Pacific Journal of Mathematics

ON THE L^p THEORY OF HANKEL TRANSFORMS

George Milton Wing

Vol. 1, No. 2 December 1951

ON THE L^p THEORY OF HANKEL TRANSFORMS

G. M. WING

1. Introduction. Under suitable restrictions on $f(x)$ and ν, the Hankel transform $g(t)$ of $f(x)$ is defined by the relation

$$g(t) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(x)\, dx. \tag{1}$$

The inverse is then given formally by

$$f(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) g(t)\, dt. \tag{2}$$

These integrals represent generalizations of the Fourier sine and cosine transforms to which they reduce when $\nu = \pm 1/2$. The L^p theory for the Fourier case has been studied in considerable detail. In this note we present some results concerning the inversion formula (2) in the L^p_x case.

It is clear that if $f(x) \in L$ and $\Re(\nu) \geq -1/2$ then the integral in (1) exists. It has been shown [3,6] that if $f(x) \in L^p$, $1 < p \leq 2$, then

$$g_a(t) = \int_0^a (xt)^{1/2} J_\nu(xt) f(x)\, dx \tag{3}$$

converges strongly to a function $g(t)$ in $L^{p'}$. For this case Kober has obtained the inversion formula,

$$f(x) = x^{-1/2-\nu} \frac{d}{dx} \left\{ x^{\nu+1/2} \int_0^\infty \frac{(xt)^{1/2} J_{\nu+1}(xt)}{t} g(t)\, dt \right\},$$

which holds for almost all x. In her investigation of Watson transforms, Busbridge [1] has given analogous results for more general kernels. Except when $p = 2$ the question of the strong convergence of the inversion integral has apparently been considered only in the Fourier case [2]. We now investigate this problem

Received January 10, 1951.
Pacific J. Math. 1 (1951), 313-319.

for the Hankel transforms. We assume throughout that $\Re(\nu) \geq -1/2$.

2. Theorem. We shall establish the following result.

THEOREM 1. *Let $f(x) \in L^p$, $1 < p \leq 2$, and let $g(t)$ be the limit in mean of $g_a(t)$, $g(t) = \text{l.i.m.}\ g_a(t)$, where $g_a(t)$ is defined by* (3). *If*

$$f_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt) g(t)\, dt,$$

then

$$f_a(x) \in L^p \quad \text{and} \quad f(x) = \text{l.i.m.}\ f_a(x).$$

Proof. Write

$$f_a(x,b) = \int_0^a (xt)^{1/2} J_\nu(xt) g_b(t)\, dt$$

$$= \int_0^b (xu)^{1/2} f(u)\, du \int_0^a J_\nu(ut) J_\nu(xt) t\, dt.$$

Since $g_b(t)$ converges in the mean to $g(t)$ it follows that $\lim_{b\to\infty} f_a(x,b) = f_a(x)$. Hence

$$f_a(x) = \int_0^\infty (xu)^{1/2} K(x,u,a) f(u)\, du, \tag{4}$$

where [9]

$$\begin{aligned} K(x,u,a) &= \int_0^a J_\nu(ut) J_\nu(xt) t\, dt \\ &= a\{u J_{\nu+1}(ua) J_\nu(xa) - x J_{\nu+1}(xa) J_\nu(ua)\}/(u^2 - x^2). \end{aligned} \tag{5}$$

An integral very similar to (4) has been studied in a previous paper [10]. The same methods may be used here to show that $\| f_a(x) \|_p < M_p \| f(x) \|_p$. Our theorem will now follow in the usual way if we can prove it for step functions which vanish outside a finite interval. Let $\phi(x)$ be a step function, $\phi(x) = 0$ for $x > A$, and let $\phi_a(x)$ correspond to it as in (4). Choose $\xi > 2A$, $a > A$, to get

$$\int_\xi^\infty |\phi_a(x) - \phi(x)|^p\, dx = \int_\xi^\infty dx \left| \int_0^A \phi(u) (xu)^{1/2} K(x,u,a)\, du \right|^p.$$

From the relations

$$(6)\qquad x^{1/2} J_\nu(x) = (2/\pi)^{1/2} \{\cos(x+\delta_\nu) + x^{-1} A_\nu \sin(x+\delta_\nu)\} + O(x^{-2}) \qquad (x \longrightarrow \infty),$$

where

$$A_\nu = (1-4\nu^2)/8, \qquad \delta_\nu = -(2\nu+1)\pi/4,$$

and

$$(7)\qquad J_\nu(x) = O(x^{\nu_1}) \qquad (x \longrightarrow 0),$$

where $\nu_1 = \Re(\nu)$, it is easy to see that

$$(xu)^{1/2} |K(x,u,a)| < M/|u-x|,$$

so that we have

$$\int_\xi^\infty |\phi_a(x) - \phi(x)|^p\, dx < M \int_\xi^\infty \frac{dx}{|x-A|^p} \int_0^A |\phi(u)|^p\, du < \epsilon$$

for ξ sufficiently large. Now

$$\|\phi_a(x) - \phi(x)\|_p^p = \int_0^\xi + \int_\xi^\infty |\phi_a(x)-\phi(x)|^p\, dx$$

$$\leq M \left\{ \int_0^\xi |\phi_a(x) - \phi(x)|^2\, dx \right\}^{p/2} + \epsilon.$$

As $a \longrightarrow \infty$ the integral goes to zero by the L^2 theory for Hankel transforms (see [7, Chapter 8]). This completes the proof.

3. The case $p = 1$. Theorem 1 fails to hold in the case $p = 1$. The proof, similar to that given by Hille and Tamarkin in the Fourier case [2] , will only be sketched.

THEOREM 2. *There exists a function $h(t)$, the Hankel transform of a function $\psi(x) \in L$, such that if*

$$(8)\qquad \psi_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt) h(t)\, dt$$

then l.i.m. *$\psi_a(x)$ fails to exist.*

Proof. Let $h(t) = t^{1/2} J_\nu(t)/\log(t + 2)$. Two integrations of (8) by parts and use of formulas (5), (6), and (7) yield

$$\psi_a(x) = \frac{ax^{3/2} J_\nu(a) J_{\nu+1}(ax)}{(x^2-1)\log(a+2)} + O(x^{-2}) \tag{9}$$

for large x.

Now define $\psi(x) = \lim_{a\to\infty} \psi_a(x)$. It is evident from (8) that $\psi(x)$ is continuous except perhaps at $x = 1$, while (9) shows that $\psi(x) = O(x^{-2})$. To show that $\psi(x) \in L$ it suffices to consider the neighborhood of $x = 1$. Formula (6) yields, after some calculation,

$$\psi(x) = \int_0^\infty \frac{\cos(1-x)t}{\log(t+2)}\,dt + \alpha(x),$$

where $\alpha(x)$ is continuous near $x = 1$. Thus

$$\int_{1+\epsilon}^2 \{\psi(x) - \alpha(x)\}\,dx = -\int_0^\infty \frac{\sin t}{t\log(2+t/\epsilon)}\,dt + \int_0^\infty \frac{\sin t}{t\log(2+t)}\,dt.$$

The first integral on the right tends to zero as $\epsilon \to 0^+$. Since $\psi(x) - \alpha(x)$ is positive (see [2]) it follows that $\psi(x) - \alpha(x)$ is integrable over (1,2) [8, p.342]. The interval (0,1) may be handled similarly. Hence $\psi(x) \in L$.

That $h(t)$ is indeed the Hankel transform of $\psi(x)$ is a consequence of a result of P. M. Owen [5, p. 310]. But it may be seen from (9) that $\psi_a(x)$ is not in L, so that l.i.m. $\psi_a(x)$ surely fails to exist.

4. A summability method. It is natural to try to include the case $p = 1$ into the theory by introducing a suitable summability method. Our interest will be confined to the Cesàro method. If $f(x) \in L$ and $g(t)$ is its Hankel transform then we shall define

$$f_a(x) = \int_0^a (1-t/a)^k (xt)^{1/2} J_\nu(xt) g(t)\,dt \tag{10}$$

$$= \int_0^\infty f(y) C_k(x,y,a)\,dy,$$

where

$$C_k(x,y,a) = \int_0^a (xy)^{1/2} uJ_\nu(xu)J_\nu(yu)(1-u/a)^k \, du. \tag{11}$$

Offord [4] has studied the local convergence properties of $f_a(x)$ for $k = 1$. We are able to extend his results to the case $k > 0$, but the estimates required are too long and tedious for presentation here. Instead we investigate the strong convergence.

THEOREM 3. *Let* $f(x) \in L$, $k > 0$. *If* $f_a(x)$ *is defined by* (10), *then* $f_a(x)$ *converges strongly to* $f(x)$.

Proof. We shall first prove that $C_k(x,y,a) \in L$ and $\|C_k(x,y,a)\| < M$, where the norm is taken with respect to x and the bound M is independent of y and a. An integration by parts and a change of variable in (11) give

$$C_k(x,y,a) = -\frac{ka}{2} \int_0^1 (1-s)^{k-1} s(xy)^{1/2} Q \, ds \tag{12}$$

where

$$Q = \frac{J_{\nu+1}(ays)J_\nu(axs) - J_\nu(ays)J_{\nu+1}(axs)}{y-x} + \frac{J_{\nu+1}(ays)J_\nu(axs) + J_\nu(ays)J_{\nu+1}(axs)}{y+x}.$$

Consider

$$I = \int_{|y-x|>1/a} \frac{dx}{|y-x|} \left| \int_0^1 (1-s)^{k-1}(ays)^{1/2} J_{\nu+1}(ays)(axs)^{1/2} J_\nu(axs)\, ds \right|$$

$$= \int_{|ay-z|>1} \frac{dz}{|ay-z|} \left| \int_0^\infty G(a,y,s)(zs)^{1/2} J_\nu(zs)\, ds \right|,$$

where

$$G(a,y,s) = \begin{cases} (1-s)^{k-1}(ays)^{1/2} J_{\nu+1}(ays) & (0 \le s < 1), \\ 0 & (s \ge 1). \end{cases}$$

Now, as a function of s, $G(a, y, s) \in L^p$ for some $p > 1$ so that

$$F(a, y, z) = \int_0^\infty G(a, y, s)(sz)^{1/2} J_\nu(sz)\, ds$$

is in $L^{p'}$ as a function of z [3]. Also

$$\left\{\int_0^\infty |F(a, y, z)|^{p'} dz\right\}^{1/p'} \leq A_p \left\{\int_0^\infty |G(a, y, s)|^p ds\right\}^{1/p} < M,$$

where M is a constant independent of a and y. Thus

$$I \leq \left\{\int_{|ay-z|>1} \frac{dz}{|ay-z|^p}\right\}^{1/p} \left\{\int_0^\infty |F(a, y, z)|^{p'} dz\right\}^{1/p'} < M.$$

The other parts of (12) may be cared for similarly, so that we have

$$\int_{|y-x|>1/a} |C_k(x, y, a)|\, dx < M.$$

The range $|y - x| \leq 1/a$ is easily handled since, by (11), for this range we have $|C_k(x,y,a)| < Ma$. Hence $\|C_k(x, y, a)\| < M$. We see at once from (10) that

$$\int_0^\infty |f_a(x)|\, dx = \int_0^\infty dx \left|\int_0^\infty f(y)\, C_k(x, y, a)\, dy\right|$$

$$\leq \int_0^\infty |f(y)|\, dy \int_0^\infty |C_k(x, y, a)|\, dx,$$

so $\|f_a(x)\| < M\|f(x)\|$. The proof may now be completed by the methods of Theorem 1.

REFERENCES

1. I. W. Busbridge, *A theory of general transforms for functions of the class* $L^p(0, \infty)$ $(1 < p \leq 2)$, Quart. J. Math., Oxford Ser. 9 (1938), 148-160.

2. E. Hille and J. D. Tamarkin, *On the theory of Fourier transforms*, Bull. Amer. Math. Soc. 39 (1933), 768-774.

3. H. Kober, *Hankelsche Transformationen*, Quart. J. Math., Oxford Ser. 8 (1937), 186-199.

4. A. C. Offord, *On Hankel transforms*, Proc. London Math. Soc. (2) 39 (1935), 49-67.

5. P. M. Owen, *The Riemannian theory of Hankel transforms*, Proc. London Math. Soc. (2) 39 (1935), 295-320.

6. E. C. Titchmarsh, *A note on Hankel transforms*, J. London Math. Soc. 1 (1926), 195-196.

7. ———, *Introduction to the theory of Fourier integrals*, University Press, Oxford, 1937.

8. ———, *The theory of functions*, University Press, Oxford, 1932.

9. G. N. Watson, *Theory of Bessel functions*, University Press, Cambridge, England, 1922.

10. G. M. Wing, *The mean convergence of orthogonal series*, Amer. J. Math. 72 (1950), 792-807.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $8.00; single issues, $2.50. Special price to individual faculty members of supporting institutions and to members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics

Vol. 1, No. 2 December, 1951