CONVOLUTION TRANSFORMS WITH COMPLEX KERNELS

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1. Introduction. In the present paper we shall consider the inversion of a class of convolution transforms with kernel $G(t)$ of the form

\begin{align}
G(t) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} [E(s)]^{-1} e^{st} ds \quad (-\infty < t < \infty), \\
E(s) &= \prod_{1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/b_k},
\end{align}

where $a_k = b_k + ic_k$ ($k = 1, 2, \cdots$) being a sequence of complex numbers such that

\begin{align}
\sum_{k=1}^{\infty} \frac{1}{b_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{c_k}{b_k} < \infty.
\end{align}

This class of kernels is more extensive than that treated previously by the authors, see [4], [5], [6], and [7]; however the results obtained here are slightly less precise than those which it was possible to obtain there. We shall show essentially that if

\begin{align}
f(x) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t),
\end{align}

and if $x_1$ and $x_2$ are points of continuity of $\alpha(t)$, then

\begin{align}
\lim_{m \to \infty} \int_{x_1}^{x_2} \left[ \prod_{k=1}^{m} \left(1 - \frac{D}{a_k}\right) e^{D/b_k}\right] f(x) dx = \alpha(x_2) - \alpha(x_1).
\end{align}

Here $D$ is the operation of differentiation, and $e^{D/\alpha}$ that of translation through the
distance $1/a$, so that, for example,

$$
\left(1 - \frac{D}{a_1}\right)e^{D/b_1} \left(1 - \frac{D}{a_2}\right)e^{D/b_2} f(x) = f \left(x + \frac{1}{b_1} + \frac{1}{b_2}\right) - \left(\frac{1}{a_1} + \frac{1}{a_2}\right)f' \left(x + \frac{1}{b_1} + \frac{1}{b_2}\right) + \frac{1}{a_1a_2}f'' \left(x + \frac{1}{b_1} + \frac{1}{b_2}\right).
$$

If we replace equation (1.2) and inequalities (1.3) by the more special relations

$$
E(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right),
$$

$$
\lim_{k \to \infty} b_k/k = \Omega > 0, \quad \sum_{k=1}^{\infty} \frac{c_k}{b_k} < \infty,
$$

we have in addition the complex inversion formula,

$$
\lim_{\lambda \to 1^-} \int_{x_1}^{x_2} dx \frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x)K(w) dw = \alpha(x_2) - \alpha(x_1),
$$

where

$$
K(w) = \int_0^\infty E(s) e^{-sw} ds
$$

and $C_\lambda$ is a closed rectifiable curve encircling the segment $[-i\Omega, i\Omega]$ and lying in the strip $|\Delta w| < \Omega/\lambda$. The inner integral in formula (1.8) is to be taken in the counterclockwise direction.

As one example we may take

$$
E(s) = \frac{\Gamma(1/2 + \nu/2)^2}{\Gamma(1/2 + \nu/2 - s/2) \Gamma(1/2 + \nu/2 + s/2)},
$$

$$
G(t) = \frac{(e^t + e^{-t})^{-\nu-1}}{\Gamma(1/2 + \nu/2)^2}.
$$
where $\Re \nu > -1$. If

$$f(x) = \int_{-\infty}^{\infty} [e^{(x-t)} + e^{-(x-t)}]^{-1} \Gamma(1/2 + \nu/2)^{-2} d \alpha(t),$$

then

$$\lim_{n \to \infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^{n} \left[ 1 - \left( \frac{D}{-1/2 + \nu/2 + k} \right)^2 \right] f(x) \right\} dx = \alpha(x_2) - \alpha(x_1);$$

and if $\Re \nu > 0$, then

$$\lim_{\nu \to 1} \frac{2^{\nu}(1/2 + \nu/2)^2}{\pi \Gamma(\nu)} \int_{x_1}^{x_2} \int_{-\pi/2}^{\pi/2} f(x + i \lambda w) [\cos \omega]^{\nu-1} dw \lambda$$

$$= \alpha(x_2) - \alpha(x_1).$$

See [7] and [8], and [9]. A second example is

$$E(s) = \pi 2^s \left[ \cos \frac{\pi s}{2} - \Gamma \left( \frac{1}{2} + \nu - s \right) \Gamma \left( \frac{1}{2} + \nu + s \right) \right]^{-1},$$

$$G(t) = \frac{2}{\pi} \cos \frac{\pi s}{2} e^{t K

for $-1 < \Re \nu < 1$. If

$$f(x) = \int_{-\infty}^{\infty} e^{x-t} K_{\nu}(e^{x-t}) \left( \frac{2}{\pi} \cos \frac{\nu \pi}{2} \right) d \alpha(t),$$

then

$$\lim_{n \to \infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^{n} \left( 1 - \frac{D}{-1/2 - \nu/2 + k} \right) \left( 1 + \frac{D}{-1/2 + \nu/2 + k} \right) f(x) \right\} dx$$

$$= \alpha(x_2) - \alpha(x_1).$$

See [2].

2. Inversion of a class of convolution transforms. We assume as given throughout this section a sequence, $\{a_k\}_{k=1}^{\infty}$, of complex numbers $a_k = b_k + ic_k$ subject
to the restrictions

\[ (2.1) \sum_{k=1}^{\infty} \frac{1}{b_k^2} < \infty, \quad \sum_{k=1}^{\infty} \frac{c_k}{b_k^2} < \infty. \]

We define the entire functions

\[ (2.2) E_{m,n}(s) = \prod_{k=m+1}^{n} (1 - s/a_k) e^{s/b_k}, \]
\[ E_m(s) = \prod_{k=m+1}^{\infty} (1 - s/a_k) e^{s/b_k}, \]
\[ F_m(s) = \prod_{k=m+1}^{\infty} \left| b_k/a_k \right| (1 - s/b_k) e^{s/b_k}. \]

The definition of \( E_m(s) \) is significant because

\[ E_m(s) = \left\{ \prod_{m+1}^{\infty} \left( 1 - \frac{s}{a_k} \right) e^{s/a_k} \right\} \left\{ \exp \sum_{m+1}^{\infty} \frac{ic_k s}{b_k (b_k + ic_k)} \right\}, \]

and because the series \( \sum_{m+1}^{\infty} |a_k|^2 \), \( \sum_{m+1}^{\infty} c_k/b_k(b_k + ic_k) \) converge as a consequence of (2.1) and Schwarz's inequality. Similarly, \( F_m(s) \) is well defined. We define

\[ (2.3) P_m(D) = \prod_{k=1}^{m} (1 - D/a_k) e^{D/b_k} \quad (m = 0, 1, \cdots). \]

We also set

\[ (2.4) \beta_1(m) = \max_{\begin{subarray}{c} b_k < 0 \\ k > m \end{subarray}} (b_k, -\infty), \quad \beta_2(m) = \min_{\begin{subarray}{c} b_k > 0 \\ k > m \end{subarray}} (b_k, \infty), \]

**Theorem 2a.** Let

\[ G_m(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [E_m(s)]^{-1} e^{st} ds \quad (-\infty < t < \infty; m = 0, 1, 2, \cdots); \]
then we have

A. \[ \int_{-\infty}^{\infty} G_m(t) e^{-st} dt = 1/\mathcal{E}_m(s), \quad \beta_1(m) < \Re s < \beta_2(m); \]

B. \[ \int_{-\infty}^{\infty} |G_m(t)| e^{-\sigma t} dt \leq 1/\mathcal{F}_m(\sigma), \quad \beta_1(m) < \sigma < \beta_2(m); \]

C. \[ P_m(b)G_0(t) = G_m(t); \]

D. \[ (d/dt)^k G_m(t) = O(e^{\gamma_1 t}), \quad t \to +\infty, \]
\[ = O(e^{\gamma_2 t}), \quad t \to -\infty \quad (k = 0, 1, \ldots), \]

for \( \gamma_1 > \beta_1(m) \) and \( \gamma_2 < \beta_2(m) \).

Conclusion A is an immediate consequence of Hamburger’s theorem; see [4, pp. 141-144]. We define \( g(u) = e^{u-1} \) for \(-\infty < u \leq 1\), and \( g(u) = 0 \) for \( 1 < u < \infty \), and we set

\[ g_k(t) = a_k \text{ sgn } b_k \{\exp[i\alpha_k(t - b_k^1)]\} g(b_k t). \]

It is immediately verifiable that

\[ \int_{-\infty}^{\infty} e^{-st} g_k(t) dt = \left( 1 - \frac{s}{a_k} \right) e^{s/b_k} \left[ \frac{1}{\mathcal{E}_m(s)} \right]^{-1}, \]

for \(-\infty < \Re s < b_k \) if \( b_k > 0 \), and for \( b_k < \Re s < \infty \) if \( b_k < 0 \). Let

\[ g_1 \ast g_2(t) = \int_{-\infty}^{\infty} g_1(t - u)g_2(u) du, \]

and so on; then by the convolution theorem for the bilateral Laplace transform we have

\[ \int_{-\infty}^{\infty} g_{m+1} \ast g_{m+2} \ast \cdots \ast g_n(t)e^{-st} dt = \left[ \mathcal{E}_{m,n}(s) \right]^{-1} \]

for \( \beta_1(m) < \Re s < \beta_2(m) \). From the complex inversion formula for the bilateral Laplace transform we obtain

\[ g_{m+1} \ast g_{m+2} \ast \cdots \ast g_n(t) = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} \left[ \mathcal{E}_{m,n}(s) \right]^{-1} e^{st} ds. \]
Since
\[ \lim_{n \to \infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( [E_m(s)]^{-1} - [E_m,n(s)]^{-1} \right) e^{st} \, ds = 0 \]
for \(-\infty < t < \infty\), it follows that
\[ \lim_{n \to \infty} g_{m+1} * \cdots * g_n(t) = G_m(t) \quad (-\infty < t < \infty). \]

See [4; pp. 139-145]. It is easily seen that
\[ \int_{-\infty}^{\infty} |g_k(t)| e^{-st} \, dt = \left[ (1 - s/b_k)e^{s/b_k}|b_k/a_k| \right]^{-1}, \]
for \(-\infty < R_s < b_k\) if \(b_k > 0\), or for \(b_k < R_s < \infty\) if \(b_k < 0\). By Fatou's lemma we have
\[ \int_{-\infty}^{\infty} |G_m(t)| e^{-st} \, dt \leq \liminf_{n \to \infty} \int_{-\infty}^{\infty} |g_{m+1} * \cdots * g_n(t)| e^{-st} \, dt, \]
so that conclusion B is established.

Conclusion C follows from the identity
\[ P_a(b)e^{st} = e^{st} \prod_{k=1}^{m} (1 - sa_k^{-1}) e^{s/a_k}. \]

Conclusion D may be established by shifting the line of integration in the integral defining \(G_m(t)\) to \(R_s = \gamma_1\) and \(R_s = \gamma_2\). See [4; pp. 152-154].

In what follows we shall write \(G(t)\) for \(G_0(t)\).

**Theorem 2b.** If
\begin{enumerate}
  \item \((a)\) \(G(t)\) is defined as in Theorem 2a,
  \item \((b)\) \(\beta_1(0) < c < \beta_2(0), \quad c + \gamma_1 > \beta_1(0), \quad c + \gamma_2 < \beta_2(0), \)
\end{enumerate}
(c) $\alpha(t)$ is of bounded variation on every finite interval, $\alpha(t) = O(e^{\gamma_1 t})$ as $t \to -\infty$, $\alpha(t) = O(e^{\gamma_2 t})$ as $t \to +\infty$,

(d) $P_m(D)$ is defined as in equation (2.3),

(e) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$,

(f) $x_1$ and $x_2$ are points of continuity of $\alpha(t)$,

then

$$\lim_{n \to \infty} \int_{x_1}^{x_2} e^{-cx}[P_n(D)f(x)]dx = \alpha(x_2) - \alpha(x_1).$$

From assumption (c) and from conclusion D of Theorem 2a we may show, using integration by parts, that each of the integrals

$$\int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\alpha(t)$$

converges uniformly for $x$ in any finite interval. Since $P_n(D)G(t) = G_m(t)$ by conclusion C of Theorem 2a, it follows (see [4; pp. 167-170]) that

$$(2.5) \quad P_n(D)f(x) = \int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\alpha(t), \quad (-\infty < x < \infty).$$

Multiplying by $e^{-cx}$ and integrating by parts, we have

$$e^{-cx}P_n(D)f(x) = -\int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial t} [G_m(x-t)e^{-c(x-t)}] \right] \alpha(t) dt$$

$$= \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right] \alpha(t) dt.$$

Since this integral converges uniformly for $x$ in any finite interval, we obtain

$$\int_{x_1}^{x_2} e^{-cx}P_n(D)f(x) dx$$

$$= \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right] \alpha(t) dt$$

$$= \int_{-\infty}^{\infty} \alpha(t) dt \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right] dx.$$
We thus need only show that if $x$ is a point of continuity of $\alpha(t)$ we have
\begin{equation}
\lim_{m \to \infty} \int_{-\infty}^{\infty} G_m(x - t)e^{-ct(\mu - t)} \alpha(t) dt = \alpha(x) .
\end{equation}

We shall first show that for any $\epsilon > 0$ we have
\begin{equation}
\lim_{m \to \infty} \int_{|t| \geq \epsilon} G_m(t)e^{-ct(\mu - t)} \alpha(t) dt = 0 .
\end{equation}

Using assumptions (a) and (b) we see that it is enough to prove that for any $\delta$ with $\beta_1(0) < \delta < \beta_2(0)$, we have
\begin{equation}
\lim_{m \to \infty} \int_{|t| \geq \delta} |G_m(t)|e^{-\delta t} dt = 0 .
\end{equation}

Choose $\eta > 0$ so small that $\beta_1(0) < \delta < \beta_2(0)$. For $|t| \geq \epsilon$ we have
\[
e^{-\delta t} \leq \frac{e^{-\delta t} (\sinh \eta t)^2}{(\sinh \epsilon t)^2} ,
\]
so that it is enough to prove that
\begin{equation}
\lim_{m \to \infty} \int_{-\infty}^{\infty} |G_m(t)|e^{-\delta t} [\sinh \eta t]^2 dt = 0 .
\end{equation}

Using conclusions A and B of Theorem 2a we see that
\[
\int_{-\infty}^{\infty} |G_m(t)|e^{-\delta t} [\sinh \eta t]^2 dt \\
\leq \frac{1}{4} \left[ \frac{1}{F_m(\delta + 2\eta)} + \frac{1}{F_m(\delta - 2\eta)} - \frac{2}{E_m(\delta)} \right] = o(1) \quad (m \to +\infty) ,
\]
and equation (2.8) follows from this. We assert that
\begin{equation}
\lim_{m \to \infty} \int_{-\infty}^{\infty} G_m(t)e^{-\delta t} dt = 1
\end{equation}
\begin{equation}
(2.10) \quad \lim_{m \to -\infty} \sup \int_{-\infty}^{\infty} |G_m(t)| e^{-ct} dt = 1.
\end{equation}

These results are immediate consequences of conclusions A and B of Theorem 2a. Now \( x \) being fixed and \( \eta > 0 \) being given, let us choose \( \varepsilon > 0 \) so small that 
\[ |\alpha(t) - \alpha(x)| \leq \eta \text{ for } |t - x| \leq \varepsilon. \]
We have
\[ \int_{-\infty}^{\infty} G_m(x - t)e^{-c(x-t)} \alpha(t) dt - \alpha(x) = I_1 + I_2 + I_3, \]
where
\[ I_1 = \alpha(x) \left[ \int_{-\infty}^{\infty} G_m(x - t)e^{-c(x-t)} dt - 1 \right] \]
\[ I_2 = \int_{|t| \geq \varepsilon} G_m(t)e^{-ct}[\alpha(x - t) - \alpha(x)] dt \]
\[ I_3 = \int_{|t| \leq \varepsilon} G_m(t)e^{-ct}[\alpha(x - t) - \alpha(x)] dt. \]

We have \( \lim_{m \to -\infty} I_1 = 0 \) by equation (2.9), \( \lim_{m \to -\infty} I_2 = 0 \) by equation (2.7), and 
\( \limsup_{m \to -\infty} |I_3| \leq \eta \) by equation (2.10). Since \( \eta \) is arbitrary our demonstration is complete.

3. Complex inversion formulas. In this section we restrict our attention to a much more special class of kernels. We suppose that
\begin{equation}
(3.1) \quad b_k > 0, \quad b_k \sim \frac{\Omega_k}{\eta}, \quad \sum_{k=1}^{\infty} \left( \frac{c_k}{b_k} \right)^2 < \infty.
\end{equation}

We define
\begin{equation}
(3.2) \quad E(s) = \prod_{k=1}^{\infty} \left( 1 - \frac{s}{\alpha_k^2} \right)^2,
\end{equation}
\begin{equation}
(3.3) \quad H(\lambda, s) = \prod_{k=1}^{\infty} \left[ \lambda^2 + (1 - \lambda^2) \frac{|\alpha_k| b_k}{b_k^2 - s^2} \right] \quad (0 \leq \lambda < 1).
\end{equation}
The product \((3.3)\) is defined for \(s \neq b_k (k = 1, 2, \ldots)\) since it can be rewritten as

\[
H(\lambda, s) = \prod_{k=1}^{\infty} \left[ 1 - \frac{\lambda^2 s^2}{b_k^2} + (1 - \lambda^2) \frac{|a_k| - b_k}{b_k} \right] \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{b_k^2} \right),
\]

and assumption (a) implies that \(\Sigma_1^\infty \left[ |a_k| - b_k \right] b_k^{-1}\) is convergent. We define

\[
(3.4) \quad \beta = \min b_k.
\]

**Theorem 3a.** If

\[
G(\lambda, w) = \frac{1}{2\pi i} \int_{i \infty}^{i \infty} \frac{e^{sw}E(\lambda s)}{E(s)} ds \quad (0 \leq \lambda < 1),
\]

then

A. \(G(\lambda, w)\) is analytic for \(|w| < \Omega(1 - \lambda)\);

B. \(\int_{-\infty}^{\infty} G(\lambda, t)e^{-st} dt = E(\lambda s)/E(s), \quad -\beta \leq \Re s < \beta\);

C. \((d/dw)^k G(\lambda, w) = 0(e^{\gamma_1 u}) \quad (u \to +\infty)\)

\[
= 0(e^{\gamma_2 u}) \quad (u \to -\infty) \quad (k = 0, 1, \ldots),
\]

where \(\gamma_1 > -\beta, \quad \gamma_2 < \beta\), uniformly for \(|v| < \Omega(1 - \lambda) - \epsilon, \quad \epsilon > 0\). (Here \(w = u + iv.\))

D. \(\int_{-\infty}^{\infty} |G(\lambda, t)| e^{-\sigma t} dt \leq H(\lambda, \sigma), \quad -\beta < \sigma < \beta\).

We shall write \(G(t)\) for \(G(0, t)\).

We assert that
(3.5) \[ \log |E(\sigma + i\tau)| \sim \Omega |	au| \quad (\tau \to \pm \infty) \]

uniformly for $\sigma$ in any finite interval. We define

\[ E \ast (s) = \prod_{k=1}^{\infty} (1 - s^2 b_k^{-2}). \]

We have

\[ \frac{E(s)}{E \ast (s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \prod_{k=1}^{\infty} \left(1 + \frac{a_k^2 - b_k^2}{b_k^2 - s^2}\right), \]

from which it follows that

(3.6) \[ \lim_{s \to \infty} \frac{E(s)}{E \ast (s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \]

uniformly for $0 < \epsilon \leq |\arg s| \leq \pi - \epsilon$. From [1, pp. 267-279] we have that \( \log |E \ast (\sigma + i\tau)| \sim \Omega |	au| \) as $\tau \to \pm \infty$, uniformly for $\sigma$ in any finite interval. Relation (3.5) now follows.

Conclusion A follows immediately from (3.5) and the definition of $G(h,w)$. Conclusion B is a consequence of (3.5) and Hamburger’s Theorem. The two conclusions C are obtained by shifting the line of integration in the integral defining $G(\lambda, \tau)$ to $Rs = \gamma_1$, and to $Rs = \gamma_2$, respectively. See [6, pp. 688-691]. To establish conclusion D we introduce the functions

\[ G_n(\lambda, t) = \frac{1}{2\pi i} \int_{i\omega}^{i\infty} e^{st} \prod_{k=1}^{n} \left(1 - \frac{\lambda^2 s^2}{a_k^2}\right) \prod_{k=n+1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right) ds. \]

It is immediate that

\[ \lim_{n \to \infty} G_n(\lambda, t) = G(\lambda, t) \quad (-\infty < t < \infty). \]
We define

$$h_k(\lambda, t) = \lambda^2 j(t) + (1 - \lambda^2) \frac{a_k}{2} \int_{-\infty}^{t} e^{-a_k |u|} \, du,$$

where \( j(t) = 0 \) for \(-\infty < t < 0\); \( j(0) = 1/2 \); \( j(t) = 1 \) for \( 0 < t < \infty \).

It is easily verified that for \(-b_k < \sigma < b_k\) we have

$$\int_{-\infty}^{\infty} e^{-st} \, dh_k(\lambda, t) = \frac{1 - \lambda^2 s^2/a_k^2}{1 - s^2/a_k^2}.$$  

Just as in §2 we may show that

$$G_n(\lambda, t) = \lim \frac{d}{dt} \left[ h_1(\lambda, t) * \cdots * h_n(\lambda, t) * h_{n+1}(0, t) * \cdots * h_m(0, t) \right].$$

Here \( h_1 * h_2(t) = \int_{-\infty}^{\infty} h_1(t - u) \, dh_2(u) \). Note that this differs from the convention employed in §2. By Fatou's lemma,

$$\int_{-\infty}^{\infty} e^{-\sigma t} |G_n(\lambda, t)| \, dt \leq \lim \inf_{m \to \infty} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_1(\lambda, t) * \cdots * h_n(\lambda, t) * h_{n+1}(0, t) * \cdots * h_m(0, t) \right|$$

$$\leq \lim \inf_{m \to \infty} \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_k(\lambda, t) \right| \prod_{k=n+1}^{m} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_k(0, t) \right|$$

$$\leq \prod_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_k(\lambda, t) \right| \prod_{k=n+1}^{\infty} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_k(0, t) \right| .$$

By Fatou's lemma, again,

$$\int_{-\infty}^{\infty} e^{-\sigma t} |G(\lambda, t)| \, dt \leq \lim \inf_{m \to \infty} \int_{-\infty}^{\infty} e^{-\sigma t} |G_n(\lambda, t)| \, dt \leq \prod_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{-\sigma t} \left| dh_k(\lambda, t) \right|$$

$$\leq \prod_{k=1}^{\infty} \left[ \lambda^2 + (1 - \lambda^2) \frac{|a_k| b_k}{b_k^2 - \sigma^2} \right].$$
This completes the proof of the theorem.

We define

\[ K(w) = \int_{0}^{\infty} E(s) e^{-sw} \, ds . \]

It follows from relations (3.1) that given \( \varepsilon > 0 \), for all sufficiently large \( r \) we have

\[ \log |E (re^{i\theta})| \leq (\varepsilon + | \sin \theta |) \Omega r. \]

See [1, pp. 267-279]. From equation (3.6) it follows that

\[ \log |E (re^{i\theta})| \leq (\varepsilon + | \sin \theta |) \Omega r \]

for \( r \) sufficiently large. Using this inequality and rotating the line of integration in the integral defining \( K(w) \) we can show that \( K(w) \) is analytic and single valued in the \( w \)-plane except on the segment \([-i\Omega , i\Omega]\). It may also be shown, see [1, pp. 295-311], that if \( C \) is a closed rectifiable curve encircling \([-i\Omega , i\Omega]\) then

\[ E(s) = \frac{1}{2\pi i} \int_{C} K(w) e^{sw} \, dw , \]

the integration proceeding in the counterclockwise direction.

**Lemma 3b.** If \( C_{\lambda} \) is a closed rectifiable curve encircling \([-i\Omega , i\Omega]\) and contained in the strip \(|v| < \Omega/\lambda\), then

\[ \frac{1}{2\pi i} \int_{C_{\lambda}} G(\lambda w + x - t) K(w) \, dw = G(\lambda, x - t) , \]

the integration proceeding in the counterclockwise direction.

We have

\[ \frac{1}{2\pi i} \int_{C_{\lambda}} G(\lambda w + x - t) K(w) \, dw \]

\[ = \frac{1}{2\pi i} \int_{C_{\lambda}} K(w) \, dw \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{s(\lambda w + x - t)} \, ds \]

\[ = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{s(x - t)} \, ds \frac{1}{2\pi i} \int_{C_{\lambda}} K(w) e^{\lambda sw} \, dw \]
\[
\int_{-1}^{1} \frac{E(\lambda s)}{E(s)} e^{s(x-t)} \, ds
\]

\[= G(\lambda, x - t).\]

**Theorem 3c.** If

(a) \( G(t) \) is defined as in Theorem 3a

(b) \(-\beta < c < \beta, \ -\beta < c + \gamma_1, \ c + \gamma_2 < \beta\)

(c) \( \alpha(t) \) is of bounded variation on every finite interval and

\[
\alpha(t) = (e^{\gamma_1 t}) \quad (t \to +\infty), \quad \alpha(t) = (e^{\gamma_2 t}) \quad (t \to -\infty)
\]

(d) \( f(w) = \int_{-\infty}^{\infty} G(w - t) e^{\alpha(t)} \, d\alpha(t) \)

(e) \( K(w) \) is defined as in equation (3.7)

(f) \( C_\lambda \) is defined as in Lemma 3b

(g) \( x_1 \) and \( x_2 \) are points of continuity of \( \alpha(t) \), then

\[
\lim_{\lambda \to 1^-} \int_{x_1}^{x_2} e^{-\epsilon x} \, dx \cdot \frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x) K(w) \, dw = \alpha(x_2) - \alpha(x_1).
\]

It follows from assumption (c) and from conclusion C of Theorem 3a that the integral defining \( f(w) \) converges uniformly for \( w \) in any compact set contained in the strip \( |\Im w| < \Omega \). Hence

\[
\frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x) K(w) \, dw
\]

\[= \int_{-\infty}^{\infty} e^{\epsilon t} d\alpha(t) \cdot \frac{1}{2\pi i} \int_{C_\lambda} G(\lambda w + x - t) K(w) \, dw
\]

\[= \int_{-\infty}^{\infty} G(\lambda, x - t) e^{\epsilon t} \, d\alpha(t)
\]

by Lemma 3b. The proof may now be completed exactly in the manner of Theorem 2b.

4. **Remark.** If it is assumed that the roots of \( E(s) \) occur in conjugate pairs, then equation (1.5) can be established under conditions less restrictive than (1.3). A discussion of this case is given in the Master's thesis of Mr. A. O. Garder [3], written under the direction of one of us.
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