ON A TAUBERIAN THEOREM FOR ABEL SUMMABILITY

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1. Introduction. In 1928 the author proved the following theorem [2, Section 2]:

**Theorem A.** If \( p > 1 \) and

\[
\sum_{\nu=1}^{n} \nu^p |a_{\nu}|^p = O(n), \quad n \to \infty,
\]

then Abel summability of the series \( \sum_{n=0}^{\infty} a_n \) to \( s \) implies its convergence to \( s \).

The theorem is the more general the smaller \( p \) is; it does not hold for \( p = 1 \) [2, Section 1; 1, pp.119,122]. However, for this case Rényi proved the following theorem:

**Theorem B.** If

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} \nu |a_{\nu}| = l < \infty
\]

exists, then Abel summability of \( \sum_{n=0}^{\infty} a_n \) to \( s \) implies convergence of the series to \( s \).

2. Generalization. We give a simpler proof and at the same time a slight generalization of Theorem B.

**Theorem 1.** Assume that

\[
V_n = \sum_{\nu=1}^{n} \nu |a_{\nu}| = O(n),
\]

and that

\[
\frac{1}{m} V_m - \frac{1}{n} V_n \to 0,
\]

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for every sequence \( m = m_n \), such that \( m_n/n \to 1 \) as \( n \to \infty \). Then Abel summability to \( s \) of \( \sum_{n=0}^{\infty} a_n \) implies its convergence to \( s \).

Property (2.2) is called slow oscillation of the sequence \( V_n/n \).

Proof of Theorem 1. We write

\[
\sum_{\nu=0}^{n} a_\nu = s_n , \quad \sum_{\nu=0}^{n} s_\nu = (n + 1) \sigma_n .
\]

It is easy to verify that, for \( k = 0, 1, 2, \ldots \), we have

(2.3) \[ s_{n-1} - \sigma_{n+k} = \frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) - \frac{1}{k+1} \sum_{\nu=0}^{k} (k + 1 - \nu) a_{n+\nu} . \]

It is known [see 2, Section 2] that if for a finite \( s \) we have

\[
\lim_{x \to 1} \sum_{n=0}^{\infty} a_n x^n = s ,
\]

then (2.1) implies \( \sigma_n \to s \); thus, if

(2.4) \[ \text{l.u.b.} \quad |\sigma_{n-1} - \sigma_{n+k}| = \epsilon_n , \]

then \( \epsilon_n \to 0 \).

We now choose

(2.5) \[ k = k_n = \left\lfloor n \epsilon_n^{1/2} \right\rfloor , \quad \text{so that} \quad k \leq n \epsilon_n^{1/2} < k + 1 ; \]

it follows, in view of (2.4), that

\[
\frac{n}{k+1} |\sigma_{n-1} - \sigma_{n+k}| < \epsilon_n^{1/2} .
\]

In view of (2.3) our theorem will be proved if we show that

\[
\frac{1}{k+1} \sum_{\nu=0}^{k} (k + 1 - \nu) a_{n+\nu} \to 0 , \quad n \to \infty .
\]
Now

\[
\frac{1}{k+1} \left| \sum_{\nu=0}^{k} (k+1-\nu) a_{n+\nu} \right|
\]

\[
\leq \frac{1}{k+1} \sum_{\nu=0}^{k} (n+\nu) \left| a_{n+\nu} \right| \frac{k+1-\nu}{n+\nu} \leq \frac{1}{n} (V_{n+k} - V_{n-1}) ,
\]

and

\[
(2.6) \quad \frac{1}{n} (V_{n+k} - V_{n-1}) = \frac{V_{n+k}}{n+k} \cdot \frac{n+k}{n} - \frac{V_{n-1}}{n-1} \cdot \frac{n-1}{n}
\]

\[
= \frac{V_{n+k}}{n+k} \cdot \frac{V_{n-1}}{n-1} + \frac{k}{n} \frac{V_{n+k}}{n+k} + \frac{1}{n} \frac{V_{n-1}}{n-1} ;
\]

using (2.2) and (2.5), we see that

\[
(2.7) \quad \frac{1}{n} (V_{n+k} - V_{n-1}) \rightarrow 0 \quad \text{as} \quad \frac{k}{n} \rightarrow 0 \quad \text{and} \quad n \rightarrow \infty ,
\]

and thus Theorem 1 is proved.

Rényi observed that the Theorems A and B are overlapping. We now show that Theorem 1 includes not only Theorem B, but also Theorem A. Clearly (2.1) follows from (1.1) by Hölder's inequality. Furthermore,

\[
V_{n+k} - V_n = \sum_{\nu=n+1}^{n+k} \nu |a_{\nu}| \leq k^{(p-1)/p} \left( \sum_{\nu=n+1}^{n+k} \nu^p |a_{\nu}|^p \right)^{1/p}
\]

\[
= k^{(p-1)/p} O \left[ (n+k)^{1/p} \right] ;
\]

hence,

\[
\frac{1}{n} (V_{n+k} - V_n) = \frac{k}{n} O \left( \left( \frac{n}{k} \right)^{1/p} \right) = O \left( \left( \frac{k^{(p-1)/p}}{n} \right) \right) \rightarrow 0 \quad \text{as} \quad \frac{k}{n} \rightarrow 0 .
\]

It now follows from (2.6) that (2.2) holds; thus (1.1) implies (2.1) and (2.2), which proves our assertion.

An example of a sequence $V_n > 0$, and increasing, for which (2.2) holds,
while \( n^{-1} V_n \uparrow \infty \), is
\[ V_n = n \log n, \quad n \geq 2, \]
because
\[ \frac{V_{n+k}}{n+k} - \frac{V_n}{n} = \log \left(1 + \frac{k}{n}\right) \to 0, \quad \text{as} \quad \frac{k}{n} \to 0, \quad n \to \infty. \]

3. A more general result. A generalization of Theorem A is the following [see 5, p.56]:

**Theorem A'.** If for some \( p > 1 \), we have
\[ (3.1) \quad \sum_{\nu=1}^{n} \nu^p (|a_\nu| - a_\nu)^p = O(n), \quad n \to \infty, \]
then the Abel summability of \( \sum_{n=1}^{\infty} a_n \) implies its convergence to the same value.

An analogue to Theorem 1 is the theorem:

**Theorem 2.** Assume that
\[ (3.2) \quad U_n = \sum_{\nu=1}^{n} \nu (|a_\nu| - a_\nu) = O(n), \]
and that
\[ (3.3) \quad \frac{1}{m} U_m - \frac{1}{n} U_n \to 0 \quad \text{as} \quad \frac{m}{n} \to 1, \quad n \to \infty. \]

If now \( \sum_{n=1}^{\infty} a_n \) is Abel summable to \( s \), then it converges to \( s \).

**Proof of Theorem 2.** We have
\[ - \sum_{\nu=1}^{n} \nu a_\nu \leq \sum_{\nu=1}^{n} \nu (|a_\nu| - a_\nu) = O(n); \]
hence [see 5, the Lemma on p.52] Abel summability of \( \sum_{n=0}^{\infty} a_n \) implies its summability \((C,1)\). From (2.3) we have
from (2.4) and (2.5) we obtain
\[
\frac{n}{k + 1} (\sigma_{n+k} - \sigma_{n-1}) < \epsilon_n^{1/2}.
\]
Using the same argument as in the proof of Theorem 1, replacing \( V_n \) by \( U_n \), we find that
\[
(3.4) \quad \limsup_{n \to \infty} s_n \leq s.
\]
We next employ the identity, similar to (2.3),
\[
\begin{align*}
s_n - \sigma_{n-k-1} & = \frac{n + 1}{k + 1} (\sigma_n - \sigma_{n-k-1}) \\
& + \frac{1}{k + 1} \sum_{\nu=0}^{k} (k - \nu) a_{n-\nu}, \quad k = 0, 1, 2, \ldots,
\end{align*}
\]
and the inequality
\[
a_\nu \geq a_\nu - |a_\nu|.
\]
The same reasoning as before now yields
\[
(3.5) \quad \liminf_{n \to \infty} s_n \geq s.
\]
Finally (3.4) and (3.5) prove Theorem 2.

It is clear from the proof that condition (3.3) can be replaced by
\[
\frac{1}{n} (U_m - U_n) \to 0, \quad \text{as} \quad \frac{m}{n} \to 1, \quad n \to \infty.
\]

4. An equivalent result. A glance at the proof of Theorem 1 shows that the following lemma holds:
Lemma 1. If $V_n$ is positive and monotone increasing, and if
\begin{equation}
V_n = O(n), \quad \text{as } n \to \infty,
\end{equation}
and (2.2) holds, then
\begin{equation}
\frac{1}{n} (V_m - V_n) \to 0, \quad \text{as } \frac{m}{n} \to 1, \quad n \to \infty.
\end{equation}

We now prove the inverse:

Lemma 2. If $V_n > 0$, and increasing, and if (4.2) holds, then (4.1) and (2.2) hold.

Proof. We write

\begin{align*}
V_n = n \omega_n, \quad \omega_n \geq 0,
\end{align*}
and

\begin{equation}
\frac{1}{n} (V_m - V_n) = \omega_m - \omega_n + \left( \frac{m}{n} - 1 \right) \omega_m.
\end{equation}

Let

\begin{align*}
\max_{\nu \leq n} \omega_\nu = \rho_n;
\end{align*}

then $\rho_n \uparrow \rho \leq \infty$. If $\rho < \infty$, then $V_n = O(n)$. Suppose now that $\rho = \infty$; then there are infinitely many indices $m = m_\nu$, so that $\omega_m = \rho_m$ for $m = m_\nu, \nu = 1, 2, 3, \ldots$.

For these $m$ and for $n < m$, from (4.3) we get
\begin{equation}
\frac{1}{n} (V_m - V_n) > \left( \frac{m}{n} - 1 \right) \rho_m.
\end{equation}

We now choose

\begin{align*}
n &= \frac{m \rho_m^{1/2}}{1 + \rho_m^{1/2}} < m,
\end{align*}

so that

\begin{align*}
\frac{m}{n} &= \frac{1 + \rho_m^{1/2}}{\rho_m^{1/2}} \to 1;
\end{align*}
then, using (4.4), we have

$$\frac{1}{n} (V_m - V_n) > \gamma_n^{1/2} \to \infty,$$

in contradiction to the assumption (4.2). It follows that (4.1) holds; finally (2.2) follows from (4.1), (4.2), and (4.3). This proves Lemma 2.

We now prove the following theorem:

**Theorem 3.** Let $U_n = \sum_{\nu=1}^n \nu(|a_\nu| - a_\nu)$; if

$$\frac{1}{n} (U_m - U_n) \to 0, \quad \text{as} \quad \frac{m}{n} \to 1, \quad n \to \infty,$$

and if $\sum_{n=0}^\infty a_n$ is Abel summable, then $\sum_{n=0}^\infty a_n$ is convergent to the same value.

**Proof of Theorem 3.** In view of Lemma 2, Theorem 3 includes Theorem 2; it also includes Theorem 1, because of Lemma 2, and of the inequality

$$U_m - U_n \leq 2 (V_m - V_n), \quad m > n.$$ 

Conversely, by Lemma 2, (4.5) implies (3.2) and (3.3), so that Theorem 3 is equivalent to Theorem 2, and is thus valid.

To show that Theorem 1 is actually more general than Theorem B we give an example of a sequence $\omega_n$ so that $n \omega_n$ is increasing, $\omega_n$ is slowly oscillating and $\omega_n = O(1)$, but $\lim \omega_n$ does not exist. Let

$$\omega_n = \sum_{\nu=1}^n \nu^{-1} \varepsilon_\nu, \quad \text{where} \quad \varepsilon_\nu = \pm 1;$$

choose $\varepsilon_\nu = +1$ as long as $\omega_n \leq 3; \nu = 1, 2, \cdots, n_1, \text{say}$. Choose $\varepsilon_\nu = -1$ as long as $\omega_n \geq 2; \nu = 1 + n_1, 2 + n_1, \cdots, n_2, \text{say};$ and so on. It is clear that $\omega_n = O(1)$, and that $\lim \omega_n$ does not exist. Furthermore, for $n \leq n_1, \omega_n \uparrow,$ for $n_1 < n \leq n_2, \omega_n \downarrow,$ and so on. Now

$$(n + 1) \omega_{n+1} - n \omega_n = n(\omega_{n+1} - \omega_n) + \omega_{n+1} \geq \frac{3}{2} - 1 = \frac{1}{2},$$

hence $n \omega_n \uparrow$. Finally

$$|\omega_m - \omega_n| \leq \sum_{\nu=n+1}^m \frac{1}{\nu} \leq \frac{m - n}{n} \to 0, \quad \text{for} \quad \frac{m}{n} \to 1.$$
hence $\omega_n$ is slowly oscillating.

5. Another equivalent result. We first establish the following lemma.

Lemma 3. Suppose that $U_n \geq 0$ and increasing, with $U_0 = 0$, and let

\begin{equation}
(5.1) \quad b_n = \frac{1}{n} (U_n - U_{n-1}), \quad n \geq 1, \quad b_0 = 0;
\end{equation}

\begin{equation}
(5.2) \quad B_n = \sum_{\nu=0}^{n} b_{\nu}, \quad n \geq 0.
\end{equation}

Then whenever $k = k(n)$ is so chosen that $k/n \to 0$, as $n \to \infty$, the two statements

\begin{equation}
(5.3) \quad \frac{1}{n} (U_{n+k} - U_n) \to 0
\end{equation}

and

\begin{equation}
(5.4) \quad B_{n+k} - B_n \to 0
\end{equation}

are equivalent.

Proof. From (5.1) we have

\[ U_n = \sum_{\nu=0}^{n} \nu b_{\nu}, \quad U_{n+k} - U_n = \sum_{\nu=n+1}^{n+k} \nu b_{\nu}. \]

Now

\[ B_{n+k} - B_n = \sum_{\nu=n+1}^{n+k} b_{\nu} \leq \frac{1}{n} \sum_{\nu=n+1}^{n+k} \nu b_{\nu} = \frac{1}{n} (U_{n+k} - U_n); \]

thus (5.3) implies (5.4). Furthermore,

\[ B_{n+k} - B_n \geq \frac{1}{n+k} (U_{n+k} - U_n); \]

hence (5.4) implies (5.3). This proves the lemma.

We note that
\( B_n = \frac{1}{n} U_n + \sum_{\nu=1}^{n-1} \frac{1}{\nu (\nu + 1)} U_\nu \),

and

\( U_n = nB_n - \sum_{\nu=0}^{n-1} B_\nu \).

It is an immediate consequence of Lemma 3 that Theorem 3 is equivalent to the following theorem (for a direct proof see [4, Theorem IV]).

**Theorem 4.** If

\[
\sum_{\nu=n+1}^{n+k} (|a_\nu| - a_\nu) \to 0, \quad \text{as} \quad \frac{k}{n} \to 0, \quad n \to \infty,
\]

then Abel summability of \( \sum_{n=0}^{\infty} a_n \) implies convergence of the series to the same value.

A generalization of this theorem to Dirichlet series and to Laplace integrals, on different lines, is given in [3].

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