

# Pacific Journal of Mathematics

**ON THE NUMBER OF INTEGERS IN THE SUM OF TWO SETS  
OF POSITIVE INTEGERS**

HENRY B. MANN

# ON THE NUMBER OF INTEGERS IN THE SUM OF TWO SETS OF POSITIVE INTEGERS

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1. **Introduction.** Let  $A, B, \dots$  be sets of nonnegative integers. We define  $A + B = \{a + b\}_{a \in A, b \in B}$ . By  $A^0, B^0, \dots$  we shall denote the union of  $A, B, \dots$  and the number 0, by  $A(n)$  the number of positive  $a$ 's that do not exceed  $n$ . We further put

$$(1) \quad \text{g.l.b.} \frac{A(n)}{n} = \alpha ,$$

$$(2) \quad \text{g.l.b.} \frac{A(n)}{n + 1} = \alpha^* ,$$

$$(3) \quad \liminf \frac{A(n)}{n} = \bar{\alpha} .$$

If  $1, 2, \dots, k-1 \in A, k \notin A$ , we further put

$$(4) \quad \text{g.l.b.}_{n \geq k} \frac{A(n)}{n + 1} = \alpha_1 .$$

The real number  $\alpha$  is called the *density* of  $A$ ,  $\alpha_1$  the *modified density*, and  $\bar{\alpha}$  the *asymptotic density* of  $A$ . Densities of  $A, B, C, \dots$  will be denoted by the corresponding Greek letters  $\alpha, \beta, \gamma, \dots$ .

Besicovitch [1] introduced  $\alpha^*$ , and Erdős [2]  $\alpha_1$ .

The author [3] proved: If  $C = A^0 + B$  for  $B \ni 1$  and  $A^0 + B^0$  otherwise, then for all  $n \notin C$  we have

$$(5) \quad C(n) \geq \alpha^* n + B(n) .$$

It was also shown [3] that in (5),  $\alpha^*$  cannot be replaced by  $\alpha$ .

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It is the purpose of the present note to improve (5) to the relation

$$(6) \quad C(n) \geq \alpha_1 n + B(n).$$

The proof of (6) requires only a modification of the proof of (5), but will be given in full to make the present note self-sufficient.

The inequality (6) immediately yields

$$(7) \quad \bar{\gamma} \geq \alpha_1 + \bar{\beta}$$

if  $C$  has infinitely many gaps.

Now (7) is sometimes better and sometimes not as good as Erdős' [2] inequality

$$(8) \quad \bar{\gamma} \geq \bar{\alpha} + \bar{\beta}/2$$

for the case  $\alpha > \beta$ ,  $B \ni 1$ ,  $C = A^0 + B^0$ . (To establish (8) it is really sufficient to assume that there is at least one  $b^0$  such that  $b^0 + 1 \in B$ .) However (7) holds also for  $C = A^0 + B$  if  $B \ni 1$ , and for  $C = A^0 + B^0$  without any restriction on  $B$ .

**2. Proof.** We shall now give a proof of (6) for the case  $C = A^0 + B$ ,  $B \ni 1$ , and then shall indicate the changes which have to be made if nothing is assumed about  $B$  but if  $C = A^0 + B^0$ . By  $a, b, c, \dots$  we shall denote unspecified integers in  $A, B, C, \dots$ .

Let  $n_1 < n_2 < \dots$  be all the gaps in  $C$ . Put  $n_r = n$ ,  $n - n_i = d_i$  for  $i < r$ . If there is one  $e \in B$  such that

$$(9) \quad a + e + d_i = n_j,$$

form all numbers  $e + d_t$  for which

$$(10) \quad a + e + d_t = n_s, \quad t < r, \quad s < r.$$

Let  $T$  be the set of indices occurring in (10). Put  $B^* = \{e + d_s\}_{s \in T}$ .

It is not difficult to prove the following propositions.

PROPOSITION 1. *The intersection  $B \cap B^*$  is empty.*

PROPOSITION 2. *The integer  $n$  is not of the form  $a + e + d_s$  for any  $s$ .*

Since (10) also implies

$$(10') \quad a + e + d_s = n_t,$$

it follows that  $B^*$  contains as many numbers as there are gaps in  $C$  which precede  $n$  and which are not gaps in  $A + B \cup B^*$ . Hence we have the following result.

PROPOSITION 3. *If  $B \cup B^* = B_1$ ,  $A + B_1 = C_1$ , then*

$$(11) \quad C_1(n) - C(n) = B_1(n) - B(n) .$$

Thus we have proved the following lemma.

LEMMA. *If there is at least one equation of the form  $a + b + d_i = n_j$ , then there exists a  $B_1 \supset B$  such that  $C_1 = A + B_1$  does not contain  $n$ , and such that*

$$(12) \quad C_1(n) - C(n) = B_1(n) - B(n) > 0 .$$

Now let  $C = A^0 + B$ ,  $B \ni 1$ . Clearly,  $n_1 > 1$ . The numbers smaller than  $n_1$  are either in  $B$ , or of the form  $n_1 - a$ , or of neither of these two sorts. Also  $n_1 \notin B$ , since  $C \supset B$ . Hence we have

$$(13) \quad C(n_1) = n_1 - 1 \geq A(n_1 - 1) + B(n_1) .$$

Since  $B \ni 1$ , we must have  $n_1 - 1 \notin A$ ,  $(n_1 - 1) \geq k$ . Thus, we obtain

$$(14) \quad C(n_1) \geq \alpha_1 n_1 + B(n_1) .$$

We proceed by induction and assume (6) proved, when  $n$  is the  $j$ th gap,  $j < r$ . We distinguish two cases.

Case 1:  $d_{r-1} < n_1$ . Then

$$C \ni n_1 - d_{r-1} = a + b .$$

We now apply the lemma. Let  $n$  be the  $j$ th gap in  $C_1$ . Then  $j < r$ , and we have, by induction,

$$(15) \quad C_1(n) \geq \alpha_1 n + B_1(n) ,$$

and, by the lemma,

$$(16) \quad C_1(n) - C(n) = B_1(n) - B(n) .$$

Subtracting (16) from (15), we obtain (6).

Case 2:  $d_{r-1} \geq n_1$ . Now

$$n - n_{r-1} - 1 \geq n_1 - 1 \notin A .$$

Hence we have

$$A(n - n_{r-1} - 1) \geq \alpha_1(n - n_{r-1}) .$$

The numbers between  $n_{r-1}$  and  $n$  are either of the form  $n - a$ , or in  $B$ , or of neither of these two sorts. But  $n \notin B$ ; hence,

$$(17) \quad \begin{aligned} n - n_{r-1} - 1 &\geq A(n - n_{r-1} - 1) + B(n) - B(n_{r-1}) \\ &\geq \alpha_1(n - n_{r-1}) + B(n) - B(n_{r-1}) . \end{aligned}$$

By induction we have

$$(18) \quad C(n_{r-1}) = n_{r-1} - (r - 1) \geq \alpha_1 n_{r-1} + B(n_{r-1}) .$$

Adding (17) and (18), we obtain (6).

From the proof it is evident that we may obtain the even stronger inequality

$$(6') \quad C(n) \geq \alpha_1 n + B(n) + \min_{n_i \leq n} \left[ \frac{A(n_i - 1)}{n_i} - \alpha_1 \right] n_i .$$

To establish (6) for  $C = A^0 + B^0$  without the restriction  $B \ni 1$ , we first remark that in (13) the term  $A(n_i - 1)$  can be replaced by  $A(n_i)$ . The cases to be distinguished are  $d_{r-1} \leq n_1$  and  $d_{r-1} > n_1$ . The proof of Case 1 is then word by word the same when we replace  $B$  by  $B^0$  and  $B_1$  by  $B_1^0$ . In Case 2 we have

$$n - n_{r-1} - 1 \geq n_1 \geq k ,$$

so that  $A(n - n_{r-1} - 1) \geq \alpha_1(n - n_{r-1})$ ; the remainder of the argument remains unchanged. For  $C = A^0 + B^0$ , we can obtain the even stronger inequality

$$(6'') \quad C(n) \geq \alpha_1 n + B(n) + \min_{n_i \leq n} \left[ \frac{A(n_i)}{n_i} - \alpha_1 \right] n_i ,$$

which again implies the even stronger result

$$\begin{aligned} C(n) &\geq \max \left\{ \alpha_1 n + B(n) + \left[ \frac{A(n_1)}{n_1} - \alpha_1 \right] n_1 , \right. \\ &\quad \left. A(n) + \beta_1 n + \min_{n_i \leq n} \left[ \frac{B(n_i)}{n_i} - \beta_1 \right] n_i \right\} . \end{aligned}$$

To establish (7), it is sufficient to show that for any set  $S$  we have

$$\frac{S(m)}{m} > \frac{S(n)}{n}$$

if  $m > n$ ,  $n \notin S$ ,  $S(m) - S(n) = m - n$ . However, this can easily be verified. Thus if  $S$  has infinitely many gaps, then

$$\bar{\sigma} = \liminf \frac{S(m)}{m} = \liminf_{n \notin S} \frac{S(n)}{n}.$$

It thus appears that in (7) we may replace  $\bar{\beta}$  by

$$\liminf_{n \notin C} \frac{B(n)}{n} \geq \bar{\beta}.$$

If  $C = A^0 + B^0$ , we may of course write

$$\bar{\gamma} \geq \max (\alpha_1 + \bar{\beta}, \bar{\alpha} + \beta_1).$$

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