ON THE $L^p$ THEORY OF HANKEL TRANSFORMS

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1. Introduction. Under suitable restrictions on $f(x)$ and $\nu$, the Hankel transform $g(t)$ of $f(x)$ is defined by the relation

$$g(t) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(x) \, dx.$$  

The inverse is then given formally by

$$f(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) g(t) \, dt.$$  

These integrals represent generalizations of the Fourier sine and cosine transforms to which they reduce when $\nu = \pm 1/2$. The $L^p$ theory for the Fourier case has been studied in considerable detail. In this note we present some results concerning the inversion formula (2) in the $L^p$ case.

It is clear that if $f(x) \in L$ and $R(\nu) \geq -1/2$ then the integral in (1) exists. It has been shown [3,6] that if $f(x) \in L^p$, $1 < p \leq 2$, then

$$g_a(t) = \int_0^a (xt)^{1/2} J_\nu(xt) f(x) \, dx$$

converges strongly to a function $g(t)$ in $L^p'$. For this case Kober has obtained the inversion formula,

$$f(x) = x^{-1/2-\nu} \frac{d}{dx} \left[ x^{\nu+1/2} \int_0^\infty (xt)^{1/2} J_{\nu+1}(xt) \frac{g(t)}{t} \, dt \right],$$

which holds for almost all $x$. In her investigation of Watson transforms, Busbridge [1] has given analogous results for more general kernels. Except when $p = 2$ the question of the strong convergence of the inversion integral has apparently been considered only in the Fourier case [2]. We now investigate this problem.

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for the Hankel transforms. We assume throughout that $R(\nu) \geq -1/2$.

2. Theorem. We shall establish the following result.

**Theorem 1.** Let $f(x) \in L^p$, $1 < p \leq 2$, and let $g(t)$ be the limit in mean of $g_a(t)$, $g(t) = \lim_{a \to \infty} g_a(t)$, where $g_a(t)$ is defined by (3). If

$$f_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt)g(t) \, dt,$$

then

$$f_a(x) \in L^p \quad \text{and} \quad f(x) = \lim_{a \to \infty} f_a(x).$$

**Proof.** Write

$$f_a(x, b) = \int_0^a (xt)^{1/2} J_\nu(xt)g_b(t) \, dt$$

$$= \int_0^b (xu)^{1/2} f(u) \, du \int_0^a J_\nu(ut)J_\nu(xt) t \, dt.$$

Since $g_b(t)$ converges in the mean to $g(t)$ it follows that $\lim_{b \to \infty} f_a(x, b) = f_a(x)$. Hence

$$f_a(x) = \int_0^\infty (xu)^{1/2} K(x, u, a) f(u) \, du,$$

where [9]

(5) $K(x, u, a) = \int_0^a J_\nu(ut)J_\nu(xt) t \, dt$

$$= \{a u J_{\nu+1}(ua) J_\nu(xa) - x J_{\nu+1}(xa) J_\nu(ua)\} / (u^2 - x^2).$$

An integral very similar to (4) has been studied in a previous paper [10]. The same methods may be used here to show that $\|f_a(x)\|_p < M_p \|f(x)\|_p$. Our theorem will now follow in the usual way if we can prove it for step functions which vanish outside a finite interval. Let $\phi(x)$ be a step function, $\phi(x) = 0$ for $x > A$, and let $\phi_a(x)$ correspond to it as in (4). Choose $\xi > 2A$, $a > A$, to get

$$\int_\xi^\infty |\phi_a(x) - \phi(x)|^p \, dx = \int_\xi^\infty \left| \int_0^A \phi(u)(xu)^{1/2} K(x, u, a) \, du \right|^p.$$
From the relations

\[(6) \quad x^{1/2} J_\nu(x) = (2/\pi)^{1/2} \{\cos (x + \delta_\nu) + x^{-1} A_\nu \sin (x + \delta_\nu)\} + O(x^{-2}) \quad (x \to \infty),\]

where

\[A_\nu = (1 - 4 \nu^2)/8, \quad \delta_\nu = -(2 \nu + 1) \pi/4,\]

and

\[(7) \quad J_\nu(x) = O(x^{\nu_1}) \quad (x \to 0),\]

where \(\nu_1 = \Re(\nu)\), it is easy to see that

\[(xu)^{1/2} |K(x, u, a)| < M/|u - x|,\]

so that we have

\[
\int_\xi^\infty |\phi_a(x) - \phi(x)|^p dx < M \int_\xi^\infty \frac{dx}{|x - A|^p} \int_0^A |\phi(u)|^p du < \epsilon
\]

for \(\xi\) sufficiently large. Now

\[
\|\phi_a(x) - \phi(x)\|^p = \int_0^\xi + \int_\xi^\infty |\phi_a(x) - \phi(x)|^p dx
\]

\[
\leq M \left\{ \int_0^\xi |\phi_a(x) - \phi(x)|^2 dx \right\}^{p/2} + \epsilon.
\]

As \(a \to \infty\) the integral goes to zero by the \(L^2\) theory for Hankel transforms (see [7, Chapter 8]). This completes the proof.

3. The case \(p = 1\). Theorem 1 fails to hold in the case \(p = 1\). The proof, similar to that given by Hille and Tamarkin in the Fourier case [2], will only be sketched.

**Theorem 2.** There exists a function \(h(t)\), the Hankel transform of a function \(\psi(x) \in L\), such that if

\[(8) \quad \psi_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt)h(t) dt\]

then l.i.m. \(\psi_a(x)\) fails to exist.
Proof. Let \( h(t) = t^{1/2} J_\nu(t)/\log(t + 2) \). Two integrations of (8) by parts and use of formulas (5), (6), (7) yield

\[
\psi_a(x) = \frac{ax^{3/2} J_\nu(a) J_{\nu+1}(ax)}{(x^2 - 1) \log (a + 2)} + O(x^{-2})
\]

for large \( x \).

Now define \( \psi(x) = \lim_{a \to \infty} \psi_a(x) \). It is evident from (8) that \( \psi(x) \) is continuous except perhaps at \( x = 1 \), while (9) shows that \( \psi(x) = O(x^{-2}) \). To show that \( \psi(x) \in L \) it suffices to consider the neighborhood of \( x = 1 \). Formula (6) yields, after some calculation,

\[
\psi(x) = \int_0^\infty \frac{\cos \left(1 - x \right) t}{\log (2 + t)} \, dt + \alpha(x),
\]

where \( \alpha(x) \) is continuous near \( x = 1 \). Thus

\[
\int_{1+\epsilon}^{2} \left[ \psi(x) - \alpha(x) \right] \, dx = - \int_0^\infty \frac{\sin t}{t \log (2 + t / \epsilon)} \, dt + \int_0^\infty \frac{\sin t}{t \log (2 + t)} \, dt.
\]

The first integral on the right tends to zero as \( \epsilon \to 0^+ \). Since \( \psi(x) - \alpha(x) \) is positive (see [2]) it follows that \( \psi(x) - \alpha(x) \) is integrable over \((1, 2) \) \( [8, \text{p. 342}] \). The interval \((0, 1)\) may be handled similarly. Hence \( \psi(x) \in L \).

That \( h(t) \) is indeed the Hankel transform of \( \psi(x) \) is a consequence of a result of P. M. Owen [5, p. 310]. But it may be seen from (9) that \( \psi_a(x) \) is not in \( L \), so that \( \text{l.i.m. } \psi_a(x) \) surely fails to exist.

4. A summability method. It is natural to try to include the case \( p = 1 \) into the theory by introducing a suitable summability method. Our interest will be confined to the Cesàro method. If \( f(x) \in L \) and \( g(t) \) is its Hankel transform then we shall define

\[
f_a(x) = \int_0^\infty (1 - t/a)^k(xt)^{1/2} J_\nu(xt) g(t) \, dt
\]

\[= \int_0^\infty f(y) C_k(x, y, a) \, dy,
\]
where

\begin{equation}
C_k(x, y, a) = \int_0^a (xy)^{1/2} u J_{\nu}(xu) J_{\nu}(yu) \left(1 - u/a\right)^k \, du.
\end{equation}

Offord [4] has studied the local convergence properties of \( f_a(x) \) for \( k = 1 \). We are able to extend his results to the case \( k > 0 \), but the estimates required are too long and tedious for presentation here. Instead we investigate the strong convergence.

**Theorem 3.** Let \( f(x) \in L' \), \( k > 0 \). If \( f_a(x) \) is defined by (10), then \( f_a(x) \) converges strongly to \( f(x) \).

**Proof.** We shall first prove that \( C_k(x, y, a) \in L \) and \( \| C_k(x, y, a) \| < M \), where the norm is taken with respect to \( x \) and the bound \( M \) is independent of \( y \) and \( a \). An integration by parts and a change of variable in (11) give

\begin{equation}
C_k(x, y, a) = -\frac{ka}{2} \int_0^1 (1 - s)^{k-1} s (xy)^{1/2} Q \, ds
\end{equation}

where

\[
Q = \frac{J_{\nu+1}(ays) J_{\nu}(axs) - J_{\nu}(ays) J_{\nu+1}(axs)}{y - x} + \frac{J_{\nu+1}(ays) J_{\nu}(axs) + J_{\nu}(ays) J_{\nu+1}(axs)}{y + x}.
\]

Consider

\[
I = \int_{|y-x|>1/a} \left| \int_0^1 (1 - s)^{k-1} (ays)^{1/2} J_{\nu+1}(ays)(axs)^{1/2} J_{\nu}(axs) \, ds \right| \frac{dx}{|y-x|}
\]

\[
= \int_{|ay-z|>1} \left| \int_0^\infty G(a, y, s)(zs)^{1/2} J_{\nu}(zs) \, ds \right| \frac{dz}{|ay-z|},
\]

where

\[
G(a, y, s) = \begin{cases} (1 - s)^{k-1} (ays)^{1/2} J_{\nu+1}(ays) & (0 \leq s < 1), \\ 0 & (s \geq 1). \end{cases}
\]
Now, as a function of $s$, $G(a, y, s) \in L^p$ for some $p > 1$ so that

$$F(a, y, z) = \int_0^\infty G(a, y, s)(sz)^{1/2} J_\nu(sz) \, ds$$

is in $L^{p'}$ as a function of $z$ [3]. Also

$$\left( \int_0^\infty |F(a, y, z)|^{p'} \, dz \right)^{1/p'} \leq A_p \left( \int_0^\infty |G(a, y, s)|^p \, ds \right)^{1/p} < M,$$

where $M$ is a constant independent of $a$ and $y$. Thus

$$I \leq \left\{ \int_{|ay - z| > 1} \frac{dz}{|ay - z|^p} \right\}^{1/p} \left( \int_0^\infty |F(a, y, z)|^{p'} \, dz \right)^{1/p'} < M.$$

The other parts of (12) may be cared for similarly, so that we have

$$\int_{|y-x| > 1/a} |C_k(x, y, a)| \, dx < M.$$

The range $|y - x| \leq 1/a$ is easily handled since, by (11), for this range we have $|C_k(x, y, a)| < Ma$. Hence $\|C_k(x, y, a)\| < M$. We see at once from (10) that

$$\int_0^\infty |f_a(x)| \, dx = \int_0^\infty dx \left| \int_0^\infty f(y) C_k(x, y, a) \, dy \right|$$

$$\leq \int_0^\infty |f(y)| \, dy \, \int_0^\infty |C_k(x, y, a)| \, dx,$$

so $\|f_a(x)\| < M \|f(x)\|$. The proof may now be completed by the methods of Theorem 1.

**References**


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