ON DEDEKIND'S FUNCTION $\eta(\tau)$

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1. Introduction. A transformation of the form

$$(1.1) \qquad \tau' = \frac{a\tau + b}{c\tau + d} \quad ,$$

where a, b, c, d are rational integers satisfying

$$(1.2) \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 1 \; ,$$

is called a *modular transformation*. Without loss of generality we may assume $c \geq 0$. A function $f(\tau)$, analytic in the upper halfplane $\Im(\tau) > 0$, and satisfying the functional equation

$$(1.3) \qquad f(\tau) = (c\tau + d)^k \; f\!\left(\frac{a\tau + b}{c\tau + d}\right),$$

is called a *modular form of dimension k*. An example of a modular form is the discriminant

$$(1.4) \qquad \Delta(\tau) = \exp\{2\pi i\tau\} \prod_{m=1}^{\omega} \left(1 - \exp\{2\pi i m\tau\}\right)^{24} \; ,$$

which is of dimension -12; that is, it satisfies the equation*

$$(1.5) \qquad \Delta(\tau') = (c\tau + d)^{12}\Delta(\tau) \; .$$

An important role in the theory of modular functions is played by the function

$$(1.6) \qquad \eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \prod_{m=1}^{\infty} \left(1 - \exp\{2\pi i m\tau\}\right) \; ,$$

which is the 24th root of $\Delta(\tau)$. The transformation formula for this function may be obtained from (1.5) and is conveniently written as:

$$(1.7) \qquad \eta(\tau') = \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{-i(c\tau + d)}\ \eta(\tau)\ .$$

Since we have assumed $c \geq 0$ and $\mathfrak{I}(\tau) > 0$, the radicand has a nonnegative real part. By the square root we always mean the principal branch; that is, $\mathfrak{R}(\sqrt{\ }) > 0$. The ϵ appearing in (1.7) is a 24th root of unity. The purpose of the present paper is to determine this ϵ completely.

Investigations concerning this root of unity were carried out first by Dedekind [2] and later by Tannery and Molk [10] and Rademacher [8; 9]. However, they use the theory of log $\eta(\tau)$, which requires much more than is needed for this purpose. Hurwitz discusses only $[\Delta(\tau)]^{1/12}$ and remarks that the transformation formula of $\eta(\tau)$ can be obtained by means of θ-functions. The investigations of Hermite [5] are likewise not sufficient for our purpose, because he discusses only $\eta^3(\tau)$, and therefore a third root of unity remains still undetermined.

In the following, we shall approach the determination of ϵ directly by investigations of the function $\eta(\tau)$, which, by a well-known formula due to Euler, can be written as the following sum:

$$(1.8) \quad \eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\{\pi i \tau \lambda(3\lambda - 1)\}$$

$$= \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\left\{3\pi i \tau\left(\lambda - \frac{1}{6}\right)^2\right\}\ .$$

Our starting point is formula (1.8); our principal tools are a Poisson transformation formula and Gaussian sums.

2. Application of a Poisson formula. We introduce a new variable z with $\mathfrak{R}(z) > 0$ by the substitution*

$$(2.1) \qquad\qquad \tau' = \frac{iz}{c} + \frac{a}{c}\ , \qquad\qquad c > 0;\ (a, c) = 1\ ,$$

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*This requires $c \neq 0$, but the case $c = 0$ is trivial.

and obtain, from (1.8)

$$(2.2) \qquad \eta\!\left(\frac{a}{c} + \frac{iz}{c}\right) = \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda}\, \exp\left\{\frac{3\pi i}{c}\,(a + iz)\left(\lambda - \frac{1}{6}\right)^{2}\right\}$$

$$= \sum_{j \bmod 2c} \exp \pi i\left\{ j + \frac{3a}{c}\left(j - \frac{1}{6}\right)^{2}\right\}$$

$$\times \sum_{q=-\infty}^{+\infty} \exp\left\{-\frac{3\pi z}{c}\left(2cq + j - \frac{1}{6}\right)^{2}\right\}\ .$$

To the inner sum,

$$F_{c}\,(z) = \sum_{q=-\infty}^{+\infty} \exp\left\{-12\,\pi c z\!\left(q + \frac{6j-1}{12c}\right)^{2}\right\},$$

we apply Poisson's formula (cf. [11]),

$$\sum_{m=-\infty}^{+\infty} \exp\{-\pi\,(m + \alpha)^{2}t\} = \frac{1}{\sqrt{t}}\sum_{m=-\infty}^{+\infty} \exp\left\{2\pi i m\alpha - \frac{\pi m^{2}}{t}\right\}, \qquad \Re\,(t) > 0,$$

and obtain

$$F_{c}\,(z) = \frac{1}{2\sqrt{3\,cz}}\sum_{q=-\infty}^{+\infty} \exp\left\{2\pi i q\,\frac{6j-1}{12c} - \frac{\pi q^{2}}{12 cz}\right\}\ .$$

Putting this in (2.2), we get:

$$(2.3) \qquad \eta\!\left(\frac{a}{c} + \frac{iz}{c}\right) = \frac{1}{\sqrt{3\,cz}}\sum_{q=-\infty}^{+\infty} \exp\left\{\frac{-\pi q^{2}}{12 cz}\right\} T_{q}\,(c)\ ,$$

where

$$T_{q}\,(c) = \frac{1}{2}\sum_{j \bmod 2c} \exp \pi i\left\{ j + \frac{3a}{c}\left(j - \frac{1}{6}\right)^{2} + q\,\frac{6j-1}{6c}\right\}$$

$$= \frac{1}{2}\,\exp \pi i\left\{\frac{a - 2q}{12c}\right\}\left[1 + \exp \pi i\{3ac + c - a + q\}\right]$$

$$\times \sum_{j=1}^{c} \exp\left\{\frac{\pi i}{c}\,[3aj^{2} + j\,(c - a + q)]\right\}\right\}.$$

But, a and c being coprime, and thus

$$3ac + c - a \equiv 1 \quad (\text{mod } 2) ,$$

only the T_q with odd subscripts actually appear so that we have

$$(2.4) \quad T_{2r+1}(c) = \exp \pi i \left\{ \frac{a - 4r - 2}{12c} \right\} \sum_{j=1}^{c} \exp\left\{ \frac{\pi i}{c} \left[3aj^2 + j(c - a + 1 + 2r) \right] \right\} .$$

In order to have a complete square in the exponent we multiply each term of the sum by

$$\exp \pi i \left\{ j \frac{ad - 1}{c} (c - 1 + 2r) \right\} = \exp \pi i \left\{ jb(c + 1) \right\} .$$

As we do not wish to change T_{2r+1} by this multiplication, we have to assume that, for c even, b also is even. Using the abbreviation

$$(2.5) \qquad\qquad\qquad \beta = cd + d - 1 ,$$

we obtain from (2.4):

$$(2.6) \quad T_{2r+1}(c) = \exp \pi i \left\{ \frac{a - 4r - 2}{12c} \right\} \sum_{j=1}^{c} \exp\left\{ \frac{\pi i a}{12c} \left[36j^2 + 12j(cd + d - 1 + 2rd) \right] \right\}$$

$$= \exp \pi i \left\{ \frac{a - a\beta^2 - 2}{12c} - \frac{r}{3c} (ad^2 r + ad\beta + 1) \right\}$$

$$\times \sum_{j=1}^{c} \exp\left\{ \frac{\pi i a}{12c} (6j + \beta + 2rd)^2 \right\} .$$

In the sum appearing here, j can be taken as running over any full residue system mod c, because $\beta \equiv c \pmod{2}$ and therefore the sum remains unchanged if j is replaced by $j + c$. Consequently, β can be chosen arbitrarily, mod 6, and $T_{2r+1}(c)$ can be simplified by the substitution $r = 3\mu + \nu$. We note that

$$\exp \pi i \left\{ \frac{-r}{3c} (ad^2 r + ad\beta + 1) \right\}$$

$$= \exp \pi i \left\{ \frac{-\mu}{c} (3\mu d + 3\mu bcd + 2d\nu + bc\beta + cd + d) \right.$$

$$\left. - \frac{\nu}{3c} (d\nu + bcd\nu + bc\beta + cd + d) \right\} ;$$

and considering

$$\exp\{-\pi i\mu\,(b\beta+d+3\mu bd)\} = \exp\{-\pi i\mu\,(bcd-b+d)\} = \exp\{\pi i\mu\}\,,$$

we obtain

$$T_{6\mu+2\nu+1}(c) = \exp\pi i\left\{\frac{a-a\beta^2-2}{12c} - \frac{\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}\,(\nu+1)\right]\right.$$
$$\left.-\frac{\mu}{c}\,[3\mu d+d(1+2\nu)+c]\right\}H_{a,c}\,(\beta+2\nu d)$$

with the abbreviation

$$(2.7)\qquad H_{a,c}\,(\beta) = \sum_{j\bmod c}\exp\left\{\frac{\pi i a}{12c}\,(6j+\beta)^2\right\},\qquad\qquad \beta\equiv c\;(\bmod\;2)\,.$$

Looking back to (2.3), we see that the result we have obtained so far may be written as:

$$(2.8)\;\; \eta\!\left(\frac{a}{c}+\frac{iz}{c}\right) = \frac{1}{\sqrt{3cz}}\,\exp\pi i\left\{\frac{a-a\beta^2-2}{12c}\right\}$$
$$\times\sum_{\nu=0}^{2}\exp\left\{\frac{-\pi i\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}\,(\nu+1)\right]\right\}U_\nu(z)\,H_{a,c}(\beta+2d\nu),$$

with

$$U_\nu(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\,\mu^2-\frac{d}{c}\,\mu\,(2\nu+1)\right]-\frac{3\pi}{cz}\left(\mu+\frac{2\nu+1}{6}\right)^2\right\}\,.$$

These expressions are easy to sum, since, according to (1.8), we have

$$U_0(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\Big(\mu^2+\frac{\mu}{3}\Big)\right]-\frac{3\pi}{cz}\left(\mu+\frac{1}{6}\right)^2\right\}$$
$$= \exp\left\{\frac{\pi id}{12c}\right\}\eta\!\left(-\frac{d}{c}+\frac{i}{cz}\right);$$

and, replacing μ by $-\mu-1$, we see that

$$U_1(z) = -U_1(z), \quad \text{or} \quad U_1(z) = 0 \ ,$$

$$U_2(z) = -\exp\left\{\pi i \ \frac{2d}{c}\right\} \ U_0(z) \ .$$

Now, by the meaning of z in (2.1), we get

$$-\frac{d}{c} + \frac{i}{cz} = \frac{-d\tau' + b}{c\tau' - a} = \tau \ ,$$

and have therefore:

(2.9)
$$\eta(\tau') = \frac{1}{\sqrt{3c}} \ \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\}$$

$$\times \left[H_{a,c}\,(\beta) \ - \ \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\}\right.$$

$$\left. \times H_{a,c}\,(\beta + 4d)\right] \sqrt{-i(c\tau + d)} \ \ \eta(\tau) \ .$$

Comparing this with (1.7), we see that we have obtained so far:

(2.91)
$$\epsilon = \frac{1}{\sqrt{3c}} \ \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\}$$

$$\times \left[H_{a,c}(\beta) \ - \ \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} H_{a,c}\,(\beta + 4d)\right]$$

$$= \frac{1}{\sqrt{3c}} \ \exp \pi i \left\{\frac{bd(1-c^2)-cd}{12} + \frac{(1-d)(b+ad)}{6}\right\}$$

$$\times \left[H_{a,c}\,(\beta) \ - \ \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} H_{a,c}\,(\beta + 4d)\right]$$

and it remains to be shown that this is a root of unity.

3. Reduction to Gaussian sums. The sums $H_{a,c}\,(\beta)$ which appear in (2.91) are defined in (2.7) only for $\beta \equiv c \pmod 2$. In this section, however, it will be more convenient to consider the more general sums*

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*We have used the letters h and k instead of a and c in order to indicate that the investigations of this section are independent from our previous results.

(3.1) $$H_{h,k}(\gamma) = \frac{1}{2} \sum_{j \bmod 2k} \exp\left\{\frac{\pi i h}{12k}(6j+\gamma)^2\right\},$$

with no restriction on γ. These sums can be expressed in terms of Gaussian sums

(3.2) $$G(h,k) = \sum_{j \bmod k} \exp\left\{\frac{2\pi i h}{k}j^2\right\}.$$

Comparing the definitions (3.1) and (3.2) one finds immediately that:

$$H_{h,k}(0) + H_{h,k}(1) + H_{h,k}(2) + H_{h,k}(3) + H_{h,k}(4) + H_{h,k}(5) = \frac{1}{4}G(h,24k),$$

$$H_{h,k}(0) + H_{h,k}(2) + H_{h,k}(4) = \frac{1}{2}G(h,6k),$$

$$H_{h,k}(0) + H_{h,k}(3) = \frac{1}{4}G(3h,8k).$$

If we consider that

$$H_{h,k}(-\gamma) = H_{h,k}(\gamma) = H_{h,k}(\gamma+6n),$$

we get the following relations:

(3.31) $$H_{h,k}(0) = \frac{1}{2}G(3h,2k),$$

(3.32) $$H_{h,k}(3) = \frac{1}{4}G(3h,8k) - \frac{1}{2}G(3h,2k),$$

(3.33) $$H_{h,k}(2) = \frac{1}{4}G(h,6k) - \frac{1}{4}G(3h,2k),$$

(3.34) $$H_{h,k}(1) = \frac{1}{8}G(h,24k) - \frac{1}{8}G(3h,8k) - \frac{1}{4}G(h,6k) + \frac{1}{4}G(3h,2k).$$

In order to obtain the sums $H_{h,k}(\gamma)$ explicitly, the following rules concerning Gaussian sums may be useful.*

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* For the formulas (3.41)–(3.47) see [1] or [3]; (3.46) may also be found in [7].

As elementary consequences of the definition (3.2) we have:

(3.41) $G(mh, mk) = mG(h, k)$ $m > 0$

(3.42) $G(h, k_1 k_2) = G(hk_1, k_2) \, G(hk_2, k_1)$ $(k_1, k_2) = 1$

(3.43) $G(m^2 h, k) = G(h, k)$ $(m, k) = 1$

(3.44) $G(h, m^2 k) = mG(h, k)$ $(m, h) = 1 ;$ $m > 0$ and odd.

The following results, due to Gauss [4], are a little deeper:

(3.45) $G(h_1 h_2, k) = \left(\dfrac{h_1}{k} \right) G(h_2, k)$ $(h_1 h_2, k) = 1 ,$ k odd

(3.46) $G(1, k) = \sqrt{k} \; i^{\left[(k-1)/2 \right]^2}$ k odd

(3.47) $G(h, 2^\alpha) = \begin{cases} 0 \\ 2^{(\alpha+1)/2} \left(\dfrac{2}{h} \right)^{\alpha+1} e^{\pi i h/4} \end{cases}$ $\begin{array}{l} h \text{ odd}, \quad \alpha = 1 \\[1em] h \text{ odd}, \quad \alpha \geq 2 . \end{array}$

The symbol $\left(\dfrac{h}{k} \right)$ is the Jacobi symbol.

The following discussion may be restricted to the case $\gamma \equiv k \pmod 2$, which will be sufficient for our purpose. Furthermore, we put* throughout $k = 2^\lambda k_1$ (k_1 being odd), and have then to distinguish whether 3 does or does not divide k_1.

Assume first $3 \mid k_1$. Then we find, using (3.41) and (3.44), that

(3.51) $H_{h,k}(1) = 0 ,$ $H_{h,k}(2) = 0 ;$

and, applying (3.41), (3.42), (3.44), (3.45), and (3.47), we obtain:

(3.52) $H_{h,k}(0) = 2^{\lambda/2} \left(\dfrac{2}{h} \right)^\lambda \exp\left\{ \dfrac{3}{4} \pi i h k_1 \right\} G(2h, 3k_1) ,$

(3.53) $H_{h,k}(3) = \exp\left\{ \dfrac{3}{4} \pi i h k \right\} G(2h, 3k) .$

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*We do this in order to avoid the reciprocity law for Gaussian sums which would require additional distinctions concerning the sign of h.

As a consequence of (3.46) we have:

$$G(1, 3k) = \sqrt{3k} \ \exp\left\{\frac{\pi i}{8} (3k-1)^2\right\} = -\sqrt{3} \ \exp\left\{\frac{-\pi i k}{2}\right\} G(1, k) ,$$

and therefore, according to (3.45),

$$G(2h, 3k) = \left(\frac{2h}{3k}\right) G(1, 3k) = -\left(\frac{2h}{3}\right) \sqrt{3} \ \exp\left\{\frac{-\pi i k}{2}\right\} G(2h, k) .$$

This formula enables us to express (3.52) and (3.53) in the single formula:

$$(3.6) \quad H_{h,k}(k) = \sqrt{3} \ 2^{\lambda/2} \left(\frac{h}{3}\right) \exp \pi i \left\{\frac{k_1(h-1)}{2} + \frac{hk_1}{4} + \lambda\frac{h^2-1}{8}\right\} G(2h, k_1) .$$

In case $3 \nmid k_1$, by use of (3.42) and (3.43) we can express the more complicated sums $H_{h,k}(1)$ and $H_{h,k}(2)$ by $H_{h,k}(3)$ and $H_{h,k}(0)$, respectively:

$$(3.71) \qquad\qquad H_{h,k}(1) = \exp\left\{\frac{4}{3} \pi ihk\right\} H_{h,k}(3) ,$$

$$(3.72) \qquad\qquad H_{h,k}(2) = \exp\left\{\frac{4}{3} \pi ihk\right\} H_{h,k}(0) .$$

More generally, the following recursion formula holds:

$$(3.73) \qquad\qquad H_{h,k}(\gamma + 2n) = \exp\left\{\frac{\pi i}{3} (\gamma+n) nhk\right\} H_{h,k}(\gamma) .$$

In order to compute $H_{h,k}(0)$ and $H_{h,k}(3)$, we apply (3.42), (3.43), (3.45), and (3.47) to obtain:

$$H_{h,k}(3) = \left(\frac{k}{3}\right) \exp \pi i \left\{\frac{k-1}{2} + \frac{3hk}{4}\right\} G(2h, k) ,$$

$$H_{h,k}(0) = \left(\frac{k}{3}\right) 2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda} \exp \pi i \left\{\frac{k_1-1}{2} + \frac{3hk_1}{4}\right\} G(2h, k_1) .$$

Applying this on (3.71) and (3.72), and considering

$$\exp \pi i \left\{\frac{4}{3} hk + \frac{3}{4} hk_1\right\} = \exp \pi i \left\{\frac{hk}{12} + \frac{3}{4} h(k_1 - k)\right\} ,$$

we can combine (3.71) and (3.72) into:

$$(3.8) \quad H_{h,k}(k) = 2^{\lambda/2} \left(\frac{k}{3} \right)$$

$$\times \exp \pi i \left\{ \frac{hk}{12} + \frac{3}{4} h(k_1 - k) + \frac{k_1 - 1}{2} + \lambda \frac{h^2 - 1}{8} \right\} G(2h, k_1) \ .$$

4. Determination of the root of unity. Now we go back to our result (2.9) and consider the following expression:

$$(4.1) \quad \rho = \frac{1}{\sqrt{3}c} \exp \left\{ \frac{\pi i}{6} (1 - d)(b + ad) \right\}$$

$$\times \left[H_{a,c}(\beta) - \exp \left\{ \frac{-2\pi i}{3} (d + 2bd + b\beta) \right\} H_{a,c}(\beta + 4d) \right] \ .$$

According to the results of the preceding section, we have to distinguish whether c is divisible by 3 or not and to keep in mind that $c = 2^{\lambda} c_1$, c_1 odd.

Let us assume first $3 \mid c$; according to (3.51) we know that:

$$H_{a,c}(\beta) = H_{a,c}(dc + d - 1) = 0 \qquad\qquad \text{if} \quad d \equiv -1 \pmod{3} ,$$

$$H_{a,c}(\beta + 4d) = H_{a,c}(dc + 5d - 1) = 0 \qquad\qquad \text{if} \quad d \equiv +1 \pmod{3} .$$

Therefore we have:

$$(4.2) \quad \rho = \left(\frac{d}{3} \right) \frac{1}{\sqrt{3}c} \exp \pi i \left\{ \frac{1}{6} (1-d)(b+ad) + \frac{2}{3} (d-1)(1+b) \right\} H_{a,c}(c)$$

$$= \left(\frac{a}{3} \right) \frac{1}{\sqrt{3}c} \exp \left\{ \frac{\pi i}{2} (d-1)(b+ad) \right\} H_{a,c}(c) \ .$$

Considering that

$$\exp \left\{ \frac{\pi i}{2} [(d - 1)(b + ad + c) + (a - 1)(c_1 - c)] \right\} = 1 ,$$

and therefore that

$$\exp \pi i \left\{ \frac{1}{2} (d-1)(b+ad) + \frac{1}{2} (a-1) c_1 \right\}$$

$$= \exp \left\{ \frac{\pi i}{2} \left[(d-1)(b+ad+c) + (a-1)(c_1-c) - c(d-a) \right] \right\}$$

$$= \exp \left\{ \frac{\pi i}{6} c(d-a) \right\} ,$$

we get from (4.2) and (3.6):

$$(4.3) \qquad \rho = \frac{1}{\sqrt{c_1}} \exp \pi i \left\{ \frac{a}{4} (c_1-c) + \frac{cd}{6} + \frac{ac}{12} + \lambda \frac{a^2-1}{8} \right\} G(2a, c_1) .$$

In case $3 \not/ c$, we can apply (3.73), which gives us

$$H_{a,c} (\beta + 4d) = \exp \left\{ \frac{2\pi i}{3} (\beta + 2d) \, acd \right\} H_{a,c} (\beta)$$

$$= \exp \left\{ \frac{2\pi i}{3} (b\beta + 2bd + d - c) \right\} H_{a,c} (\beta) ,$$

and obtain from (4.1):

$$\rho = \frac{1}{\sqrt{3c}} \exp \left\{ \frac{\pi i}{6} (1-d)(b+ad) \right\} \left[1 - \exp \left\{ \frac{-2\pi i c}{3} \right\} \right] H_{a,c} (\beta)$$

$$= \frac{1}{\sqrt{c}} \left(\frac{c}{3} \right) \exp \pi i \left\{ \frac{1}{6} (1-d)(b+ad) - \frac{1}{2} + \frac{2c}{3} \right\} H_{a,c} (\beta) .$$

Now we apply (3.37) once more, putting

$$H_{a,c} (\beta) = H_{a,c} (c + \beta - c) = \exp \left\{ \frac{\pi i}{3} \left(c + \frac{\beta-c}{2} \right) \frac{\beta-c}{2} ac \right\} H_{a,c} (c)$$

$$= \exp \left\{ \frac{\pi i}{12} (\beta^2 - c^2) \, ac \right\} H_{a,c} (c) .$$

Using (3.8) and considering

$$\exp\left\{\frac{\pi i}{12}\ (\beta^2 - c^2)\ ac\right\}$$

$$= \exp\left\{\frac{\pi i}{12}\ [ac(c^2 - 1)(d^2 - 1) + 2ac(d - 1)(cd + d)]\right\}$$

$$= \exp\left\{\frac{\pi i}{6}\ (d - 1)(bc + c + b + c^2)\right\}$$

$$= \exp \pi i\left\{\frac{1}{6}\ (d - 1)(b + ad) - \frac{1}{2}\ (d - 1)(c^2 - 1) + \frac{c}{6}\ (d - 1)\right\},$$

$$\exp\left\{\frac{\pi i}{2}\ [(a - 1)(c_1 - c) - (d - 1)(c^2 - 1)]\right\} = 1,$$

we see that the expression for ρ becomes again (4.3). Therefore, we have in all cases:

$$(4.4) \qquad \epsilon = \exp \pi i\left\{\frac{1}{12}\ [bd(1 - c^2) + c(a + d)] + a\ \frac{c_1 - c}{4} + \lambda\frac{a^2 - 1}{8}\right\}$$

$$\times \frac{1}{\sqrt{c_1}}\ G(2a, c_1)\ ,$$

with the only restriction that, for even c, b also has to be even.

In order to show that our formula (4.4) holds even if this condition is not satisfied, we put

$$\tau' = \frac{a\tau + b}{c\tau + d}\ , \qquad\qquad\qquad\qquad c\ \text{even,}\quad b\ \text{odd,}$$

$$\tau^* = \frac{(a + c)\ \tau + (b + d)}{c\tau + d} = \tau' + 1\ .$$

Then, for τ^*, formula (4.4) holds; considering

$$\eta(\tau + 1) = \exp\left\{\frac{-\pi i}{12}\right\}\ \eta(\tau)\ ,$$

which is an immediate consequence of (1.6), we find:

$$\eta(\tau^*) = \epsilon^*\eta(\tau) = \exp\left\{\frac{-\pi i}{12}\right\} \quad \eta(\tau') = \exp\left\{\frac{-\pi i}{12}\right\} \epsilon\,\eta(\tau)$$

(4.5)
$$\epsilon = \exp\left\{\frac{\pi i}{12}\right\} \epsilon^*\,.$$

Now, if we compute ϵ^* by means of (4.4), and then ϵ, using (4.5), the result will be exactly the same as we get computing ϵ directly by means of (4.4).

Finally, we can omit the Gaussian sums in (4.3) and, using (3.45) and (3.46), obtain:

(4.6)
$$\epsilon = \left(\frac{a}{c_1}\right)$$
$$\times \exp \pi i\left\{\frac{1}{12}\left[bd(1-c^2)+c(a+d)\right] + \frac{1-c_1}{4} + a\,\frac{c-c_1}{4} + \lambda\frac{a^2-1}{8}\right\}.$$

This formula agrees with the one given by Tannery and Molk [10, p. 112] .

References

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Vari-Type Composition by
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Printed in the United States of America by
Edwards Brothers, Inc., Ann Arbor, Michigan