ON DEDEKIND’S FUNCTION $\eta(\tau)$

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1. Introduction. A transformation of the form

\[ \tau' = \frac{a\tau + b}{c\tau + d}, \]

where \(a, b, c, d\) are rational integers satisfying

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 1, \]

is called a modular transformation. Without loss of generality we may assume \(c > 0\). A function \(f(\tau)\), analytic in the upper halfplane \( \Im(\tau) > 0\), and satisfying the functional equation

\[ f(\tau) = (c\tau + d)^k f\left(\frac{a\tau + b}{c\tau + d}\right), \]

is called a modular form of dimension \(k\). An example of a modular form is the discriminant

\[ \Delta(\tau) = \exp\{2\pi i\tau\} \prod_{m=1}^{\infty} (1 - \exp\{2\pi im\tau\})^{24}, \]

which is of dimension \(-12\); that is, it satisfies the equation*

\[ \Delta(\tau') = (c\tau + d)^{12}\Delta(\tau). \]

An important role in the theory of modular functions is played by the function

\[ \eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \prod_{m=1}^{\infty} (1 - \exp\{2\pi im\tau\}), \]

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* Cf. Hurwitz [6]; however, he gives this formula only in homogeneous coordinates.

which is the 24th root of $\Delta(\tau)$. The transformation formula for this function may
be obtained from (1.5) and is conveniently written as:

$$(1.7) \quad \eta(\tau') = \eta\left(\frac{a \tau + b}{c \tau + d}\right) = \epsilon \sqrt{-i \left(c \tau + d\right)} \eta(\tau).$$

Since we have assumed $c \geq 0$ and $\Re(\tau) > 0$, the radicand has a nonnegative real
part. By the square root we always mean the principal branch; that is, $\Re(\sqrt{\cdot}) > 0$.
The $\epsilon$ appearing in (1.7) is a 24th root of unity. The purpose of the present paper
is to determine this $\epsilon$ completely.

Investigations concerning this root of unity were carried out first by Dedekind
[2] and later by Tannery and Molk [10] and Rademacher [8; 9]. However, they
use the theory of $\log \eta(\tau)$, which requires much more than is needed for this
purpose. Hurwitz discusses only $[\Delta(\tau)]^{1/12}$ and remarks that the transformation
formula of $\eta(\tau)$ can be obtained by means of $\theta$-functions. The investigations of
Hermite [5] are likewise not sufficient for our purpose, because he discusses
only $\eta^3(\tau)$, and therefore a third root of unity remains still undetermined.

In the following, we shall approach the determination of $\epsilon$ directly by investi-
gations of the function $\eta(\tau)$, which, by a well-known formula due to Euler, can
be written as the following sum:

$$(1.8) \quad \eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\left\{\pi i \tau \lambda (3 \lambda - 1)\right\}$$

$$\quad = \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\left\{3 \pi i \tau \left(\frac{\lambda - \frac{1}{6}}{6}\right)^2\right\}.$$

Our starting point is formula (1.8); our principal tools are a Poisson transforma-
tion formula and Gaussian sums.

2. Application of a Poisson formula. We introduce a new variable $z$ with
$\Re(z) > 0$ by the substitution*

$$\tau' = \frac{i z}{c} + \frac{a}{c}, \quad c > 0; \quad (a, c) = 1,$$

*This requires $c \neq 0$, but the case $c = 0$ is trivial.
and obtain, from (1.8)

\[ \eta \left( \frac{a}{c} + \frac{iz}{c} \right) = \sum_{\lambda = -\infty}^{+\infty} (-1)^\lambda \exp \left\{ \frac{3\pi i}{c} \left( a + iz \right) \left( \lambda - \frac{1}{6} \right) \right\} \]

\[ = \sum_{j \equiv \lambda \mod 2c} \exp \pi i \left\{ j + \frac{3a}{c} \left( j - \frac{1}{6} \right)^2 \right\} \times \sum_{q = -\infty}^{+\infty} \exp \left\{ -\frac{3\pi z}{c} \left( 2cq + j - \frac{1}{6} \right)^2 \right\}. \]

To the inner sum,

\[ F_c(z) = \sum_{q = -\infty}^{+\infty} \exp \left\{ -12\pi cz \left( q + \frac{6j - 1}{12c} \right)^2 \right\}, \]

we apply Poisson's formula (cf. [11]),

\[ \sum_{m = -\infty}^{+\infty} \exp \left\{ -\pi \left( m + \omega \right)^2 t \right\} = \frac{1}{\sqrt{t}} \sum_{m = -\infty}^{+\infty} \exp \left\{ \frac{2\pi im\omega - \pi \omega^2}{t} \right\}, \quad \Re(t) > 0, \]

and obtain

\[ F_c(z) = \frac{1}{2\sqrt{3cz}} \sum_{q = -\infty}^{+\infty} \exp \left\{ 2\pi i q \frac{6j - 1}{12c} - \frac{\pi q^2}{12cz} \right\}. \]

Putting this in (2.2), we get:

\[ \eta \left( \frac{a}{c} + \frac{iz}{c} \right) = \frac{1}{\sqrt{3cz}} \sum_{q = -\infty}^{+\infty} \exp \left\{ -\pi q^2 \frac{12c}{12cz} \right\} T_q(c), \]

where

\[ T_q(c) = \frac{1}{2} \sum_{j \equiv \lambda \mod 2c} \exp \pi i \left\{ j + \frac{3a}{c} \left( j - \frac{1}{6} \right)^2 + q \frac{6j - 1}{6c} \right\} \]

\[ = \frac{1}{2} \exp \pi i \left\{ \frac{a - 2q}{12c} \right\} \left[ 1 + \exp \pi i \left\{ 3ac + c - a + q \right\} \right] \times \sum_{j = 1}^{c} \exp \left\{ \frac{\pi i}{c} \left[ 3aj^2 + j(c - a + q) \right] \right\}. \]

But, \( a \) and \( c \) being coprime, and thus
only the $T_q$ with odd subscripts actually appear so that we have

$$(2.4) \ T_{2r+1}(c) = \exp \pi i \left\{ \frac{a-4r-2}{12c} \right\} \sum_{j=1}^{c} \exp \left\{ \frac{\pi i}{c} [3aj^2 + j(c-a+1+2r)] \right\}.$$ 

In order to have a complete square in the exponent we multiply each term of the sum by

$$\exp \pi i \left\{ j \frac{ad-1}{c} (c-1+2r) \right\} = \exp \pi i \{jb(c+1)\}.$$ 

As we do not wish to change $T_{2r+1}$ by this multiplication, we have to assume that, for $c$ even, $b$ also is even. Using the abbreviation

$$(2.5) \ \beta = cd + d - 1,$$ 

we obtain from (2.4):

$$(2.6) \ T_{2r+1}(c) = \exp \pi i \left\{ \frac{a-4r-2}{12c} \right\} \sum_{j=1}^{c} \exp \left\{ \frac{\pi i}{12c} [36j^2 + 12j(cd+d-1+2rd)] \right\} \times \sum_{j=1}^{c} \exp \left\{ \frac{\pi i}{12c} (6j + \beta + 2rd)^2 \right\}.$$ 

In the sum appearing here, $j$ can be taken as running over any full residue system mod $c$, because $\beta \equiv c \ (mod \ 2)$ and therefore the sum remains unchanged if $j$ is replaced by $j + c$. Consequently, $\beta$ can be chosen arbitrarily, mod 6, and $T_{2r+1}(c)$ can be simplified by the substitution $r = 3\mu + \nu$. We note that

$$\exp \pi i \left\{ \frac{-r}{3c} (ad^2r + ad\beta + 1) \right\}$$

$$= \exp \pi i \left\{ \frac{-\mu}{c} (3\mu d + 3\mu bcd + 2d\nu + bc\beta + cd + d) \right\}$$

$$- \frac{\nu}{3c} (d\nu + bcd\nu + bc\beta + cd + d) \right\};$$
and considering
\[ \exp\{-\pi i \mu (b \beta + d + 3 \mu b d)\} = \exp\{-\pi i \mu (b c d - b + d)\} = \exp\{\pi i \mu\}, \]
we obtain
\[ T_{\mu+2\nu+1}(c) = \exp \pi i \left\{ \frac{a-a \beta^2 - 2}{12c} - \frac{\nu}{3} \left[ bd\nu + d + b \beta + \frac{d}{c} (\nu + 1) \right] \right. \]
\[ \left. - \frac{\mu}{c} \left[ 3 \mu d + d (1 + 2 \nu) + c \right] \right\} H_{a,c} (\beta + 2 \nu d) \]
with the abbreviation
\[ (2.7) \quad H_{a,c} (\beta) = \sum_{j \mod c} \exp \left\{ \frac{\pi i a}{12c} (6j + \beta)^2 \right\}, \quad \beta \equiv c \pmod{2}. \]

Looking back to (2.3), we see that the result we have obtained so far may be written as:

\[ (2.8) \quad \eta \left( \frac{a}{c} + \frac{iz}{c} \right) = \frac{1}{\sqrt{3cz}} \exp \pi i \left\{ \frac{a-a \beta^2 - 2}{12c} \right\} \]
\[ \times \sum_{\nu=0}^{2} \exp \left\{ \frac{-\pi i \nu}{3} \left[ bd\nu + d + b \beta + \frac{d}{c} (\nu + 1) \right] \right\} U_{\nu}(z) H_{a,c} (\beta + 2 \nu d), \]
with
\[ U_{\nu}(z) = \sum_{\mu=-\infty}^{+\infty} \exp \left\{ \pi i \left[ \mu - \frac{3d}{c} \mu^2 - \frac{d}{c} \mu (2 \nu + 1) \right] - \frac{3\pi}{cz} \left( \mu + \frac{2 \nu + 1}{6} \right)^2 \right\}. \]

These expressions are easy to sum, since, according to (1.8), we have
\[ U_{0}(z) = \sum_{\mu=-\infty}^{+\infty} \exp \left\{ \pi i \left[ \mu - \frac{3d}{c} \left( \mu^2 + \frac{\mu}{3} \right) \right] - \frac{3\pi}{cz} \left( \mu + \frac{1}{6} \right)^2 \right\} \]
\[ = \exp \left\{ \frac{\pi id}{12c} \right\} \eta \left( \frac{d}{c} + \frac{i}{cz} \right); \]
and, replacing \( \mu \) by \( -\mu - 1 \), we see that
\[ U_1(z) = -U_1(z), \quad \text{or} \quad U_1(z) = 0, \]
\[ U_2(z) = -\exp \left\{ \pi i \frac{2d}{c} \right\} U_0(z). \]

Now, by the meaning of \( z \) in (2.1), we get
\[-\frac{d}{c} + \frac{i}{cz} = \frac{-d \tau' + b}{c \tau' - a} = \tau,\]
and have therefore:

\[ \eta(\tau') = \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{a(1-\beta^2) - 2 + d}{12c} \right\} \]
\[ \times \left[ H_{a,c}(\beta) - \exp \left( -\frac{2\pi i}{3} (d + 2bd + b\beta) \right) \right] \]
\[ \times \left[ H_{a,c}(\beta + 4d) \right] \eta(\tau). \]

Comparing this with (1.7), we see that we have obtained so far:

\[ (2.91) \quad \epsilon = \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{a(1-\beta^2) - 2 + d}{12c} \right\} \]
\[ \times \left[ H_{a,c}(\beta) - \exp \left( -\frac{2\pi i}{3} (d + 2bd + b\beta) \right) H_{a,c}(\beta + 4d) \right] \]
\[ = \frac{1}{\sqrt{3c}} \exp \pi i \left\{ \frac{bd(1-c^2) - cd}{12} + \frac{(1-d)(b+ad)}{6} \right\} \]
\[ \times \left[ H_{a,c}(\beta) - \exp \left( -\frac{2\pi i}{3} (d + 2bd + b\beta) \right) H_{a,c}(\beta + 4d) \right] \]
and it remains to be shown that this is a root of unity.

3. Reduction to Gaussian sums. The sums \( H_{a,c}(\beta) \) which appear in (2.91) are defined in (2.7) only for \( \beta \equiv c \) (mod 2). In this section, however, it will be more convenient to consider the more general sums*

\* We have used the letters \( h \) and \( k \) instead of \( a \) and \( c \) in order to indicate that the investigations of this section are independent from our previous results.
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(3.1) \[ H_{h,k}(\gamma) = \frac{1}{2} \sum_{j \mod 2k} \exp \left( \frac{\pi i h}{12k} (6j + \gamma)^2 \right), \]

with no restriction on $\gamma$. These sums can be expressed in terms of Gaussian sums

(3.2) \[ G(h,k) = \sum_{j \mod k} \exp \left( \frac{2\pi i h}{k} j^2 \right). \]

Comparing the definitions (3.1) and (3.2) one finds immediately that:

\[ H_{h,k}(0) + H_{h,k}(1) + H_{h,k}(2) + H_{h,k}(3) + H_{h,k}(4) + H_{h,k}(5) = \frac{1}{4} G(h,24k), \]

\[ H_{h,k}(0) + H_{h,k}(2) + H_{h,k}(4) = \frac{1}{2} G(h,6k), \]

\[ H_{h,k}(0) + H_{h,k}(3) = \frac{1}{4} G(3h,8k). \]

If we consider that

\[ H_{h,k}(-\gamma) = H_{h,k}(\gamma) = H_{h,k}(\gamma + 6n), \]

we get the following relations:

(3.31) \[ H_{h,k}(0) = \frac{1}{2} G(3h,2k), \]

(3.32) \[ H_{h,k}(3) = \frac{1}{4} G(3h,8k) - \frac{1}{2} G(3h,2k), \]

(3.33) \[ H_{h,k}(2) = \frac{1}{4} G(h,6k) - \frac{1}{4} G(3h,2k), \]

(3.34) \[ H_{h,k}(1) = \frac{1}{8} G(h,24k) - \frac{1}{8} G(3h,8k) - \frac{1}{4} G(h,6k) + \frac{1}{4} G(3h,2k). \]

In order to obtain the sums $H_{h,k}(\gamma)$ explicitly, the following rules concerning Gaussian sums may be useful.*

* For the formulas (3.41)-(3.47) see [1] or [3]; (3.46) may also be found in [7].
As elementary consequences of the definition (3.2) we have:

(3.41) \[ G(mh, mk) = mG(h, k) \quad m > 0 \]

(3.42) \[ G(h, k_1 k_2) = G(hk_1, k_2) G(hk_2, k_1) \quad (k_1, k_2) = 1 \]

(3.43) \[ G(m^2 h, k) = G(h, k) \quad (m, k) = 1 \]

(3.44) \[ G(h, m^2 k) = mG(h, k) \quad (m, h) = 1; \quad m > 0 \text{ and odd.} \]

The following results, due to Gauss [4], are a little deeper:

(3.45) \[ G(h_1 h_2, k) = \left(\frac{h_1}{k}\right) G(h_2, k) \quad (h_1 h_2, k) = 1, \quad k \text{ odd} \]

(3.46) \[ G(1, k) = \sqrt{k} i^{[(k-1)/2]^2} \quad k \text{ odd} \]

(3.47) \[ G(h, 2^\alpha) = \begin{cases} 0 & h \text{ odd, } \alpha = 1 \\ 2^{(\alpha+1)/2} \left(\frac{2}{h}\right)^{\alpha+1} e^{n i h/4} & h \text{ odd, } \alpha \geq 2. \end{cases} \]

The symbol \( \left(\frac{h}{k}\right) \) is the Jacobi symbol.

The following discussion may be restricted to the case \( \gamma \equiv k \pmod{2} \), which will be sufficient for our purpose. Furthermore, we put* throughout \( k = 2^\lambda k_1 \) (\( k_1 \) being odd), and have then to distinguish whether 3 does or does not divide \( k_1 \).

Assume first \( 3 \mid k_1 \). Then we find, using (3.41) and (3.44), that

(3.51) \[ H_{h, k}(1) = 0, \quad H_{h, k}(2) = 0; \]

and, applying (3.41), (3.42), (3.44), (3.45), and (3.47), we obtain:

(3.52) \[ H_{h, k}(0) = 2^{\lambda/2} \left(\frac{2}{h}\right)^\lambda \exp\left\{\frac{3}{4} \pi i h k_1\right\} G(2h, 3k_1) , \]

(3.53) \[ H_{h, k}(3) = \exp\left\{\frac{3}{4} \pi i h k\right\} G(2h, 3k) . \]

*We do this in order to avoid the reciprocity law for Gaussian sums which would require additional distinctions concerning the sign of \( h \).
As a consequence of (3.46) we have:

\[ G(1, 3k) = \sqrt{3k} \exp \left\{ \frac{\pi i}{8} (3k - 1)^2 \right\} = -\sqrt{3} \exp \left\{ -\frac{\pi i k}{2} \right\} G(1, k) , \]

and therefore, according to (3.45),

\[ G(2h, 3k) = \left( \frac{2h}{3k} \right) G(1, 3k) = -\left( \frac{2h}{3} \right) \sqrt{3} \exp \left\{ -\frac{\pi i k}{2} \right\} G(2h, k) . \]

This formula enables us to express (3.52) and (3.53) in the single formula:

\[ (3.6) \quad H_{h,k} (k) = \sqrt{3} \, 2^{\lambda/2} \left( \frac{h}{3} \right) \exp \pi i \left\{ \frac{k_1 (h-1)}{2} + \frac{hk_1}{4} + \frac{h^2-1}{8} \right\} G(2h, k_1) . \]

In case \(3/k_1\), by use of (3.42) and (3.43) we can express the more complicated sums \(H_{h,k}(1)\) and \(H_{h,k}(2)\) by \(H_{h,k}(3)\) and \(H_{h,k}(0)\), respectively:

\[ (3.71) \quad H_{h,k} (1) = \exp \left\{ \frac{4}{3} \pi i h k \right\} H_{h,k} (3) , \]

\[ (3.72) \quad H_{h,k} (2) = \exp \left\{ \frac{4}{3} \pi i h k \right\} H_{h,k} (0) . \]

More generally, the following recursion formula holds:

\[ (3.73) \quad H_{h,k} (\gamma + 2n) = \exp \left\{ \frac{\pi i}{3} (\gamma + n) nhk \right\} H_{h,k} (\gamma) . \]

In order to compute \(H_{h,k}(0)\) and \(H_{h,k}(3)\), we apply (3.42), (3.43), (3.45), and (3.47) to obtain:

\[ H_{h,k} (3) = \left( \frac{k}{3} \right) \exp \pi i \left\{ \frac{k-1}{2} + \frac{3hk}{4} \right\} G(2h, k) , \]

\[ H_{h,k} (0) = \left( \frac{k}{3} \right) 2^{\lambda/2} \left( \frac{2}{h} \right)^{\lambda} \exp \pi i \left\{ \frac{k_1 - 1}{2} + \frac{3hk_1}{4} \right\} G(2h, k_1) . \]

Applying this on (3.71) and (3.72), and considering

\[ \exp \pi i \left\{ \frac{4}{3} h k + \frac{3}{4} h k_1 \right\} = \exp \pi i \left\{ \frac{hk}{12} + \frac{3}{4} h (k_1 - k) \right\} , \]
we can combine (3.71) and (3.72) into:

\[(3.8) \quad \Pi_{h,k} (k) = 2^{\lambda/2} \left( \frac{k}{3} \right) \times \exp \pi i \left( \frac{hk}{12} + \frac{3}{4} h(k_1 - k) + \frac{k_1 - 1}{2} + \frac{\lambda h^2 - 1}{8} \right) G(2h, k_1). \]

4. Determination of the root of unity. Now we go back to our result (2.9) and consider the following expression:

\[(4.1) \quad \rho = \frac{1}{\sqrt{3c}} \exp \left( \frac{\pi i}{6} (1 - d)(b + ad) \right) \times \left[ H_{a,c}(\beta) - \exp \left( \frac{-2\pi i}{3} (d + 2bd + b\beta) \right) H_{a,c}(\beta + 4d) \right]. \]

According to the results of the preceding section, we have to distinguish whether \(c\) is divisible by 3 or not and to keep in mind that \(c = 2^\lambda c_1, c_1 \text{ odd.}\)

Let us assume first \(3 \mid c\); according to (3.51) we know that:

- \(H_{a,c}(\beta) = H_{a,c}(dc + d - 1) = 0\) if \(d \equiv -1 \pmod{3}\),
- \(H_{a,c}(\beta + 4d) = H_{a,c}(dc + 5d - 1) = 0\) if \(d \equiv +1 \pmod{3}\).

Therefore we have:

\[(4.2) \quad \rho = \left( \frac{d}{3} \right) \frac{1}{\sqrt{3c}} \exp \left( \frac{\pi i}{6} (1 - d)(b + ad) + \frac{2}{3} (d - 1)(1 + b) \right) H_{a,c}(c)
\]

\[= \left( \frac{a}{3} \right) \frac{1}{\sqrt{3c}} \exp \left( \frac{\pi i}{2} (d - 1)(b + ad) \right) H_{a,c}(c). \]

Considering that

\[\exp \left( \frac{\pi i}{2} \left[ (d - 1)(b + ad + c) + (a - 1)(c_1 - c) \right] \right) = 1,\]

and therefore that
\[
\exp \pi i \left\{ \frac{1}{2} (d-1)(b+ad) + \frac{1}{2} (a-1) \ c_1 \right\} \\
= \exp \left\{ \frac{\pi i}{2} \left[ (d-1)(b+ad+c) + (a-1)(c_1-c) - c(d-a) \right] \right\} \\
= \exp \left\{ \frac{\pi i}{6} c(d-a) \right\},
\]
we get from (4.2) and (3.6):

\[
(4.3) \quad \rho = \frac{1}{\sqrt{c_1}} \exp \pi i \left\{ \frac{a}{4} (c_1-c) + \frac{cd}{6} + \frac{ac}{12} + \lambda \frac{a^2-1}{8} \right\} G(2a, c_1).
\]

In case 3/c, we can apply (3.73), which gives us

\[
H_{a,c} (\beta + 4d) = \exp \left\{ \frac{2 \pi i}{3} (\beta + 2d) \ acd \right\} H_{a,c} (\beta)
\]
\[
= \exp \left\{ \frac{2 \pi i}{3} (b\beta + 2bd + d - c) \right\} H_{a,c} (\beta),
\]
and obtain from (4.1):

\[
\rho = \frac{1}{\sqrt{3c}} \exp \left\{ \frac{\pi i}{6} (1-d)(b+ad) \right\} \left[ 1 - \exp \left\{ \frac{-2\pi ic}{3} \right\} \right] H_{a,c} (\beta)
\]
\[
= \frac{1}{\sqrt{c}} \left( \frac{c}{3} \right) \exp \pi i \left\{ \frac{1}{6} (1-d)(b+ad) - \frac{1}{2} + \frac{2c}{3} \right\} H_{a,c} (\beta).
\]

Now we apply (3.37) once more, putting

\[
H_{a,c} (\beta) = H_{a,c} (c + \beta - c) = \exp \left\{ \frac{\pi i}{3} \left( c + \frac{\beta-c}{2} \right) \frac{\beta-c}{2} \ ac \right\} H_{a,c} (c)
\]
\[
= \exp \left\{ \frac{\pi i}{12} (\beta^2 - c^2) \ ac \right\} H_{a,c} (c).
\]
Using (3.8) and considering

\[ \exp \left( \frac{\pi i}{12} (\beta^2 - c^2) ac \right) \]

\[ = \exp \left( \frac{\pi i}{12} \left[ ac(c^2 - 1) - 1 + 2ac(d - 1)(cd + d) \right] \right) \]

\[ = \exp \left( \frac{\pi i}{6} (d - 1)(b + c + b + c^2) \right) \]

\[ = \exp \left( \frac{\pi i}{2} [(a - 1)(c_1 - c) - (d - 1)(c^2 - 1)] \right) = 1 , \]

we see that the expression for \( \rho \) becomes again (4.3). Therefore, we have in all cases:

\[ (4.4) \quad \epsilon = \exp \pi i \left\{ \frac{1}{12} \left[ bd(1 - c^2) + c(a + d) \right] + a \frac{c_1 - c}{4} + \lambda \frac{a^2 - 1}{8} \right\} \]

\[ \times \frac{1}{\sqrt{c_1}} G(2a, c_1) , \]

with the only restriction that, for even \( c, b \) also has to be even.

In order to show that our formula (4.4) holds even if this condition is not satisfied, we put

\[ \tau' = \frac{a \tau + b}{c \tau + d} , \quad c \text{ even, } b \text{ odd,} \]

\[ \tau^* = \frac{(a + c) \tau + (b + d)}{c \tau + d} = \tau' + 1 . \]

Then, for \( \tau^* \), formula (4.4) holds; considering

\[ \eta(\tau + 1) = \exp \left( -\frac{\pi i}{12} \right) \eta(\tau) , \]

which is an immediate consequence of (1.6), we find:
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\[ \eta(\tau^*) = e^* \eta(\tau) = \exp\left(\frac{-\pi i}{12}\right) \eta(\tau') = \exp\left(\frac{-\pi i}{12}\right) \epsilon \eta(\tau) \]

(4.5)

\[ \epsilon = \exp\left(\frac{\pi i}{12}\right) e^*. \]

Now, if we compute $\epsilon^*$ by means of (4.4), and then $\epsilon$, using (4.5), the result will be exactly the same as we get computing $\epsilon$ directly by means of (4.4).

Finally, we can omit the Gaussian sums in (4.3) and, using (3.45) and (3.46), obtain:

\[ \epsilon = \left(\frac{a}{c_1}\right) \times \exp\pi i \left[ \frac{1}{12} \left[ bd(1-c^2) + c(a + d) \right] + \frac{1-c_1}{4} + \frac{c - c_1}{4} + \frac{a^2 - 1}{8} \right]. \]

This formula agrees with the one given by Tannery and Molk [10, p. 112].

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