H_{h,k}(1) = \frac{1}{8} G(h, 24k) - \frac{1}{8} G(3h, 8k) - \frac{1}{4} G(h, 6k) + \frac{1}{4} G(3h, 2k).$$

In order to obtain the sums $H_{h,k}(\gamma)$ explicitly, the following rules concerning Gaussian sums may be useful.*

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* For the formulas (3.41)–(3.47) see [1] or [3]; (3.46) may also be found in [7].

As elementary consequences of the definition (3.2) we have:

(3.41) $G(mh, mk) = mG(h, k)$ $\qquad m > 0$

(3.42) $G(h, k_1k_2) = G(hk_1, k_2)\ G(hk_2, k_1)$ $\qquad (k_1, k_2) = 1$

(3.43) $G(m^2h, k) = G(h, k)$ $\qquad (m, k) = 1$

(3.44) $G(h, m^2k) = mG(h, k)$ $\qquad (m, h) = 1;\quad m > 0$ and odd.

The following results, due to Gauss [4], are a little deeper:

(3.45) $$G(h_1h_2, k) = \left(\frac{h_1}{k}\right) G(h_2, k) \qquad (h_1h_2, k) = 1,\quad k \text{ odd}$$

(3.46) $$G(1, k) = \sqrt{k}\ i^{[(k-1)/2]^2} \qquad k \text{ odd}$$

(3.47) $$G(h, 2^{\alpha}) = \begin{cases} 0 & h \text{ odd},\quad \alpha = 1 \\ 2^{(\alpha+1)/2} \left(\dfrac{2}{h}\right)^{\alpha+1} e^{\pi i h/4} & h \text{ odd},\quad \alpha \geq 2. \end{cases}$$

The symbol $\left(\frac{h}{k}\right)$ is the Jacobi symbol.

The following discussion may be restricted to the case $\gamma \equiv k \pmod 2$, which will be sufficient for our purpose. Furthermore, we put* throughout $k = 2^{\lambda}k_1$ (k_1 being odd), and have then to distinguish whether 3 does or does not divide k_1.

Assume first $3 \mid k_1$. Then we find, using (3.41) and (3.44), that

(3.51) $$H_{h,k}(1) = 0, \qquad H_{h,k}(2) = 0;$$

and, applying (3.41), (3.42), (3.44), (3.45), and (3.47), we obtain:

(3.52) $$H_{h,k}(0) = 2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda} \exp\left\{\frac{3}{4}\pi i h k_1\right\} G(2h, 3k_1),$$

(3.53) $$H_{h,k}(3) = \exp\left\{\frac{3}{4}\pi i h k\right\}\ G(2h, 3k).$$

*We do this in order to avoid the reciprocity law for Gaussian sums which would require additional distinctions concerning the sign of h.

As a consequence of (3.46) we have:

$$G(1,\,3k)=\sqrt{3k}\;\exp\left\{\frac{\pi i}{8}\,(3k-1)^2\right\}=-\sqrt{3}\;\exp\left\{\frac{-\pi ik}{2}\right\}\,G(1,k)\,,$$

and therefore, according to (3.45),

$$G(2h,3k)=\left(\frac{2h}{3k}\right)\,G(1,3k)=-\left(\frac{2h}{3}\right)\,\sqrt{3}\;\exp\left\{\frac{-\pi ik}{2}\right\}\,G(2h,k)\,.$$

This formula enables us to express (3.52) and (3.53) in the single formula:

$$\text{(3.6)}\quad H_{h,k}\,(k)=\sqrt{3}\;2^{\lambda/2}\,\left(\frac{h}{3}\right)\,\exp\pi i\left\{\frac{k_1(h-1)}{2}+\frac{hk_1}{4}+\lambda\frac{h^2-1}{8}\right\}\,G(2h,k_1)\,.$$

In case $3\nmid k_1$, by use of (3.42) and (3.43) we can express the more complicated sums $H_{h,k}(1)$ and $H_{h,k}(2)$ by $H_{h,k}(3)$ and $H_{h,k}(0)$, respectively:

$$\text{(3.71)}\qquad H_{h,k}\,(1)=\exp\left\{\frac{4}{3}\,\pi\,ihk\right\}\,H_{h,k}\,(3)\,,$$

$$\text{(3.72)}\qquad H_{h,k}\,(2)=\exp\left\{\frac{4}{3}\,\pi\,ihk\right\}\,H_{h,k}\,(0)\,.$$

More generally, the following recursion formula holds:

$$\text{(3.73)}\qquad H_{h,k}\,(\gamma+2n)=\exp\left\{\frac{\pi i}{3}\,(\gamma+n)\,nhk\right\}\,H_{h,k}\,(\gamma)\,.$$

In order to compute $H_{h,k}(0)$ and $H_{h,k}(3)$, we apply (3.42), (3.43), (3.45), and (3.47) to obtain:

$$H_{h,k}\,(3)=\left(\frac{k}{3}\right)\exp\pi i\left\{\frac{k-1}{2}+\frac{3hk}{4}\right\}\,G(2h,k)\,,$$

$$H_{h,k}\,(0)=\left(\frac{k}{3}\right)\,2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda}\exp\pi i\left\{\frac{k_1-1}{2}+\frac{3hk_1}{4}\right\}\,G(2h,k_1)\,.$$

Applying this on (3.71) and (3.72), and considering

$$\exp\pi i\left\{\frac{4}{3}\,hk+\frac{3}{4}\,hk_1\right\}=\exp\pi i\left\{\frac{hk}{12}+\frac{3}{4}\,h(k_1-k)\right\}\,,$$

we can combine (3.71) and (3.72) into:

$$(3.8)\quad H_{h,k}(k) = 2^{\lambda/2}\left(\frac{k}{3}\right) \times \exp \pi i \left\{\frac{hk}{12} + \frac{3}{4} h(k_1 - k) + \frac{k_1 - 1}{2} + \lambda \frac{h^2 - 1}{8}\right\} G(2h, k_1) .$$

4. Determination of the root of unity. Now we go back to our result (2.9) and consider the following expression:

$$(4.1)\qquad \rho = \frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{6}(1 - d)(b + ad)\right\} \times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}(d + 2bd + b\beta)\right\} H_{a,c}(\beta + 4d)\right] .$$

According to the results of the preceding section, we have to distinguish whether c is divisible by 3 or not and to keep in mind that $c = 2^{\lambda} c_1$, c_1 odd.

Let us assume first $3 \mid c$; according to (3.51) we know that:

$$H_{a,c}(\beta) = H_{a,c}(dc + d - 1) = 0 \qquad \text{if } d \equiv -1 \pmod 3 ,$$

$$H_{a,c}(\beta + 4d) = H_{a,c}(dc + 5d - 1) = 0 \qquad \text{if } d \equiv +1 \pmod 3 .$$

Therefore we have:

$$(4.2)\qquad \rho = \left(\frac{d}{3}\right) \frac{1}{\sqrt{3c}} \exp \pi i \left\{\frac{1}{6}(1 - d)(b + ad) + \frac{2}{3}(d - 1)(1 + b)\right\} H_{a,c}(c)$$

$$= \left(\frac{a}{3}\right) \frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{2}(d - 1)(b + ad)\right\} H_{a,c}(c) .$$

Considering that

$$\exp\left\{\frac{\pi i}{2}\left[(d - 1)(b + ad + c) + (a - 1)(c_1 - c)\right]\right\} = 1 ,$$

and therefore that

$$\exp \pi i \left\{\frac{1}{2}(d-1)(b+ad)+\frac{1}{2}(a-1)\ c_1\right\}$$

$$=\exp\left\{\frac{\pi i}{2}\ [(d-1)(b+ad+c)+(a-1)(c_1-c)-c(d-a)]\right\}$$

$$=\exp\left\{\frac{\pi i}{6}\ c(d-a)\right\},$$

we get from (4.2) and (3.6):

$$\rho=\frac{1}{\sqrt{c_1}}\ \exp \pi i\left\{\frac{a}{4}(c_1-c)+\frac{cd}{6}+\frac{ac}{12}+\lambda\frac{a^2-1}{8}\right\} G(2a,c_1)\ . \tag{4.3}$$

In case $3\nmid c$, we can apply (3.73), which gives us

$$H_{a,c}(\beta+4d)=\exp\left\{\frac{2\pi i}{3}(\beta+2d)\ acd\right\} H_{a,c}(\beta)$$

$$=\exp\left\{\frac{2\pi i}{3}(b\beta+2bd+d-c)\right\} H_{a,c}(\beta)\ ,$$

and obtain from (4.1):

$$\rho=\frac{1}{\sqrt{3c}}\exp\left\{\frac{\pi i}{6}(1-d)(b+ad)\right\}\left[1-\exp\left\{\frac{-2\pi i c}{3}\right\}\right] H_{a,c}(\beta)$$

$$=\frac{1}{\sqrt{c}}\left(\frac{c}{3}\right)\ \exp \pi i\left\{\frac{1}{6}(1-d)(b+ad)-\frac{1}{2}+\frac{2c}{3}\right\} H_{a,c}(\beta)\ .$$

Now we apply (3.37) once more, putting

$$H_{a,c}(\beta)=H_{a,c}(c+\beta-c)=\exp\left\{\frac{\pi i}{3}\left(c+\frac{\beta-c}{2}\right)\frac{\beta-c}{2}\ ac\right\} H_{a,c}(c)$$

$$=\exp\left\{\frac{\pi i}{12}(\beta^2-c^2)\ ac\right\} H_{a,c}(c)\ .$$

Using (3.8) and considering

$$\exp\left\{\frac{\pi i}{12}\ (\beta^2-c^2)\ ac\right\}$$

$$= \exp\left\{\frac{\pi i}{12}\ \left[ac(c^2-1)(d^2-1)+2ac(d-1)(cd+d)\right]\right\}$$

$$= \exp\left\{\frac{\pi i}{6}\ (d-1)(bc+c+b+c^2)\right\}$$

$$= \exp \pi i \left\{\frac{1}{6}\ (d-1)(b+ad) - \frac{1}{2}\ (d-1)(c^2-1)\ + \frac{c}{6}\ (d-1)\right\},$$

$$\exp\left\{\frac{\pi i}{2}\ \left[(a-1)(c_1-c)\ -\ (d-1)(c^2-1)\right]\right\}\ = 1\ ,$$

we see that the expression for ρ becomes again (4.3). Therefore, we have in all cases:

$$\epsilon = \exp \pi i \left\{\frac{1}{12}\ \left[bd(1-c^2)+c(a+d)\right]+a\ \frac{c_1-c}{4}+\lambda\frac{a^2-1}{8}\right\} \times \frac{1}{\sqrt{c_1}}\, G(2a, c_1)\ , \tag{4.4}$$

with the only restriction that, for even c, b also has to be even.

In order to show that our formula (4.4) holds even if this condition is not satisfied, we put

$$\tau' = \frac{a\tau + b}{c\tau + d}\ , \qquad c \text{ even}, \quad b \text{ odd},$$

$$\tau^* = \frac{(a+c)\ \tau + (b+d)}{c\tau + d} = \tau' + 1\ .$$

Then, for τ^*, formula (4.4) holds; considering

$$\eta(\tau+1) = \exp\left\{\frac{-\pi i}{12}\right\}\ \eta(\tau)\ ,$$

which is an immediate consequence of (1.6), we find:

$$\eta(\tau^*) = \epsilon^*\eta(\tau) = \exp\left\{\frac{-\pi i}{12}\right\} \eta(\tau') = \exp\left\{\frac{-\pi i}{12}\right\} \epsilon\eta(\tau)$$

$$\epsilon = \exp\left\{\frac{\pi i}{12}\right\} \epsilon^* . \tag{4.5}$$

Now, if we compute ϵ^* by means of (4.4), and then ϵ, using (4.5), the result will be exactly the same as we get computing ϵ directly by means of (4.4).

Finally, we can omit the Gaussian sums in (4.3) and, using (3.45) and (3.46), obtain:

$$\epsilon = \left(\frac{a}{c_1}\right) \times \exp\pi i\left\{\frac{1}{12}\left[bd(1-c^2)+c(a+d)\right] + \frac{1-c_1}{4} + a\,\frac{c-c_1}{4} + \lambda\frac{a^2-1}{8}\right\} . \tag{4.6}$$

This formula agrees with the one given by Tannery and Molk [10, p. 112] .

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UNIVERSITY OF PENNSYLVANIA

THE HEAVY SPHERE SUPPORTED BY A CONCENTRATED FORCE

WERNER LEUTERT

1. Introduction. In the linear three-dimensional theory of elasticity only a few particular solutions are known which describe the action of a concentrated force on an isotropic homogeneous solid. The fundamental particular solution which expresses the displacement due to a force at a point within an indefinitely extended solid was given first by Lord Kelvin [5]. It was found again at a later date by Boussinesq [1] along with other particular solutions which can be derived from it and which lead to the solution of the problem of a concentrated force acting on an infinite solid bounded by a plane. Michell [4] obtained the displacements and stresses in an infinite cone acted on by a concentrated force at the vertex by using Boussinesq's results. The solids considered by these authors all extend to infinity.

In this paper a particular solution describing the action of a concentrated force on a finite solid will be considered.

2. The problem. Let there be given an isotropic homogeneous sphere of radius a, which is supported by a radial concentrated force at the south pole. Our problem is the determination of the displacement vector at any point of the sphere in the case of equilibrium, that is, in the case in which the magnitude of the force is equal to the total weight of the sphere.

3. General theory. In the linear theory of elasticity for an isotropic homogeneous medium, the components u, v, w of the displacement vector $\mathfrak{u}$ with respect to a cartesian coordinate system x, y, z satisfy the differential equations of Lamé [2],

$$\Delta \mathfrak{u} + \alpha \operatorname{grad} \operatorname{div} \mathfrak{u} + \beta \mathfrak{X} = 0 , \tag{1}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

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The vector $\mathbf{X}$ with components X, Y, Z respectively denotes the body force per unit volume, and

$$\alpha = \frac{\lambda + G}{G}, \qquad \beta = \frac{1}{G}$$

are two constants depending on the material considered. The first component of (1) is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right] + \beta X = 0,$$

which explains the vector notation used. We restrict our attention to the physically important case

$$1 < \alpha, \qquad 0 < \beta \,.$$

The components F_x, F_y, F_z of the distributed force per unit surface area $\mathbf{F}$ which is necessary to maintain the displacement $\mathbf{u}$ throughout the solid are given by

$$\beta F_x = \left(\frac{\partial \mathbf{u}}{\partial x}, \mathbf{n}\right) + \frac{\partial u}{\partial n} + (\alpha - 1)\, n_x \operatorname{div} \mathbf{u}$$

(2)
$$\beta F_y = \left(\frac{\partial \mathbf{u}}{\partial y}, \mathbf{n}\right) + \frac{\partial v}{\partial n} + (\alpha - 1)\, n_y \operatorname{div} \mathbf{u}$$

$$\beta F_z = \left(\frac{\partial \mathbf{u}}{\partial z}, \mathbf{n}\right) + \frac{\partial w}{\partial n} + (\alpha - 1)\, n_z \operatorname{div} \mathbf{u} \,;$$

n_x, n_y, n_z are the components of the exterior unit normal $\mathbf{n}$. The first line in (2) may be written in the form

$$\beta F_x = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial v}{\partial x} + n_z \frac{\partial w}{\partial x} + n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial z} + (\alpha - 1)\, n_x \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right].$$

4. **Particular solutions when no body forces are present.** Boussinesq showed [1] that particular solutions of (1) for $X = 0$ may be obtained from a scalar function $\phi(x, y, z)$ by putting

$$u = \frac{\partial^2\phi}{\partial x\partial z} \qquad v = \frac{\partial^2\phi}{\partial y\partial z} \qquad w = \frac{\partial^2\phi}{\partial z^2} - \frac{\alpha + 1}{\alpha}\Delta\phi, \tag{3}$$

provided ϕ is a biharmonic function,

$$\Delta\Delta\phi = 0 . \tag{4}$$

Let

$$r^2 = x^2 + y^2 + z^2 ;$$

then

$$\phi = r \tag{5}$$

represents the action of a concentrated force in the z direction at the origin within an infinite solid [5].

The function

$$\phi = z \log(r + z) - r \tag{6}$$

leads to Boussinesq's solution [1] for an infinite solid bounded by the (x, y)-plane and acted upon by a concentrated force at the origin in the z direction. Michell's solution [4] can be obtained by a linear combination of (5) and (6).

The function

$$\phi = (r^2 - 3z^2) \log(r + z) + 3zr \tag{7}$$

was used by the author [3] to describe the displacements in a spherical shell under concentrated radial forces.

Since for $X = 0$ the system (1) is linear homogeneous with constant coefficients, particular solutions can be obtained from (5)–(7) by partial differentiation.

If (5) and (6) are differentiated with respect to z, two new particular solutions

$$\phi = \frac{z}{r} , \tag{8}$$

$$\phi = \log(r + z) \tag{9}$$

result. From (9) the particular solution

$$\phi = \frac{1}{r} \tag{10}$$

can be derived. A linear combination of (10) and the derivative of (8) with respect to z yields

$$\phi = \frac{z^2}{r^3}\,. \tag{11}$$

5. A particular solution for constant body force in the z-direction. If u, v, w are computed from

$$\phi = \frac{P\alpha\beta}{64\pi(\alpha+1)(3\alpha-1)a^3}\,[-(2\alpha-1)\,r^4 - 6r^2z^2 + (4\alpha+7)\,z^4 - 16(\alpha+1)\,az^3 + 8(\alpha+1)\,ar^2z\,]\,, \tag{12}$$

according to (3), it can be verified that (1) is satisfied for

$$X = Y = 0\,, \qquad Z = -\frac{3P}{4\pi a^3} \tag{13}$$

Equation (4) is no longer valid for the ϕ of (12).

6. Solution of the problem. The south pole of the sphere is taken as the origin of the coordinate system, with the z axis directed vertically upward. The sphere is then represented by the equation

$$r^2 \leq 2az\,. \tag{14}$$

The components of the exterior unit normal n are

$$an_x = x\,, \qquad an_y = y\,, \qquad an_z = z - a\,.$$

It can be verified that the function

$$\phi = \frac{P\alpha\beta}{64\pi(\alpha+1)(3\alpha-1)\,a^3}\,[-(2\alpha-1)\,r^4 - 6r^2z^2 \tag{15}$$

$$+ (4\alpha + 7)\ z^4 - 16(\alpha + 1)\ az^3 + 8(\alpha + 1)\ ar^2 z\ ]$$

$$+ \frac{P\beta}{96\ \pi(\alpha + 1)\ a r^3}\left[9(\alpha + 1)\ zr^4 - 12(2\alpha + 1)\ a r^4\right.$$

$$\left. - 48\alpha a^2 zr^2 + 16(\alpha - 1)\ a^3 r^2 + 16\alpha a^3 z^2\right]$$

$$+ \frac{P\beta}{32\pi a}\left[r^2 - 3z^2 + 4az - \frac{16a^2}{\alpha + 1}\right] \log(r + z)$$

satisfies (1) provided the body forces are distributed according to (13). The particular solution (14) consists of a linear combination of (5)–(11) added to (12). On the surface of the sphere $r^2 = 2az$ it is found that the distributed force F per unit surface area (2) vanishes on the whole surface except at the origin, where the particular solution (15) has a singularity.

Because the resulting body force must be in equilibrium with the resulting exterior force, it follows from (13) that the latter is radial upward and of magnitude P.

Since ϕ in (15) possesses a singularity at the origin, the corresponding displacements and stresses will be infinite at that point. To avoid this difficulty we can imagine the material near the origin cut out and the concentrated force P replaced by the statically equivalent forces distributed over the surface of the small cavity.

The displacements belonging to (15) can be computed by using (3).

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UNIVERSITY OF MARYLAND

ON THE DEFINITION OF NORMAL NUMBERS

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1. Introduction. Let R be a real number with fractional part $.x_1 x_2 x_3 \cdots$ when written to scale r. Let $N(b, n)$ denote the number of occurrences of the digit b in the first n places. The number R is said to be *simply normal* to scale r if

$$\lim_{n \to \infty} \frac{N(b, n)}{n} = \frac{1}{r} \tag{1}$$

for each of the r possible values of b; R is said to be *normal* to scale r if all the numbers $R, rR, r^2R, \cdots$ are simply normal to all the scales $r, r^2, r^3, \cdots$. These definitions, for $r = 10$, were introduced by Émile Borel [1], who stated (p. 261) that "la propriété caractéristique" of a normal number is the following: that for any sequence B whatsoever of v specified digits, we have

$$\lim_{n \to \infty} \frac{N(B, n)}{n} = \frac{1}{r^v}, \tag{2}$$

where $N(B, n)$ stands for the number of occurrences of the sequence B in the first n decimal places.

Several writers, for example Champernowne [2], Koksma [3, p. 116], and Copeland and Erdös [4], have taken this property (2) as the definition of a normal number. Hardy and Wright [5, p. 124] state that property (2) is equivalent to the definition, but give no proof. It is easy to show that a normal number has property (2), but the implication in the other direction does not appear to be so obvious. If the number R has property (2) then any sequence of digits

$$B = b_1 b_2 \cdots b_v$$

appears with the appropriate frequency, but will the frequencies all be the same for $i = 1, 2, \cdots, v$ if we count only those occurrences of B such that b_1 is an $i, i + v, i + 2v, \cdots$ *-th* digit? It is the purpose of this note to show that this is

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so, and thus to prove the equivalence of property (2) and the definition of normal number.

2. Notation. In addition to the notation already introduced, we shall use the following:

S_α is the first α digits of R.

BXB is the totality of sequences of the form $b_1 b_2 \cdots b_v x x \cdots x b_1 b_2 \cdots b_v$, where $xx \cdots x$ is any sequence of t digits.

$k_i(\alpha)$ is the number of times that B occurs in S_α with b_1 in a place congruent to $i \pmod v$.

$$g(\alpha) = \sum_{i=0}^{v-1} k_i(\alpha) .$$

$\theta_t(\alpha)$ is the number of occurrences of BXB in S_α.

$$k_{i,j}(\alpha) = k_i(\alpha) - k_j(\alpha) , \qquad i \neq j .$$

B^* is any block of digits of length from $v+1$ to $2v-1$ whose first v digits are B and whose last v digits are B. Such a block need not exist.

3. Proof. We shall assume that the number R has the property (2), so that we have

$$\lim_{n\to\infty} \frac{g(n)}{n} = \frac{1}{r^v} \tag{3}$$

and

$$\lim_{n\to\infty} \frac{\theta_t(n)}{n} = \frac{1}{r^{2v}} \tag{4}$$

for each fixed t, and we prove that

$$\lim_{n\to\infty} \frac{k_{i,j}(n)}{n} = 0 , \tag{5}$$

from which it follows that R is a normal number.

Now $k_i(\alpha + s) - k_i(\alpha)$ is the number of B with b_1 in a place congruent to to $i \pmod v$ that are in $S_{\alpha+s}$ but not entirely in S_α. Therefore

$$\sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \{k_i(\alpha+s)-k_i(\alpha)\}\{k_j(\alpha+s)-k_j(\alpha)\}$$

counts the number of BXB and B^* that occur in $S_{\alpha+s}$ such that the first B is not contained entirely in S_α. Here the number t of digits in X runs through all values $\not\equiv 0 (\text{mod } v)$ with $0 \le t \le s - v - 1$. We take $n > s$ and sum the above expression to get

$$\sigma = \sum_{\alpha=0}^{n-s} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \{k_i(\alpha+s)-k_i(\alpha)\}\{k_j(\alpha+s)-k_j(\alpha)\}. \tag{6}$$

Considering S_n and any BXB contained in it with $t \le s - v - 1$, we see that BXB is counted in σ a certain number of times. In fact if BXB is not too near either end of S_n it is counted just $s - t - v$ times and it is never counted more than this many times. Furthermore if BXB is preceded by at least $s - t - 2v$ digits and is followed in S_n by at least $s - t - v - 1$ digits then BXB is counted exactly $s - t - v$ times. Therefore we have, ignoring any B^* blocks which may be counted by σ,

$$\sigma \ge \sum_{\substack{t=0 \\ t\not\equiv 0 (\text{mod } v)}}^{s-v-1} (s-t-v)\{\theta_t(n-s)-\theta_t(s)\}. \tag{7}$$

Using (4) we find

$$\lim_{n\to\infty} \frac{\theta_t(n-s)}{n} = \frac{1}{r^{2v}}$$

for any fixed s; hence, from (7), we have

$$\lim_{n\to\infty} \frac{\sigma}{n} \ge \sum_{\substack{t=0 \\ t\not\equiv 0 (\text{mod } v)}}^{s-v-1} (s-t-v)\frac{1}{r^{2v}}.$$

It is now convenient to take s, which is otherwise arbitrary, to be congruent to

$0 \pmod v$. Then the above formula reduces to

$$\lim_{n\to\infty} \frac{\sigma}{n} \geq \frac{(v-1)(s-v)^2}{2v} \cdot \frac{1}{r^{2v}} . \tag{8}$$

In a similar manner we count the BXB in S_n where the number t of digits of X is congruent to $0 \pmod v$. This gives us

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \frac{1}{2} \{k_i(\alpha+s) - k_i(\alpha)\}\{k_i(\alpha+s) - k_i(\alpha) - 1\} \tag{9}$$

$$= \sum_{\substack{t=0 \\ t\not\equiv 0 \pmod v}}^{s-v-1} (s-t-v) \frac{1}{r^{2v}} = \frac{s(s-v)}{2v} \cdot \frac{1}{r^{2v}} .$$

Now, by (3) we have

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\} = \lim_{n\to\infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \{g(\alpha+s) - g(\alpha)\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{2n} \sum_{\alpha=n-s+1}^{n} g(\alpha+s) - \frac{1}{2n} \sum_{\alpha=0}^{s-1} g(\alpha) \right\} = \frac{s}{2r^v} ,$$

and (9) reduces to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\}^2 = \frac{s}{r^v} + \frac{s(s-v)}{vr^{2v}} . \tag{10}$$

From (6), (8), and (10) we find that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \{[k_i(\alpha+s) - k_i(\alpha)] - [k_j(\alpha+s) - k_j(\alpha)]\}^2 \tag{11}$$

$$\leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

for any fixed $s \equiv 0 \pmod v$. Using the inequality

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$$

we obtain

$$\sum_{\alpha=0}^{n-s} \{[k_i(\alpha+s)-k_i(\alpha)]-[k_j(\alpha+s)-k_j(\alpha)]\}^2$$

$$\geq \frac{1}{n-s+1}\left\{\sum_{\alpha=0}^{n-s}[k_i(\alpha+s)-k_i(\alpha)-k_j(\alpha+s)+k_j(\alpha)]\right\}^2$$

$$=\frac{1}{n-s+1}\left\{\sum_{\alpha=0}^{n-s}[k_{i,j}(\alpha+s)-k_{i,j}(\alpha)]\right\}^2$$

$$=\frac{1}{n-s+1}\left\{\sum_{\alpha=0}^{s-1}k_{i,j}(n-\alpha)-\sum_{\alpha=0}^{s-1}k_{i,j}(\alpha)\right\}^2 .$$

This with (11) implies

$$\overline{\lim_{n\to\infty}}\frac{1}{n(n-s+1)}\sum_{\substack{i<j\\ i=0,1,\cdots,v-2\\ j=1,2,\cdots,v-1}}\left\{\sum_{\alpha=0}^{s-1}k_{i,j}(n-\alpha)-\sum_{\alpha=0}^{s-1}k_{i,j}(\alpha)\right\}^2 \leq \frac{(v-1)s}{r^v}+\frac{(v-1)(s-v)}{r^{2v}} . \tag{12}$$

From the definition we have $|k_{i,j}(\alpha)| < \alpha$ and hence

$$\lim_{n\to\infty}\frac{1}{n(n-s+1)}\left\{\sum_{\alpha=0}^{s-1}k_{i,j}(\alpha)\right\}^2=0$$

and

$$\lim_{n\to\infty}\frac{1}{n(n-s+1)}\sum_{\alpha=0}^{s-1}k_{i,j}(n-\alpha)\sum_{\alpha=0}^{s-1}k_{i,j}(\alpha)=0$$

for fixed s.

Therefore (12) implies

$$\overline{\lim_{n\to\infty}} \frac{1}{n(n-s+1)} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) \right\}^2 \le \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}},$$

which can be written in the form

$$\overline{\lim_{n\to\infty}} \frac{1}{n(n-s+1)} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \left\{ s k_{i,j}(n) + \sum_{\alpha=0}^{s-1} \left[k_{i,j}(n-\alpha) - k_{i,j}(n) \right] \right\}^2 \le \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}.$$

But $|k_{i,j}(n-\alpha) - k_{i,j}(n)| < 2\alpha$ so that this implies

$$\overline{\lim_{n\to\infty}} \frac{1}{n(n-s+1)} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} s^2 \{k_{i,j}(n)\}^2 \le \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

or

$$\overline{\lim_{n\to\infty}} \sum_{\substack{i<j \\ i=0,1,\cdots,v-2 \\ j=1,2,\cdots,v-1}} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \le \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}.$$

From this we have

$$\overline{\lim_{n\to\infty}} \frac{\{k_{i,j}(n)\}^2}{n^2} = \overline{\lim_{n\to\infty}} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \le \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}$$

for any fixed $s \equiv 0 \pmod{v}$. Since the right member can be made arbitrarily small, we have

$$\lim_{n\to\infty} \frac{|k_{i,j}(n)|}{n} = 0$$

or

$$\lim_{n\to\infty} \frac{k_i(n)}{n} = \lim_{n\to\infty} \frac{k_j(n)}{n} .$$

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UNIVERSITY OF OREGON AND
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COMPLETE MAPPINGS OF FINITE GROUPS

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1. Introduction. A *complete mapping* of a group, loop, or quasigroup G is a biunique mapping $x \longrightarrow \theta(x)$ of G upon G such that $x \times \theta(x) \equiv \eta(x)$ is a biunique mapping of G upon G. This concept was introduced by H.B.Mann [3]; other applications have been indicated by R.H.Bruck [2], and Paige [6]. However, the determination of all groups which possess a complete mapping is still an open question. For abelian groups and groups of infinite order the problem has been answered in [1] and [5].

The first part of the present paper considers the question of complete mappings for finite non-abelian groups; the latter part is devoted to an application of complete mappings in the construction of orthogonal Latin squares.

2. Complete mappings. We shall consider finite groups G written multiplicatively, the identity element being $g_1 = 1$. A group G will be called an *admissible group* if there exists a complete mapping for G; otherwise G is said to be non-admissible.

It should be noted that all groups of odd order are admissible by letting $\theta(x) = x$.

THEOREM 1. *A necessary condition that G be an admissible group is that there exist an ordering of the elements of G such that* $g_1 \times g_2 \times \cdots \times g_n = 1$.

COROLLARY. *If G is an admissible group, the product of the elements of G in any order is an element of the commutator subgroup of G.*

Proof. Assume that $x \longrightarrow \theta(x)$ is a complete mapping for G. Without loss of generality we can take $\theta(1) = \eta(1) = 1$. Now consider $g_2 \times \theta(g_2)$; here $g_2^{-1} \neq \theta(g_2)$, so that $\theta(g_2)^{-1}$ occurs among the remaining elements of G. Then let $\theta(g_2)^{-1} = g_3$ and form the product $g_2 \times \theta(g_2) \times g_3 \times \theta(g_3)$. We continue in this manner and ultimately reach a product

$$g_2 \times \theta(g_2) \times g_3 \times \theta(g_3) \times \cdots \times g_s \times \theta(g_s) = 1, \tag{1}$$

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where $\theta(g_{i-1}) = g_i^{-1} (i = 3, \cdots, s)$ and $\theta(g_s) = g_2^{-1}$.

If $s < n$, we repeat the process beginning with $g_{s+1} \times \theta(g_{s+1})$ and finally we arrive at a series of cycles similar to (1) whose product is the identity. Thus, $\eta(g_1) \times \eta(g_2) \times \cdots \times \eta(g_n) = 1$, completing the proof and yielding the corollary as a consequence.

We note that in the cycle represented by (1), the elements

$$g_2 \times \theta(g_2) \times \cdots \times g_i \times \theta(g_i) = \eta(g_2) \times \cdots \times \eta(g_i) \qquad (i \leq s),$$

are all distinct; for the equality of two such products would imply $\theta(g_i) = \theta(g_j)$ or $i = j$. Hence, we have the following result.

THEOREM 2. *A necessary condition that G be admissible is that there exist an ordering of the elements of G into subsets, such that in each subset, the elements*

$$(2) \qquad g_{i_2},\ g_{i_2} \times g_{i_3},\ \cdots,\ g_{i_2} \times g_{i_3} \times \cdots \times g_{i_s} = 1$$

are all distinct.

In the most favorable case where G possesses a single subset of $n - 1$ nonidentity elements which satisfy condition (2), we may prove that G is an admissible group. To do this, let g_2 be the element that is not represented in the set of elements (2). Construct the mapping $\theta(x)$ as follows: $\theta(1) = 1$, $\theta(g_2)$ is the solution of the equation $g_2 \times x = g_{i_2}$, and successively let $g_{i+1} = \theta(g_i)^{-1}$, and let $\theta(g_{i+1})$ be the solution of the equation $g_{i+1} \times x = g_{i_{i+1}}$. All the $\theta(x)$'s are are distinct and different from 1; for if $\theta(g_k) = \theta(g_s)$, $k \neq s$, we would have

$$g_2 \times \theta(g_k) = g_{i_2} \times \cdots \times g_{i_k} = g_{i_2} \times \cdots \times g_{i_s} = g_2 \times \theta(g_s),$$

the inner equality being contrary to hypothesis for $k \neq s$. Moreover if $\theta(g_k) = 1$, we would have $g_2 = g_{i_2} \times \cdots \times g_{i_k}$, contrary to the selection of g_2. Thus, we have proved the following theorem.

THEOREM 3. *A sufficient condition that G be an admissible group is that there exist an ordering of the nonidentity elements of G, such that the elements*

$$g_2 \times \cdots \times g_i \qquad \text{for} \quad (i = 2, \cdots, n)$$

are all distinct and $g_2 \times \cdots \times g_n = 1$.

For abelian groups, Theorem 1 is also a sufficient condition that G be admissible and we conjecture that this is likewise the case for non-abelian groups.

However, the best we have been able to prove are theorems of the following type.

THEOREM 4. *Let H be a normal subgroup of G. If G/H admits a complete mapping θ_1, H a complete mapping θ_2, then G is an admissible group.*

Proof. Let $G/H \cong K$, the elements of K being $e, p, q, \cdots$. Let u_p be an element of G that maps upon $p \in K$. Every element of G has the form $u_p \times h$ or $h \times u_p$, where $p \in K$, $h \in H$. The equality of $u_p \times h$ and $u_q \times h'$ implies $p = q$ and $h = h'$.

Define $\theta(u_p \times h) = \theta_2(h)\, u_{\theta_1(p)}$. Obviously this mapping is biunique of G upon G. Consider

$$u_p \times h \times \theta_2(h) u_{\theta_1(p)} = u_q h' \theta_2(h') u_{\theta_1(q)} \ . \tag{3}$$

This implies

$$u_p \times u_{\theta_1(p)} \times H = u_q \times u_{\theta_1(q)} \times H \qquad \text{or} \quad u_{p\times\theta_1(p)} \times H = u_{q\times\theta_1(q)} \times H \ ,$$

whence $p \times \theta_1(p) = q \times \theta_1(q)$ or $p = q$, since θ_1 is a complete mapping for K. It then follows from (3) that $h = h'$ and θ is a complete mapping for G.

THEOREM 5. *If G is a group containing a subgroup H of odd order such that G/H is a nonadmissible abelian group, then G is nonadmissible.*

Proof. If G/H is a nonadmissible abelian group, G/H possesses a single element of order 2 [6; p.49]. Let this coset be $g_2 \times H$. Considering the product of the elements of G modulo H, we have $\prod_{i=1}^{n} g_i \equiv g_2 \bmod H$. Since g_2 is not in H, the product of the elements of G in any order is not in H. However, H contains the commutator subgroup of G and it follows from Corollary 1 of Theorem 1 that G is not admissible.

The two preceding theorems may be used to establish the admissibility or nonadmissibility of many groups. Often it is necessary to develop other techniques, as for example in groups of order 2^n. Here we are able to argue modulo the commutator subgroup and establish by mathematical induction the admissibility of those groups whose commutator subgroups are not cyclic. The remaining cases have been analyzed by Bruck and found to be admissible except in the obvious case where G is cyclic of order 2^n.

3. Orthogonal Latin squares. Recalling the definition of a Latin square [3, p.418], we see that the multiplication table of a quasigroup, loop, or group G is a Latin square. Indeed, any Latin square of order m may be used to define a quasigroup of order m. Mann [3, 4] has shown how Latin squares, orthogonal to a group G, may be constructed by means of complete mappings. (A Latin square L is said to be orthogonal to a group G if L is orthogonal to the multiplication table of G.) We may extend these results in the following manner.

For convenience we shall assume henceforth that the elements of a group or quasigroup G are $1, 2, \cdots, n$.

THEOREM 6. *Let G be a quasigroup. Let $\theta_1, \theta_2, \cdots, \theta_n$ be n complete mappings of G with the following property*:

$$\theta_i(g) \neq \theta_j(g), \qquad \text{for} \quad i \neq j, \qquad \text{all} \quad g \in G. \tag{4}$$

Construct a Latin square S by placing j in the kth row and $\theta_j(k)$th column. Then S is orthogonal to G.

Moreover, all Latin squares S, orthogonal to G, may be represented in this manner.

Proof. Obviously the square S is a Latin square and it is orthogonal to G since the number pairs $[k \times \theta_j(k), j]$ assume n^2 distinct values.

Conversely, let S be any Latin square orthogonal to G. Let j occupy the row and column positions $(1, i_{j,1}), \cdots, (n, i_{j,n})$ in S, where $(i_{j,1}, \cdots, i_{j,n})$ is, of necessity, some permutation of $(1, 2, \cdots, n)$. Let $\theta_j(k)$ be defined by $\theta_j(k) = i_{j,k}$. The assumption that $k \times \theta_j(k) = h \times \theta_j(h) = m$ for $k \neq h$ leads to a contradiction, in that the number pair (m, j) would occur twice in the orthogonal Latin squares G and S. Since $i_{r,k} \neq i_{s,k}$ for $r \neq s$, property (4) is satisfied, and this completes the proof.

Although anticipated in part by Theorem 2 of [3], we may improve upon the previous result for a group G.

THEOREM 7. *A necessary and sufficient condition that there exist a Latin square orthogonal to a group G is that there exist a complete mapping $\theta(x)$ for G.*

Proof. The necessity follows trivially from Theorem 6. The sufficiency is evident from the fact that, given one complete mapping $\theta(x)$ of G, we may define n complete mappings of G satisfying (4) by letting $\theta(x) \times i = \theta_i(x)$, $i = 1, 2, \cdots, n$.

A more convenient method of obtaining a Latin square orthogonal to a group G is to apply the following theorem.

THEOREM 8. *Let G be a group, $\theta(x)$ a complete mapping for G. Construct a*

Latin square S as follows: In the ith row and kth column place $i \times k \times \theta(k)$. Then S is a Latin square orthogonal to G.

Proof. Trivially, S is a Latin square. In the orthogonal squares the number pairs are $[i \times k,\ i \times k \times \theta(k)]$; and every pair (r, s), $(r, s = 1, 2, \cdots, n)$, exists since the equations $i \times k = r$, $i \times k \times \theta(k) = s$ have a unique solution. Thus the Latin square S is orthogonal to G.

Theorem 8 is a variation of the method employed by Mann [4, p.253] and is simpler to compute.

The problem of finding more than two mutually orthogonal Latin squares has its basis in investigations of finite plane geometries [4] and nets [2]. Theorem 6 yields easily formulated but involved results in this connection. The representation of Theorem 8 yields more interesting results. Consider the case of two Latin squares S_1 and S_2 represented in the manner of Theorem 8 and orthogonal to a group G. Then S_1 will be orthogonal to S_2 if and only if the number pairs

$$[i \times k \times \theta_1(k),\ i \times k \times \theta_2(k)] \qquad (i, k = 1, 2, \cdots, n)$$

take on every value (r, s), $(r, s = 1, 2, \cdots, n)$. Hence, we can conclude immediately that a necessary and sufficient condition for S_1 to be orthogonal to S_2 is that the equation

$$r \times \theta_1(k)^{-1} = s \times \theta_2(k)^{-1} \tag{5}$$

have a unique solution k for all pairs (r, s). The generalization to any number of mutually orthogonal Latin squares of this type should be apparent.

We note from (5) that if $\theta_2(x) = \theta_1(x) \times x$ is a complete mapping, our condition is trivially satisfied. In the case that G is abelian of order $2^n (n > 1)$ and every element of order 2, $\theta_2(x) = \theta_1(x) \times x$ is a complete mapping. Thus for this group it is always possible to find at least two Latin squares mutually orthogonal to G. This brings us to an interesting question that we have been unable to answer: For a given group G, what is the maximum number of mutually orthogonal Latin squares orthogonal to G?

In conclusion, we would like to conjecture that there exist no Latin squares orthogonal to a symmetric group.

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ON A TAUBERIAN THEOREM FOR ABEL SUMMABILITY

OTTO SZÁSZ

1. Introduction. In 1928 the author proved the following theorem [2, Section 2]:

THEOREM A. *If* $p > 1$ *and*

$$(1.1) \qquad \sum_{\nu=1}^{n} \nu^p |a_\nu|^p = O(n), \qquad n \longrightarrow \infty,$$

then Abel summability of the series $\sum_{n=0}^{\infty} a_n$ *to* s *implies its convergence to* s.

The theorem is the more general the smaller p is; it does not hold for $p = 1$ [2, Section 1; 1, pp. 119, 122]. However, for this case Rényi proved the following theorem:

THEOREM B. *If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \nu |a_\nu| = l < \infty$$

exists, then Abel summability of $\sum_{n=0}^{\infty} a_n$ *to* s *implies convergence of the series to* s.

2. Generalization. We give a simpler proof and at the same time a slight generalization of Theorem B.

THEOREM 1. *Assume that*

$$(2.1) \qquad V_n = \sum_{\nu=1}^{n} \nu |a_\nu| = O(n),$$

and that

$$(2.2) \qquad \frac{1}{m} V_m - \frac{1}{n} V_n \longrightarrow 0,$$

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for every sequence $m = m_n$, *such that* $m_n/n \longrightarrow 1$ *as* $n \longrightarrow \infty$. *Then Abel summability to* s *of* $\sum_{n=0}^{\infty} a_n$ *implies its convergence to* s.

Property (2.2) is called *slow oscillation* of the sequence V_n/n.

Proof of Theorem 1. We write

$$\sum_{\nu=0}^{n} a_\nu = s_n \,, \qquad \sum_{\nu=0}^{n} s_\nu = (n+1)\,\sigma_n \,.$$

It is easy to verify that, for $k = 0, 1, 2, \cdots$, we have

$$s_{n-1} - \sigma_{n+k} = \frac{n}{k+1}\,(\sigma_{n+k} - \sigma_{n-1}) - \frac{1}{k+1} \sum_{\nu=0}^{k} (k+1-\nu)\, a_{n+\nu} \,. \tag{2.3}$$

It is known [see 2, Section 2] that if for a finite s we have

$$\lim_{x \to 1} \sum_{n=0}^{\infty} a_n x^n = s \,,$$

then (2.1) implies $\sigma_n \longrightarrow s$; thus, if

$$\text{l.u.b.}_{k \geq 0} \,|\sigma_{n-1} - \sigma_{n+k}| = \epsilon_n \,, \tag{2.4}$$

then $\epsilon_n \longrightarrow 0$.

We now choose

$$k = k_n = [n\,\epsilon_n^{1/2}] \,, \qquad \text{so that} \quad k \leq n\,\epsilon_n^{1/2} < k+1 \,; \tag{2.5}$$

it follows, in view of (2.4), that

$$\frac{n}{k+1}\,|\sigma_{n-1} - \sigma_{n+k}| < \epsilon_n^{1/2} \,.$$

In view of (2.3) our theorem will be proved if we show that

$$\frac{1}{k+1} \sum_{\nu=0}^{k} (k+1-\nu)\, a_{n+\nu} \longrightarrow 0 \,, \qquad n \longrightarrow \infty.$$

Now

$$\frac{1}{k+1}\left|\sum_{\nu=0}^{k}(k+1-\nu)\,a_{n+\nu}\right|$$

$$\leq \frac{1}{k+1}\sum_{\nu=0}^{k}(n+\nu)\,|a_{n+\nu}|\,\frac{k+1-\nu}{n+\nu} \leq \frac{1}{n}\,(V_{n+k}-V_{n-1})\,,$$

and

$$\frac{1}{n}\,(V_{n+k}-V_{n-1}) = \frac{V_{n+k}}{n+k}\cdot\frac{n+k}{n} - \frac{V_{n-1}}{n-1}\cdot\frac{n-1}{n} \tag{2.6}$$

$$= \frac{V_{n+k}}{n+k} - \frac{V_{n-1}}{n-1} + \frac{k}{n}\,\frac{V_{n+k}}{n+k} + \frac{1}{n}\,\frac{V_{n-1}}{n-1}\;;$$

using (2.2) and (2.5), we see that

$$\frac{1}{n}\,(V_{n+k}-V_{n-1}) \longrightarrow 0 \qquad \text{as} \quad \frac{k}{n} \longrightarrow 0 \quad \text{and} \quad n \longrightarrow \infty, \tag{2.7}$$

and thus Theorem 1 is proved.

Rényi observed that the Theorems A and B are overlapping. We now show that Theorem 1 includes not only Theorem B, but also Theorem A. Clearly (2.1) follows from (1.1) by Hölder's inequality. Furthermore,

$$V_{n+k}-V_n = \sum_{\nu=n+1}^{n+k}\nu\,|a_\nu| \leq k^{(p-1)/p}\left(\sum_{\nu=n+1}^{n+k}\nu^p\,|a_\nu|^p\right)^{1/p}$$

$$= k^{(p-1)/p}\,O[(n+k)^{1/p}]\;;$$

hence,

$$\frac{1}{n}\,(V_{n+k}-V_n) = \frac{k}{n}\,O\left[\left(\frac{n}{k}\right)^{1/p}\right] = O\left[\left(\frac{k}{n}\right)^{(p-1)/p}\right] \longrightarrow 0 \qquad \text{as} \quad \frac{k}{n} \longrightarrow 0\,.$$

It now follows from (2.6) that (2.2) holds; thus (1.1) implies (2.1) and (2.2), which proves our assertion.

An example of a sequence $V_n > 0$, and increasing, for which (2.2) holds, while $n^{-1} V_n \uparrow \infty$, is

$$V_n = n \log n , \qquad n \geq 2 ,$$

because

$$\frac{V_{n+k}}{n+k} - \frac{V_n}{n} = \log\left(1 + \frac{k}{n}\right) \longrightarrow 0 , \qquad as \ \frac{k}{n} \longrightarrow 0, \qquad n \longrightarrow \infty .$$

3. A more general result. A generalization of Theorem A is the following [see 5, p.56]:

THEOREM A'. *If for some* $p > 1$, *we have*

$$\sum_{\nu=1}^{n} \nu^p (|a_\nu| - a_\nu)^p = O(n) , \qquad n \longrightarrow \infty , \tag{3.1}$$

then the Abel summability of $\sum_{n=0}^{\infty} a_n$ *implies its convergence to the same value.*

An analogue to Theorem 1 is the theorem:

THEOREM 2. *Assume that*

$$U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu) = O(n) , \tag{3.2}$$

and that

$$\frac{1}{m} U_m - \frac{1}{n} U_n \longrightarrow 0 \qquad \text{as } \frac{m}{n} \longrightarrow 1 , \qquad n \longrightarrow \infty . \tag{3.3}$$

If now $\sum_{n=0}^{\infty} a_n$ *is Abel summable to s, then it converges to s.*

Proof of Theorem 2. We have

$$- \sum_{\nu=1}^{n} \nu a_\nu \leq \sum_{\nu=1}^{n} \nu (|a_\nu| - a_\nu) = O(n) ;$$

hence [see 5, the Lemma on p.52] Abel summability of $\sum_{n=0}^{\infty} a_n$ implies its summability $(C, 1)$. From (2.3) we have

$$s_{n-1} - \sigma_{n+k} \leq \frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1})$$

$$+ \frac{1}{k+1} \sum_{\nu=0}^{k} (k+1-\nu)(|a_{n+\nu}| - a_{n+\nu}) ;$$

from (2.4) and (2.5) we obtain

$$\frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) < \epsilon_n^{1/2} .$$

Using the same argument as in the proof of Theorem 1, replacing V_n by U_n, we find that

$$\limsup_{n\to\infty} s_n \leq s . \tag{3.4}$$

We next employ the identity, similar to (2.3),

$$s_n - \sigma_{n-k-1} = \frac{n+1}{k+1} (\sigma_n - \sigma_{n-k-1})$$

$$+ \frac{1}{k+1} \sum_{\nu=0}^{k} (k-\nu)\, a_{n-\nu} , \qquad k = 0, 1, 2, \cdots,$$

and the inequality

$$a_\nu \geq a_\nu - |a_\nu| .$$

The same reasoning as before now yields

$$\liminf_{n\to\infty} s_n \geq s . \tag{3.5}$$

Finally (3.4) and (3.5) prove Theorem 2.

It is clear from the proof that condition (3.3) can be replaced by

$$\frac{1}{n} (U_m - U_n) \longrightarrow 0 , \qquad \text{as } \frac{m}{n} \longrightarrow 1 , \qquad n \longrightarrow \infty .$$

4. An equivalent result. A glance at the proof of Theorem 1 shows that the following lemma holds:

LEMMA 1. *If V_n is positive and monotone increasing, and if*

$$V_n = O(n)\,, \qquad \text{as } n \longrightarrow \infty, \tag{4.1}$$

and (2.2) *holds, then*

$$\frac{1}{n}\,(V_m - V_n) \longrightarrow 0\,, \qquad \text{as } \frac{m}{n} \longrightarrow 1\,, \quad n \longrightarrow \infty. \tag{4.2}$$

We now prove the inverse:

LEMMA 2. *If $V_n > 0$, and increasing, and if* (4.2) *holds, then* (4.1) *and* (2.2) *hold.*

Proof. We write

$$V_n = n\,\omega_n\,, \qquad \omega_n \geq 0\,,$$

and

$$\frac{1}{n}\,(V_m - V_n) = \omega_m - \omega_n + \left(\frac{m}{n} - 1\right)\omega_m. \tag{4.3}$$

Let

$$\max_{\nu \leq n} \omega_\nu = \rho_n\;;$$

then $\rho_n \uparrow \rho \leq \infty$. If $\rho < \infty$, then $V_n = O(n)$. Suppose now that $\rho = \infty$; then there are infinitely many indices $m = m_\nu$, so that $\omega_m = \rho_m$ for $m = m_\nu, \nu = 1,2,3,\cdots$. For these m and for $n < m$, from (4.3) we get

$$\frac{1}{n}\,(V_m - V_n) > \left(\frac{m}{n} - 1\right)\rho_m\,. \tag{4.4}$$

We now choose

$$n = \frac{m\rho_m^{1/2}}{1 + \rho_m^{1/2}} < m\,,$$

so that

$$\frac{m}{n} = \frac{1 + \rho_m^{1/2}}{\rho_m^{1/2}} \longrightarrow 1\,;$$

then, using (4.4), we have

$$\frac{1}{n}(V_m - V_n) > \rho_m^{1/2} \longrightarrow \infty,$$

in contradiction to the assumption (4.2). It follows that (4.1) holds; finally (2.2) follows from (4.1), (4.2), and (4.3). This proves Lemma 2.

We now prove the following theorem:

THEOREM 3. *Let* $U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu)$; *if*

$$\frac{1}{n}(U_m - U_n) \longrightarrow 0, \qquad \text{as } \frac{m}{n} \longrightarrow 1, \quad n \longrightarrow \infty, \tag{4.5}$$

and if $\sum_{n=0}^{\infty} a_n$ *is Abel summable, then* $\sum_{n=0}^{\infty} a_n$ *is convergent to the same value.*

Proof of Theorem 3. In view of Lemma 2, Theorem 3 includes Theorem 2; it also includes Theorem 1, because of Lemma 2, and of the inequality

$$U_m - U_n \leq 2(V_m - V_n), \qquad m > n.$$

Conversely, by Lemma 2, (4.5) implies (3.2) and (3.3), so that Theorem 3 is equivalent to Theorem 2, and is thus valid.

To show that Theorem 1 is actually more general than Theorem B we give an example of a sequence ω_n so that $n\omega_n$ is increasing, ω_n is slowly oscillating and $\omega_n = O(1)$, but $\lim \omega_n$ does not exist. Let

$$\omega_n = \sum_{\nu=1}^{n} \nu^{-1}\epsilon_\nu, \qquad \text{where } \epsilon_\nu = \pm 1;$$

choose $\epsilon_\nu = +1$ as long as $\omega_n \leq 3$; $\nu = 1, 2, \cdots, n_1$, say. Choose $\epsilon_\nu = -1$ as long as $\omega_n \geq 2$; $\nu = 1 + n_1, 2 + n_1, \cdots, n_2$, say; and so on. It is clear that $\omega_n = O(1)$, and that $\lim \omega_n$ does not exist. Furthermore, for $n \leq n_1$, $\omega_n \uparrow$, for $n_1 < n \leq n_2$, $\omega_n \downarrow$, and so on. Now

$$(n+1)\,\omega_{n+1} - n\,\omega_n = n(\omega_{n+1} - \omega_n) + \omega_{n+1} \geq \frac{3}{2} - 1 = \frac{1}{2},$$

hence $n\omega_n \uparrow$. Finally

$$|\omega_m - \omega_n| \leq \sum_{\nu=n+1}^{m} \frac{1}{\nu} < \frac{m-n}{n} \longrightarrow 0, \qquad \text{for } \frac{m}{n} \longrightarrow 1,$$

hence ω_n is slowly oscillating.

5. Another equivalent result. We first establish the following lemma.

LEMMA 3. *Suppose that* $U_n \geq 0$ *and increasing, with* $U_0 = 0$, *and let*

$$b_n = \frac{1}{n}(U_n - U_{n-1}), \qquad n \geq 1, \qquad b_0 = 0; \tag{5.1}$$

$$B_n = \sum_{\nu=0}^{n} b_\nu, \qquad n \geq 0. \tag{5.2}$$

Then whenever $k = k(n)$ *is so chosen that* $k/n \longrightarrow 0$, *as* $n \longrightarrow \infty$, *the two statements*

$$\frac{1}{n}(U_{n+k} - U_n) \longrightarrow 0 \tag{5.3}$$

and

$$B_{n+k} - B_n \longrightarrow 0 \tag{5.4}$$

are equivalent.

Proof. From (5.1) we have

$$U_n = \sum_{\nu=0}^{n} \nu b_\nu, \qquad U_{n+k} - U_n = \sum_{\nu=n+1}^{n+k} \nu b_\nu.$$

Now

$$B_{n+k} - B_n = \sum_{\nu=n+1}^{n+k} b_\nu \leq \frac{1}{n} \sum_{\nu=n+1}^{n+k} \nu b_\nu = \frac{1}{n}(U_{n+k} - U_n);$$

thus (5.3) implies (5.4). Furthermore,

$$B_{n+k} - B_n \geq \frac{1}{n+k}(U_{n+k} - U_n);$$

hence (5.4) implies (5.3). This proves the lemma.

We note that

$$B_n = \frac{1}{n} U_n + \sum_{\nu=1}^{n-1} \frac{1}{\nu(\nu+1)} U_\nu ,$$

and

$$U_n = nB_n - \sum_{\nu=0}^{n-1} B_\nu .$$

It is an immediate consequence of Lemma 3 that Theorem 3 is equivalent to the following theorem (for a direct proof see [4, Theorem IV]).

THEOREM 4. *If*

$$\sum_{\nu=n+1}^{n+k} (|a_\nu| - a_\nu) \longrightarrow 0 , \qquad as \quad \frac{k}{n} \longrightarrow 0 , \qquad n \longrightarrow \infty ,$$

then Abel summability of $\sum_{n=0}^{\infty} a_n$ *implies convergence of the series to the same value.*

A generalization of this theorem to Dirichlet series and to Laplace integrals, on different lines, is given in [3].

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CLASSES OF MATRICES AND QUADRATIC FIELDS

OLGA TAUSSKY

1. Introduction. In a recent paper [1] a correspondence between classes of matrices with rational integral elements and ideal classes in algebraic number fields was discussed. This is now studied in more detail in the case of quadratic fields. In particular the ideal classes of order 2 are discussed and the significance of the sign of the norm of the fundamental unit in real quadratic fields is displayed in an example; further results in this connection will be published elsewhere.

For completeness the result of [1] is repeated:

Let $f(x) = 0$ be an irreducible algebraic equation of degree n with integral coefficients, α one of its algebraic roots, $A = (a_{ik})$ an $n \times n$ matrix with rational integers as elements which satisfies $f(x) = 0$, and S a matrix with rational integers as elements and determinant ± 1. It was shown that the matrix classes $S^{-1}AS$ are in one-to-one correspondence with the ideal classes in the ring generated by α. The correspondence can be expressed in the following way: If $\alpha_1, \cdots, \alpha_n$ is a module base for an ideal in the ring and A the matrix for which

$$\alpha(\alpha_1, \cdots, \alpha_n) = A(\alpha_1, \cdots, \alpha_n) \tag{1}$$

then the ideal class determined by $(\alpha_1, \cdots, \alpha_n)$ corresponds to the matrix class determined by A.

2. Inverse classes. Let m be a square-free positive or negative integer. Consider the quadratic field generated by $m^{1/2}$ or $(1/2)(-1 + m^{1/2})$ according as $m \equiv 2, 3\ (4)$ or $\equiv 1\ (4)$. The first result to be proved is the following.

THEOREM 1. *The inverse of an ideal class corresponds to the class determined by the transpose of the matrix class which corresponds to the ideal class.*

Proof. We treat the two cases separately.

(a) The case $m \equiv 2, 3\,(4)$. Here choose $\alpha = m^{1/2}$. Let α_1, α_2 be a module base for an ideal $\mathfrak{a}$. If

$$\alpha_1 = a + bm^{1/2}, \qquad \alpha_2 = c + dm^{1/2}$$

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then norm $(\alpha_1, \alpha_2) = |ad - bc|$. Put $ad - bc = \Delta$. On the other hand, norm $\mathfrak{a} = \alpha \cdot \alpha'$ when α' is the conjugate of α; hence,

$$\text{norm } \mathfrak{a} = [b^2m - a^2, \quad d^2m - c^2, \quad ac - bdm - \alpha(ad - bc), \quad ac - bdm + \alpha(ad - bc)].$$

In order to find the matrix

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}$$

which corresponds to the ideal $\mathfrak{a}$, we use the fact that

$$\alpha\alpha_1 = bm + a\alpha = \lambda_1(a + b\alpha) + \lambda_2(c + d\alpha),$$

$$\alpha\alpha_2 = dm + c\alpha = \mu_1(a + b\alpha) + \mu_2(c + d\alpha).$$

Hence,

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} = \begin{pmatrix} \dfrac{bdm - ac}{\Delta} & \dfrac{a^2 - b^2m}{\Delta} \\ \dfrac{d^2m - c^2}{\Delta} & \dfrac{ac - bdm}{\Delta} \end{pmatrix}.$$

The elements in this matrix are rational integers.

The ideal which corresponds to the transpose of this matrix is, by (1),

$$\left(\frac{ac - bdm}{ad - bc} - \alpha, \quad \frac{b^2m - a^2}{ad - bc}\right)$$

which is equivalent to

$$\mathfrak{b} = [ac - bdm - \alpha(ad - bc), \quad b^2m - a^2].$$

It will now be shown that this is an ideal inverse to $\mathfrak{a}$. For this purpose we show that the product $\mathfrak{ab}$ is a principal ideal, namely, the ideal $(ad - bc)\alpha_1$. For,

$$\mathfrak{ab} = \{[ac - bdm - \alpha(ad - bc)]\alpha_1, \quad [ac - bdm - \alpha(ad - bc)]\alpha_2, \quad (b^2m - a^2)\alpha_1, \quad (b^2m - a^2)\alpha_2\}.$$

The number $(b^2m - a^2)\alpha_2$ can be expressed in the following form:

$$-(bm^{1/2} + a)(a - bm^{1/2})(c + dm^{1/2}) = -\alpha_1[ac - bdm + \alpha(ad - bc)].$$

Similarly,

$$[ac - bdm - \alpha(ad - bc)]\,\alpha_2 = (d^2 m - c^2)\,\alpha_1 \;.$$

Hence, it follows that

$$\alpha\bar{\delta} = \alpha_1 \cdot \text{norm } \mathfrak{a} \;.$$

(b) Case $m \equiv 1\,(4)$. Here we choose $\alpha = (1/2)(-1 + m^{1/2})$.
Let

$$\alpha_1 = a + b\alpha = \frac{2a - b}{2} + \frac{bm^{1/2}}{2} \;,$$

$$\alpha_2 = c + d\alpha = \frac{2c - d}{2} + \frac{dm^{1/2}}{2} \;.$$

Then

$$\text{norm } \alpha_1 = a(a - b) - b^2 \frac{m-1}{4}, \quad \text{norm } \alpha_2 = c(c - d) - d^2 \frac{m-1}{4} \;,$$

$$\text{norm } \mathfrak{a} = \Big[\text{norm } \alpha_1 , \quad \text{norm } \alpha_2 , \quad a(c - d) - bd\,\frac{m-1}{4} + \alpha(bc - ad),$$

$$a(c - d) - bd\,\frac{m-1}{4} - \alpha(bc - ad)\Big].$$

It follows that

$$\alpha\alpha_1 = b\,\frac{m-1}{4} + \alpha(a - b) = \lambda_1(a + b\alpha) + \lambda_2(c + d\alpha) \;,$$

$$\alpha\alpha_2 = d\,\frac{m-1}{4} + \alpha(c - d) = \mu_1(a + b\alpha) + \mu_2(c + d\alpha) \;.$$

Hence,

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} = \begin{pmatrix} \dfrac{bd\,\frac{m-1}{4} - c(a - b)}{\Delta} & \dfrac{d^2\frac{m-1}{4} - c(c - d)}{\Delta} \\[2ex] \dfrac{a(a - b) - b^2\,\frac{m-1}{4}}{\Delta} & \dfrac{a(c - d) - bd\,\frac{m-1}{4}}{\Delta} \end{pmatrix} .$$

Again, all the numbers in this matrix are rational integers.

The ideal which corresponds to the transposed matrix is equivalent to:

$$\mathfrak{b} = \left[a(c-d) - bd\,\frac{m-1}{4} - \alpha(ad-bc)\,, \qquad a(a-b) - b^2\,\frac{m-1}{4}\right].$$

The product ideal $\mathfrak{a}\mathfrak{b}$ is again shown to be the principal ideal $(ad - bc)\alpha_1$. For it is

$$\left\{(\alpha_1 \text{ norm } \alpha_1\,, \qquad \alpha_2 \text{ norm } \alpha_1, \qquad \alpha_1\left[a(c-d) - bd\,\frac{m-1}{4} - \alpha(ad-bc)\right],\right.$$

$$\left.\alpha_2\left[a(c-d) - bd\,\frac{m-1}{4} - \alpha(ad-bc)\right]\right\}.$$

We have

$$\alpha_2 \text{ norm } \alpha_1 = (a + b\alpha)(a + b\alpha')(c + d\alpha)$$

where α' is the conjugate of α. Further,

$$(a + b\alpha')(c + d\alpha) = \frac{[2a + b(-1-m^{1/2})][2c + d(-1 + m^{1/2})]}{4}$$

$$= a(c-d) - bd\,\frac{m-1}{4} + \alpha(ad-bc)\,.$$

Similarly,

$$\alpha_2\left[a(c-d) - bd\,\frac{m-1}{4} - \alpha(ad-bc)\right] = \alpha_1 \text{ norm } \alpha_2\,.$$

This shows that again, $\mathfrak{a}\mathfrak{b} = \alpha_1$ norm $\mathfrak{a}$.

3. Classes of order two. From Theorem 1 it follows that a matrix which corresponds to an ideal class of order 2 is equivalent to its transpose. The question arises, when does the class to which this matrix belongs contain a symmetric matrix? A result in this direction is the following.

THEOREM 2. *A matrix class which corresponds to an ideal class of order two contains a symmetric matrix if and only if every matrix in the class is transformed into its transpose by a unimodular matrix of the form* XX'. *In particular the transforming matrix must be of determinant* $+1$.

Proof. Let A be a matrix equivalent with its transpose; that is,

$$A' = SAS^{-1}$$

when S is unimodular. Let T also be unimodular and assume that $T^{-1}AT$ is symmetric. We then have

$$T^{-1}AT = T'A'T'^{-1}$$

or

$$T'^{-1}T^{-1}ATT' = A' \ .$$

Hence, it is possible to transform A into its transpose by a matrix of the form XX'. Conversely, if

$$A' = X'^{-1}X^{-1}AXX'$$

we have

$$X'A'X'^{-1} = X^{-1}AX \ .$$

Hence $X^{-1}AX$ is symmetric.

The question arises, are both cases possible, the one when the matrix class contains symmetric matrices and the one when it does not? Both cases, in fact, are possible and it can even happen that the same field contains ideal classes of order 2, some of which correspond to symmetric matrices, while others do not. An example is the field generated by $(410)^{1/2}$. An ideal of order 2 in this field is $[7, 19 + (410)^{1/2}]$, and a matrix which corresponds to it is

$$\begin{pmatrix} -19 & 7 \\ 7 & 19 \end{pmatrix} ,$$

which is clearly symmetric. Another ideal of order 2 in the same field is the ideal $[2, 20 + (410)^{1/2}]$, a corresponding matrix being

$$\begin{pmatrix} -20 & 2 \\ 5 & 20 \end{pmatrix} .$$

Any matrix which transforms the latter into its transpose is of the form

$$\begin{pmatrix} \dfrac{-40b + 5d}{2} & b \\ b & d \end{pmatrix}$$

where b, d are parameters. In order to have integral coefficients we put $d = 2d'$. The matrix will then be

$$\begin{pmatrix} -20b + 5d' & b \\ b & 2d' \end{pmatrix} .$$

This will be unimodular if

$$-40bd' + 10d'^2 - b^2 = 410d'^2 - (b + 20d')^2 = \pm 1 .$$

This equation for +1 is impossible, since the fundamental unit of the field generated by $(410)^{1/2}$ has the norm +1. Hence, the matrix class which corresponds to this ideal does not contain any symmetric matrix.

A symmetric matrix can correspond only to ideals in real fields since such a matrix can have only real characteristic roots. It can further be seen easily that, in this case, m has to be a sum of two squares.

Reference

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National Bureau of Standards

THE ASYMPTOTIC EXPANSION OF A RATIO OF GAMMA FUNCTIONS

F. G. TRICOMI AND A. ERDÉLYI

1. Introduction. Many problems in mathematical analysis require a knowledge of the asymptotic behavior of the quotient $\Gamma(z+\alpha)/\Gamma(z+\beta)$ for large values of $|z|$. Examples of such problems are the study of integrals of the Mellin-Barnes type, and the investigation of the asymptotic behavior of confluent hypergeometric functions when the variable and one of the parameters become very large simultaneously.

Stirling's series can be used to find a first approximation for our quotient for very large $|z|$, it being understood that α and β are bounded. Without too much algebra one finds

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O(|z|^{-2})\right] \tag{1}$$

as $z \to \infty$, under conditions which will be stated later; but the determination of the coefficients of $z^{-2}, z^{-3}, \cdots$, in the asymptotic expansion of which (1) gives the first two terms, is a very laborious process, and the determination of the general term from Stirling's series is a well-nigh hopeless task.

The present paper originated when the first-named author (F.G. Tricomi) noticed that the asymptotic expansion of $\Gamma(z+\alpha)/\Gamma(z+\beta)$ can be obtained by methods similar to those which he used in a recent investigation of the asymptotic behavior of Laguerre polynomials [3]. The first proof given in this paper, and the detailed investigation of the coefficients A_n and C_n, are entirely due to him. Afterwards, the second named author (A. Erdélyi) pointed out that a shorter proof can be given by using Watson's lemma. His contributions to the present paper are the second proof, the generating function (18) of the coefficients, and their expression in terms of generalized Bernoulli polynomials.

We may mention that the same quotient was recently investigated by J.S. Frame [1]; but there is no overlapping with the results presented here.

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2. The case $\beta = 0$. Let us begin with the particular case $\beta = 0$ (after which the general case will easily be treated), starting from the well-known formula

$$\int_0^\infty \frac{x^{u-1}}{(1+x)^v}\,dx = \frac{\Gamma(u)\,\Gamma(v-u)}{\Gamma(v)}, \qquad (0 < \Re u < \Re v), \tag{2}$$

where each power has its *principal value.*

Putting

$$u = z + \alpha, \quad v = z\,; \quad \frac{\Gamma(z+\alpha)}{\Gamma(z)} = F(\alpha, z); \quad x = \frac{z}{t}, \quad \zeta = e^{i \arg z},$$

from the previous equality, under the hypothesis

$$0 < \Re(\alpha + z) < \Re z$$

we obtain

$$\Gamma(-\alpha)\,F(\alpha, z) = z^{\alpha} \int_0^{\zeta\infty} e^{-z\log(1+t/z)}\, t^{-\alpha-1}\,dt\,.$$

But as long as $|t| < |z|$ we have

$$e^{-z\log(1+t/z)} = e^{-t} \exp\left[\frac{t^2}{z}\left(\frac{1}{2} - \frac{1}{3}\frac{t}{z} + \frac{1}{4}\frac{t^2}{z^2} - \cdots\right)\right]$$

$$= e^{-t} \sum_{m=0}^{\infty} \frac{t^{2m} z^{-m}}{m!}\left(\frac{1}{2} - \frac{1}{3}\frac{t}{z} + \frac{1}{4}\frac{t^2}{z^2} - \cdots\right)^m.$$

Hence, if we put generally*

$$\left(\frac{1}{2} + \frac{1}{3}w + \frac{1}{4}w^2 + \cdots\right)^m = \sum_{k=0}^{\infty} c_k^{(m)} z^k, \qquad (m = 0, 1, 2, \cdots), \tag{3}$$

and in particular

$$c_0^{(m)} = \frac{1}{2^m}, \qquad c_1^{(m)} = \frac{m}{2^{m-1}\cdot 3}, \qquad c_2^{(m)} = \frac{m(4m+5)}{2^{m+1}\cdot 3^2}, \cdots; \tag{3'}$$

*The repeated use of the coefficients of the (formal) mth power of a power series is one of the features of the methods of the paper quoted [3].

with the help of the substitution $k + m = n$, we obtain

$$e^{-z \log(1 + t/z)} = \sum_{k,m=0}^{\infty} \frac{(-1)^k}{m!} c_k^{(m)} t^{2m+k} z^{-m-k}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} \sum_{m=0}^{n} \frac{(-1)^m}{m!} c_{n-m}^{(m)} t^{n+m} .$$

This shows that our quotient $F(\alpha, z) = \Gamma(z + \alpha)/\Gamma(z)$ admits *at least formally* the negative-powers expansion

$$(4) \qquad \sum_{n=0}^{\infty} A_n(\alpha) z^{\alpha-n} ;$$

where, for the sake of brevity, we put

$$A_n(\alpha) = \frac{(-1)^n}{\Gamma(-\alpha)} \sum_{m=0}^{n} \frac{(-1)^m}{m!} c_{n-m}^{(m)} \int_0^{\zeta\infty} e^{-t} t^{n+m-\alpha-1} dt .$$

Better still, because

$$\frac{1}{\Gamma(-\alpha)} \int_0^{\zeta\infty} e^{-t} t^{n+m-\alpha-1} dt = \frac{\Gamma(n + m - \alpha)}{\Gamma(-\alpha)} = (-1)^{n+m} \binom{\alpha}{n + m} (n + m)!$$

since $\Re\zeta > 0$, we can also write

$$(5) \qquad A_n(\alpha) = \sum_{m=0}^{n} \binom{\alpha}{n + m} \frac{(n + m)!}{m!} c_{n-m}^{(m)} .$$

In particular, we have

$$(5') \qquad A_0(\alpha) = 1, \quad A_1(\alpha) = \binom{\alpha}{2}, \quad A_2(\alpha) = \frac{3\alpha - 1}{4} \binom{\alpha}{3} ,$$

$$A_3(\alpha) = \binom{\alpha}{2} \cdot \binom{\alpha}{4} , \cdots .$$

3. Relations connecting the coefficients $A_n(\alpha)$. The infinite series (4) is generally *divergent* because otherwise the function F would be the product of z^{α} by a function regular at infinity, in contradiction with the fact that, as long as α is not an integer, the function F has an infinite number of poles at $z = 0, -1, -2, \cdots$, with the condensation point $z = \infty$. In spite of its divergence, the series (4) represents the function F asymptotically (in the sense of Poincaré); that is, we have

$$F(\alpha, z) = \frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim \sum_{n=0}^{\infty} A_n(\alpha)\, z^{\alpha-n}, \tag{6}$$

at least as long as

$$0 < -\Re\alpha < \Re z, \tag{7}$$

because for any positive integer N we obviously have

$$e^{-z\log(1+t/z)} = \sum_{n=0}^{N} \frac{(-1)^n}{z^n} \sum_{m=0}^{n} \frac{(-1)^n}{m!}\, c_{n-m}^{(m)}\, t^{n+m} + O(|z|^{-N-1}).$$

Let us now establish some relations connecting the coefficients $A_n(\alpha)$ together; these arise from the unicity theorem for the asymptotic expansions, and from the functional equations

$$F(\alpha+1, z) = (\alpha+z)\, F(\alpha, z), \quad F(\alpha, z+1) = \left(1+\frac{\alpha}{z}\right) F(\alpha, z), \tag{8}$$

which are obviously satisfied by the function F.

Precisely from the first equation (8) it follows immediately that

$$A_n(\alpha+1) = A_n(\alpha) + \alpha A_{n-1}(\alpha), \qquad (n = 1, 2, 3, \cdots), \tag{9}$$

while from the second one it follows that

$$\left(1+\frac{\alpha}{z}\right)\sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m} \sim \sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m}\left(1+\frac{1}{z}\right)^{\alpha-m}$$

$$\sim \sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m} \sum_{k=0}^{\infty}\binom{\alpha-m}{k} z^{-k} \sim \sum_{n=0}^{\infty} z^{\alpha-n} \sum_{m=0}^{n-1}\binom{\alpha-m}{n-m} A_m(\alpha).$$

This shows that

$$A_n(\alpha) + \alpha A_{n-1}(\alpha) = \sum_{m=0}^{n}\binom{\alpha-m}{n-m} A_m(\alpha) = \sum_{m=0}^{n-2}\binom{\alpha-m}{n-m} A_m(\alpha) + (\alpha-n+1);$$

simplifying and changing n into $n+1$, we thus obtain the important recurrence relation

$$(10) \qquad A_n(\alpha) = \frac{1}{n} \sum_{m=0}^{n-1} \binom{\alpha - m}{n-m+1} A_m(\alpha), \qquad (n = 1, 2, \cdots).$$

From the manner of deduction, it may seem that the validity of (9) and (10) is conditioned by $\Re\alpha < 0$; but since these equalities are equalities between certain analytic functions of α (even polynomials!), there is no doubt that, as a matter of fact, both equations are true for *any* value of α.

4. On the condition (7). By use of the functional equations (8) and the relations (9) and (10) between the coefficients, it would be possible to weaken progressively the conditions (7) by passing successively from α to $\alpha-1$, $\alpha-2$, $\cdots$, and from z to $z+1$, $z+2$, $\cdots$. But we do not need to enter into the details of this reasoning because the method of Section 7 will give us directly the end results free of unnecessary restrictions. Nevertheless, we state explicitly that *the asymptotic expansion* (6) *is valid for any* α *(real or complex) on the whole complex z-plane cut along any curve connecting* $z = 0$ *with* $z = \infty$*, *provided that, in going to* ∞, z *avoids the points* $z = 0, -1, -2, \cdots$ *and* $z = -\alpha, -\alpha-1, -\alpha-2, \cdots$.

For example, when α is real and positive the expansion (6) is surely valid if

$$-\pi + \epsilon < \arg z < \pi - \epsilon ,$$

where ϵ is an arbitrarily small positive number.

5. The asymptotic expansion. Now in order to obtain the asymptotic expansion of the quotient indicated at the beginning, it is sufficient to observe that

$$\Phi(z) \equiv \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = F(\alpha - \beta, \; z + \beta)$$

Precisely, putting

$$\alpha - \beta = \alpha' ,$$

*This with regard to the many-valuedness of the power z^{α}.

we find thus

$$\Phi(z) \sim \sum_{m=0}^{\infty} A_m(\alpha')(z+\beta)^{\alpha'-m} \sim \sum_{m=0}^{\infty} A_m(\alpha')\, z^{\alpha'-m} \sum_{k=0}^{\infty} \binom{\alpha'-m}{k} \left(\frac{\beta}{z}\right)^k$$

$$\sim \sum_{n=0}^{\infty} z^{\alpha'-n} \sum_{m=0}^{n} \binom{\alpha'-m}{n-m} A_m(\alpha')\, \beta^{n-m} .$$

In other words, if we put

$$C_n(\alpha',\beta) = \sum_{m=0}^{n} \binom{\alpha'-m}{n-m} A_m(\alpha')\, \beta^{n-m} , \qquad (n = 0,1,2,\cdots) , \tag{11}$$

on the whole z-plane cut along any curve connecting $z = 0$ with $z = \infty$, we have

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{n=0}^{\infty} C_n(\alpha-\beta,\beta)\, z^{\alpha-\beta-n} , \tag{12}$$

provided that z avoids the points $z = -\alpha, -\alpha-1, -\alpha-2, \cdots$ and $z = -\beta, -\beta-1, -\beta-2, \cdots$.

The coefficients C_n are given by (11), which shows in particular that

$$C_0 = 1, \quad C_1 = \frac{1}{2}\alpha'(\alpha'+2\beta-1) = \frac{1}{2}(\alpha-\beta)(\alpha+\beta-1) ,$$

$$C_2 = \frac{1}{12}\binom{\alpha'}{2}[(\alpha'-2)(3\alpha'-1)+12\beta(\alpha'+\beta-1)]$$

$$= \frac{1}{12}\binom{\alpha-\beta}{2}[3(\alpha+\beta-1)^2-\alpha+\beta-1] , \cdots .$$

6. The coefficients C_n. The calculation of the coefficients C_n by means of (11) is quite easy, but in spite of this it may be useful to know that for such coefficients there is also a recursion formula of the kind (10). Precisely, in a similar manner as in Section 3, we notice first that the function $\Phi(z)$ satisfies the functional equation

$$\Phi(z+1) = \frac{z+\alpha}{z+\beta}\Phi(z) = \left(1+\frac{\alpha}{z}\right)\left(1+\frac{\beta}{z}\right)^{-1}\Phi(z) .$$

Consequently, since

$$\left(1+\frac{\beta}{z}\right)^{-1} \sim 1 - \frac{\beta}{z} + \frac{\beta^2}{z^2} - \cdots ,$$

we obtain

$$\left(1+\frac{\beta}{z}\right)^{-1}\Phi(z)\sim\sum_{m=0}^{\infty}C_m\,z^{\alpha'-m}\sum_{k=0}^{\infty}(-1)^k\beta^k z^{-k}$$

$$\sim\sum_{n=0}^{\infty}(-1)^n\,z^{\alpha'-n}\sum_{m=0}^{n}(-1)^m\,\beta^{n-m}C_m\ ,$$

and further

$$\left(1+\frac{\alpha}{z}\right)\left(1+\frac{\beta}{z}\right)^{-1}\Phi(z)$$

$$\sim\sum_{n=0}^{\infty}\left[C_n+(-1)^n(\beta-\alpha)\sum_{m=0}^{n}(-1)^m\,\beta^{n-m-1}\,C_m\right]z^{\alpha'-n}$$

on the other hand,

$$\Phi(z+1)\sim\sum_{m=0}^{\infty}C_m\,z^{\alpha'-m}\sum_{k=0}^{\infty}\binom{\alpha'-m}{k}z^{-k}\sim\sum_{n=0}^{\infty}z^{\alpha'-n}\sum_{m=0}^{n}\binom{\alpha'-m}{n-m}C_m\ .$$

By comparing the two results we thus obtain

$$\sum_{m=0}^{n}\binom{\alpha'-m}{n-m}C_m=C_n-(-1)^n\,\alpha'\sum_{m=0}^{n}(-1)^m\,\beta^{n-m-1}C_m\ ;$$

that is,

$$\sum_{m=0}^{n-1}\left[\binom{\alpha'-m}{n-m}+(-1)^{n+m}\,\alpha'\,\beta^{n-m-1}\right]C_m=0\ . \tag{13}$$

In other words, detaching the last term of the sum and changing n into $n+1$, we have the recurrence relation

$$C_n(\alpha',\beta)=\frac{1}{n}\sum_{m=0}^{n-1}\left[\binom{\alpha'-m}{n-m+1}-(-1)^{n+m}\alpha'\beta^{n-m}\right]C_m(\alpha',\beta), \tag{14}$$

$$(n=1,2,3,\cdots)\,.$$

7. An alternate proof. If we put $u = \exp(-v)$ in Euler's integral of the first kind,

$$\int_0^1 u^{z+\alpha-1}\,(1-u)^{\beta-\alpha-1}\,du = \frac{\Gamma(z+\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(z+\beta)}\,,$$

we have the integral representation

$$(15)\qquad \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{1}{\Gamma(\beta-\alpha)}\int_0^\infty e^{-(z+\alpha)v}\,(1-e^{-v})^{\beta-\alpha-1}dv\,.$$

We shall now show that an alternative proof of the asymptotic expansion (12) can be obtained by applying the standard technique (Watson's lemma) to this integral representation.

To begin with, (15) holds only if $\Re(\beta-\alpha) > 0$ and $\Re(z+\alpha) > 0$; but its validity can be extended by the introduction of a loop integral. We assume that $z+\alpha$ is not negative real. Then there is a δ such that

$$-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi\,, \qquad \Re\{(z+\alpha)\,e^{i\delta}\} > 0\,.$$

With such a δ, we have

$$(16)\qquad \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{1}{2\pi i}\int_{-\infty\cdot e^{i\delta}}^{(0+)} e^{zt}\,f(t)\,dt\,,$$

where

$$(17)\qquad f(t) = \Gamma(1+\alpha-\beta)\,e^{\alpha t}\,(e^t-1)^{\beta-\alpha-1}\,,$$

and for small $|t|$,

$$\delta-\pi \le \arg(e^t-1) \le \delta+\pi$$

on the loop of integration. Now (16) is valid for all α and β, with the trivial exception of $\alpha-\beta = -1, -2, \cdots$, and for all z in the complex plane slit along the line from $-\alpha$ to $-\alpha-\infty$.

Watson's lemma can be applied directly to (16). It is usual to state this lemma for an integral between 0 and ∞, but it is clear that the customary proof [4] goes through for a loop integral like (16) provided that the restriction on the growth of $f(t)$ is imposed along the whole loop, and that the expansion

$$(18)\qquad f(t) = \sum_{n=0}^{\infty} a_n\, t^{\beta-\alpha+n-1}$$

is valid in a neighborhood of $t = 0$ on the loop. Both assumptions hold good in our case, and hence a term-by-term integration of (18) leads at once to the asymptotic expansion

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum \frac{a_n \ z^{\alpha-\beta-n}}{\Gamma(\alpha-\beta-n+1)} \qquad \text{as } z \to \infty ,$$

valid for all α, β, $\alpha-\beta \neq -1, -2, \cdots$, and the complex z-plane slit from $-\alpha$ to $-\alpha-\infty$.

Comparing with (12), we see that

$$\Gamma(\alpha-\beta-n+1) \ C_n(\alpha-\beta, \beta) = a_n$$

has the generating function (17). The properties of C_n established in the earlier sections can also be derived from this generating function. It also follows from the generating function that the coefficients can be expressed in terms of generalized Bernoulli polynomials. In Nörlund's [2] notation* , we have

$$a_n = \frac{1}{n!}\Gamma(1+\alpha-\beta) \ B_n^{(\alpha-\beta+1)}(\alpha) \ . \tag{19}$$

8. **Particular cases.** Finally, we notice that in the particular case $\alpha = n$, for $n = 1, 2, \cdots$, the expansion (6) becomes

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1) \ \cdots \ (z+n-1) = \sum_{m=0}^{n-1} A_m(n) \, z^{n-m} \ ; \tag{20}$$

hence, we have

$$A_m(n) = (-1)^m \ S_n^{(m)} \ , \tag{21}$$

where $S_n^{(m)}$ denotes the sum of the products of the negative numbers $-1, -2, \cdots, -(n-1)$ taken m at a time in all the possible manners (*Stirling's numbers of the first kind*).

Another interesting particular case of (6) is the case $\alpha = 1/2$, $z = n+1$, in which we have

$$\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \sim \frac{1}{(\pi n)^{1/2}} \left(1 - \frac{1}{8n} + \frac{1}{128 n^2} - \cdots \right) . \tag{22}$$

---

* In the first instance, n in $B_\nu^{(n)}(x)$ is an integer, but Nörlund remarks (p. 146) that it may be replaced by an arbitrary complex parameter.

Among the other things we can read from (22) is the following approximation formula for π:

$$(23)\ \pi = \frac{1}{n}\left[\frac{2^n n!}{1\cdot 3\cdots(2n-1)}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\epsilon_n\right)\right]^2, \quad \epsilon_n = O(n^{-3});$$

for instance, taking $n = 20$ and neglecting the remainder ϵ_n, from (23) we obtain the good approximation $\pi = 3.141557$, with an error of only 36 millionths.

Another interesting application concerns the asymptotic evaluation of the binomial coefficient $\binom{x}{n}$ as $n \to \infty$ and x (which is not a positive integer) remains bounded. Since

$$\binom{x}{n} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)n!} = \frac{(-1)^n}{n\Gamma(-x)}\,\frac{\Gamma(n-x)}{\Gamma(n)},$$

we obtain from (6), with $z = n$ and $\alpha = -x$, the relation

$$\binom{x}{n} \sim \frac{(-1)^n}{\Gamma(-x)}\, n^{-(x+1)} \sum_{m=0}^{\infty} \frac{A_m(-x)}{n^m}$$

$$= \frac{(-1)^n}{\Gamma(-x)}\, n^{-(x+1)}\left[1 + \binom{x+1}{2}\frac{1}{n} + \binom{x+2}{3}\frac{1+3x}{4n^2} + \cdots\right].$$

This formula gives very good numerical results even for relatively small values of n, for instance for $n = 10$, provided only that x/n is small.

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ON TOTALLY DIFFERENTIABLE AND SMOOTH FUNCTIONS

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1. Introduction. H. Rademacher has proved that a function of n variables satisfying a Lipschitz condition is totally differentiable a.e. (almost everywhere) (see, for instance, Saks, [6, pp. 310-311]). It was discovered by H. Federer (though not stated as a theorem; see [2, p. 442]) that if f is totally differentiable a.e. in the bounded set P, then there is a closed set $Q \subset P$ with the measure $|P - Q|$ as small as desired, such that f is smooth (continuously differentiable) in Q; that is, the values of f in Q may be extended through space so that the resulting function g is smooth there.

Theorem 1 of the present paper strengthens the latter theorem by showing that f is approximately totally differentiable a.e. in P if and only if Q exists with the above property. The rest of the paper gives further theorems in the direction of Federer's Theorem, as follows.

Suppose the domain of definition of f were a bounded open set P. Then in applying the part (a) $\longrightarrow$ (c) of Theorem 1, we might alter f in a set $P - Q$ which included a neighborhood of the boundary of P. In applications, it might be important to keep the values of f in most of a subset close to the boundary of P, or in most of some other subset. That such can be done follows from Theorem 2.

If f satisfies a Lipschitz condition, Theorem 3 shows that g may be made to satisfy a Lipschitz condition also, with a constant which equals a number ρ_n (depending on the number n of variables only) times the constant for f; in the case of one variable, we may take $\rho_1 = 1$.

If we weaken the assumption on f, assuming only that it is measurable, then Lusin's Theorem shows that we can alter f on a set of arbitrarily small measure, giving a continuous function g. In the other direction, suppose we assume that f (defined in an open set) has continuous mth partial derivatives, and that these derivatives are totally differentiable a.e. Then Theorem 4 shows that we may alter f on a set of arbitrarily small measure, giving a function g which has continuous partial derivatives of order $m + 1$. For the case of one variable, this is essentially a theorem of Marcinkiewicz, [5, Theorem 3].

Examples show that the hypotheses in the theorems cannot be materially

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weakened without altering the conclusions. For instance, define a function ϕ of one variable as follows. Let $\phi_0(t)$ be the distance from t to the nearest integer. Using any sufficiently large integer a, set

$$\phi_i(t) = 2^i \phi_0(a^i t)/a^i, \qquad \phi(t) = \sum_{i=0}^{\infty} \phi_i(t) .$$

Then ϕ satisfies a Lipschitz condition of order $1 - \alpha$, for any $\alpha > 0$; but Property (c) of Theorem 1 is not true for it. If $\Phi(t) = \int_0^t \phi(s)\,ds$, then Φ is smooth, and its first derivative satisfies a Lipschitz condition of order $1 - \alpha$; but the conclusion of Theorem 4 (with $m = 1$) fails.

2. The theorem for bounded sets. Let $x = (x_1, \cdots, x_n)$ denote points of n-space E^n. With the unit vectors $e_1, \cdots, e_n$ of a coordinate system, any vector v can be written in the form $\Sigma v_i e_i$. The length of v is $|v| = (\Sigma v_i^2)^{1/2}$; $|y - x|$ is the distance from x to y. Given n numbers $f_1(x), \cdots, f_n(x)$, set

$$F(x) \cdot v = \sum f_k(x) v_k; \tag{2.1}$$

this is linear in v. If $f(x)$, $f(y)$, and the $f_k(x)$ are defined, set

$$e(x,y) = \frac{f(y) - f(x) - F(x) \cdot (y - x)}{|y - x|} \tag{2.2}$$

for $y \neq x$, and $e(x,x) = 0$. Let $S_z[\alpha(z)]$ denote the set of elements z with the property α. Given f, and so on, as above, set

$$H(x,\epsilon) = S_y[e(x,y) < \epsilon] . \tag{2.3}$$

The measurable function f defined in the set P is a.t.d. (approximately totally differentiable) at $x \in P$ in terms of the $f_k(x)$ (see [6, p. 300]) if for each $\epsilon > 0$ the set $H(x, \epsilon)$ has x as a point of density. (Any standard definition of density points may be used for the purposes of this paper.) If this holds, then x is a point of density of P, and the $f_k(x)$ are uniquely determined; if x is a point of density in the direction of each axis, then the $f_k(x)$ are the approximate partial derivatives of f at x. The f_k are measurable (see [6, p. 299]).

THEOREM 1. *Let f be measurable in the bounded set P. Then the four following conditions are equivalent:*

(a) *The function f is a. t. d. a. e. in P.*

(b) *The function f is approximately derivable with respect to each variable a. e. in P.*

(c) *For each $\epsilon > 0$ there is a closed set $Q \subset P$ such that $|P - Q| < \epsilon$ and f is smooth in Q.*

(d) *There is a sequence of disjoint closed sets $Q_1, Q_2, \cdots$ in P such that $|P - Q_1 \cup Q_2 \cup \cdots| = 0$ and f is smooth in each Q_i.*

REMARK. If f is assumed totally differentiable a.e. in P, the proof that (c) holds is simplified; see [2, p. 442].

Proof of Theorem 1. For the equivalence of (a) and (b), see [6, pp. 300-303]. Note that (b) is an obvious consequence of (d). We shall prove the equivalence of (a), (c) and (d).

Suppose (c) holds. We choose the disjoint closed sets $Q_1, Q_2, \cdots$ in succession so that f is smooth in each and $|P_i| \leq |P|/2^i$, where

$$P_i = P - Q_1 \cup \cdots \cup Q_i ,$$

as follows. Having found $Q_1, \cdots, Q_{i-1}$, choose a closed set Q_i' so that f is smooth in Q_i' and $|P - Q_i'| \leq |P|/2^{i+1}$. Let $U_\delta(A)$ denote the δ-neighborhood of the set A. For small enough δ, we may use

$$Q_i = Q_i' - U_\delta(Q_1 \cup \cdots \cup Q_{i-1}) .$$

Thus (d) holds.

Suppose (d) holds. Let Q_i^* be the set of points of density of Q_i, and set $Q^* = Q_1^* \cup Q_2^* \cup \cdots$. Then $|P - Q^*| = 0$. Take any $x \in Q^*$; say $x \in Q_i^*$. Since f is smooth in Q_i and x is a point of density of Q_i, f (considered now in P) is a.t.d. at x. Thus f is a.t.d. at all points of Q^*, and (a) holds.

Now given (a), we must prove (c). There is a number $a > 0$ with the following property. For any points x, y, and number r with $|y - x| \leq r$, we have

$$|U_r(x) \cap U_r(y)| \geq 2a\,|U_r(x)| .$$

For $x \in P$, set $V_i = |U_{1/i}(x)|$, and

$$\psi_i(x, \eta) = |U_{1/i}(x) - H(x, \eta)| , \tag{2.4}$$

$$\phi_i(x) = \text{g.l.b. } S_\eta[\psi_i(x, \eta) < aV_i] . \tag{2.5}$$

Since $e(x,y)$ is measurable in the pair of variables x, y, it follows that $\psi_i(x, \eta)$ is measurable for fixed η. Also, as a function of η, $\psi_i(x, \eta)$ is monotone and continuous on the left; hence

$$\phi_i(x) < \zeta \quad \text{if and only if} \quad \psi_i(x, \zeta) < aV_i . \tag{2.6}$$

Therefore ϕ_i is measurable.

Let Q_1 be the set of points where f is a.t.d.; then $f_1, \cdots, f_n$ are defined in Q_1. Given $x \in Q_1$ and $\epsilon' > 0$, we may choose $\delta > 0$ so that

$$\psi_i(x,\epsilon') < aV_i \quad \text{if} \quad 1/i < \delta ;$$

using (2.6) shows that

$$\lim_{i\to\infty} \phi_i(x) = 0 , \qquad x \in Q_1 . \tag{2.7}$$

By Lusin's and Egeroff's theorems, there is a closed set $Q \subset Q_1$ such that $|Q_1 - Q| < \epsilon$, the f_k are continuous in Q, and $\phi_i(x) \longrightarrow 0$ uniformly in Q. We now prove that for each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$e(x,y) < \epsilon' \quad \text{if} \quad x,y \in Q, \quad |y - x| < \delta . \tag{2.8}$$

Setting $\epsilon_1 = \epsilon'/6$, we may choose δ so that

$$|F(y) \cdot v - F(x) \cdot v| \le \epsilon_1 |v| \quad \text{if} \quad x,y \in Q, \quad |y - x| < 2\delta , \tag{2.9}$$

$$\phi_i(x) < \epsilon_1 \quad \text{if} \quad x \in Q, \quad 1/(i + 1) < \delta . \tag{2.10}$$

Now take any $x, y \in Q$ with $|y - x| < \delta$. Let j be the largest integer such that $1/j \ge |y - x|$, and set

$$R = U_{1/j}(x) \cap U_{1/j}(y) ; \qquad \text{then} \quad |R| \ge 2aV_j .$$

Since $1/(j + 1) < |y - x| < \delta$, (2.10) and (2.6) give

$$\psi_j(x, \epsilon_1) , \quad \psi_j(y, \epsilon_1) < aV_j .$$

Hence there is a point z in R in neither corresponding set; that is,

$$|z - x|, \quad |z - y| < 1/j ; \qquad e(x,z) , \quad e(y,z) < \epsilon_1 .$$

Since $F(x)\cdot v$ is linear in v and $|z - x|$, $|z - y| < 2|y - x|$, we have

$$\begin{aligned} e(x, y)\,|y - x| &= |f(y) - f(x) - F(x)\cdot(y - x)| \\ &\le |f(z) - f(x) - F(x)\cdot(z - x)| \\ &\quad + |f(z) - f(y) - F(y)\cdot(z - y)| + |[F(y) - F(x)]\cdot(z - y)| \\ &\le \epsilon_1[|z - x| + |z - y| + |z - y|] < \epsilon' |y - x| \end{aligned}$$

if $y \neq x$, proving (2.8).

This fact, together with the continuity of the $f_k(x)$ in Q, shows that f is "of class C^1 in terms of the f_k in Q", as defined in [7] (the definition is given after (6.3), below). Hence, by [7, Lemma 2], we may extend f to be smooth in E^n, completing the proof. (The extension is described in Section 4, below; by use of the results of that section, it is not hard to show that f has the required properties.)

3. The theorem for unbounded sets. We remove the restriction of boundedness in Theorem 1, and give more information about the set in which f may be left unaltered.

THEOREM 2. *Let $A_1, A_2, \cdots$ be open sets in E^n such that each has points in common with, at most, a finite number of the others, and let $\epsilon_1, \epsilon_2, \cdots$ be positive numbers. Let K be the set of points x such that there is a sequence of distinct sets $A_{\mu_1}, A_{\mu_2}, \cdots$ and a sequence of points $x_1, x_2, \cdots$ with $x_i \in A_{\mu_i}$ and $x_i \longrightarrow x$. Let P be a measurable subset of $E^n - K$, and let f be a. t. d. a. e. in P in terms of the f_k. Then there is a set $Q \subset P$ such that Q is closed in $E^n - K$ and $|(P - Q) \cap A_i| < \epsilon_i$, and there is a smooth function g in $E^n - K$ such that $g(x) = f(x)$ and $\partial g(x)/\partial x_k = f_k(x)$ in Q.*

REMARKS. Clearly K is closed and $K \cap A_i = 0$ for all i. If $Q^* \subset P$, Q^* is closed in $E^n - K$, and for some positive continuous functions $\delta_1(x), \delta_2(x), \cdots$ in Q^*,

$$e(x, y) \leq 1/2^i \qquad \text{if } x \in Q^*, \qquad |y - x| < \delta_i(x),$$

the proof shows that we may make $Q \supset Q^*$. For instance, we may make Q contain any given set of points of P in which f is totally differentiable and which has no accumulation points in $E^n - K$. On the other hand, we must expect to drop out a neighborhood of the set of points where f is not totally differentiable. Further, we cannot in general keep in Q any given closed set where f is approximately totally differentiable, as is shown by the following example (in one variable):

$$f(t) = t^2 \sin(i/t)(t \neq 0), \; f(0) = 0 .$$

Proof of Theorem 2. For each pair (k, l) of positive integers, let $U_{k,l}$ be the set of points x satisfying the conditions (with a fixed x_0 in E^n)

$$k - 1 < |x - x_0| < k + 1, \qquad 1/(l - 1) > \text{dist}(x, K) > 1/(l + 1) ;$$

for $k = 1$ or $l = 1$, we drop out the first inequalities. If K is void, the index l is not needed, and the situation is simpler. The $U_{k,l}$ are bounded open sets covering $E^n - K$, and each one touches at most eight others. Arrange them in a sequence $U_1', U_2', \cdots$.

For each i, let $\lambda_{i,1}, \lambda_{i,2}, \cdots$ be the (finite or infinite) set of numbers such that $U'_{\lambda_{i,k}} \cap A_i \neq 0$. Since the $\bar{U}_j'$ are compact and in $E^n - K$, each touches at most a finite number of the A_i; hence for given j, there is at most a finite number of values of i such that $\lambda_{i,k} = j$ for some k. Let ϵ_j' be the smallest of the numbers $\epsilon_i/2^k$, using these values of i and corresponding k.

Considering f and the f_k in $P \cap U_j'$ alone, apply the proof of Theorem 1 to find a closed set $Q_j \subset P \cap U_j'$ such that $|P \cap U_j' - Q_j| < \epsilon_j'$, and such that f is of class C^1 in terms of the f_k in Q_j. Set

$$V_j = U_j' - Q_j, \qquad V = \cup_j V_j, \qquad Q = E^n - K - V .$$

Then V is open, Q is closed in $E^n - K$, and $Q \cap U_j' \subset Q_j$. Now

$$(P - Q) \cap A_i = P \cap V \cap A_i = \cup_j (P \cap V_j \cap A_i),$$
$$|P \cap V_j| = |P \cap U'_j - Q_j| < \epsilon'_j .$$

Since $V_j \subset U_j'$, $P \cap V_j \cap A_i$ is void unless $j = \lambda_{i,k}$ for some k. Hence

$$|(P - Q) \cap A_i| \leq \sum_j |P \cap V_j \cap A_i| < \sum_k \epsilon'_{\lambda_{i,k}} \leq \sum_k \epsilon_i/2^k = \epsilon_i .$$

Since each U_j' is open and $Q \cap U_j' \subset Q_j$, f is clearly of class C^1 in terms of the f_k in Q. Hence, as before, we may extend the values of f in Q through $E^n - K$, as required. (We are applying [7, Lemma 2] in an open set; the change required in the proof is very simple. Or we could use [7, Theorem III].)

4. The theorem for Lipschitz functions. The following theorem has two parts, corresponding to the two theorems above.

THEOREM 3. *For each positive integer n there is a number* ρ_n (we may take $\rho_1 = 1$) *with the following properties.*

(a) *Let f be defined and satisfy a Lipschitz condition in the bounded closed set* $P \subset E^n$:

$$(4.1) \qquad |f(y) - f(x)| \leq N|y - x|, \qquad x, y \in P .$$

Then for each $\epsilon > 0$ *there is a closed set* $Q \subset P$ *such that* $|P - Q| < \epsilon$, *and there is a smooth function g in* E^n *satisfying a Lipschitz condition* (see (4.15)) *with the constant* $\rho_n N$, *such that* $g = f$ *in* Q.

(b) *Let the* A_i, ϵ_i, *and* K *be as in Theorem* 2. *Let* P *be closed in* $E^n - K$ (*it may have accumulation points in* K). *Let* f *be defined in* P *and satisfy* (4.1). *Then there is a set* $Q \subset P$ *which is closed in* P (*and hence in* $E^n - K$) *such that* $|(P - Q) \cap A_i| < \epsilon_i$; *and there is a function* g *satisfying* (4.15) *in* E^n *which is smooth in* $E^n - Q^*$, *where* $Q^* = \bar{Q} - Q$, *such that* $g = f$ *in* Q.

(c) *We may take* Q [*in either* (a) *or* (b)] *so that* f *is totally differentiable in* Q *in terms of functions* $f_1, \cdots, f_n$; *we may then take* g *so that* $\partial g/\partial x_k = f_k$ *in* $Q (k = 1, \cdots, n)$.

(d) *Given a positive continuous function* $\eta(x)$ in $E^n - K$ [*in* E^n, *for case* (a)], *we may make*

$$(4.2) \qquad |g(x) - f(x)| < \eta(x) , \qquad x \in P .$$

REMARKS. It is no restriction to take P closed (or closed in $E^n - K$). For if P is not closed, it is easily seen that we may extend f (uniquely) over $\bar{P}$ so that it is continuous there; then (4.1) now holds in $\bar{P}$. (We can in fact extend f to satisfy (4.1) in E^n; see [3] or [4].) Note that, in (b), $Q^* \subset \bar{P} - P$; if K is void, then Q^* is void, and g is smooth in E^n. As an immediate consequence of (4.15), we have

$$(4.3) \qquad |\Sigma v_k \partial g(x)/\partial x_k| \le \rho_n N \quad \text{if} \quad x \in E^n - Q^* , \ |v| = 1 .$$

The hypothesis of total differentiability a.e. in P, together with

$$|\Sigma v_k f_k(x)| \le N|v|$$

where the f_k are defined, is not enough to give the theorem (unless, for instance, $P = E^n$), as simple examples show. (Compare the examples in H. Whitney [8].) If we wish to prove (4.3) rather than (4.15), the proof may be slightly simplified; of course (4.15) follows from (4.3) if $Q^* = 0$ (hence if $K = 0$). See also the remarks following Theorem 2.

Proof of Theorem 3. To prove the theorem, we first note that (a) is contained in (b); use $A_1 = E^n$, $\epsilon_1 = \epsilon$. Next, (d) will follow at once from (4.1) and (4.15) if we make sure that each point of P is sufficiently close to some point of Q; this will clearly be the case if, in applying the proof of Theorem 2, we take the ϵ'_j small enough. Also, just as in Theorem 2, (c) will hold. It remains to show that we can obtain the properties in (b), using the proof of Theorem 2. We do this here, except for showing that we can make $\rho_1 = 1$.

We must examine the proof of [7, Lemma 2]. First, since f is totally differentiable a.e. in P [6, p. 311], we may choose Q as in the proof of Theorem 2;

recall that the f_k are continuous in Q. We shall use a cubical subdivision of $E^n - \bar{Q}$, essentially as in [7]. For each integer s (in [7], only $s \geq 0$ was used), let K_s' be the set of all cubes of edge length $1/2^s$, the coordinates of whose corners are integral multiples of $1/2^s$. Let K_s'' consist of the cubes of K_s' whose distances from $\bar{Q}$ are at least $6n^{1/2}/2^s$. Let K_s consist of the cubes of K_s'' which are not in cubes of K_{s-1}''. Take any cube $C \in K_s$; suppose $C \subset C'$, $C' \in K_{s-1}'$. Then $\operatorname{dist}(C', \bar{Q}) < 6n^{1/2}/2^{s-1}$. Therefore, clearly

$$(4.4) \qquad 6n^{1/2}/2^s \leq \operatorname{dist}(C, \bar{Q}) < 13n^{1/2}/2^s, \qquad C \in K_s.$$

Take $C \in K_s$, $C' \in K_{s+2}$. Then each point of C' is within

$$n^{1/2}/2^{s+2} + 13n^{1/2}/2^{s+2}$$

from $\bar{Q}$; hence

$$(4.5) \qquad \operatorname{dist}(C, C') \geq (5/2)n^{1/2}/2^s, \qquad C \in K_s, \quad C' \in K_{s+2}.$$

Let $y^1, y^2, \cdots$ be the set of all corners of all cubes of all K_s. Choose $x^\nu \in \bar{Q}$ with $|x^\nu - y^\nu| = \operatorname{dist}(y^\nu, \bar{Q})$. Let b_ν be the largest length of edge of any cube of any K_s with y^ν as a corner, and let I_ν be the cube defined by

$$|x_i - y_i^\nu| \leq b_\nu \ (i = 1, \cdots, n).$$

Let ϕ_0' be a smooth function which is positive within a fixed unit cube and is zero outside; by a translation and similarity transformation, define ϕ_ν', positive within I_ν and zero outside. Set $\phi_\nu = \phi_\nu'/\Sigma\phi_\lambda'$; then ϕ_ν is positive within I_ν and zero outside, and $\Sigma\phi_\nu = 1$ in $E^n - \bar{Q}$. Since there is at most some fixed number of shapes of cubes (of some K_s, and K_{s+1} perhaps) forming any I_ν, there is clearly a number $M_n \geq 1$ with the following property (compare [7, Section 10]): taking $|v| = 1$,

$$(4.6) \quad |\Sigma v_i \, \partial\phi_\nu/\partial x_i| < 2^s M_n \quad \text{if} \quad \phi_\nu(y) \neq 0 \quad \text{for some} \quad y \in C \in K_s.$$

Extend f to be continuous in $\bar{P}$ (if $\bar{P} \neq P$); (4.1) still holds. For any $x^* \in \bar{Q}$ and any $x \in E^n$, set

$$(4.7) \qquad \psi(x; x^*) = f(x^*) + \sum f_i(x^*)(x_i - x_i^*);$$

this is the value at x of the linear function approximating to f at x^*. Then set

$$(4.8) \qquad g(x) = \sum \phi_\nu(x) \ \psi(x; x^\nu), \qquad x \in E^n - \bar{Q}.$$

It is not hard to show that if $g = f$ in $\bar{Q}$, then g is smooth in $E^n - Q^*$ and $\partial g/\partial x_i = f_i$ in Q; see the proof of [7, Lemma 2]. We must still prove (4.15).

Take first any x and x' in $E^n - \bar{Q}$; say for definiteness that

(4.9) $$x \in C \in K_s, \quad x' \in C' \in K_{s'}, \quad s \geq s' .$$

Let x^* be a point of $\bar{Q}$ nearest to x. Since $\Sigma\phi_\nu(x) = 1$, we may write

$$g(x) = f(x^*) + \sum_\nu \phi_\nu(x)[f(x^\nu) - f(x^*)] + \sum_{\nu, i} \phi_\nu(x) f_i(x^\nu)(x_i - x_i^\nu) .$$

Hence

(4.10) $$g(x') - g(x) = \sum_\nu [\phi_\nu(x') - \phi_\nu(x)][f(x^\nu) - f(x^*)]$$

$$+ \sum_{\nu, i} [\phi_\nu(x') - \phi_\nu(x)] f_i(x^\nu)(x_i' - x_i^\nu) + \sum_{\nu, i} \phi_\nu(x) f_i(x^\nu)(x_i' - x_i) .$$

We shall find a bound for each non-zero term. First we show that

(4.11) $$|\phi_\nu(x') - \phi_\nu(x)| \; |f(x^\nu) - f(x^*)| \leq 64 N M_n n^{1/2} \; |x' - x| .$$

Consider first any ν such that $\phi_\nu(x) \neq 0$. Then by (4.6),

(4.12) $$|\phi_\nu(x') - \phi_\nu(x)| < 2^s M_n |x' - x| .$$

Also, since

$$|x^* - x| \leq \operatorname{diam}(C) + \operatorname{dist}(C, \bar{Q}) \leq 14 n^{1/2}/2^s ,$$

$$|y^\nu - x| \leq 2 \operatorname{diam}(C) \leq 2 n^{1/2}/2^s ,$$

$$|y^\nu - x^\nu| \leq |x^* - y^\nu| \leq 16 n^{1/2}/2^s ,$$

we have

$$|x^\nu - x^*| \leq 32 n^{1/2}/2^s ,$$

and hence

$$|f(x^\nu) - f(x^*)| \leq 32 N n^{1/2}/2^s .$$

These relations give (4.11). Next consider any ν such that $\phi_\nu(x') \neq 0$. Then (using inequalities like those above) we obtain

$$|x^\nu - x^*| \leq |x^\nu - x'| + |x' - x| + |x - x^*|$$

$$\leq 18n^{1/2}/2^{s'} + |x' - x| + 14n^{1/2}/2^s \leq 32n^{1/2}/2^{s'} + |x' - x| .$$

In the present case, (4.12) holds with s'. Suppose first that

$$|x' - x| \leq 32n^{1/2}/2^{s'} .$$

Then

$$|f(x^\nu) - f(x^*)| \leq N \cdot 64n^{1/2}/2^{s'} ,$$

and (4.11) follows. If $|x' - x| > 32n^{1/2}/2^{s'}$, then $|x^\nu - x^*| < 2\,|x' - x|$, and since $|\phi_\nu(x') - \phi_\nu(x)| \leq 2$ and $4N < 64NM_n n^{1/2}$, (4.11) follows again.

Next we show that

$$(4.13) \qquad |\phi_\nu(x') - \phi_\nu(x)|\ |f_i(x^\nu)|\ |x'_i - x^\nu_i| \leq 40NM_n n^{1/2}|x' - x| .$$

We may suppose that $\phi_\nu(x') \neq 0$, in which case $|x' - x^\nu| \leq 18n^{1/2}/2^{s'}$, or $\phi_\nu(x) \neq 0$, in which case $|x' - x^\nu| \leq 18n^{1/2}/2^s + |x' - x|$; in either case,

$$|x'_i - x^\nu_i| \leq |x' - x^\nu| \leq 18n^{1/2}/2^{s'} + |x' - x| .$$

First suppose that $|x' - x| \leq 2n^{1/2}/2^{s'}$. Then, by (4.5), $s \leq s' + 1$. Hence, using (4.6) with s or s' we get

$$|\phi_\nu(x') - \phi_\nu(x)| \leq 2^{s'+1}M_n|x' - x| ;$$

since $|f_i(x^\nu)| \leq N$, (4.13) follows. Next suppose that $|x' - x| > 2n^{1/2}/2^{s'}$. Then $|\phi_\nu(x') - \phi_\nu(x)| \leq 2$, and $|x'_i - x^\nu_i| < 10|x' - x|$, giving (4.13) again.

Finally, we have

$$(4.14) \qquad |\phi_\nu(x)\, f_i(x^\nu)(x'_i - x_i)| \leq N|x' - x| .$$

There is clearly a number c_n such that for any x, there are at most c_n values of ν such that $\phi_\nu(x) \neq 0$. In the three groups of terms in (4.10), there are at most $2c_n$, $2c_n n$, and $c_n n$ non-zero terms respectively. Hence, by (4.11), (4.13) and (4.14), we have

$$|g(x') - g(x)| \leq c_n N M_n n^{1/2} (128 + 80n + n) |x' - x| ,$$

which gives

$$(4.15) \qquad |g(x') - g(x)| \leq \rho_n N |x' - x| , \qquad \rho_n = 209\, c_n M_n n^{3/2} .$$

If x and x' are in $\bar{Q}$, (4.15) follows from (4.1), since $g = f$ (or the extended f) in $\bar{Q}$. Suppose finally that $x \in \bar{Q}$, $x' \in E^n - \bar{Q}$ (or *vice versa*). Let x'' be the last point of the segment xx' in $\bar{Q}$. Then (4.1) holds for x and x'', and (4.15) holds for x''' and x', with x''' in $x''x'$ and arbitrarily close to x''; hence (4.15) holds in all cases, and the proof is complete.

5. Lipschitz functions of one variable. We must prove Theorem 3, with $n = 1$, $\rho_1 = 1$. The proof is elementary in nature; we do not need [7]. Find a closed subset Q_1 of $E^1 - K$ (or of E^1, in case (a)) as in the proof of Theorem 2 (or Theorem 1, if we are only using (a)). Now (4.1) holds in Q_1, f_1 is continuous in Q_1, and f is smooth in terms of f_1 in Q_1 (see Section 3, above), that is, for each $x \in Q_1$ and each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$(5.1) \qquad |f(x'') - f(x') - (x'' - x') f_1(x')| \leq \epsilon' |x'' - x'|$$
$$\text{if} \quad |x'' - x|, \qquad |x' - x| < \delta , \quad x', x'' \in Q_1 .$$

Let primes on functions denote differentiation. We shall find a set Q which is closed in $E^1 - K$, with $|Q_1 - Q| < \epsilon^*$ in case (a) or $|(Q_1 - Q) \cap A_i| < \epsilon_i^*$ in case (b), and a function g which satisfies (4.1) in E^1 and is smooth in $E^1 - K$, and such that $g = f$ and $g' = f_1$ in Q; for ϵ^* or the ϵ_i^* small enough, Q and g have the required properties.

Let $I_1, I_2, \cdots$ be the closed intervals whose interiors fill out $E^1 - Q_1 \cup K$. Extend f through E^1 so that (4.1) holds there; see [3] or [4]. Set $g_0 = f$ in $Q_1 \cup K$, and let g_0 be linear in the I_k, so that g_0 is continuous in the closed intervals. Then g_0 is continuous in E^1, and satisfies (4.1) there.

We shall need the following lemma.

LEMMA 1. *Let ϕ be defined and satisfy (4.1) in the closed interval $[a^*, b]$, and let ϕ be linear in the subinterval $[a, b]$. Then there is an arbitrarily small* *interval* $[a', a'']$ *about a such that* $\phi'(a')$ *exists, and there is a function* ψ *in* $[a^*, b]$ *which equals* ϕ *in* $[a^*, a'] \cup [a'', b]$ *and is smooth in* $[a', b]$, *and is such that* $\psi'(a') = \phi'(a')$, *and for* x *in* $[a', a'']$, $\psi'(x)$ *lies between* $\phi'(a')$ *and* $\phi'(a'')$.

We use the notation $\Delta\phi(x, y) = [\phi(y) - \phi(x)]/(y - x)$. If ϕ is linear in some interval $[x_0, b]$ with $x_0 < a$, we may set $\psi = \phi$. If not, we may choose $c < a$ arbitrarily close to a so that $\Delta\phi(c, a) \neq \Delta\phi(a, b)$. Suppose for definiteness that $\Delta\phi(c, a) < \Delta\phi(a, b)$. Take $a'' > a$ arbitrarily close to a. Because of (4.1), ϕ is absolutely continuous, and $\phi(a'') - \phi(c) = \int_c^{a''} \phi'(x)\,dx$. Hence there is a point a' in $[c, a]$ such that $\phi'(a')$ exists and

$$\phi'(a') < \Delta\phi(a', a'') < \Delta\phi(a'', b) = \phi'(a'');$$

that is, the tangents at a' and a'' intersect at a point x' between a' and a''. Using these tangents except near x', and smoothing near x', gives the required ψ.

We return to the theorem. Let $x_1, x_2, \cdots$ be the set of end points of the intervals I_k. Let I_1'' be an interval about x_1, of length $< \epsilon_1''$ for some ϵ_1'' (see below), with one end point interior to an interval I_k with x_1 as end point. Apply the lemma (or the lemma with x replaced by $-x$) to find an interior interval $I_1' = [a_1', a_1'']$ about x_1, and using $\phi = g_0$ in I_1'', define $\psi = g$ in I_1'. We may require that neither a_1' nor a_1'' is any x_i. In general, having found disjoint intervals $I_1', \cdots, I_{j-1}'$, let x_h be the first point of the sequence which is in none of these, and let I_j'' be an interval about x_h, of length $< \epsilon_j''$, disjoint from the preceding I_i'. Apply the lemma as before to find I_j', and define g in I_j'. Set $g = g_0$ elsewhere in E^1. Let Q be the set of points of Q_1 interior to no I_j'. For small enough ϵ_j'', the inequalities with ϵ^* or ϵ_i^* hold. We shall show that g is smooth in $E^1 - K$ and $g' = f_1$ in Q; the other properties of g are clear.

Clearly g' is continuous in a neighborhood of any point interior to an I_j' or an I_k, that is, in $E^1 - K \cup Q$. Now take any $x \in Q$; we shall show that $g'(x) = f_1(x)$ and g' is continuous at x, considering only points $x' \geq x$ for which $g'(x')$ is defined. The same fact holds for $x' \leq x$, and this will complete the proof. By definition of g, this is true if x is the left hand end point of some I_j'; suppose this is not the case.

Given $\epsilon' > 0$, choose δ so that (5.1) holds, and so that $|f_1(x') - f_1(x)| < \epsilon'$ for $x' \in Q_1$, $|x' - x| < \delta$. Choose $y > x$ in Q_1 within δ of x. Now any difference quotient of f, with points in $[x, y] \cap Q_1$, is within $2\epsilon'$ of $f_1(x)$; hence clearly any difference quotient of g_0 in $[x, y]$ is within $2\epsilon'$ of $f_1(x)$. Hence, for any x' in $[x, y]$ such that $g_0'(x')$ exists, $|g_0'(x') - f_1(x)| < 2\epsilon'$. Because of the last property in the lemma, $|g'(x') - f_1(x)| < 2\epsilon'$ if $g'(x')$ exists. Since $g(x') - g(x) = \int_x^{x'} g'(t)\,dt$, this shows that $g'(x)$ exists (as a right hand derivative) and equals $f_1(x)$, and proves the required continuity.

6. **Functions with totally differentiable** m**th partial derivatives.** We shall prove a theorem corresponding to (a) $\longrightarrow$ (c) in Theorem 1; the extension to the case corresponding to Theorem 2 is clear.

THEOREM 4. *Let f and its partial derivatives of order $\leq m$ be defined in a bounded open set $P \subset E^n$, and let each mth partial derivative be totally differentiable a.e. in P. Then for each $\epsilon > 0$ there is a closed set $Q \subset P$ such that $|P - Q| < \epsilon$, and there is a function g with continuous $(m + 1)$th partial derivatives in E^n such that all partial derivatives of f of order $\leq m + 1$ exist in Q and equal those of g there. In particular, $g = f$ in Q.*

Because of Theorem 1, we may suppose $m \geq 1$. We use the notation of [7]; thus

$$\frac{f_k(x)}{k!}(x' - x)^k = \frac{f_{k_1 \cdots k_n}(x_1, \cdots, x_n)}{k_1! \cdots k_n!}(x_1' - x_1)^{k_1} \cdots (x_n' - x_n)^{k_n}$$

[do not confuse with the earlier $f_k(x)$], $\sigma_k = k_1 + \cdots + k_n$, and so on. Also

$$f_{k_1 \cdots k_n} = \partial^{\sigma_k} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n}$$

where defined.

Take any k with $\sigma_k = m - 1$, and any integers i and j. Since $\partial f_k / \partial x_i$ and $\partial f_k / \partial x_j$ are defined in P and are totally differentiable a.e. in P, it follows that their partial derivatives $\partial^2 f_k / \partial x_i \partial x_j$ and $\partial^2 f_k / \partial x_j \partial x_i$ exist a.e. in P; by a theorem of Currier [1], these are equal a.e. in P. Where this is so for all i, j, it is clear that we may define f_k with $\sigma_k = m + 1$ uniquely. Let P' be the subset of P in which the f_k exist for $\sigma_k \leq m + 1$, and each f_k $(\sigma_k = m)$ is totally differentiable in terms of the f_l $(l_i \geq k_i, \sigma_l = m + 1)$; then $|P - P'| = 0$. As seen in Section 2, the f_k are measurable.

As in [7], let $\psi_k(x'; x)$, for $\sigma_k \leq m$, be the value at x' of the polynomial of degree at most $m - \sigma_k$ which has the same value and partial derivatives of order $\leq m - \sigma_k$ at x as f_k. Then

$$\psi_k(x'; x) = \sum_{\sigma_l \leq m - \sigma_k} \frac{f_{k+l}(x)}{l!}(x' - x)^l .$$

Let $R_k(x'; x)$ be the corresponding remainder in Taylor's expansion:

$$R_k(x'; x) = f_k(x') - \psi_k(x'; x), \qquad \sigma_k \leq m. \tag{6.2}$$

Define ψ_k' and R_k' similarly for $x \in P'$, with m replaced by $m + 1$. We shall say a remainder R_k' is *of order* m' *at* x^0 if the following is true. For each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$|R_k'(x; x^0)| \leq \epsilon' |x - x^0|^{m'} \qquad \text{if} \quad |x - x^0| < \delta. \tag{6.3}$$

Recall from [7] that in a closed set, f is of class C^{m+1} in terms of the $f_k (\sigma_k \leq m + 1)$ if and only if each R_k' is of order $m + 1 - \sigma_k$ uniformly in a neighborhood of each point.

With the help of Lemma 2 below, we prove Theorem 4 as follows. By Lusin's Theorem, there is a closed set $Q' \subset P'$ with $|P' - Q'| < \epsilon/2$ such that each $f_k(x)$ with $\sigma_k = m + 1$ is continuous in Q'; that is, $R_k'(\sigma_k = m + 1)$ is of order 0 in Q'. For each integer i and each $x^0 \in Q'$, let $\delta_i(x^0)$ be the upper bound of numbers $\delta \leq 1$ such that (6.3) holds with $m' = m - \sigma_k + 1$, $\epsilon' = 1/2^i$, for all k with $\sigma_k \leq m + 1$. Then by the lemma, $\delta_i(x) > 0$ in Q'. As in Section 2, we see that the $\delta_i(x)$ are measurable. Find sets Q_i as in Section 2, and set $Q = Q_1 \cap Q_2 \cap \cdots$. Then clearly f is of class C^{m+1} in Q in terms of the f_k, and hence [7, Lemma 2] f may be extended from Q over E^n so that

$$\partial^{\sigma_k} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n} = f_k$$

in Q. This extension is the required g. There remains to prove

LEMMA 2. *Let P be open, let*

$$\partial^{\sigma_k} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n} = f_k$$

in P for $\sigma_k \leq m$, let $f_k(x^0)$ be defined for $\sigma_k = m + 1$, and let the $f_k(\sigma_k = m)$ be totally differentiable in terms of the $f_l(l_i \geq k_i,\ \sigma_l = m + 1)$ at x^0. Define $R_k'(x; x^0)$ as above. Then R_k' is of order $m - \sigma_k + 1$ at x^0 if $\sigma_k \leq m - 1$.

Note that the hypothesis shows that R_k' for $\sigma_k = m$ is of order 1 at x^0.

Suppose we have proved Lemma 2 for the case that $f_k(x^0) = 0$ for all k, $\sigma_k \leq m + 1$. Then it holds for the general case. For set

$$\bar{f}_k(x) = f_k(x) - \psi_k'(x; x^0) = R_k'(x; x^0) \qquad (\sigma_k \leq m + 1);$$

then $\bar{f}_k(x^0) = 0$. Also, since $\bar{\psi}_k'(x; x^0) = 0$ (using the $\bar{f}_l$), $\bar{R}_k'(x; x^0) = R_k'(x; x^0)$.

Since the $\bar{R}'_k$ are of order $m - \sigma_k + 1$ at x^0 for $\sigma_k = m$, the lemma shows that this is true also for $\sigma_k < m$. Thus $R'_k = \bar{R}'_k$ is of order $m - \sigma_k + 1$ at x^0, as required.

We shall need Taylor's Theorem with exact remainder:

LEMMA 3. *Let ϕ be a function of one variable such that $\phi^{(h)} = d^h\phi/dx^h$ exists for $h \leq m'$ in an interval and is bounded. Then*

$$\phi(t_1) = \sum_{h=0}^{m'} \frac{\phi^{(h)}(t_0)}{h!}(t_1 - t_0)^h$$
$$+ \frac{1}{(m'-1)!}\int_{t_0}^{t_1}(t_1 - s)^{m'-1}[\phi^{(m')}(s) - \phi^{(m')}(t_0)]\,ds\,.$$

Since $\phi^{(m')}$ is bounded, $\phi^{(m'-1)}$ satisfies a Lipschitz condition; hence for any smooth α, $\beta = \alpha\phi^{(m'-1)}$ is absolutely continuous, and

$$\int_a^b (d\beta/dt)dt = \beta(b) - \beta(a)\,.$$

Therefore the usual proof applies.

We return to Lemma 2, assuming $f_k(x^0) = 0$ ($\sigma_k \leq m+1$). Set

$$x^i = (x_1, \cdots, x_i, x^0_{i+1}, \cdots, x^0_n)\,;$$

then $x^n = x$. Take any $i > 0$, and any k with $\sigma_k \leq m-1$. Set $m' = m - \sigma_k$, $k(i) = (k_1, \cdots, k_i + m', \cdots, k_n)$, and

$$x^i(s) = (x_1, \cdots, x_{i-1}, s, x^0_{i+1}, \cdots, x^0_n)\,.$$

Then $x^i(x^0_i) = x^{i-1}$, $x^i(x_i) = x^i$. For some $\delta_1 > 0$, the $f_k(x)$ ($\sigma_k \leq m$) are bounded for $|x - x^0| < \delta_1$. Lemma 3 gives

$$f_k(x^i) = \psi_k(x^i; x^{i-1})$$
$$+ \frac{1}{(m'-1)!}\int_{x^0_i}^{x_i}(x_i - s)^{m'-1}\{f_{k(i)}[x^i(s)] - f_{k(i)}(x^{i-1})\}\,ds\,.$$

Since $f_l(x^i(s)) = R'_l(x^i(s); x^0)$, and so on, the definition of R_k gives

$$R_k(x^i; x^{i-1}) = \frac{1}{(m'-1)!}\int_{x^0_i}^{x_i}(x_i - s)^{m'-1}$$
$$\times \{R'_{k(i)}[x^i(s); x^0] - R'_{k(i)}(x^{i-1}; x^0)\}\,ds\,.$$

For a certain ϵ^* chosen below, choose $\delta \leq \delta_1$ so that

$$|R_l^{\prime}(x'; x^0)| \leq \epsilon^* |x' - x^0| \qquad \text{if} \quad \sigma_l = m, \quad |x' - x^0| < \delta .$$

Then if $|x - x^0| < \delta$, using $|x^i(s) - x^0| \leq |x - x^0|$ for $x_i^0 \leq s \leq x_i$, and so on, gives

$$|R_k(x^i; x^{i-1})| \leq \frac{2\epsilon^* |x - x^0|}{(m' - 1)!} \int_{x_i^0}^{x_i} (x_i - s)^{m'-1} ds \leq \frac{2\epsilon^* |x - x^0|^{m'+1}}{m'!} .$$

Now in [7, (6.3)], subtract $f_k(x'')$ from both sides, and change x, x', x'' to x^{i-1}, x^i, x respectively; this gives

$$R_k(x; x^{i-1}) - R_k(x; x^i) = \sum_{\sigma_l \leq m - \sigma_k} \frac{R_{k+l}(x^i; x^{i-1})}{l!} (x - x^i)^l .$$

Hence,

$$|R_k(x; x^{i-1}) - R_k(x; x^i)| \leq 2\epsilon^* |x - x^0|^{m - \sigma_k + 1} \sum_{\sigma_l \leq m - \sigma_k} \frac{1}{(m - \sigma_k - \sigma_l)!\, l!} .$$

Let A_k denote the sum, and let A be the largest A_k. Since $R_k^{\prime}(x; x^0) = R_k(x; x^0)$ for the case at hand, adding the inequalities for $i = 1, \cdots, n$ gives

$$|R_k^{\prime}(x; x^0)| \leq 2nA\,\epsilon^* |x - x^0|^{m - \sigma_k + 1} .$$

Given $\epsilon' > 0$, set $\epsilon^* = \epsilon'/(2nA)$, and choose δ accordingly; this inequality then completes the proof.

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