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ON THE DEFINITION OF NORMAL NUMBERS

IVAN NIVEN AND H. ZUCKERMAN

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1. Introduction. Let R be a real number with fractional part $.x_1x_2x_3 \cdots$ when written to scale r . Let $N(b, n)$ denote the number of occurrences of the digit b in the first n places. The number R is said to be *simply normal* to scale r if

$$(1) \quad \lim_{n \rightarrow \infty} \frac{N(b, n)}{n} = \frac{1}{r}$$

for each of the r possible values of b ; R is said to be *normal* to scale r if all the numbers R, rR, r^2R, \cdots are simply normal to all the scales r, r^2, r^3, \cdots . These definitions, for $r = 10$, were introduced by Émile Borel [1], who stated (p.261) that "la propriété caractéristique" of a normal number is the following: that for any sequence B whatsoever of v specified digits, we have

$$(2) \quad \lim_{n \rightarrow \infty} \frac{N(B, n)}{n} = \frac{1}{r^v},$$

where $N(B, n)$ stands for the number of occurrences of the sequence B in the first n decimal places.

Several writers, for example Champernowne [2], Koksma [3, p.116], and Copeland and Erdős [4], have taken this property (2) as the definition of a normal number. Hardy and Wright [5, p.124] state that property (2) is equivalent to the definition, but give no proof. It is easy to show that a normal number has property (2), but the implication in the other direction does not appear to be so obvious. If the number R has property (2) then any sequence of digits

$$B = b_1b_2 \cdots b_v$$

appears with the appropriate frequency, but will the frequencies all be the same for $i = 1, 2, \cdots, v$ if we count only those occurrences of B such that b_1 is an $i, i + v, i + 2v, \cdots$ -th digit? It is the purpose of this note to show that this is

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so, and thus to prove the equivalence of property (2) and the definition of normal number.

2. Notation. In addition to the notation already introduced, we shall use the following:

S_α is the first α digits of R .

BXB is the totality of sequences of the form $b_1 b_2 \cdots b_\nu x x \cdots x b_1 b_2 \cdots b_\nu$, where $x x \cdots x$ is any sequence of t digits.

$k_i(\alpha)$ is the number of times that B occurs in S_α with b_1 in a place congruent to $i \pmod{\nu}$.

$$g(\alpha) = \sum_{i=0}^{\nu-1} k_i(\alpha).$$

$\theta_t(\alpha)$ is the number of occurrences of BXB in S_α .

$$k_{i,j}(\alpha) = k_i(\alpha) - k_j(\alpha), \quad i \neq j.$$

B^* is any block of digits of length from $\nu + 1$ to $2\nu - 1$ whose first ν digits are B and whose last ν digits are B . Such a block need not exist.

3. Proof. We shall assume that the number R has the property (2), so that we have

$$(3) \quad \lim_{n \rightarrow \infty} \frac{g(n)}{n} = \frac{1}{r^\nu}$$

and

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\theta_t(n)}{n} = \frac{1}{r^{2\nu}}$$

for each fixed t , and we prove that

$$(5) \quad \lim_{n \rightarrow \infty} \frac{k_{i,j}(n)}{n} = 0,$$

from which it follows that R is a normal number.

Now $k_i(\alpha + s) - k_i(\alpha)$ is the number of B with b_1 in a place congruent to $i \pmod{\nu}$ that are in $S_{\alpha+s}$ but not entirely in S_α . Therefore

$$\sum_{\substack{i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1 \\ i < j}} \{k_i(\alpha + s) - k_i(\alpha)\} \{k_j(\alpha + s) - k_j(\alpha)\}$$

counts the number of BXB and B^* that occur in $S_{\alpha+s}$ such that the first B is not contained entirely in S_α . Here the number t of digits in X runs through all values $\not\equiv 0 \pmod v$ with $0 \leq t \leq s - v - 1$. We take $n > s$ and sum the above expression to get

$$(6) \quad \sigma = \sum_{\alpha=0}^{n-s} \sum_{\substack{i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1 \\ i < j}} \{k_i(\alpha + s) - k_i(\alpha)\} \{k_j(\alpha + s) - k_j(\alpha)\}.$$

Considering S_n and any BXB contained in it with $t \leq s - v - 1$, we see that BXB is counted in σ a certain number of times. In fact if BXB is not too near either end of S_n it is counted just $s - t - v$ times and it is never counted more than this many times. Furthermore if BXB is preceded by at least $s - t - 2v$ digits and is followed in S_n by at least $s - t - v - 1$ digits then BXB is counted exactly $s - t - v$ times. Therefore we have, ignoring any B^* blocks which may be counted by σ ,

$$(7) \quad \sigma \geq \sum_{\substack{t=0 \\ t \not\equiv 0 \pmod v}}^{s-v-1} (s - t - v) \{ \theta_t(n - s) - \theta_t(s) \}.$$

Using (4) we find

$$\lim_{n \rightarrow \infty} \frac{\theta_t(n - s)}{n} = \frac{1}{r^{2v}}$$

for any fixed s ; hence, from (7), we have

$$\lim_{n \rightarrow \infty} \frac{\sigma}{n} \geq \sum_{\substack{t=0 \\ t \not\equiv 0 \pmod v}}^{s-v-1} (s - t - v) \frac{1}{r^{2v}}.$$

It is now convenient to take s , which is otherwise arbitrary, to be congruent to

$0 \pmod{v}$. Then the above formula reduces to

$$(8) \quad \lim_{n \rightarrow \infty} \frac{\sigma}{n} \geq \frac{(v-1)(s-v)^2}{2v} \cdot \frac{1}{r^{2v}}.$$

In a similar manner we count the BXB in S_n where the number t of digits of X is congruent to $0 \pmod{v}$. This gives us

$$(9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \frac{1}{2} \{k_i(\alpha+s) - k_i(\alpha)\} \{k_i(\alpha+s) - k_i(\alpha) - 1\} \\ = \sum_{\substack{t=0 \\ t \not\equiv 0 \pmod{v}}}^{s-v-1} (s-t-v) \frac{1}{r^{2v}} = \frac{s(s-v)}{2v} \cdot \frac{1}{r^{2v}}.$$

Now, by (3) we have

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\} = \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \{g(\alpha+s) - g(\alpha)\} \\ = \lim_{n \rightarrow \infty} \left\{ \frac{1}{2n} \sum_{\alpha=n-s+1}^n g(\alpha+s) - \frac{1}{2n} \sum_{\alpha=0}^{s-1} g(\alpha) \right\} = \frac{s}{2r^v},$$

and (9) reduces to

$$(10) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\}^2 = \frac{s}{r^v} + \frac{s(s-v)}{vr^{2v}}.$$

From (6), (8), and (10) we find that

$$(11) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \{[k_i(\alpha+s) - k_i(\alpha)] - [k_j(\alpha+s) - k_j(\alpha)]\}^2 \\ \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

for any fixed $s \equiv 0 \pmod{v}$. Using the inequality

$$\sum_{i=1}^n x_i^2 \geq \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2$$

we obtain

$$\begin{aligned} & \sum_{\alpha=0}^{n-s} \{ [k_i(\alpha+s) - k_i(\alpha)] - [k_j(\alpha+s) - k_j(\alpha)] \}^2 \\ & \geq \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{n-s} [k_i(\alpha+s) - k_i(\alpha) - k_j(\alpha+s) + k_j(\alpha)] \right\}^2 \\ & = \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{n-s} [k_{i,j}(\alpha+s) - k_{i,j}(\alpha)] \right\}^2 \\ & = \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) - \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2. \end{aligned}$$

This with (11) implies

$$(12) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) - \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}.$$

From the definition we have $|k_{i,j}(\alpha)| < \alpha$ and hence

$$\lim_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2 = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) = 0$$

for fixed s .

Therefore (12) implies

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) \right\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}},$$

which can be written in the form

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ s k_{i,j}(n) + \sum_{\alpha=0}^{s-1} [k_{i,j}(n-\alpha) - k_{i,j}(n)] \right\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}.$$

But $|k_{i,j}(n-\alpha) - k_{i,j}(n)| < 2\alpha$ so that this implies

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} s^2 \{k_{i,j}(n)\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

or

$$\overline{\lim}_{n \rightarrow \infty} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \leq \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}.$$

From this we have

$$\overline{\lim}_{n \rightarrow \infty} \frac{\{k_{i,j}(n)\}^2}{n^2} = \overline{\lim}_{n \rightarrow \infty} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \leq \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}$$

for any fixed $s \equiv 0 \pmod{v}$. Since the right member can be made arbitrarily small, we have

$$\lim_{n \rightarrow \infty} \frac{|k_{i,j}(n)|}{n} = 0$$

or

$$\lim_{n \rightarrow \infty} \frac{k_i(n)}{n} = \lim_{n \rightarrow \infty} \frac{k_j(n)}{n}.$$

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