

Pacific Journal of Mathematics

ON A TAUBERIAN THEOREM FOR ABEL SUMMABILITY

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1. Introduction. In 1928 the author proved the following theorem [2, Section 2]:

THEOREM A. *If $p > 1$ and*

$$(1.1) \quad \sum_{\nu=1}^n \nu^p |a_\nu|^p = O(n), \quad n \rightarrow \infty,$$

then Abel summability of the series $\sum_{n=0}^{\infty} a_n$ to s implies its convergence to s .

The theorem is the more general the smaller p is; it does not hold for $p = 1$ [2, Section 1; 1, pp.119,122]. However, for this case Rényi proved the following theorem:

THEOREM B. *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\nu=1}^n \nu |a_\nu| = l < \infty$$

exists, then Abel summability of $\sum_{n=0}^{\infty} a_n$ to s implies convergence of the series to s .

2. Generalization. We give a simpler proof and at the same time a slight generalization of Theorem B.

THEOREM 1. *Assume that*

$$(2.1) \quad V_n = \sum_{\nu=1}^n \nu |a_\nu| = O(n),$$

and that

$$(2.2) \quad \frac{1}{m} V_m - \frac{1}{n} V_n \rightarrow 0,$$

Received April 10, 1950. The preparation of this paper was sponsored (in part) by the Office of Naval Research.

Pacific J. Math. 1 (1951), 117-125.

for every sequence $m = m_n$, such that $m_n/n \rightarrow 1$ as $n \rightarrow \infty$. Then Abel summability to s of $\sum_{n=0}^{\infty} a_n$ implies its convergence to s .

Property (2.2) is called *slow oscillation* of the sequence V_n/n .

Proof of Theorem 1. We write

$$\sum_{\nu=0}^n a_{\nu} = s_n, \quad \sum_{\nu=0}^n s_{\nu} = (n+1) \sigma_n.$$

It is easy to verify that, for $k = 0, 1, 2, \dots$, we have

$$(2.3) \quad s_{n-1} - \sigma_{n+k} = \frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) - \frac{1}{k+1} \sum_{\nu=1}^k (k+1-\nu) a_{n+\nu}.$$

It is known [see 2, Section 2] that if for a finite s we have

$$\lim_{x \rightarrow 1} \sum_{n=0}^{\infty} a_n x^n = s,$$

then (2.1) implies $\sigma_n \rightarrow s$; thus, if

$$(2.4) \quad \text{l.u.b.}_{k \geq 0} |\sigma_{n-1} - \sigma_{n+k}| = \epsilon_n,$$

then $\epsilon_n \rightarrow 0$.

We now choose

$$(2.5) \quad k = k_n = [n \epsilon_n^{1/2}], \quad \text{so that} \quad k \leq n \epsilon_n^{1/2} < k+1;$$

it follows, in view of (2.4), that

$$\frac{n}{k+1} |\sigma_{n-1} - \sigma_{n+k}| < \epsilon_n^{1/2}.$$

In view of (2.3) our theorem will be proved if we show that

$$\frac{1}{k+1} \sum_{\nu=0}^k (k+1-\nu) a_{n+\nu} \rightarrow 0, \quad n \rightarrow \infty.$$

Now

$$\begin{aligned} & \frac{1}{k+1} \left| \sum_{\nu=0}^k (k+1-\nu) a_{n+\nu} \right| \\ & \leq \frac{1}{k+1} \sum_{\nu=0}^k (n+\nu) |a_{n+\nu}| \frac{k+1-\nu}{n+\nu} \leq \frac{1}{n} (V_{n+k} - V_{n-1}), \end{aligned}$$

and

$$\begin{aligned} (2.6) \quad \frac{1}{n} (V_{n+k} - V_{n-1}) &= \frac{V_{n+k}}{n+k} \cdot \frac{n+k}{n} - \frac{V_{n-1}}{n-1} \cdot \frac{n-1}{n} \\ &= \frac{V_{n+k}}{n+k} - \frac{V_{n-1}}{n-1} + \frac{k}{n} \frac{V_{n+k}}{n+k} + \frac{1}{n} \frac{V_{n-1}}{n-1}; \end{aligned}$$

using (2.2) and (2.5), we see that

$$(2.7) \quad \frac{1}{n} (V_{n+k} - V_{n-1}) \rightarrow 0 \quad \text{as } \frac{k}{n} \rightarrow 0 \quad \text{and } n \rightarrow \infty,$$

and thus Theorem 1 is proved.

Rényi observed that the Theorems A and B are overlapping. We now show that Theorem 1 includes not only Theorem B, but also Theorem A. Clearly (2.1) follows from (1.1) by Hölder's inequality. Furthermore,

$$\begin{aligned} V_{n+k} - V_n &= \sum_{\nu=n+1}^{n+k} \nu |a_\nu| \leq k^{(p-1)/p} \left(\sum_{\nu=n+1}^{n+k} \nu^p |a_\nu|^p \right)^{1/p} \\ &= k^{(p-1)/p} O[(n+k)^{1/p}]; \end{aligned}$$

hence,

$$\frac{1}{n} (V_{n+k} - V_n) = \frac{k}{n} O\left[\left(\frac{n}{k}\right)^{1/p}\right] = O\left[\left(\frac{k}{n}\right)^{(p-1)/p}\right] \rightarrow 0 \quad \text{as } \frac{k}{n} \rightarrow 0.$$

It now follows from (2.6) that (2.2) holds; thus (1.1) implies (2.1) and (2.2), which proves our assertion.

An example of a sequence $V_n > 0$, and increasing, for which (2.2) holds,

while $n^{-1}V_n \uparrow \infty$, is

$$V_n = n \log n, \quad n \geq 2,$$

because

$$\frac{V_{n+k}}{n+k} - \frac{V_n}{n} = \log \left(1 + \frac{k}{n} \right) \rightarrow 0, \quad \text{as } \frac{k}{n} \rightarrow 0, \quad n \rightarrow \infty.$$

3. A more general result. A generalization of Theorem A is the following [see 5, p.56]:

THEOREM A'. *If for some $p > 1$, we have*

$$(3.1) \quad \sum_{\nu=1}^n \nu^p (|a_\nu| - a_\nu)^p = O(n), \quad n \rightarrow \infty,$$

then the Abel summability of $\sum_{n=0}^{\infty} a_n$ implies its convergence to the same value.

An analogue to Theorem 1 is the theorem:

THEOREM 2. *Assume that*

$$(3.2) \quad U_n = \sum_{\nu=1}^n \nu (|a_\nu| - a_\nu) = O(n),$$

and that

$$(3.3) \quad \frac{1}{m} U_m - \frac{1}{n} U_n \rightarrow 0 \quad \text{as } \frac{m}{n} \rightarrow 1, \quad n \rightarrow \infty.$$

If now $\sum_{n=0}^{\infty} a_n$ is Abel summable to s , then it converges to s .

Proof of Theorem 2. We have

$$- \sum_{\nu=1}^n \nu a_\nu \leq \sum_{\nu=1}^n \nu (|a_\nu| - a_\nu) = O(n);$$

hence [see 5, the Lemma on p.52] Abel summability of $\sum_{n=0}^{\infty} a_n$ implies its summability $(C, 1)$. From (2.3) we have

$$s_{n-1} - \sigma_{n+k} \leq \frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) \\ + \frac{1}{k+1} \sum_{\nu=0}^k (k+1-\nu) (|a_{n+\nu}| - a_{n+\nu});$$

from (2.4) and (2.5) we obtain

$$\frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) < \epsilon_n^{1/2}.$$

Using the same argument as in the proof of Theorem 1, replacing V_n by U_n , we find that

$$(3.4) \quad \limsup_{n \rightarrow \infty} s_n \leq s.$$

We next employ the identity, similar to (2.3),

$$s_n - \sigma_{n-k-1} = \frac{n+1}{k+1} (\sigma_n - \sigma_{n-k-1}) \\ + \frac{1}{k+1} \sum_{\nu=0}^k (k-\nu) a_{n-\nu}, \quad k = 0, 1, 2, \dots,$$

and the inequality

$$a_\nu \geq a_\nu - |a_\nu|.$$

The same reasoning as before now yields

$$(3.5) \quad \liminf_{n \rightarrow \infty} s_n \geq s.$$

Finally (3.4) and (3.5) prove Theorem 2.

It is clear from the proof that condition (3.3) can be replaced by

$$\frac{1}{n} (U_m - U_n) \rightarrow 0, \quad \text{as } \frac{m}{n} \rightarrow 1, \quad n \rightarrow \infty.$$

4. An equivalent result. A glance at the proof of Theorem 1 shows that the following lemma holds:

LEMMA 1. If V_n is positive and monotone increasing, and if

$$(4.1) \quad V_n = O(n), \quad \text{as } n \rightarrow \infty,$$

and (2.2) holds, then

$$(4.2) \quad \frac{1}{n} (V_m - V_n) \rightarrow 0, \quad \text{as } \frac{m}{n} \rightarrow 1, \quad n \rightarrow \infty.$$

We now prove the inverse:

LEMMA 2. If $V_n > 0$, and increasing, and if (4.2) holds, then (4.1) and (2.2) hold.

Proof. We write

$$V_n = n \omega_n, \quad \omega_n \geq 0,$$

and

$$(4.3) \quad \frac{1}{n} (V_m - V_n) = \omega_m - \omega_n + \left(\frac{m}{n} - 1\right) \omega_m.$$

Let

$$\max_{\nu \leq n} \omega_\nu = \rho_n;$$

then $\rho_n \uparrow \rho \leq \infty$. If $\rho < \infty$, then $V_n = O(n)$. Suppose now that $\rho = \infty$; then there are infinitely many indices $m = m_\nu$, so that $\omega_m = \rho_m$ for $m = m_\nu, \nu = 1, 2, 3, \dots$. For these m and for $n < m$, from (4.3) we get

$$(4.4) \quad \frac{1}{n} (V_m - V_n) > \left(\frac{m}{n} - 1\right) \rho_m.$$

We now choose

$$n = \frac{m \rho_m^{1/2}}{1 + \rho_m^{1/2}} < m,$$

so that

$$\frac{m}{n} = \frac{1 + \rho_m^{1/2}}{\rho_m^{1/2}} \rightarrow 1;$$

then, using (4.4), we have

$$\frac{1}{n} (V_m - V_n) > \rho_m^{1/2} \rightarrow \infty,$$

in contradiction to the assumption (4.2). It follows that (4.1) holds; finally (2.2) follows from (4.1), (4.2), and (4.3). This proves Lemma 2.

We now prove the following theorem:

THEOREM 3. *Let $U_n = \sum_{\nu=1}^n \nu(|a_\nu| - a_\nu)$; if*

$$(4.5) \quad \frac{1}{n} (U_m - U_n) \rightarrow 0, \quad \text{as } \frac{m}{n} \rightarrow 1, \quad n \rightarrow \infty,$$

and if $\sum_{n=0}^\infty a_n$ is Abel summable, then $\sum_{n=0}^\infty a_n$ is convergent to the same value.

Proof of Theorem 3. In view of Lemma 2, Theorem 3 includes Theorem 2; it also includes Theorem 1, because of Lemma 2, and of the inequality

$$U_m - U_n \leq 2(V_m - V_n), \quad m > n.$$

Conversely, by Lemma 2, (4.5) implies (3.2) and (3.3), so that Theorem 3 is equivalent to Theorem 2, and is thus valid.

To show that Theorem 1 is actually more general than Theorem B we give an example of a sequence ω_n so that $n\omega_n$ is increasing, ω_n is slowly oscillating and $\omega_n = O(1)$, but $\lim \omega_n$ does not exist. Let

$$\omega_n = \sum_{\nu=1}^n \nu^{-1} \epsilon_\nu, \quad \text{where } \epsilon_\nu = \pm 1;$$

choose $\epsilon_\nu = +1$ as long as $\omega_n \leq 3$; $\nu = 1, 2, \dots, n_1$, say. Choose $\epsilon_\nu = -1$ as long as $\omega_n \geq 2$; $\nu = 1 + n_1, 2 + n_1, \dots, n_2$, say; and so on. It is clear that $\omega_n = O(1)$, and that $\lim \omega_n$ does not exist. Furthermore, for $n \leq n_1$, $\omega_n \uparrow$, for $n_1 < n \leq n_2$, $\omega_n \downarrow$, and so on. Now

$$(n + 1) \omega_{n+1} - n \omega_n = n(\omega_{n+1} - \omega_n) + \omega_{n+1} \geq \frac{3}{2} - 1 = \frac{1}{2},$$

hence $n\omega_n \uparrow$. Finally

$$|\omega_m - \omega_n| \leq \sum_{\nu=n+1}^m \frac{1}{\nu} < \frac{m-n}{n} \rightarrow 0, \quad \text{for } \frac{m}{n} \rightarrow 1,$$

hence ω_n is slowly oscillating.

5. Another equivalent result. We first establish the following lemma.

LEMMA 3. *Suppose that $U_n \geq 0$ and increasing, with $U_0 = 0$, and let*

$$(5.1) \quad b_n = \frac{1}{n} (U_n - U_{n-1}), \quad n \geq 1, \quad b_0 = 0;$$

$$(5.2) \quad B_n = \sum_{\nu=0}^n b_\nu, \quad n \geq 0.$$

Then whenever $k = k(n)$ is so chosen that $k/n \rightarrow 0$, as $n \rightarrow \infty$, the two statements

$$(5.3) \quad \frac{1}{n} (U_{n+k} - U_n) \rightarrow 0$$

and

$$(5.4) \quad B_{n+k} - B_n \rightarrow 0$$

are equivalent.

Proof. From (5.1) we have

$$U_n = \sum_{\nu=0}^n \nu b_\nu, \quad U_{n+k} - U_n = \sum_{\nu=n+1}^{n+k} \nu b_\nu.$$

Now

$$B_{n+k} - B_n = \sum_{\nu=n+1}^{n+k} b_\nu \leq \frac{1}{n} \sum_{\nu=n+1}^{n+k} \nu b_\nu = \frac{1}{n} (U_{n+k} - U_n);$$

thus (5.3) implies (5.4). Furthermore,

$$B_{n+k} - B_n \geq \frac{1}{n+k} (U_{n+k} - U_n);$$

hence (5.4) implies (5.3). This proves the lemma.

We note that

$$B_n = \frac{1}{n} U_n + \sum_{\nu=1}^{n-1} \frac{1}{\nu(\nu+1)} U_\nu,$$

and

$$U_n = nB_n - \sum_{\nu=0}^{n-1} B_\nu.$$

It is an immediate consequence of Lemma 3 that Theorem 3 is equivalent to the following theorem (for a direct proof see [4, Theorem IV]).

THEOREM 4. *If*

$$\sum_{\nu=n+1}^{n+k} (|a_\nu| - a_\nu) \rightarrow 0, \quad \text{as } \frac{k}{n} \rightarrow 0, \quad n \rightarrow \infty,$$

then Abel summability of $\sum_{n=0}^{\infty} a_n$ implies convergence of the series to the same value.

A generalization of this theorem to Dirichlet series and to Laplace integrals, on different lines, is given in [3].

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Printed in the United States of America by
Edwards Brothers, Inc., Ann Arbor, Michigan

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Ralph Palmer Agnew, <i>Ratio tests for convergence of series</i>	1
Richard Arens and James Dugundji, <i>Topologies for function spaces</i>	5
B. Arnold, <i>Distributive lattices with a third operation defined</i>	33
R. Bing, <i>Concerning hereditarily indecomposable continua</i>	43
David Dekker, <i>Generalizations of hypergeodesics</i>	53
A. Dvoretzky, A. Wald and J. Wolfowitz, <i>Relations among certain ranges of vector measures</i>	59
Paul Erdős, F. Herzog and G. Pirani, <i>Schlicht Taylor series whose convergence on the unit circle is uniform but not absolute</i>	75
Whilhelm Fischer, <i>On Dedekind's function $\eta(\tau)$</i>	83
Werner Leutert, <i>The heavy sphere supported by a concentrated force</i>	97
Ivan Niven and H. Zuckerman, <i>On the definition of normal numbers</i>	103
L. Paige, <i>Complete mappings of finite groups</i>	111
Otto Szász, <i>On a Tauberian theorem for Abel summability</i>	117
Olga Taussky, <i>Classes of matrices and quadratic fields</i>	127
F. Tricomi and A. Erdélyi, <i>The asymptotic expansion of a ratio of gamma functions</i>	133
Hassler Whitney, <i>On totally differentiable and smooth functions</i>	143