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**ON THE DEFINITION OF NORMAL NUMBERS**

IVAN NIVEN AND H. ZUCKERMAN

# ON THE DEFINITION OF NORMAL NUMBERS

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**1. Introduction.** Let  $R$  be a real number with fractional part  $.x_1x_2x_3 \cdots$  when written to scale  $r$ . Let  $N(b, n)$  denote the number of occurrences of the digit  $b$  in the first  $n$  places. The number  $R$  is said to be *simply normal* to scale  $r$  if

$$(1) \quad \lim_{n \rightarrow \infty} \frac{N(b, n)}{n} = \frac{1}{r}$$

for each of the  $r$  possible values of  $b$ ;  $R$  is said to be *normal* to scale  $r$  if all the numbers  $R, rR, r^2R, \cdots$  are simply normal to all the scales  $r, r^2, r^3, \cdots$ . These definitions, for  $r = 10$ , were introduced by Émile Borel [1], who stated (p.261) that "la propriété caractéristique" of a normal number is the following: that for any sequence  $B$  whatsoever of  $v$  specified digits, we have

$$(2) \quad \lim_{n \rightarrow \infty} \frac{N(B, n)}{n} = \frac{1}{r^v},$$

where  $N(B, n)$  stands for the number of occurrences of the sequence  $B$  in the first  $n$  decimal places.

Several writers, for example Champernowne [2], Koksma [3, p.116], and Copeland and Erdős [4], have taken this property (2) as the definition of a normal number. Hardy and Wright [5, p.124] state that property (2) is equivalent to the definition, but give no proof. It is easy to show that a normal number has property (2), but the implication in the other direction does not appear to be so obvious. If the number  $R$  has property (2) then any sequence of digits

$$B = b_1b_2 \cdots b_v$$

appears with the appropriate frequency, but will the frequencies all be the same for  $i = 1, 2, \cdots, v$  if we count only those occurrences of  $B$  such that  $b_1$  is an  $i, i + v, i + 2v, \cdots$ -th digit? It is the purpose of this note to show that this is

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so, and thus to prove the equivalence of property (2) and the definition of normal number.

**2. Notation.** In addition to the notation already introduced, we shall use the following:

$S_\alpha$  is the first  $\alpha$  digits of  $R$ .

$BXB$  is the totality of sequences of the form  $b_1 b_2 \cdots b_v xx \cdots x b_1 b_2 \cdots b_v$ , where  $xx \cdots x$  is any sequence of  $t$  digits.

$k_i(\alpha)$  is the number of times that  $B$  occurs in  $S_\alpha$  with  $b_1$  in a place congruent to  $i \pmod v$ .

$$g(\alpha) = \sum_{i=0}^{v-1} k_i(\alpha).$$

$\theta_t(\alpha)$  is the number of occurrences of  $BXB$  in  $S_\alpha$ .

$$k_{i,j}(\alpha) = k_i(\alpha) - k_j(\alpha), \quad i \neq j.$$

$B^*$  is any block of digits of length from  $v + 1$  to  $2v - 1$  whose first  $v$  digits are  $B$  and whose last  $v$  digits are  $B$ . Such a block need not exist.

**3. Proof.** We shall assume that the number  $R$  has the property (2), so that we have

$$(3) \quad \lim_{n \rightarrow \infty} \frac{g(n)}{n} = \frac{1}{r^v}$$

and

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\theta_t(n)}{n} = \frac{1}{r^{2v}}$$

for each fixed  $t$ , and we prove that

$$(5) \quad \lim_{n \rightarrow \infty} \frac{k_{i,j}(n)}{n} = 0,$$

from which it follows that  $R$  is a normal number.

Now  $k_i(\alpha + s) - k_i(\alpha)$  is the number of  $B$  with  $b_1$  in a place congruent to  $i \pmod v$  that are in  $S_{\alpha+s}$  but not entirely in  $S_\alpha$ . Therefore

$$\sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \{k_i(\alpha + s) - k_i(\alpha)\} \{k_j(\alpha + s) - k_j(\alpha)\}$$

counts the number of  $BXB$  and  $B^*$  that occur in  $S_{\alpha+s}$  such that the first  $B$  is not contained entirely in  $S_\alpha$ . Here the number  $t$  of digits in  $X$  runs through all values  $\not\equiv 0 \pmod v$  with  $0 \leq t \leq s - v - 1$ . We take  $n > s$  and sum the above expression to get

$$(6) \quad \sigma = \sum_{\alpha=0}^{n-s} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \{k_i(\alpha + s) - k_i(\alpha)\} \{k_j(\alpha + s) - k_j(\alpha)\}.$$

Considering  $S_n$  and any  $BXB$  contained in it with  $t \leq s - v - 1$ , we see that  $BXB$  is counted in  $\sigma$  a certain number of times. In fact if  $BXB$  is not too near either end of  $S_n$  it is counted just  $s - t - v$  times and it is never counted more than this many times. Furthermore if  $BXB$  is preceded by at least  $s - t - 2v$  digits and is followed in  $S_n$  by at least  $s - t - v - 1$  digits then  $BXB$  is counted exactly  $s - t - v$  times. Therefore we have, ignoring any  $B^*$  blocks which may be counted by  $\sigma$ ,

$$(7) \quad \sigma \geq \sum_{\substack{t=0 \\ t \not\equiv 0 \pmod v}}^{s-v-1} (s - t - v) \{ \theta_t(n - s) - \theta_t(s) \}.$$

Using (4) we find

$$\lim_{n \rightarrow \infty} \frac{\theta_t(n - s)}{n} = \frac{1}{r^{2v}}$$

for any fixed  $s$ ; hence, from (7), we have

$$\lim_{n \rightarrow \infty} \frac{\sigma}{n} \geq \sum_{\substack{t=0 \\ t \not\equiv 0 \pmod v}}^{s-v-1} (s - t - v) \frac{1}{r^{2v}}.$$

It is now convenient to take  $s$ , which is otherwise arbitrary, to be congruent to

$0 \pmod{v}$ . Then the above formula reduces to

$$(8) \quad \lim_{n \rightarrow \infty} \frac{\sigma}{n} \geq \frac{(v-1)(s-v)^2}{2v} \cdot \frac{1}{r^{2v}}.$$

In a similar manner we count the  $BXB$  in  $S_n$  where the number  $t$  of digits of  $X$  is congruent to  $0 \pmod{v}$ . This gives us

$$(9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \frac{1}{2} \{k_i(\alpha+s) - k_i(\alpha)\} \{k_i(\alpha+s) - k_i(\alpha) - 1\} \\ = \sum_{\substack{t=0 \\ t \neq 0 \pmod{v}}}^{s-v-1} (s-t-v) \frac{1}{r^{2v}} = \frac{s(s-v)}{2v} \cdot \frac{1}{r^{2v}}.$$

Now, by (3) we have

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\} = \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{\alpha=0}^{n-s} \{g(\alpha+s) - g(\alpha)\} \\ = \lim_{n \rightarrow \infty} \left\{ \frac{1}{2n} \sum_{\alpha=n-s+1}^n g(\alpha+s) - \frac{1}{2n} \sum_{\alpha=0}^{s-1} g(\alpha) \right\} = \frac{s}{2r^v},$$

and (9) reduces to

$$(10) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{i=0}^{v-1} \{k_i(\alpha+s) - k_i(\alpha)\}^2 = \frac{s}{r^v} + \frac{s(s-v)}{vr^{2v}}.$$

From (6), (8), and (10) we find that

$$(11) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\alpha=0}^{n-s} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \{[k_i(\alpha+s) - k_i(\alpha)] - [k_j(\alpha+s) - k_j(\alpha)]\}^2 \\ \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

for any fixed  $s \equiv 0 \pmod{v}$ . Using the inequality

$$\sum_{i=1}^n x_i^2 \geq \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2$$

we obtain

$$\begin{aligned} & \sum_{\alpha=0}^{n-s} \{ [k_i(\alpha + s) - k_i(\alpha)] - [k_j(\alpha + s) - k_j(\alpha)] \}^2 \\ & \geq \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{n-s} [k_i(\alpha + s) - k_i(\alpha) - k_j(\alpha + s) + k_j(\alpha)] \right\}^2 \\ & = \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{n-s} [k_{i,j}(\alpha + s) - k_{i,j}(\alpha)] \right\}^2 \\ & = \frac{1}{n-s+1} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) - \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2 . \end{aligned}$$

This with (11) implies

$$\begin{aligned} (12) \quad & \overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) - \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2 \\ & \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}} . \end{aligned}$$

From the definition we have  $|k_{i,j}(\alpha)| < \alpha$  and hence

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) \right\}^2 = 0$$

and

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) \sum_{\alpha=0}^{s-1} k_{i,j}(\alpha) = 0$$

for fixed  $s$ .

Therefore (12) implies

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ \sum_{\alpha=0}^{s-1} k_{i,j}(n-\alpha) \right\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}},$$

which can be written in the form

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \left\{ s k_{i,j}(n) + \sum_{\alpha=0}^{s-1} [k_{i,j}(n-\alpha) - k_{i,j}(n)] \right\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}.$$

But  $|k_{i,j}(n-\alpha) - k_{i,j}(n)| < 2\alpha$  so that this implies

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n(n-s+1)} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} s^2 \{k_{i,j}(n)\}^2 \leq \frac{(v-1)s}{r^v} + \frac{(v-1)(s-v)}{r^{2v}}$$

or

$$\overline{\lim}_{n \rightarrow \infty} \sum_{\substack{i < j \\ i=0, 1, \dots, v-2 \\ j=1, 2, \dots, v-1}} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \leq \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}.$$

From this we have

$$\overline{\lim}_{n \rightarrow \infty} \frac{\{k_{i,j}(n)\}^2}{n^2} = \overline{\lim}_{n \rightarrow \infty} \frac{\{k_{i,j}(n)\}^2}{n(n-s+1)} \leq \frac{v-1}{sr^v} + \frac{(v-1)(s-v)}{s^2 r^{2v}}$$

for any fixed  $s \equiv 0 \pmod{v}$ . Since the right member can be made arbitrarily small, we have

$$\lim_{n \rightarrow \infty} \frac{|k_{i,j}(n)|}{n} = 0$$

or

$$\lim_{n \rightarrow \infty} \frac{k_i(n)}{n} = \lim_{n \rightarrow \infty} \frac{k_j(n)}{n} .$$

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Ralph Palmer Agnew, <i>Ratio tests for convergence of series</i> .....	1
Richard Arens and James Dugundji, <i>Topologies for function spaces</i> .....	5
B. Arnold, <i>Distributive lattices with a third operation defined</i> .....	33
R. Bing, <i>Concerning hereditarily indecomposable continua</i> .....	43
David Dekker, <i>Generalizations of hypergeodesics</i> .....	53
A. Dvoretzky, A. Wald and J. Wolfowitz, <i>Relations among certain ranges of vector measures</i> .....	59
Paul Erdős, F. Herzog and G. Pirani, <i>Schlicht Taylor series whose convergence on the unit circle is uniform but not absolute</i> .....	75
Whilhelm Fischer, <i>On Dedekind's function <math>\eta(\tau)</math></i> .....	83
Werner Leutert, <i>The heavy sphere supported by a concentrated force</i> .....	97
Ivan Niven and H. Zuckerman, <i>On the definition of normal numbers</i> .....	103
L. Paige, <i>Complete mappings of finite groups</i> .....	111
Otto Szász, <i>On a Tauberian theorem for Abel summability</i> .....	117
Olga Taussky, <i>Classes of matrices and quadratic fields</i> .....	127
F. Tricomi and A. Erdélyi, <i>The asymptotic expansion of a ratio of gamma functions</i> .....	133
Hassler Whitney, <i>On totally differentiable and smooth functions</i> .....	143