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COMPLETENESS OF SETS OF TRANSLATED COSINES

R. P. Boas, Jr.

1. Introduction. Conditions for the completeness on $(0,\pi)$ of sets $\{\cos \lambda_n x\}$ are well known. Here we shall consider sets $\{\cos(\lambda_n x + q_n)\}$. Such sets seem first to have been considered by Ditkin [3], who proved that $\{\cos(nx + q_n)\}_0^\infty$ is L-complete in $(0,\pi)$ if $0 \le q_n < \pi/2$.

Ditkin's very simple proof uses Fourier series and does not seem capable of extension to the more general sets considered here. Our principal object is to show how the problem may be attacked by complex-variable methods; we shall not attempt an exhaustive discussion.

As a specimen we quote the following case. If $\lambda_n \ge 0$ and $|\lambda_n - n| \le \delta < 1/2$, then the sets $\{\cos(\lambda_n x + q_n)\}_0^\infty$ and $\{\sin(\lambda_n x + q_n)\}_1^\infty$ are L-complete in $(0,\pi)$ if $\pi\delta/2 \le q_n < \pi(1-\delta)/2$. (The statement "$\{f_n(x)\}$ is L^p-complete" means that the only functions of L^p which are orthogonal to all $f_n(x)$ are almost everywhere zero.) A further result, not covered by the present paper, has been given by Bitsadze [1], who showed that every function satisfying a Hölder condition admits a uniformly convergent expansion in terms of the set $\{\cos(nx + \pi/4)\}$; he indicates an application of this result to the Tricomi partial differential equation.

We remark that although Ditkin's set $\{\cos(nx + q_n)\}_0^\infty$ remains complete when all $q_n = \pi/2$, it may fail to be complete if some but not all $q_n = \pi/2$. In fact, the set $\{1, \sin x, \cos 2x, \cos 3x, \cdots\}$ is orthogonal to $\cos x$. However, we shall show that not only is the set $\{\sin(nx + q_n)\}_0^\infty$ complete if $0 \le q_n < \pi/2$, but even the set $\{\sin(nx + q_n)\}_1^\infty$ is complete.

By applying the completeness theorem of Paley and Wiener [5, p. 100] to the equivalent set $\{\cos nx + a_n \sin nx\}$, $0 \le |a_n| < 1$, we can show at once that $\{\cos(nx + q_n)\}_0^\infty$ is L^2-complete if either $0 \le |q_n| \le \delta < \pi/4$ for all n or else $\pi/4 < \delta \le |q_n| \le \pi/2$ for all n. The problem of necessary and sufficient conditions for the completeness of $\{\cos(nx + q_n)\}$ remains open.

2. A general theorem. We shall obtain our results on $\{\cos(\lambda_n x + q_n)\}$ as

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corollaries of a theorem on a related set of more artificial appearance.

THEOREM. *Let* $\{\lambda_n\}_0^\infty$ *be an increasing unbounded sequence of nonnegative numbers; let* $N_1(r)$ *and* $N_2(r)$ *denote respectively the number of* λ_{2n} *and of* λ_{2n+1} *not exceeding* r. *If both*

$$\int_1^r t^{-1} N_1(t)\,dt > \frac{1}{2} r - \gamma \log r - \text{constant}, \tag{2.1}$$

and

$$\int_1^r t^{-1} N_2(t)\,dt > \frac{1}{2} r - \left(\gamma + \frac{1}{2}\right) \log r - \text{constant}, \tag{2.2}$$

where $\gamma = 1/(2p')$ *if* $1 \le p < \infty$, $p' = p/(p-1)$, *and* $\gamma < 1/2$ *if* $p = \infty$, *then the set*

$$\begin{cases} \cos \lambda_{2n} t + a_{2n} \sin \lambda_{2n} t, \\ -a_{2n+1} \cos \lambda_{2n+1} t + \sin \lambda_{2n+1} t \end{cases} \tag{2.3}$$

is L^p*-complete on* $(-\pi/2, \pi/2)$ *if the* a_n *are real numbers all of the same sign.*

COROLLARY 1. *The set* (2.3), *with the* a_n *all of the same sign, is* L^p*-complete on* $(-\pi/2, \pi/2)$ *if* $0 \le \lambda_n \le n + 1 + 1/p'$, $1 \le p < \infty$; *it is* L^∞*-complete if* $0 \le \lambda_n \le n + \delta$, $\delta < 2$.

COROLLARY 2. *If* $\lambda_n \ge 0$ *and*

$$|\lambda_n - n| \le \delta < \frac{1}{2}, \quad \frac{\pi\delta}{2} \le q_n < \frac{\pi(1-\delta)}{2},$$

then the set $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is L-complete on* $(0, \pi)$.

For $\delta = 0$, Corollary 2 reduces to Ditkin's theorem; for $\delta \ne 0$, the range of q_n is more restricted. If the λ_n are confined to one side of n, a sharper result is true.

COROLLARY 3. *If* $n \le \lambda_n \le n + \delta$, $0 \le \delta < 1$, *and* $0 \le q_n < \pi(1-\delta)/2$, $n \ge 0$; *or if* $n - \delta \le \lambda_n \le n$ *for* $n > 0$, $0 \le \delta < 1$, *and* $\pi(1-\delta)/2 < q_n \le 0$, *then* $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is L-complete in* $(0, \pi)$.

The following result on sets of sines includes the fact that $\{\sin(nx + q_n)\}_1^\infty$

is L-complete on $(0,\pi)$ if $0 \le q_n < \pi/2$.

COROLLARY 4. *If* $|n+1-\lambda_n| \le \delta < 1/2$ *and* $\pi\delta/2 \le q_n < \pi(1-\delta)/2$, *then the set* $\{\sin(\lambda_n x + q_n)\}_0^\infty$ *is L-complete on* $(0,\pi)$.

By demanding only L^p-completeness instead of L-completeness, we can allow the λ_n to be larger than in Corollary 2.

COROLLARY 5. *If* $1 < p < \infty$ *and* $n+2-\delta < \lambda_n < n+2-1/p$, $1/p < \delta < 1$, *then the set* $\{\cos(\lambda_n x + q_n)\}_0^\infty$ *is L^p-complete on* $(0,\pi)$ *if* $\pi\delta/2 \le q_n < \pi/2$.

3. Proof of the general theorem. We now prove the theorem stated above. We must show that if $f(x) \in L^p$ and if

$$\begin{aligned}(3.1)\qquad &\int_{-\pi/2}^{\pi/2} (\cos \lambda_{2n} t + a_{2n} \sin \lambda_{2n} t) f(t)\, dt \\ &= \int_{-\pi/2}^{\pi/2} (-a_{2n+1} \cos \lambda_{2n+1} t + \sin \lambda_{2n+1} t) f(t)\, dt \\ &= 0 \qquad\qquad (n = 0, 1, 2, \cdots),\end{aligned}$$

where all a_n satisfy $a_n \ge 0$ or else all a_n satisfy $a_n \le 0$, then $f(x) = 0$ almost everywhere.

Write

$$(3.2)\qquad F(z) = \int_{-\pi/2}^{\pi/2} f(t) \cos zt\, dt, \qquad G(z) = \int_{-\pi/2}^{\pi/2} f(t) \sin zt\, dt;$$

then (3.1) is

$$\begin{aligned}(3.3)\qquad &F(\lambda_{2n}) + a_{2n} G(\lambda_{2n}) = 0, \\ &-a_{2n+1} F(\lambda_{2n+1}) + G(\lambda_{2n+1}) = 0.\end{aligned}$$

Let $H(z) = F(z)G(z)$; then $H(0) = 0$; if $\lambda_0 = 0$, then $H'(0) = H''(0) = 0$; and $H(\lambda_{2n})H(\lambda_{2n+1}) \le 0$. Note that $H(z)$ is an odd function. Let $N(t) = N_1(t) + N_2(t)$, and let $\Lambda(t)$ denote the number of zeros of $H(z)$ in $0 \le |z| \le t$.

We prove first that

$$(3.4)\qquad \Lambda(r) \ge 2N(r) + 1.$$

To begin with, if $\lambda_0 = 0$, we have, for $0 \le r < \lambda_1$, the relations $N(t) = 1$, $\Lambda(r) \ge 3$; if $\lambda_0 > 0$, we have $N(t) = 0$ for $0 \le r < \lambda_0$, $\Lambda(r) = 1$. We proceed by induction. Suppose that (3.4) is true for $r \le \lambda_k$. Then it remains true for $r < \lambda_{k+1}$, since $N(r)$ does not change in $\lambda_k \le r < \lambda_{k+1}$. If $H(\lambda_k)H(\lambda_{k+1}) \ne 0$, then $H(\lambda_k)$ and $H(\lambda_{k+1})$ have opposite signs and so $\Lambda(\lambda_{k+1}) \ge \Lambda(\lambda_k) + 2 \ge 2N(\lambda_k) + 3 = 2N(\lambda_{k+1}) + 1$, so that (3.4) is true for $r = \lambda_{k+1}$. If $H(\lambda_{k+1}) = 0$, then (3.4) is true for $r = \lambda_{k+1}$ since $\Lambda(r)$ increases by 2 at $r = \lambda_{k+1}$ while $N(r)$ increases by 1. Finally, suppose $H(\lambda_k) = 0, H(\lambda_{k+1}) \ne 0$. If $H(\lambda_j) = 0$ for $j = 0, 1, 2, \cdots, k$, then $\Lambda(\lambda_{k+1}) \ge \Lambda(\lambda_k) \ge 2k + 3 = 2N(\lambda_{k+1}) + 1$, and (3.4) is verified for $r = \lambda_{k+1}$. Otherwise there is a largest $j < k$ for which $H(\lambda_j) \ne 0$, and $\Lambda(\lambda_j) \ge 2N(\lambda_j) + 1$; there are at least $k - j$ zeros of $H(z)$ in $\lambda_j < x \le \lambda_{k+1}$; but the number of zeros in this interval is even if $k - j + 1$ is even [since $H(\lambda_{k+1})$ and $H(\lambda_j)$ then have the same sign], odd if $k - j + 1$ is odd; so the number of zeros cannot be $k - j$ and hence must be at least $k - j + 1$. This completes the proof of (3.4).

By combining (3.4) with (2.1) and (2.2), we see that

$$\int_1^r t^{-1}\Lambda(t)\,dt > 2r - 4\gamma \log r - \text{constant}, \tag{3.5}$$

where $4\gamma = 2/p'$ if $1 \le p < \infty$, $4\gamma < 2$ if $p = \infty$.

We now appeal to a modification of a result of Levinson [4, pp. 7-9] to show that $H(z) \equiv 0$. This is as follows.

LEMMA. *Let $\{x_n\}_{-\infty}^{\infty}$ be a sequence of real numbers arranged in nondecreasing order, and let $H(z)$ be an entire function which is known to vanish at all x_n; if $H(z)$ is known to have a multiple zero at some x_n, that x_n is to be repeated, according to its multiplicity, in the sequence. Let $\nu(r)$ denote the number of x_n such that $|x_n| \le r$ and suppose that*

$$\int_1^r t^{-1}\nu(t)\,dt \ge 2r - \alpha \log r - \text{constant}.$$

Suppose finally that

$$|H(x+iy)| \le \left\{\int_0^{\pi/2} h(t)\,e^{t|y|}\,dt\right\}^2,$$

where $h(t) \ge 0$, $h(t) \in L^p(0, \pi/2)$, $1 \le p < \infty$. Then $H(z) \equiv 0$ if $\alpha \le 2/p'$, $p' = p/(p-1)$. If $p = \infty$, then $H(z) \equiv 0$ if $\alpha < 2$.

The proof of the lemma is parallel to that given by Boas and Pollard [2] for a similar result, and we omit it.

Since $H(z) \equiv 0$, we have either $F(z) \equiv 0$ or $G(z) \equiv 0$. If $F(z) \equiv 0$, (3.3) shows that $G(\lambda_{2n+1}) = 0$; if $G(z) \equiv 0$, (3.3) shows that $F(\lambda_{2n}) = 0$.

We first consider the case when $F(z) \equiv 0$. Then, in particular, we have

$$\int_{-\pi/2}^{\pi/2} f(t)\,dt = 0\,,$$

and

$$\int_{-\pi/2}^{\pi/2} f(t) \cos \lambda_{2n+1} t \, dt = 0 \qquad (n = 0, 1, 2, \cdots),$$

$$\int_{-\pi/2}^{\pi/2} f(t) \sin \lambda_{2n+1} t \, dt = 0 \qquad (n = 0, 1, 2, \cdots);$$

hence

$$\int_{-\pi/2}^{\pi/2} f(t)\, e^{i\mu_n t}\, dt = 0 \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{3.6}$$

where

$$\mu_0 = 0\,, \quad \mu_n = \lambda_{2n-1} \ (n > 0)\,, \quad \mu_n = -\lambda_{-2n-1} \ (n < 0)\,. \tag{3.7}$$

A result of Levinson [4, p. 6], reduced to the interval $(-\pi/2, \pi/2)$, is that $\{e^{i\mu_n t}\}$ is L^p-complete if $M(t)$, the number of $|\mu_n| \le t$, satisfies

$$\int_1^r t^{-1} M(t)\,dt > r - (1/p') \log r - \text{constant}, \tag{3.8}$$

$1 \le p < \infty$; his proof also shows that L^∞-completeness follows from (3.8) if $1/p'$ is replaced by any number less than 1. Since $M(t) = 2N_2(t) + 1$, (3.8) is true in virtue of (2.2). Thus (2.2) implies $f(t) = 0$ almost everywhere if $F(z) \equiv 0$.

Now suppose that $G(z) \equiv 0$. In the same way we have

$$\int_{-\pi/2}^{\pi/2} f(t)\, e^{i\mu_n t}\, dt = 0\,,$$

where now

$$\mu_n = \lambda_{2n} \ (n \ge 0)\,, \quad \mu_n = -\lambda_{-2n-2} \ (n < 0)\,. \tag{3.9}$$

In this case $M(t) = 2N_1(t)$ and (3.8) follows from (2.1). The rest of the argument is as before.

4. **Proof of Corollary 1.** To prove Corollary 1 we have to show that (2.1) and (2.2) follow from $0 \le \lambda_n \le n + \delta$ $(n = 0,1,2,\cdots)$, where $\delta = 1 + 1/p'$, $1 \le p < \infty$. In the interval $2k + \delta \le u < 2k + \delta + 2$, where $k = 0,1,2,\cdots$, we have $N_1(u) \ge k + 1$. Let $x > 1$ and define n by $2n + \delta \le x < 2n + \delta + 2$. Then

$$\begin{aligned}
\int_\delta^x \frac{N_1(u)}{u}\,du &\ge \int_\delta^{2+\delta}\frac{du}{u} + \int_{2+\delta}^{4+\delta}\frac{2\,du}{u} + \cdots + \int_{2n-2+\delta}^{2n+\delta}\frac{n}{u}\,du \\
&= \sum_{k=1}^{n} k \log\left(1 + \frac{2}{2k+\delta-2}\right) \\
&\ge \sum_{k=1}^{n} k\left\{\frac{2}{2k+\delta-2} - \frac{1}{2}\left(\frac{2}{2k+\delta-2}\right)^2\right\} \\
&\ge \sum_{k=2}^{n}\left\{1 + \frac{2-\delta}{2k} - \frac{1}{2(k-1)}\right\} \\
&= n + \left(1 - \frac{1}{2}\delta - \frac{1}{2}\right)\log n + O(1) \\
&= \frac{1}{2}x + \frac{1-\delta}{2}\log x + O(1) = \frac{1}{2}x - \frac{1}{2p'}\log x + O(1).
\end{aligned}$$

On the other hand, in the interval $2k + 1 + \delta \le u < 2k + 3 + \delta$ $(k = 0,1,2,\cdots)$, we have $N_2(u) \ge k + 1$. Thus

$$\begin{aligned}
\int_1^x \frac{N_2(u)}{u}\,du &\ge \int_{1+\delta}^{3+\delta}\frac{du}{u} + \cdots + \int_{2n-1+\delta}^{2n+1+\delta}\frac{n}{u}\,du \\
&= \sum_{k=1}^{n} k\log\left(1 + \frac{2}{2k-1+\delta}\right) \\
&\ge \sum_{k=1}^{n} k\left\{\frac{2}{2k-1+\delta} - \frac{1}{2}\left(\frac{2}{2k-1+\delta}\right)^2\right\}
\end{aligned}$$

$$\geq \sum_{k=1}^{n} \left\{ 1 + \frac{1-\delta}{2k+1} - \frac{1}{2k-1} \right\}$$

$$= x + \frac{1}{2}(1-\delta) - \frac{1}{2} + O(1)$$

$$= x - \frac{1}{2p'} - \frac{1}{2} + O(1) .$$

5. Proof of Corollaries 2–5. In proving Corollaries 2–5, it is convenient to write $-a_n$ instead of a_n, and $t = x - \pi/2$, so that (2.3) becomes

$$\begin{cases} \cos(\lambda_n x - \lambda_n\pi/2) - a_n \sin(\lambda_n x - \lambda_n\pi/2) & (n \text{ even}) ; \\ a_n(\cos \lambda_n x - \lambda_n\pi/2) + \sin(\lambda_n x - \lambda_n\pi/2) & (n \text{ odd}) . \end{cases}$$

Put $a_n(1 + a_n^2)^{-1/2} = \sin b_n$, $(1 + a_n^2)^{-1/2} = \cos b_n$, $0 \leq b_n < \pi/2$ or $-\pi/2 < b_n \leq 0$, according as $a_n \geq 0$ or $a_n \leq 0$. Then the completeness of (2.3) is equivalent to that of

$$\begin{cases} \cos(\lambda_n x - \lambda_n\pi/2)\cos b_n - \sin(\lambda_n x - \lambda_n\pi/2)\sin b_n & (n \text{ even}) ; \\ \sin(\lambda_n x - \lambda_n\pi/2)\cos b_n + \cos(\lambda_n x - \lambda_n\pi/2)\sin b_n & (n \text{ odd}) ; \end{cases}$$

that is, to the completeness of

$$\begin{cases} \cos(\lambda_n x - \lambda_n\pi/2 + b_n) & (n \text{ even}) ; \\ \sin(\lambda_n x - \lambda_n\pi/2 + b_n) & (n \text{ odd}) . \end{cases}$$

Now let $\lambda_n = m - 2\epsilon_n/\pi$, where m is an integer of the same parity as n. Then the completeness of (2.3) is equivalent to that of

$$\cos(\lambda_n x + \epsilon_n + b_n) \qquad (n = 0, 1, 2, \cdots) . \tag{5.1}$$

Thus a set

$$\cos(\lambda_n x + q_n) \tag{5.2}$$

is equivalent to a set of the form (2.3) if for all n either

$$\epsilon_n \leq q_n < \pi/2 + \epsilon_n \tag{5.3}$$

or

$$-\pi/2 + \epsilon_n < q_n \leq \epsilon_n . \tag{5.4}$$

We may satisfy (5.3) or (5.4) in various ways. For example, (5.3) is certainly true if $|n - \lambda_n| < \delta\ (n = 0, 1, 2, \cdots)$, with $\delta < 1/2$ and $\pi\delta/2 \leq q_n < \pi(1 - \delta)/2$; this establishes Corollary 2, since the condition of Corollary 1 is certainly satisfied in this case. Corollary 1 requires only that $\lambda_n \leq n + 1$ if $p = 1$; if we restrict λ_n to lie always on one side of n we can therefore obtain a stronger result than Corollary 2. In fact, if $n \leq \lambda_n \leq n + 1$ we have $\epsilon_n \leq 0$, and (5.3) is satisfied if $0 \leq q_n < \pi/2 + \epsilon_n$, hence certainly if $n \leq \lambda_n \leq n + \delta$, $\delta < 1$, and $0 \leq q_n < \pi(1 - \delta)/2$. On the other hand, if $n - 1 \leq \lambda_n \leq n$ $(n > 0)$, we have $\epsilon_n \geq 0$ and (5.4) is satisfied if $n - \delta \leq \lambda_n < n\ (n > 0)$, $\delta < 1$, and $-\pi(1 - \delta)/2 < q_n \leq 0$.

If we let $\lambda_n = m - 2\epsilon_n/\pi$, where m has opposite parity to n, (2.3) reduces to $\{\sin(\lambda_n x + \epsilon_n + b_n)\}$; by taking $m = n + 1$ we obtain Corollary 4. Finally, Corollary 5 is obtained by taking $m = n + 2$. Further theorems of the same character are readily written down.

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NORTHWESTERN UNIVERSITY

MATRICES OF QUATERNIONS

J. L. BRENNER

1. Introduction. In this note, some theorems which concern matrices of complex numbers are generalized to matrices over real quaternions. First it is proved that every matrix of quaternions has a characteristic root. Next, there exist $n - 1$ mutually orthogonal unit n-vectors all orthogonal to a given vector. It is shown that Schur's lemma holds for matrices of quarternions: every matrix can be transformed into triangular form by a unitary matrix. For individual quaternions, it is known that two quaternions are similar if they have the same trace and the same norm—thus every quaternion has a conjugate $a + bj (b \geq 0)$. This fact is proved again.

The quaternion λ is called a *characteristic root* of a (square) matrix A provided a non-zero vector x exists such that $Ax = x\lambda$. Similar matrices have the same characteristic roots; if $y = Tx$, where T has an inverse, then $TAT^{-1}y = TAx = Tx\lambda = y\lambda$. Another interesting fact is that if λ is a characteristic root, then so is $\rho^{-1}\lambda\rho$; for from $Ax = x\lambda$ follows $A(x\rho) = (x\rho)\rho^{-1}\lambda\rho$; thus if the vector corresponding to the characteristic root λ is x, then $x\rho$ is the vector corresponding to the characteristic root $\rho^{-1}\lambda\rho$.

2. Lemma. We shall need the following result.

LEMMA 1. *If $A = (a_{i,j})$ is a matrix of elements from any field or fields, then a triangular matrix T exists such that $T^{-1}AT = C = (c_{i,j})$, where $c_{i,j} = 0$ whenever $i > j + 1$. The elements of T are rational functions of the elements of A.*

Proof. The proof consists in transforming A in steps so that an additional zero appears at each step. First A is transformed so that all the elements in the first column (except the first two) become zero; the transformed matrix is further transformed so that all the elements in the second column (except the first three) become zero, and so on. The formal proof is inductive; it will be sufficient to give the idea of the proof. In the first column of A, either $a_{j,1} = 0$ for all $j > 1$, or else

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$a_{j,1} \neq 0$ for some $j > 1$. In the former case, proceed directly to the second column. In the other case, assume without loss of generality that $a_{2,1} \neq 0$ (otherwise transform by a permutation matrix). Let I stand for the identity matrix, and let $e_{m,n}$ be the matrix with 1 in the (m, n)th place and 0 elsewhere. Let $w_{3,2}$ be an element of the field. The transform $B = (b_{i,j})$ of A by the matrix $I + w_{3,2}e_{3,2}$ satisfies the conditions

$$b_{2,1} = a_{2,1}, \quad b_{3,1} = a_{3,1} + w_{3,2}\, a_{2,1} \quad .$$

It is evident that if $w_{3,2}$ is suitably chosen, then the condition $b_{3,1} = 0$ will be satisfied. Further transformations by

$$I + w_{j,2}\, e_{j,2} \qquad (j = 4, \cdots, n)$$

will successively replace the elements in the first column of A (except the first two) by zeros. The second and later columns are handled in order by the same method.

The above lemma and proof follow the lines of Lemma 4.4 of [1]; in that reference, the elements of the matrix A are residue classes mod p^r, a prime power.

3. The existence of characteristic roots. We shall show that every matrix A of quaternions has a characteristic root.

Since any characteristic root of C is also a characteristic root of A, it is enough to prove that C has a characteristic root. The proof is by induction on n. There are two cases. First, suppose that $c_{j+1,j} = 0$ for some j with $j < n$. Let $C_{(j)}$ be the principal j-rowed minor of C; a non-zero vector $x_{(j)}$ and a characteristic root λ exist such that $C_{(j)}x_{(j)} = x_{(j)}\lambda$. Then λ is a characteristic root of C: the corresponding vector is obtained from the vector $x_{(j)}$ by appending $n - j$ zeros.

In the second case, it is true for each j that $c_{j+1,j} \neq 0$. There is a characteristic vector $(x_1, x_2, \cdots, x_n)$ with $x_n = 1$; it is found by solving a polynomial equation of degree n with just one term of highest degree. The fact that every such equation has a solution is proved in [5]. The equation in question comes by eliminating $x_{n-2}, x_{n-3}, \cdots, x_1$ in turn from the set $Cx = x\lambda$. This set is indeed the following:

$$(1) \qquad c_{n,n-1}\, x_{n-1} + c_{n,n} = \lambda\,,$$

$$(2) \qquad c_{n-1,n-2}\, x_{n-2} + c_{n-1,n-1}x_{n-1} + c_{n-1,n} = x_{n-1}\lambda\,,$$

$$\cdots\cdots$$

$$(n-1) \qquad c_{2,1}x_1 + c_{2,2}x_2 + \cdots + c_{2,n-1}x_{n-1} + c_{2,n} = x_2\lambda$$

$$(n) \qquad c_{1,1}x_1 + c_{1,2}x_2 + \cdots + c_{1,n-1}x_{n-1} + c_{1,n} = x_1\lambda\,.$$

First, λ must be eliminated from (2), (3), $\cdots$ (n), using (1). Call the resulting set $(2')$, $(3')$, $\cdots$, (n'). Then $(2')$ must be solved for x_{n-2}; the resulting expression is substituted into $(3')$. Next the equation so obtained is solved for x_{n-3}; the resulting expression is substituted into $(4')$, and so on. Since $c_{j+1,j} \neq 0$ for each j, these steps are meaningful. At the last stage, (n') becomes an equation of degree n in the one unknown x_{n-1}. The single term of highest degree is not zero. After x_{n-1} is determined, the values of x_{n-2}, $\cdots$, x_1 are determined from $(2')$ $\cdots$ (n'), and the value of λ is determined from (1). These values satisfy all requirements. This proves the following result.

THEOREM 1. *Every matrix of quaternions has a characteristic root.*

For an application, we note that the 2×2 matrix $(a_{i,j})$ has characteristic root $a_{1,1}$ corresponding to the vector $(1,0)$ if $a_{2,1} = 0$. If $a_{2,1} \neq 0$, a characteristic vector is $(x_1, 1)$, and the corresponding characteristic root is $\lambda = a_{2,1}x_1 + a_{2,2}$, where x_1 is a solution of $x_1 a_{2,1} x_1 = a_{1,1} x_1 - x_1 a_{2,2} + a_{1,2}$.

4. Generalization of Schur's lemma. To continue the discussion, we need:

LEMMA 2. *There exists a unitary*[1] *matrix U of quaternion elements which has a preassigned unit*[1] *vector $u_1 = (u_{1,1}, u_{1,2}, \cdots, u_{1,n})$ in the first row.*

Proof. Since the space of n-tuples over quaternions has the same dimension independent of the choice of basis [6, pp. 18-19], there is a set of n vectors $u_1, b_2, \cdots, b_n$ which are linearly independent and span the space. From these an orthonormal set $u_1, \cdots, u_n$ can be constructed by Schmidt's process of orthogonalization. The matrix which has these vectors for rows is unitary. The process is exhibited in [3, p. 10], where, however, the first displayed equation should be changed to read $b_k \cdot b_m = (b_k \cdot a_m - b_k \cdot a_m)\,\|c\|^{-1} = 0$; otherwise the reference [3, p. 21, line 2] to this equation would be inappropriate.

THEOREM 2. (Generalization of Schur's lemma.) *Every matrix of real quaternions can be transformed into triangular form by a unitary matrix.*

---

[1]A matrix U is called unitary if $UU^* = 1$. A vector (u) is called a unit vector if $uu^* = 1$.

Proof. This theorem is a direct consequence of Theorem 1 and Lemma 2. The proof, given in [9, pp. 25-26], applies with equal force when the elements of the matrices are quaternions.

5. Transformations of matrices. We shall establish several lemmas.

LEMMA 3. *Let q be a quaternion. There exists another quaternion s such that $|s| = 1$, $s^{-1}qs = A + Bj$, where $A + Bj$ is a complex number with $B \geq 0$.*

Lemma 3 is a consequence of Lemmas 4, 5, 6, 7. It is proved also in [4], which refers to [2]. Another proof is given here because this proof is so direct, and because Lemma 5 appears to be new.

LEMMA 4. *Let $q = A + Bj + Ck + Djk$, $s = E + Fj + Gk + Hjk$, $|s| = 1$. The four components of $s^{-1}qs$ are respectively*

$$A,$$

$$B[E^2 + F^2 - G^2 - H^2] + 2C[FG + EH] + 2D[FH - EG],$$

$$2B[FG - EH] + C[E^2 + G^2 - F^2 - H^2] + 2D[EF + GH],$$

$$2B[EG + FH] + 2C[GH - EF] + D[E^2 + H^2 - F^2 - G^2].$$

LEMMA 5. *If $q = A + Bj + Ck + Djk$, then $s = E + Fj$ exists such that $|s| = 1$, $s^{-1}qs$ has fourth component zero.*

Proof. If $D = 0$, take $s = 1$. If $D \neq 0$, set $s = t/|t|$, where
$$t = C - (C^2 + D^2)^{1/2} + Dj.$$

LEMMA 6. *If $q = A + Bj + Ck$, then s exists such that $|s| = 1$; $s^{-1}qs$ has third and fourth components both zero.*

Proof. If $C = 0$, take $s = 1$. If $C \neq 0$, set $J = B/C$, and take $s = t/|t|$, where $t = -1 + [J + (J^2 + 1)^{1/2}]j + k + [J - (J^2 + 1)^{1/2}]jk$.

LEMMA 7. *If $q = A + Bj$, then s exists such that $|s| = 1$; $s^{-1}qs = A - Bj$.*

Proof. Take $s = (j + jk)/\sqrt{2}$.

COROLLARY. *Every quaternion is similar to its conjugate.*

The referee outlined another proof for the fact that two quaternions with equal norms and traces can be transformed one into the other. Let the quaternions be

$$r = a_0 + a_1 j + a_2 k + a_3 jk, \quad q = b_0 + b_1 j + b_2 k + b_3 jk .$$

Consider then the equation $xr = qx$, where $x = x_0 + x_1 j + x_2 k + x_3 jk$. The four linear homogeneous equations for x_0, x_1, x_2, x_3 which are equivalent with this have as determinant an expression which under the assumption $a_0 = b_0$ reduces to

$$(a_1^2 + a_2^2 + a_3^2 - b_1^2 - b_2^2 - b_3^2)^2 ,$$

which is equal to 0 under the assumptions made.

6. **On characteristic roots.** It has already been proved that any quaternion matrix A can be transformed into triangular form T by some unitary matrix. It follows further from Lemma 3 that A can be transformed by a unitary matrix into triangular form in such a way that the diagonal elements are all of the form $A + Bj$, $B \geq 0$. Indeed this transformation can be brought about by transforming T by an appropriate unitary diagonal matrix.

The diagonal elements $A + Bj (B \geq 0)$ which appear in this last transform of A are unique; that is, any other transform of A which is in triangular form and which has numbers $A + Bj (B \geq 0)$ on the main diagonal will have the same numbers, although not necessarily in the same order.

The above fact is a consequence of general theorems concerning characteristic roots of a matrix.

THEOREM 10. *If λ is a characteristic root of A, then so is $\rho\lambda\rho^{-1}$* (see page 329).

THEOREM 11. *If A is in triangular form, then every diagonal element is a characteristic root.*

Proof. Let $A = (a_{r,s})$ be given: $a_{r,s} = 0$ when $s < r$. It is trivial that $a_{1,1}$ is a characteristic root. Suppose it has been proved that $a_{1,1}$, $a_{2,2}$, $\cdots$, $a_{t,t}$ are characteristic roots. If $a_{t+1,t+1}$ is similar to any one of these, then $a_{t+1,t+1}$ is a characteristic root in virtue of that fact alone. If $a_{t+1,t+1}$ is similar to none of the preceding diagonal elements, then the vector $(x_1, x_2, \cdots, x_{t-1}, x_t, 1, 0, 0, \cdots, 0)$ is a characteristic vector corresponding to the characteristic root $a_{t+1,t+1}$ provided all the following equations are satisfied:

$$a_{t,t}x_t + a_{t,t+1} = x_t a_{t+1,t+1} ,$$

$$a_{t-1,t-1}x_{t-1} + a_{t-1,t}x_t + a_{t-1,t+1} = x_{t-1}a_{t+1,t+1} ,$$

$$\cdots\cdots$$

$$a_{1,1}x_1 + \quad \cdots\cdots \quad = x_1 a_{t+1,t+1} .$$

Equations of the above type have been considered in [7]. It is shown there that if a, b, c are quaternions, and if a is not similar to c, then $ax + b = xc$ has a solution. Hence the above equations can be solved in serial order.

THEOREM 12. *Let a matrix of quaternions be in triangular form. Then the only characteristic roots are the diagonal elements* (*and the numbers similar to them*).

Proof. If for some λ, we have $Ax = x\lambda$, x a non-zero vector, and if A is triangular, then

$$a_{n,n}x_n = x_n \lambda ,$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = x_{n-1}\lambda ,$$

$$\cdots\cdots$$

If $x_n \neq 0$, then λ is similar to $a_{n,n}$. If

$$x_n = x_{n-1} = \cdots = x_{t+1} = 0 , \quad x_t \neq 0 ,$$

then λ is similar to $a_{t,t}$.

THEOREM 13. *Similar matrices have the same characteristic roots* (see page 329).

The determinant-like function ∇ of the matrix A, defined by Study in [10], is the product of the norms of the characteristic roots of A.

COROLLARY. *The product of the norms of the characteristic roots of a matrix of quaternions is a rational integral function of the elements and their conjugates.*

After this article was submitted for publication, the author learned of an article by H. C. Lee [8] which contains many of our results. The methods of proof there are different from ours.

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THE ASYMPTOTIC SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION IN WHICH THE COEFFICIENT OF THE PARAMETER IS SINGULAR

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1. Introduction. In this paper we are concerned with the solutions, for large values of the complex parameter λ, of the ordinary differential equation,

$$w''(s) - [\lambda^2 \sigma(s) + \tau(\lambda, s)] w(s) = 0 . \tag{1}$$

The variable s ranges over a region in the complex plane in which $\sigma(s)$ possesses a factor $(s - s_0)^{-2}$, where s_0 is some fixed point of the region. The asymptotic representations of the solutions of an equation formally identical with (1), but in which $\sigma(s)$ contains a factor $(s - s_0)^{\nu}$, $\nu > -2$, have been considered by Langer [3].

If equation (1) is considered over a region of the complex s-plane in which $\sigma(s)$ and $\tau(\lambda, s)$ are bounded, with $\sigma(s)$ bounded from zero, then it is possible to find a pair of asymptotic forms made up of elementary functions, each of these forms representing a solution over the entire region. If, however, $\sigma(s)$ becomes zero in the region under consideration, the asymptotic representations are complicated by the appearance of the Stokes' phenomenon. This necessitates abrupt but determinate changes in the asymptotic forms, if only elementary functions are used, as certain boundaries are crossed in the s- and λ-planes. The asymptotic representations of the solutions of (1) in this case have been considered by Langer [1] among others, and he has shown the Stokes' phenomenon to be quantitatively dependent upon the order of the zero of $\sigma(s)$. In a later paper [3], the theory was extended to include the cases where $\sigma(s)$ contains a factor $(s - s_0)^{\nu}$, $\nu > -2$, and $\tau(\lambda, s)$ has a pole of first or second order at s_0. He showed that the Stokes' phenomenon is engendered by and depends upon an infinity in either of the two coefficients in (1).

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It is proposed to consider in this paper solutions of equation (1) in a region which contains as the only singularity of $\sigma(s)$ a pole of second order at a point s_0, and in which $\sigma(s)$ is bounded from zero while $\tau(\lambda, s)$ has a pole of first or second order at s_0. Among the functions satisfying an equation of this type we may cite the Bessel functions and certain of the confluent hypergeometric functions.

Although the theory developed by Langer is not applicable to the case presently considered, it is nevertheless found that the broad outlines of the general methods used in the papers mentioned still apply. A differential equation is found which possesses all the essential qualities of (1), and which can be solved explicitly. The solutions of this equation are shown to give asymptotic representations of the solutions of the given equation over definable subregions of the domain in which the coefficients in (1) have the properties assumed above.

In order to arrive at the asymptotic solutions of the given equation, it is found necessary to subdivide the region of large values of λ into a finite number of subregions. For λ in each of these subregions, and for all admitted values of s, two independent asymptotic solutions are derived. Although asymptotic forms of similar structure are derivable for all subregions, the solutions which maintain these forms in the different regions are in general different functions.

2. Hypotheses and normal form of the differential equation. The equation (1) is here considered with the parameter λ ranging over any region of the complex plane in which $|\lambda|$ is unbounded. The variable s also is complex, and ranges over a bounded, simply connected domain R_s containing a point s_0 at which $\sigma(s)$ has a pole of second order. Then in some neighborhood of s_0, $\sigma(s)$ is of the form

$$\sigma(s) = \frac{\psi(s)}{(s - s_0)^2}, \tag{2}$$

where $\psi(s)$ is a single-valued, analytic function bounded from zero. The constants in the product $\lambda^2\ \psi(s)$, which appears in the first coefficient of (1), are adjusted so that $\psi(s_0) = 1$. Expanding $\psi(s)$ about the point s_0, we have

$$\psi(s) = 1 + a_1(s - s_0) + a_2(s - s_0)^2 + \cdots.$$

We assume the conditions a), b), and c) which follow in this section to be satisfied collectively by the coefficients of the differential equation, the domain R_s, and the range of values of the parameter λ. The first two of these conditions are:

a) $\psi(s)$ is a single-valued, analytic function bounded from zero.

b) The coefficient $\tau(\lambda, s)$ has the form

$$\tau(\lambda, s) \equiv \frac{A_1}{(s - s_0)^2} + \frac{B_1}{s - s_0} + C_1(\lambda, s),$$

where A_1 and B_1 are constants, and $C_1(\lambda, s)$ is an analytic function of s, uniformly bounded with respect to λ. (This condition is precisely the same imposed on $\tau(\lambda, s)$ by Langer in [3].)

The equation (1) can always be put in a more convenient form by simple changes of the dependent and independent variables.

Letting (cf. [3; p. 399])

$$s - s_0 = \frac{z^2}{4}, \quad w = z^{1/2} u,$$

we obtain the equation (1) in the form

$$u''(z) - \left[\frac{\rho^2 \phi^2(z) + A}{z^2} + \chi(\rho, z)\right] u(z) = 0, \tag{3}$$

where

$$\rho = 2\lambda, \quad A = 4A_1 + \frac{3}{4},$$

$$\chi(\rho, z) = B_1 + \frac{z^2}{4} C_1(\lambda, s),$$

$$\phi^2(z) = 1 + \frac{a_1}{4} z^2 + \frac{a_2}{16} z^4 + \cdots = 1 + z^2 \Phi(z).$$

The equation (3) is called the *normal form* of (1), and is the one we shall consider in the following discussion. It is to be observed that if the constants a_1 and B_1, appearing in the expressions for $\psi(s)$ and $\tau(\lambda, s)$ respectively, vanish, then equation (1) can be put in normal form (3) by simply translating the origin and changing notation.

Since $\psi(s)$ does not vanish in the domain R_s, $\phi^2(z) \equiv \psi(z^2/4 + s_0)$ does not vanish in the corresponding domain R_z in the z-plane. Consider the domain R_s

lying on a two-sheeted Riemann surface with branch point at s_0. Then the transformation $s - s_0 = z^2/4$ is one-to-one between the bounded, simply-connected domain R_s and the corresponding domain R_z. Denoting by $\phi(z)$ the square root of $\phi^2(z)$ which takes the value one when $z = 0$, we obtain

$$\phi(z) = 1 + b_1 z^2 + b_2 z^4 + \cdots ,$$

or

$$\phi(z) = 1 + z^2 \phi_1(z) , \tag{4}$$

where $\phi_1(z)$ is an analytic function of z in R_z. We are now ready to make the third of our hypotheses:

c) The function $ze^{\Phi_1(z)}$ is schlicht, where

$$\Phi_1(z) = \int_0^z \zeta \phi_1(\zeta)\, d\zeta .$$

Since the function $ze^{\Phi_1(z)}$ has a nonvanishing derivative at $z = 0$, it is schlicht in some neighborhood of this point. The hypothesis c) in effect restricts the z-domain under consideration (and hence R_s) to be one in which this property maintains.

3. The "related" differential equation. Throughout the considerations which follow, the quantities $(\rho^2 + 1/4 + A)^{1/2}$ and $[\phi(z)]^{1/2}$ enter frequently. It serves for notational simplification to denote the former of these by μ, that determination of the root being chosen for which $-\pi/2 < \arg \mu \leq \pi/2$ when $\rho = 0$. We determine $[\phi(z)]^{1/2}$ by the condition $[\phi(0)]^{1/2} = 1$.

In the case where equation (1) is considered over a region in which $\sigma(s)$ is bounded from zero, the asymptotic forms of a pair of solutions can be found, the leading terms of which are (cf. [2], p. 550]).

$$\frac{1}{[\sigma(s)]^{1/4}} \; e^{\pm \lambda \int [\sigma(t)]^{1/2}\, dt}$$

This suggests that, in order to find an approximating equation to equation (3), we consider the functions

$$y(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\pm\mu[\log z + \Phi_1(z)]} , \tag{5}$$

where, because of the relative complexity of our equation, it is found necessary to the following developments to replace the parameter ρ by μ. A direct calculation shows that

$$y''(z) - \frac{\rho^2\phi^2(z) + A}{z^2}\, y(z) = \omega(z)\, y(z)\,, \tag{6}$$

where

$$\omega(z) = \frac{\phi^2(z)}{4z^2} - \frac{1}{4z^2} - \frac{\phi'(z)}{2z\,\phi(z)} - \frac{\phi''(z)}{2\phi(z)} + \frac{3}{4}\left[\frac{\phi'(z)}{\phi(z)}\right]^2 + A\,\Phi(z)\,, \tag{7}$$

the quantity $\Phi(z)$ in the last term being defined by the relation $\phi^2(z) = 1 + z^2\Phi(z)$.

The differential equation (6) appears at first glance to have the same form as equation (3). However, since the denominator of each of the first three terms in the expression for $\omega(z)$ vanishes at the origin, it is necessary to consider this coefficient further. Grouping the first two terms and replacing $\phi^2(z)$ by its expression immediately above, and substituting in the third term from (4) for $\phi(z)$, we can write (7) in the form

$$\omega(z) = \frac{z^2\phi(z)}{4z^2} + \frac{z(2\phi_1(z) + z\phi_1'(z))}{2z\,\phi(z)} - \frac{1}{2}\frac{\phi''(z)}{\phi(z)} + \frac{3}{4}\left[\frac{\phi'(z)}{\phi(z)}\right]^2 + A\,\Phi(z). \tag{8}$$

Since $\phi(0) \neq 0$, it follows from (8) that if $\omega(0)$ is defined appropriately, then $\omega(z)$ is analytic throughout R_z.

In virtue of the analyticity of $\omega(z)$ over R_z, the differential equation (6) possesses all of the essential qualities of (3). Following Langer's terminology, we refer to the equation (6) as the "related" equation. The formulas (5) give explicitly a pair of independent solutions of this equation.

4. Solutions of the related equation. For convenience, let us define ξ by the formula

$$\xi = \mu[\log z + \Phi_1(z)]\,. \tag{9}$$

With this, the functions (5) which solve the related equation (6) may be written

$$y_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi}\,, \quad y_2(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{-\xi}\,. \tag{10}$$

The related equation (6) has a regular singular point at $z = 0$, with exponents

$1/2 \pm \mu$. For a fixed value of the parameter μ, it is seen that, in a neighborhood of the origin, the formulas (10) are of the form

$$(11) \qquad y_1(z) = z^{1/2+\mu}O(1) , \qquad y_2(z) = z^{1/2-\mu}O(1) ,$$

where $O(1)$ stands as usual for a bounded function of z.

From the formulas (11) it is seen that, if $\Re(\mu) > 0$, then $y_1(z)$ approaches zero as z approaches zero. The function $y_1(z)$ is in fact singled out as that solution of equation (6) which vanishes at $z = 0$ to a higher order than any other. At $z = 0$, $y_2(z)$ on the other hand vanishes or becomes infinite according as $\Re(\mu)$ is less than or greater than $1/2$.

If $\Re(\mu) < 0$, the behaviors of $y_1(z)$ and $y_2(z)$ in this respect are reversed.

5. **The transformation $\xi = \mu[\log z + \Phi_1(z)]$.** Consider the transformation

$$(12) \qquad \zeta = ze^{\Phi_1(z)} .$$

Since the function on the right of the equality sign is schlicht by hypothesis, the domain R_z is mapped conformally onto a corresponding domain which contains the origin in the ζ-plane.

Further, let w be defined by the relation

$$(13) \qquad w = \log \zeta .$$

If the ζ-domain is cut along the axis of negative real numbers, it is mapped in a one-to-one manner by the transformation (13) onto a semi-infinite strip of width $2\pi(-\pi < \Im(w) \leq \pi)$ parallel to the real axis in the w-plane.

Omitting the intermediate transformation (13), we see that the relation

$$(14) \qquad w = \log z + \Phi_1(z)$$

may be applied directly to the domain R_z. In order that (14) be a one-to-one transformation, the choice above of the strip in the w-plane imposes upon R_z a cut, the image of the upper edge of the strip, from $z = 0$ to a point on the boundary.

Let r_w denote the following subregion of the region in the w-plane: the semi-infinite, rectangular strip bounded on the right by the line $\Re(w) = K$, subject of course to the restriction that the right boundary of r_w lie in the fundamental region in the w-plane. The image in the z-plane of r_w is denoted by r_z.

The transformation (9) maps the region r_w conformally onto a region r_ξ in the ξ-plane. It is evident that the region r_ξ is obtained from r_w by a magnification with the factor $|\mu|$ coupled with a rotation about the origin through an angle arg μ.

6. **Gamma curves.** In the region r_w, denote the lower right corner by w_1^* and the upper right corner by w_2^*. In order to avoid unnecessary duplications, let us for the moment denote either of these points by w_j^*. Through every point W of r_w there passes a broken line consisting of that part of the horizontal line, $\mathfrak{I}(w) = \mathfrak{I}(W)$, contained in r_w, together with that portion of the bounding segment, $\mathfrak{R}(w) = K$, connecting this line to the point w_j^*. The images in r_z of this set of curves in r_w are referred to as the Γ-curves corresponding to w_j^*. Thus two sets of curves, corresponding to the two values of $j\,(j = 1, 2)$, are defined in r_z.

In r_z, the Γ-curves of either set are uniformly bounded in length. For by direct calculation we have

$$dz = \frac{z}{\phi(z)}\, dw .$$

From (14) it follows that

$$|z| = \left| e^{w-\Phi_1(z)} \right| ,$$

and hence that

$$|dz| \leq M \cdot |e^w| \cdot |dw| ,$$

where M is the least upper bound of

$$\left| \frac{1}{\phi(z)}\, e^{-\Phi_1(z)} \right|$$

in R_z.

As the variable point w traces out a horizontal line in r_w, $\mathfrak{I}(w)$ is constant, and with $\eta = \mathfrak{R}(w)$ we have

$$|dz| \leq Me^{\eta} |d\eta| .$$

Also, along the portion of the line $\mathfrak{R}(w) = K$ bounding r_w on the right, let $\mathfrak{I}(w) = \kappa$. Then we have

$$|dz| \leq Me^{K} |d\kappa| .$$

From the way in which the Γ-curves were defined, it follows that, if Γ denotes any one of these curves of either set, then

$$\int_{\Gamma} |dz| \leq M\int_{-\infty}^{K} e^{\eta}\, d\eta + Me^{K}\int_{-\pi}^{\pi} d\kappa = Me^{K}(1+2\pi)\,.$$

Since the term on the extreme right is independent of the particular Γ-curve chosen, the Γ-curves are uniformly bounded in length.

7. Solutions of the original equation. We have exhibited the related equation (5) which possesses all of the essential features of the equation (3), and which admits the independent solutions $y_1(z)$, $y_2(z)$ given by (10). This, as we now proceed to show, enables us to write two formal solutions of (3). The latter equation can obviously be written in the form

$$u''(z) - \left[\frac{\rho^2\phi^2(z) + A}{z^2} + \omega(z)\right] u(z) = \delta(\rho, z)\, u(z)\,, \tag{15}$$

where

$$\delta(\rho, z) \equiv \chi(\rho, z) - \omega(z)\,, \tag{16}$$

a function bounded uniformly with respect to ρ and analytic in z over the region r_z. Regarding (15) as an inhomogeneous differential equation, we see that the reduced equation coincides with (6). Thus, using a standard procedure in differential equations, we can describe a pair of independent solutions of (15) by the relations

$$u_j(z) = y_j(z) - \frac{1}{W}\int_{z_0}^{z} [y_1(z)\, y_2(z_1) - y_2(z)\, y_1(z_1)]\, \delta(\rho, z_1)\, u_j(z_1)\, dz_1 \qquad (j = 1, 2)\,. \tag{17}$$

Here W is the Wronskian of $y_1(z)$ and $y_2(z)$, direct calculation yielding $W = -2\mu$, while z_0 is any fixed point in r_z. To each solution of the equation (6), (17) relates a solution of the equation (3).

With the definitions[1]

$$Y_j(z) = z^{-1/2} e^{\mp\xi}\, y_j(z)\,, \quad U_j(z) = z^{-1/2} e^{\mp\xi}\, u_j(z)\,, \tag{18}$$

---

[1] It is convenient to use the double sign to indicate the combination of two formulas into one. The upper sign is to be associated with $j = 1$, and the lower sign with $j = 2$.

and with C denoting the path of integration in r_z, the equation (17) takes the form

$$U_j(z) = Y_j(z) + \frac{1}{2\mu} \int_C K_j(\rho, z, z_1) U_j(z_1)\, dz_1 \,, \tag{19}$$

where the kernel of this integral equation, denoted here by $K_j(\rho, z, z_1)$, has the following definition:

$$\begin{aligned} K_j(\rho, z, z_1) \\ &= \pm z_1 \delta(\rho, z_1)[Y_j(z) Y_{3-j}(z_1) - Y_{3-j}(z) Y_j(z_1) e^{\mp 2(\xi - \xi_1)}] ; \end{aligned} \tag{20}$$

ξ_1 is defined as the image of z_1 under the transformation (9).

Carrying out the process of iteration on (19), we arrive at the formal expression

$$U_j(z) = Y_j(z) + \sum_{n=1}^{\infty} Y_j^{(n)}(z) \,, \tag{21}$$

with

$$Y_j^{(n+1)}(z) = \frac{1}{2\mu} \int_C K_j(\rho, z, z_1) Y_j^{(n)}(z_1)\, dz_1 \,, \tag{22}$$

$$Y_j^{(0)}(z) = Y_j(z) \,.$$

We shall now show that for arg μ in a suitably restricted range, it is possible to choose z_0 for $j = 1, 2$ so that when $|\mu|$ is sufficiently large, the series (21) converges uniformly and hence represents an actual solution of equation (3). In accordance with this, the μ-plane will be subdivided into its four quadrants, and the asymptotic forms of the solutions derived in each quadrant. This particular choice of the subdivision of the μ-plane is in part due to the configuration of r_z, and in part due to the reversal of the behaviors of $y_1(z)$ and $y_2(z)$ as the imaginary axis in the μ-plane is crossed.

Case 1, $0 \leq \arg \mu < \pi/2$. *First Solution.* In (17) let us choose as the path of integration a curve belonging to the set of Γ-curves corresponding to w_1^*, with $z_0 = 0$. It is to be noted that upon any curve of this set, the quantity $\Re(\xi)$ increases monotonically with the arc length.

Referring to the equations (10), we observe that

$$|Y_j(z)| < M \,, \tag{23}$$

where M is a suitable large constant. This results from the fact that $\phi(z)$ is analytic in r_z and bounded from zero.

Consider the relation

$$(24) \qquad |Y_1^{(n)}(z)| < \frac{M^{n+1}}{|2\mu|^n} \qquad (n = 0, 1, 2, \cdots).$$

This, in view of (23), is evidently satisfied for $n = 0$. It can be shown in the following manner that the validity of this relation for any n implies it for $n + 1$, so that by induction the relation is established for all n.

According to (22), with Γ denoting the Γ-curve which forms the path of integration, we have

$$(25) \qquad |Y_1^{(n+1)}(z)| < \frac{M^{n+1}}{|2\mu|^{n+1}} \int_\Gamma |K_1(\rho, z, z_1)| \cdot |dz_1|.$$

Now let us consider the kernel $K_1(\rho, z, z_1)$, which is defined by the formula (20). From (16), the function $\delta(\rho, z)$ is analytic over r_z and hence bounded. The relations (23) guarantee the boundedness of Y_1 and Y_2. Furthermore, since $\Re(\xi - \xi_1) \geq 0$ on the path of integration, the exponential term is bounded. It follows that the integral on the right of (25) is bounded, and we have

$$(26) \qquad |Y_1^{(n+1)}(z)| < \frac{M^{n+2}}{|2\mu|^{n+1}} N.$$

In this it is clear that N is independent of n. Hence if we choose M at least as large as N, then we have

$$(27) \qquad |Y_1^{(n+1)}(z)| < \frac{M^{n+2}}{|2\mu|^{n+1}}.$$

This completes the induction.

In virtue of the relations (24), it is clear that the infinite series on the right of equation (21) converges uniformly for values of μ satisfying the inequality $2|\mu| > M$. Furthermore, from (21) it follows that

$$U_1(z) = Y_1(z) + \frac{O(1)}{2\mu}$$

for large values of μ. Substituting for $Y_1(z)$ and $U_1(z)$ from (18), we can write this equation in the form

$$u_1(z) = y_1(z) + z^{1/2} e^{\xi} \frac{O(1)}{2\mu} .$$

Replacing $y_1(z)$ by its expression as given in (10), we have

$$u_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi} \left[1 + \frac{O(1)}{2\mu}\right], \tag{28}$$

where $|\mu|$ is sufficiently large.

Case 1, $0 \leq \arg \mu < \pi/2$. *Second Solution.* To obtain a second solution of (3) for this range of μ, we choose as the curves of integration in (17) the same set of Γ-curves used in obtaining the first solution, but we now take $z_0 = z_1^*$, the point on the boundary of r_z which maps into w_1^* under the transformation (14). On any one of these Γ-curves, the quantity $\Re(\xi)$ is monotone decreasing with respect to the arc length.

Consider the relation

$$|Y_2^{(n)}(z)| < \frac{M^{n+1}}{|2\mu|^n}, \tag{29}$$

where M is a suitably large constant. According to the equations (23), this relation is satisfied for $n = 0$. We proceed to show by induction that it is true for all n. Assume the relation to be valid for n. From (22), it follows that

$$|Y_2^{(n+1)}(z)| < \frac{M^{n+1}}{|2\mu|^{n+1}} \int_{\Gamma} |K_2(\rho, z, z_1)| \cdot |dz_1| . \tag{30}$$

The kernel $K_2(\rho, z, z_1)$ is given by the formula (20). Arguments entirely similar to those employed in showing the boundedness of $K_1(\rho, z, z_1)$ in the relation (25) may be used here to establish the boundedness of $K_2(\rho, z, z_1)$ in (30). In fact, the only significant difference in this latter kernel is in the exponential term, which is bounded since we have $\Re(\xi - \xi_1) \leq 0$ along the path of integration. It follows that

$$|Y_2^{(n+1)}(z)| < \frac{M^{n+1}}{|2\mu|^{n+1}} N, \tag{31}$$

where N is a constant independent of n. By choosing M at least as large as N, we can write (31) in the form

(32) $$| Y_2^{(n+1)}(z) | < \frac{M^{n+2}}{|2\mu|^{n+1}} \; .$$

The induction is complete.

As in the previous solution, the infinite series appearing on the right of (21) converges uniformly for sufficiently large values of $|\mu|$. This enables us to rewrite (21), for such values of μ, in the form

$$U_2(z) = Y_2(z) + \frac{O(1)}{2\mu} \; .$$

If $Y_2(z)$ and $U_2(z)$ are replaced by their equivalent expressions given in (17), we obtain

$$u_2(z) = y_2(z) + z^{1/2} e^{-\xi} \frac{O(1)}{2\mu} \; .$$

Substituting from (10) for $y_2(z)$, we can write this equation as follows:

(33) $$u_2(z) = \left[\frac{z}{\phi(z)} \right]^{1/2} e^{-\xi} \left[1 + \frac{O(1)}{2\mu} \right],$$

for $|\mu|$ sufficiently large.

The equation (3), as was pointed out for the related differential equation, has a regular singular point at $z = 0$, with exponents $1/2 \pm \mu$. For large values of μ satisfying the condition $0 \leq \arg \mu < \pi/2$, the relations (28) and (33) give the asymptotic forms of a pair of independent solutions of (3). It is easily seen from (28) and (33), for a constant value of μ in this range, that in the neighborhood of the origin we have

(34) $$u_1(z) = O(z^{1/2 + \mu})$$

$$u_2(z) = O(z^{1/2 - \mu}) \; .$$

Since $\Re(\mu) > 0$, $u_1(z)$ is determined uniquely as that solution of the equation (3) which vanishes at $z = 0$ to a higher order than any other. The solution $u_2(z)$ either vanishes or becomes infinite at $z = 0$, according as $\Re(\mu)$ is less than or greater than $1/2$. It is evident that this behavior of $u_2(z)$ is assumed by any solution independent of $u_1(z)$.

Case 2, $\pi/2 \leq \arg \mu < \pi$. *First Solution.* For this range of arg μ, let us choose as the curves of integration in (17) the Γ-curves corresponding to w_2^*, with $z_0 = z_2^*$, the point on the boundary of r_z which is the image of w_2^* under the transformation (14). Upon any one of these curves, the quantity $\Re(\xi)$ increases monotonically with the arc length.

Carrying out an induction argument exactly like that used in obtaining the first solution of Case 1, we can establish the relation

$$|Y_1^{(n)}(z)| < \frac{M^{n+1}}{|2\mu|^n}, \tag{35}$$

for all nonnegative integral values of n. Here M is a suitably determined constant. The uniform convergence, for sufficiently large values of μ, of the series on the right of (21) follows immediately, yielding the formula

$$U_1(z) = Y_1(z) + \frac{O(1)}{2\mu}.$$

Just as in the previous case, this can be rewritten in the form

$$u_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi}\left[1 + \frac{O(1)}{2\mu}\right], \tag{36}$$

for $|\mu|$ sufficiently large.

Case 2, $\pi/2 \leq \arg \mu < \pi$. *Second Solution.* In order to find the asymptotic form of a solution independent of $u_1(z)$, we choose as the curves of integration in (17) the Γ-curves corresponding to w_2^*, with $z_0 = 0$. Along any one of these curves, $\Re(\xi)$ is monotone decreasing with respect to the arc length.

In a manner which is formally identical with the argument used to establish (29), we arrive at the analogous relation

$$|Y_2^{(n)}(z)| < \frac{M^{n+1}}{|2\mu|^n},$$

for all values of n, where M is a suitably chosen constant.

The formula (21), the right hand side of which converges uniformly for large values of μ in virtue of the preceding relation, yields the expression

$$U_2(z) = Y_2(z) + \frac{O(1)}{2\mu}.$$

By making the appropriate substitutions from (18) and (10), we obtain

$$u_2(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{-\xi}\left[1 + \frac{O(1)}{2\mu}\right], \tag{37}$$

for $|\mu|$ sufficiently large.

Since for values of μ in the second quadrant we have $\Re(\mu) < 0$, the behavior of $u_1(z)$ and $u_2(z)$ is quite different from the behavior of the solution having the same asymptotic form in the first quadrant of μ values. In fact, $u_2(z)$ is now singled out as the solution of (3) which vanishes at $z = 0$ to a higher order than any other, whereas $u_1(z)$ either vanishes or becomes infinite according as $\Re(\mu)$ is greater than or less than $-1/2$. It is to be observed that although the asymptotic forms of the two independent solutions in the second quadrant are the same as those found in the first quadrant, the solutions themselves are in general different.

Case 3, $\pi \leq \arg \mu < 3\pi/2$. For $\arg \mu$ in this range, the curves of integration in the formula (17) are chosen as the Γ-curves corresponding to w_1^*. To find the asymptotic expression for $u_1(z)$ we take $z_0 = z_1^*$, whereas to find the asymptotic form of $u_2(z)$ we choose $z_0 = 0$. Omitting the calculations, which are by now familiar, we arrive at the forms:

$$u_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi}\left[1 + \frac{O(1)}{2\mu}\right], \tag{38}$$

$$u_2(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{-\xi}\left[1 + \frac{O(1)}{2\mu}\right],$$

for $|\mu|$ sufficiently large.

The behaviors of the two independent solutions in this quadrant of the μ-plane are clearly similar to the behaviors of the corresponding solutions described in Case 2. It will be observed from the choice of z_0 that the solution $u_2(z)$ is the same in the second and third quadrants, while $u_1(z)$ is in general quite different in these two regions.

Case 4, $3\pi/2 \leq \arg \mu < 2\pi$. For values of μ in this quadrant, the Γ-curves corresponding to w_2^* are chosen as the curves of integration in the formula (17). We take $z_0 = 0$ in deriving the expression for $u_1(z)$, and $z_0 = z_2^*$ in deriving the expression for $u_2(z)$. Omitting the calculations, we arrive at the usual asymptotic forms

$$u_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi}\left[1 + \frac{O(1)}{2\mu}\right],$$

$$u_2(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{-\xi}\left[1 + \frac{O(1)}{2\mu}\right],$$

for $|\mu|$ sufficiently large.

The pair of solutions in the fourth quadrant of the μ-plane described by these forms have the same characteristics as the corresponding pair found in Case 1, and hence we omit the discussion of their behavior. It is to be noted in comparing Cases 1 and 4 that the solution $u_1(z)$ is the same, whereas $u_2(z)$ in general is different in the two quadrants considered.

We may now summarize the results of this investigation as follows:

THEOREM. *For values of* $\mu = [\rho^2 + 1/4 + A]^{1/2}$ *in a given quadrant of the complex plane,* $(j-1)\pi/2 \leq \arg \mu < j\pi/2$, $j = 1, 2, 3, 4$, *and for all* z *in* r_z, *the differential equation* (3) *admits of a pair of solutions* $u_j(z)$, $j = 1, 2$, *having the forms*

$$u_1(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{\xi}\left[1 + \frac{O(1)}{2\mu}\right],$$

$$u_2(z) = \left[\frac{z}{\phi(z)}\right]^{1/2} e^{-\xi}\left[1 + \frac{O(1)}{2\mu}\right], \qquad \xi = \mu[\log z + \Phi_1(z)],$$

for values of $|\mu|$ *sufficiently large.*

The solution with the exponent $1/2 + \mu$ relative to the origin, denoted above by $u_1(z)$, is the same in the first and fourth quadrants of admissable μ values. The solution, designated by $u_2(z)$, with the exponent $1/2 - \mu$ relative to the origin is the same in the second and third quadrants of the μ-plane. In each of these cases, the second solution is in general different in the two regions mentioned.

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AN EXTENSION OF TIETZE'S THEOREM

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1. Introduction. Let X be an arbitrary metric space, A a closed subset of X, and E^n the Euclidean n-space. Tietze's theorem asserts that any (continuous) $f: A \longrightarrow E^1$ can be extended to a (continuous) $F: X \longrightarrow E^1$. This theorem trivially implies that any $f: A \longrightarrow E^n$ and any $f: A \longrightarrow$ (Hilbert cube) can be extended; we merely decompose f into its coordinate mappings and observe that, in these cases, the continuity of each of the coordinate mappings is equivalent to that of the resultant map.

Where this equivalence is not true, for example mapping into the Hilbert space, the theorem has been neglected. We are going to prove that, in fact, Tietze's theorem is valid for continuous mappings of A into any locally convex linear space (4.1), (4.3). Two proofs of this result will be given; the second proof (4.3), although essentially the same as the first, is more direct; but it hides the geometrical motivation.

There are several immediate consequences of the above result. First we obtain a theorem on the simultaneous extension of continuous real-valued functions on a closed subset of a metric space (5.1). Secondly, we characterize completely those normed linear (not necessarily complete) spaces in which the Brouwer fixed-point theorem is true for their unit spheres (6.3). Finally, we can generalize the whole theory of locally connected spaces to arbitrary metric spaces. By way of illustration, we prove a theorem about absolute neighborhood retracts that is apparently new even in the separable metric case (7.5).

The idea of the proof of the main theorem is simple. Given A and X, we show how to replace $X - A$ by an infinite polytope; we extend f continuously first on the vertices of the polytope, and then over the entire polytope by linearity. For this we need several preliminary remarks on coverings and on polytopes.

2. On coverings and polytopes. If X is any space, a covering of X by an arbitrary collection $\{U\}$ of open sets is called a *locally finite covering* if, given any

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$x \in X$, there exists a nbd of x meeting only a finite number of the sets of $\{U\}$. If $\{U\}$, $\{V\}$ are any two coverings of X by open sets, $\{V\}$ is a *refinement* of $\{U\}$ if for each $V \in \{V\}$ there is a $U \in \{U\}$ containing it. A. H. Stone has proved [12] that every covering of an arbitrary metric space has a locally finite refinement.

2.1 LEMMA. *Let X be an arbitrary metric space, and A a closed subset of X; then there exists a covering $\{U\}$ of $X - A$ such that:*

2.11 *the covering $\{U\}$ is locally finite;*

2.12 *any* nbd *of $a \in (A -$ interior $A)$ contains infinitely many sets of $\{U\}$;*

2.13 *given any* nbd *W of $a \in A$, there exists a* nbd *W', $a \in W' \subset W$, such that $U \cap W' \neq 0$ implies $U \subset W$.*

Proof. Around each point $x \in (X - A)$, draw a nbd S_x such that diameter $S_x < (1/2)d(x, A)$, where d is the metric in X. This is a covering of $X - A$, since $X - A$ is open. By A. H. Stone's theorem, we can construct a locally finite refinement $\{U\}$. It is then evident that $\{U\}$ satisfies 2.11–2.13.

A covering of $X - A$ satisfying the conditions 2.11–2.13 will be called a *canonical covering* of $X - A$.

2.2 A *polytope P* is a point set composed of an arbitrary collection of closed Euclidean cells (higher dimensional analogs of a tetrahedron) satisfying (a) every face of a cell of the collection is itself a cell of the collection, and (b) the intersection of any two closed cells of P is a face of both of them. A *CW polytope* is a polytope with the *CW* topology of Whitehead [14]: a subset U of P is open if and only if the intersection $U \cap \bar{\sigma}$ of U with every closed cell $\bar{\sigma}$ is open in the Euclidean topology of $\bar{\sigma}$. It is easy to verify:

2.21 *a CW polytope is a Hausdorff space;*

2.22 *in a CW polytope, the star of any cell σ (the collection of all open cells having σ as a face) is an open set;*

2.23 *if Y is an arbitrary space, then $f: P \longrightarrow Y$ is continuous if and only if f is continuous on each cell.*

2.3 As a final preliminary, we need the "nerve" of a covering. Let X be a space, and $\{U\}$ a covering of X by open sets. Consider an abstract nontopologized real linear vector space R spanned by linearly independent vectors $\{p_U\}$ is a fixed one-to-one correspondence with the collection $\{U\}$; the elements of R will be called *points*. The $n+1$ points $p_{U_1}, \cdots, p_{U_n}$ determine an n-cell in the usual way if and only if the corresponding sets satisfy $U_1 \cap \cdots \cap U_n \neq 0$. The polytope determined in this way, with the CW topology, will be called the *nerve* of the covering $\{U\}$, and denoted by $N(U)$.

2.31 THEOREM. *If $\{U\}$ is a locally finite covering of a metric space X, and $N(U)$ the nerve of $\{U\}$, then there exists a continuous $K: X \longrightarrow N(U)$ such that $K^{-1}(\text{star } p_U) \subset U$ for every $U \in \{U\}$.*

Proof. (Cf. Dowker [4], where $N(U)$ is taken as a metric polytope.) Define for each $U \in \{U\}$,

$$\lambda_U(x) = \frac{d(x, X-U)}{\sum_U d(x, X-U)} \qquad (x \in X,\ d \text{ the metric in } X).$$

It is first necessary to investigate the nature of these functions. First we notice that $\Sigma_U d(x, X - U)$ is always a finite sum, since $d(x, X - U) \neq 0$ if and only if $x \in U$, and since the covering being locally finite means x lies in a finite number of U's. Further, since $\{U\}$ is a covering, we have $\Sigma_U d(x, X - U) \neq 0$ for every $x \in X$, and so $\lambda_U(x)$ is well-defined for each $x \in X$. Now each $\lambda_U(x)$ is continuous; in fact, for any $x \in X$ there is a nbd meeting only a finite number of the sets of $\{U\}$; in this nbd, $\lambda_U(x)$ is explicitly determined in terms of a finite number of continuous functions, so λ_U is continuous at each $x \in X$. Finally, it is evident that $\Sigma_U \lambda_U(x) = 1$ for each $x \in X$ and that only a finite number are not zero in some nbd of any point $x \in X$.

The mapping $K: X \longrightarrow N(U)$ is defined by setting

$$K(x) = \sum_U \lambda_U(x) p_U .$$

Now $\lambda_U(x) \neq 0$ if and only if $x \in U$; hence if $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets, then because $\Sigma_U \lambda_U(x) = 1$ for every x, $K(x)$ is the point in the interior of the cell spanned by $(p_{U_1}, \cdots, p_{U_n})$ with barycentric coordinates $\{\lambda_{U_i}(x)\}$. It follows readily that $K^{-1}(\text{star } p_U) \subset U$ for every U. Finally, K is continuous: for, given $x \in X$, let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets; then $K(x)$ is in the interior of $\bar{\sigma} = (\overline{p_{U_1}, \cdots, p_{U_n}})$. Let V be any open set containing $K(x)$; then $V \cap \bar{\sigma}$ is open in the Euclidean topology of $\bar{\sigma}$, and so the continuity of each λ_U shows the existence of an open $W \supset x$ with $K(W) \subset V \cap \bar{\sigma} \subset V$. This proves the assertion. (See also 7.4 in this connection.)

3. The replacement by polytopes. After the above preliminaries, we are ready to perform the "replacement" mentioned in the introduction.

3.1 THEOREM. *Let X be a metric space and A a closed subset of X; then there exists a space Y (not necessarily metrizable) and a continuous $\mu: X \longrightarrow Y$ with the properties:*

3.11 *$\mu | A$ is a homeomorphism and $\mu(A)$ is closed in Y;*

3.12 *$Y - \mu(A)$ is an infinite polytope, and $\mu(X - A) \subset [Y - \mu(A)]$;*

3.13 *each* nbd *of $a \in [\mu(A)$-interior $\mu(A)]$ contains infinitely many cells of $Y - \mu(A)$.*

Proof. Let $\{U\}$ be a canonical covering of $X - A$, and $N(U)$ the nerve of this covering. The set Y consists of the set A and a set of points in a one-to-one correspondence with the points of $N(U)$; to avoid extreme symbolism we denote this set Y by $A \cup N(U)$. The topology in $A \cup N(U)$ is determined as follows:

a. $N(U)$ is taken with the CW topology.

b. A subbasis for nbds of $a \in A$ in $A \cup N(U)$ is determined by selecting a nbd W of a *in* X and taking in $A \cup N(U)$ the set of points $W \cap A$ together with the star of every vertex of $N(U)$ corresponding to a set of the covering $\{U\}$ contained in W. This nbd is denoted by $\widetilde{W}$.

It is not hard to verify that $A \cup N(U)$ with this topology is a Hausdorff space, and that both A and $N(U)$, as subspaces, preserve their original topologies. We now define

$$\mu(x) = \begin{cases} x & (x \in A), \\ K(x) & [x \in (X - A)]. \end{cases}$$

Because of 2.31 and the preceding remarks, the continuity of $\mu(x)$ will be proved as soon as we show it continuous at points of $A \cap (\overline{X - A})$. Let $a \in A \cap \overline{X - A}$, and let $\widetilde{W}$ be a subbasic nbd of $\mu(a)$ in $A \cup N(U)$; this is determined by a nbd W of a in X. Now (2.13) we can determine a nbd W', $a \in W' \subset W$, such that $U \cap W' \neq 0$ implies $U \subset W$, since $\{U\}$ is canonical, and clearly $\{U \mid U \subset W' \cap (X - A)\}$ is not vacuous. We now prove $\mu(W') \subset \widetilde{W}$. In fact, if $x \in W' \cap (X - A)$ let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these; then $K(x)$ is in the interior of the cell spanned by $p_{U_1}, \cdots, p_{U_n}$, and therefore $K(x)$ is in the star of, say, p_{U_1}. But since $U_1 \cap W' \neq 0$, we have $U_1 \subset W$, and so $K(x) \in \widetilde{W}$. This shows

$$K[W' \cap (X - A)] = \mu[W' \cap (X - A)] \subset \widetilde{W}.$$

Finally, since $W' \subset W$ we have $\mu(W' \cap A) \subset W' \cap A \subset \widetilde{W}$, and so $\mu(W') \subset \widetilde{W}$. This proves that μ is continuous. The properties 3.11–3.13 now follow at once.

4. Extension of Tietze's theorem. Let X, Y be arbitrary spaces, and $A \subset X$. Let $f: A \longrightarrow Y$ be continuous. A continuous $F: X \longrightarrow Y$ is called an *extension* of f if $F(a) = f(a)$ for every $a \in A$. We now prove:

4.1 THEOREM. *Let X be an arbitrary metric space, A a closed subset of X, L a locally convex linear space* [10, p. 72], *and $f: A \longrightarrow L$ a continuous map. Then there exists an extension $F: X \longrightarrow L$ of f; furthermore, $F(X) \subset$ [convex hull of $f(A)$].*

Proof. Let us form the space $A \cup N(U)$ of Theorem 3.1. It is sufficient to prove that every continuous $f: A \longrightarrow L$ extends to a continuous $F: A \cup N(U) \longrightarrow L$. In fact, to handle the general case we first define, on $A \subset A \cup N(U)$, the map $\bar{f}(a) = f[\mu^{-1}(a)]$; extending $\bar{f}$ to $\bar{F}$ we can write $F(x) = \bar{F}[\mu(x)]$; it is evident that F is the desired extension of f.

Let then $N(U)_0$ denote the collection of all vertices of $N(U)$; we first define an extension of f to an $f_0: A \cup N(U)_0 \longrightarrow L$ as follows: in each set of $\{U\}$ select a point x_U; then choose an $a_U \in A$ such that $d(x_U, a_U) < 2d(x_U, A)$; if p_U is the vertex of $N(U)$ corresponding to U, set

$$f_0(p_U) = f(a_U)$$
$$f_0(a) = f(a) \qquad (a \in A).$$

We now prove f_0 continuous. It is clearly so on $N(U)$, since the vertices of $N(U)$ are an isolated set (the star of any one vertex excludes all the others). Thus continuity of f_0 need only be checked at A.

Select any nbd V of $f_0(a) = f(a)$; since f is continuous on A, there is a $\delta > 0$ such that $d(a,a') < \delta$ implies $f(a') \in V$. Let W be any nbd of a in X of radius $< \delta/3$. If $U \in \{U\}$ and $U \subset W$, then clearly $d(x_U, a) < \delta/3$, and so $d(a_U, a) \leq d(a_U, x_U) + d(x_U, a) < 2d(x_U, A) + \delta/3 < 2\delta/3 + \delta/3 = \delta$. Thus all vertices of $N(U)_0$ in the nbd $\widetilde{W}$ satisfy $f_0(p_U) = f(a_U) \subset V$. Hence for all $\tilde{x} \in \widetilde{W} \cap [A \cup N(U)_0]$ we have $f_0(\tilde{x}) \in V$ and continuity is proved.

We now extend linearly over each cell of $N(U)$ the mapping already given on the vertices, and thus obtain an F mapping $A \cup N(U)$ into L. This map we now prove continuous; on the basis of 2.23 we need prove F continuous only at points of A.

Let V be a convex nbd of $f(a) = F(a)$. Since f_0 is continuous at a, there is a nbd $\widetilde{W}$ with $f_0\{\widetilde{W} \cap [A \cup N(U)_0]\} \subset V$. Construct now a nbd $W' \subset W$ of a in X such that $U \cap W' \neq 0$ implies $U \subset W$. It follows that all vertices corresponding to sets in the nbd W' have images lying in the convex set V. If p_U is any vertex in the closure of the star of a vertex $p_{U'}$ with $U' \subset W'$, we observe that $U \cap W' \neq 0$ and so $p_U \subset \widetilde{W}$. Thus the vertices of any cell belonging to the closure of the star of any vertex $p_{U'}$ are sent into the convex set $V \subset L$ and therefore the linear extensions over these cells have images lying in V; this shows $F(W') \subset V$. Since L is locally convex, this result implies that F is continuous. It is evident, finally, from the construction, that $F(X) \subset$ [convex hull of $f(A)$], and that F is an extension of f. The theorem is proved.

If Y is a space with the property that, given any metric space X and any closed $A \subset X$, every continuous $f: A \longrightarrow Y$ extends to a continuous $F: X \longrightarrow Y$, we call Y an *absolute retract*. Thus Theorem 4.1 asserts that any locally convex linear space is an absolute retract. The conclusions of the theorem give a slight extension.

4.2 COROLLARY. *Let C be a convex set in a locally convex linear space L. Then C is an absolute retract.*

Proof. This is immediate from the construction of Theorem 4.1, since the extension has an image lying in the convex hull of $f(A)$, and so in C.

Note that C is not required to be closed in L.

4.3 It is possible to give an elementary direct proof of Theorem 4.1 not explicitly involving the space $A \cup N(U)$, by merely explicitly exhibiting the resulting extension that was constructed in 4.2. It has the advantage of exhibiting a certain kind of "linearity" in the constructed extension, which is sometimes more amenable to applications. In fact, using the notations of Theorem 2.31 and Theorem 4.1, we find it is simple to verify directly that

$$F(x) = \sum_U \lambda_U(x) f(a_U) \qquad [x \in (X - A)],$$

$$= f(x) \qquad (x \in A)$$

is the extension of f which we have constructed. The proof of the continuity is essentially a repetition of the last part of 4.1, and is as follows: By the considerations of 2.31, the continuity of F need be proved only at points of A. Select any convex nbd V of $F(a) = f(a)$; we are to find a nbd $W'' \supset a$ with $F(W'') \subset V$.

Since f is continuous on A, there exists a $\delta > 0$ such that $d(a, a') < \delta$ implies $f(a') \in V$. Now let W be a nbd of a in X of radius $< \delta/3$; since $\{U\}$ is canonical, we can find a nbd W', $a \in W' \subset W$, such that whenever $U \cap W' \neq 0$, then $U \in W$. It follows that for any $x_U \in W'$ we have $U \subset W$ and so $d(x_U, a) < \delta/3$; this shows that $d(a_U, a) \leq d(a_U, x_U) + d(x_U, a) < \delta$ and therefore we conclude:

(*) *Whenever* $x_U \in W'$, *then* $F(x_U) = f(a_U) \in V$.

Construct, finally, a nbd W'' such that $a \in W'' \subset W'$ and such that whenever $U \cap W'' \neq 0$, then $U \subset W'$. We are going to show that $F(W'') \subset V$.

In fact, if $x \in W'' \cap (X - A)$, let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets; since $\Sigma_U \lambda_U(x) = 1$ for every $x \in (X - A)$ and $\lambda_U(x) \neq 0$ only if $U = U_i$, $i = 1, \cdots, n$, it follows that $F(x)$ belongs to the (perhaps degenerate) cell in L spanned by $f(a_{U_1}), \cdots, f(a_{U_n})$; and since $U_i \cap W'' \neq 0$ for $i = 1, \cdots, n$, we see from (*) that $f(a_{U_i}) \subset V$, $i = 1, \cdots, n$. This means that the vertices of the cell spanned by $f(a_{U_1}), \cdots, f(a_{U_n})$ are all in the convex set V, so the linear extension lies in V also, and therefore $F(x) \in V$. Since x is arbitrary, we see that

$$F[W'' \cap (X - A)] \subset V.$$

But also, since we have diameter $W'' < \delta$, it follows that $F(W'' \cap A) = f(W'' \cap A) \subset V$, and so $F(W'') \subset V$, as stated. Since L is locally convex, this proves F continuous at points of A, and, as remarked, continuous on X. (See also Kuratowski [9]).

We note that to prove Theorem 4.1 our method requires essentially three things: (1) the existence of a canonical covering of $X - A$, (2) the possibility of mapping $X - A$ into the nerve of a canonical covering, and (3) the possibility retracting the set $\{x_U\}$ into A; for (3) allows an extension over the vertices of $N(U)$, and then with a linear extension over the cells the theorem follows at once from (1) and (2). The metric enters in obtaining (1) and (3), while the paracompactness comes into play only in establishing (2) (Dowker [4]; Stone [12]). It should be remarked that, after Theorem 4.1 was communicated to R. Arens, he was able to demonstrate that the method used here applies in the case where X is paracompact (but not metric), provided L is a Banach space. Arens' result coincides with one by Dowker (oral communication).

5. Application to the simultaneous extension of continuous functions. The explicit form of the extension given in 4.3 immediately permits us to answer a question of Borsuk [2]. Let Z be a metric space; denote by $C(Z)$ the Banach space of all bounded real-valued continuous functions on Z. We prove, as a first application:

5.1 THEOREM. *Let A be a closed subset of a metric space X; then there exists a linear operation ϕ which makes correspond to each $f \in C(A)$ an extension $\phi(f) \in C(X)$.*

Proof. With the notations of Theorem 4.1, having selected the points a_U once for all, define for every $f \in C(A)$,

$$\phi(f) = \sum_U \lambda_U(x) f(a_U) .$$

Then $\phi(f)$ is clearly an extension of f for every f (see 4.3). We have evidently

$$\phi(f + g) = \phi(f) + \phi(g),$$
$$\|\phi(f)\| = \|f\|,$$

and so ϕ is additive and continuous, hence a linear operation.

The restriction of Borsuk [2] that A be separable is thus not necessary. This result extends, naturally, to Banach space valued functions.

6. Application to normed linear spaces. To give another application, we characterize those normed linear spaces for which Brouwer's fixed-point theorem holds in their unit spheres.

6.1 LEMMA. *Let L be a normed linear space, and $C \subset L$ the set*

$$\{x \mid \|x\| = 1\}.$$

Let $\bar{\sigma}^n$ be any n-cell, and $\beta\bar{\sigma}^n$ its boundary. If C is not compact, then any $f: \beta\sigma^n \longrightarrow C$ can be extended to an $F: \bar{\sigma}^n \longrightarrow C$.

Proof. By a known theorem [1, p. 502] it is enough to show that $f(\beta\bar{\sigma}^n)$ can be contracted to a point over C. Now, since $\beta\sigma^n$ is compact and C is not, it follows that $f(\beta\bar{\sigma}^n)$ cannot cover all of C, so that there exists at least one point $x_0 \in [C - f(\beta\sigma^n)]$. Select its antipode $-x_0$ and define

$$\phi(x, t) = \frac{t(-x_0) + (1-t)f(x)}{\|t(-x_0) + (1-t)f(x)\|} \qquad (0 \le t \le 1,\ x \in \beta\sigma^n).$$

Then ϕ is continuous in x and t, since the denominator cannot vanish for any x because $-x_0$ and $f(x)$ are never antipodal. Since $\phi(x, 0) = f(x)$, $\phi(\beta\bar{\sigma}^n, 1) = -x_0$, and $\|\phi(x,t)\| = 1$ always, ϕ exhibits the desired contraction.

6.2 THEOREM. *Let L be a normed linear space, and $C = \{x \mid \|x\| = 1\}$. If C is not compact, then C is an absolute retract.*

Proof. With the notations of Theorem 4.1, let us take the space $A \cup N(U)$ and the mapping $f: A \longrightarrow C$. By the construction of Theorem 4.1, we extend f to $F: A \cup N(U) \longrightarrow L$ and notice that $F[A \cup N(U)] \subset \widetilde{C} = \{x \mid \|x\| \le 1\}$. Let $C' = \{x \mid \|x\| \le 1/2\}$; then $\widetilde{C} - C'$ is an open set and $F^{-1}(\widetilde{C} - C')$ is an open set containing A. Let us consider the totality of all closed cells contained in $F^{-1}(\widetilde{C} - C')$; this is a closed subpolytope Q of $N(U)$, and because $\{U\}$ is canonical it is easily verified that no point of A can be a limit point of $N(U) - Q$; furthermore, $A \cup Q$ is a closed subset of $A \cup N(U)$.

Let $r(l) = l/\|l\|$; then taking $rF \mid (A \cup Q)$ we observe that this is an extension of $f: A \longrightarrow C$ over the closed set $A \cup Q$, with values in C. We shall now extend $rF \mid (A \cup Q)$ over $N(U) - Q$ with values in C; this is the desired extension of f.

Define

$$\begin{aligned} \phi_0(p) &= rF(p) \qquad && (p \text{ a vertex of } N(U) - Q), \\ &= rF(x) && (x \in A \cup Q). \end{aligned}$$

Then ϕ_0 is an extension of $rF\,|\,(A \cup Q)$ over the vertices of $N(U) - Q$ with values in C; the continuity is evident since we have $rF(p) = F(p)$ for all vertices, and since F is continuous.

We proceed by induction. Let ϕ_n be an extension of ϕ_{n-1} over all $A \cup Q \cup$ [n-cells of $N(U) - Q$], with values in C. We construct ϕ_{n+1} as follows: for any $(n+1)$-cell of $N(U) - Q$, we have $\phi_n(\beta\bar{\sigma}^{n+1}) \subset C$; applying Lemma 6.1, we obtain an extension $\phi_{n+1}: \bar{\sigma}^{n+1} \longrightarrow C$; extending over every $n+1$-cell, with values in C, we obtain ϕ_{n+1}. Now, ϕ_{n+1} is continuous, in virtue of 2.23 and because no point of A is a limit point of $N(U) - Q$. Defining

$$\phi(x) = \lim_n \phi_n(x)$$

for each $x \in A \cup N(U)$, we observe that ϕ is continuous; further, ϕ is an extension with values in C of $rF\,|\,(A \cup Q)$, and hence of $f: A \longrightarrow C$. This proves the assertion.

6.3 THEOREM. *Let L be a normed linear space, and $S = \{x \,|\, \|x\| \leq 1\}$. A necessary and sufficient condition that every continuous $f: S \longrightarrow S$ have a fixed point is that S be compact.*

Proof. If S is compact, the result comes from Tychonoff's Theorem [13]. If S is not compact, it follows readily that $C: \{x \,|\, \|x\| = 1\}$ is not compact either. Let $F: S \longrightarrow C$ be an extension of the identity map $I: C \longrightarrow C$ (6.2 Theorem). Setting $\phi(x) = -F(x)$, we see that ϕ has no fixed point.

In particular (Banach, [2, p. 84]) this proves that the Brouwer fixed-point theorem for the unit sphere of any infinite dimensional Banach space is not true. This is a partial answer to a question of Kakutani [6] who showed that in the Hilbert space a fixed-point free map of the unit sphere in itself can in fact be selected to to be a homeomorphism.

6.4 COROLLARY. *Let L be a normed linear space with noncompact*

$$C: \{x \,|\, \|x\| = 1\}.$$

Then C is contractible on itself to a point.

Proof. Form the metric space $C \times I$, I the unit interval, and map $C \times 0$ by the identity, $C \times 1$ by a constant map. Since C is an absolute retract, the map on $C \times 0 \cup C \times 1 \subset C \times I$ extends to a $\phi: C \times I \longrightarrow C$, and this ϕ gives the required deformation.

7. Application to a generalization of the theory of locally connected spaces. For our final application, we show that the entire theory of locally connected spaces can be extended to arbitrary metric spaces. In this development, as in that for the separable metric spaces (Fox [5]), the role of the Hilbert cube in the classical theory is taken over by a whole class of "universal" spaces. Kuratowski [8] has shown that any metric space Z can be embedded in the Banach space $C(Z)$ of all bounded continuous real-valued functions on Z. Subsequently, Wojdyslawski [15] has pointed out that, in the Kuratowski embedding of $Z \longrightarrow C(Z)$, Z is a closed subset of its convex hull $H(Z)$. The "universal" spaces in our development are the convex sets in Banach spaces. We shall illustrate the technique by proving a theorem (7.5) about "factorization" of mappings into absolute nbd retracts.

If A is a subset of X, A is called a *retract* of X if there exists a continuous $r: X \longrightarrow A$ such that $r(a) = a$ for each $a \in A$; if X is a Hausdorff space, it follows that a retract of X is closed in X. Now we prove the following result.

7.1 THEOREM. *The following two properties of a metric space Y are equivalent:*

7.11 *In every metric space $Z \supset Y$ in which Y is closed, there is a* nbd *$V \supset Y$ of which Y is a retract.*

7.12 *If X is any metric space, A a closed subset of X, and $f: A \longrightarrow Y$, there exists a* nbd *$W \supset A$ and an extension $F: W \longrightarrow Y$ of f.*

Proof. We need only prove that 7.11 implies 7.12, the converse implication being trivial. Let Y be embedded in $H(Y)$ as a closed subset. By Corollary 4.2, we get an extension of $f: A \longrightarrow Y$ to $F: X \longrightarrow H(Y)$. Let V be a nbd of Y in $H(Y)$ which retracts onto Y, and r the retracting function. Then $F^{-1}(V) = W$ is open in X and contains A, and $rF: W \longrightarrow Y$ is an extension of f.

A metric space Y with the properties 7.11, 7.12 is called an *absolute nbd retract*, abbreviated ANR. They are thus characterized as nbd retracts of the set $H(Y)$ in $C(Y)$.

7.2 LEMMA. *Let Y be an ANR. Then given any covering $\{U\}$ of Y, there exists a refinement $\{W\}$ with the property: If X is any metric space and f_0, $f_1: X \longrightarrow Y$ are such that $f_0(x)$, $f_1(x)$ lie in a common set of $\{W\}$ for each $x \in X$, then f_0 is homotopic to f_1, and the homotopy $\phi(x,t)$, $0 \leq t \leq 1$, can be selected so that $\phi(x,I) \subset$ some U for each $x \in X$, where I denotes the* *unit interval.*

Proof. We consider Y embedded in $H(Y) \subset C(Y)$. Since Y is closed in $H(Y)$, and Y is an ANR, there is retraction r of a nbd $V \supset Y$ in $H(Y)$ onto Y. To simplify the terminology, we let a spherical nbd of $y \in H(Y)$ be the intersection of a spherical nbd of y in $C(Y)$ with $H(y)$. For each $y \in Y$, select a spherical nbd $S(y)$ in $H(y)$ such that $S(y) \subset V$ and $S(y) \cap Y \subset$ some U. Finally, for each y, select a spherical nbd $T(y) \subset S(y)$ in $H(Y)$ such that $r[T(y)] \subset S(y)$. The desired covering is $\{T(y) \cap Y\}$; it clearly refines $\{U\}$. If $f_0, f_1 : X \longrightarrow Y$ and $f_0(x), f_1(x)$ are in a common $T(y) \cap Y$ for each x, they can be joined by a line segment that lies in $T(y)$ and therefore lies in V. Letting $\phi(x, t)$ be the point $tf_0(x) + (1 - t) f_1(x)$, we see that $r\phi(x, t)$, $0 \leq t \leq 1$, gives the required homotopy.

It is not known whether this property implies that Y is an ANR. It does follow readily, however, from 7.2, that an ANR is locally contractible. The theorem also holds for LC^n metric spaces, provided $\dim X \leq n$; the property is in fact equivalent to LC^n. It should be noted that Lemma 7.2 holds also if X is any *CW* polytope, since then ϕ is still continuous (Whitehead [14]).

Our second lemma requires the following definition (Lefschetz [11]): Let Y be a space, and $\{U\}$ a covering of Y. Let P be a *CW* polytope, and Q a subpolytope of P containing all the vertices of P. An $f: Q \longrightarrow Y$ is called a *partial realization of P relative to* $\{U\}$ if, for every cell $\sigma \subset P$, we have $f(Q \cap \bar{\sigma}) \subset$ some U.

7.3 LEMMA. *Let Y be an ANR. Then given any covering $\{U\}$ of Y, there exists a refinement $\{V\}$ with the property that any partial realization of any CW polytope P relative to $\{V\}$ extends to a full realization of P relative to $\{U\}$.*

The proof given by Lefschetz [11, 10.2, p. 89] can easily be applied to yield this result, after a preliminary embedding of Y in $H(Y)$. This property is in fact equivalent to ANR; when we restrict P so that $\dim P \leq n + 1$, this property characterizes the LC^n spaces.

The final lemma required is a covering lemma.

7.4 LEMMA. *Let Y be a metric space, and $\{U\}$ a covering of Y. There exists a refinement $\{V\}$ of $\{U\}$ with the property that whenever $\bigcap_\alpha V_\alpha \neq 0$, then $\bigcup_\alpha V_\alpha \subset$ some U. The covering $\{V\}$ is called a barycentric refinement of $\{U\}$* (cf. also Dowker [4]).

Proof. Let $\{U'\}$ be a locally finite refinement of $\{U\}$, and $N(U')$ the nerve of $\{U'\}$, K the barycentric mapping (2.31) $K: Y \longrightarrow N(U')$. Let N' be the barycentric subdivision of the polytope $N(U')$ and $\{p'\}$ its vertices. We take stars in N' (the CW topology of N is a subdivision invariant); then the open sets $V = K^{-1}$ (star p') form the required covering.

We now prove the "factorization" theorem:

7.5 THEOREM. *Let Y be an ANR; then there exists a polytope P and a continuous $g: P \longrightarrow Y$ with the property that, if X is any metric space, and $f: X \longrightarrow Y$, there exists a $\mu: X \longrightarrow P$ such that $g\mu$ is homotopic to f.*

Proof. Let us take the covering of Y by Y alone, and obtain a refinement $\{W\}$ satisfying Lemma 7.2. Let $\{V'\}$ be a refinement of $\{W\}$ satisfying Lemma 7.3 relative to $\{W\}$, and $\{V\}$ a locally finite refinement of a barycentric refinement of $\{V'\}$. We now construct a mapping $g: N(V) \longrightarrow Y$, as follows: if p_v is the vertex of $N(V)$ corresponding to $V \in \{V\}$, select $y_v \in V$ and set $g(p_v) = y_v$. This is clearly a partial realization of $N(V)$. If $(p_{v_1}, \cdots, p_{v_n})$ is a cell of $N(V)$, then $V_1 \cap \cdots \cap V_n \neq 0$ so that $\bigcup_{i=1}^{n} V_i \subset$ some V'; thus all vertices are sent into a set of V'. Hence (7.3), the mapping g extends to a $g: N(V) \longrightarrow Y$. This map g and polytope $N(V)$ are those required.

Now, for any metric space X and $f: X \longrightarrow Y$, construct the covering $\{f^{-1}(V)\}$ of X, and let $\{U\}$ be a barycentric nbd-finite refinement of $\{f^{-1}(V)\}$. We take $K: X \longrightarrow N(U)$ and define $g': N(U) \longrightarrow Y$ as follows: if p_U is a vertex of $N(U)$, select $x_U \in U$ and set $g'(p_U) = f(x_U)$.

Again, as before, g' extends to a mapping of $N(U)$ into Y.

We shall first show that f is homotopic to $g'K$ by showing that for each $x, f(x)$ and $g'K(x)$ are in a common W (7.2). If $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets, then $K(x) \in (p_{U_1}, \cdots, p_{U_n})$; since $g'(p_{U_i}) = f(x_{U_i}) \in f(U_i)$ we have $\bigcup_{i=1}^{n} g'(p_{U_i}) \subset \bigcup_{i=1}^{n} f(U_i) \subset V$, so that $g'K(x)$ is in some $W \supset V$. On the other hand, $f(x) \in f(U_1 \cap \cdots \cap U_n) \subset \bigcup_{i=1}^{n} f(U_i) \subset V$ also; this shows that $g'K(x)$ and $f(x)$ are in a common set W for each x, and hence are homotopic.

Next, we map $N(U)$ into $N(V)$ simplicially as follows: if p_U is a vertex of $N(U)$, select some V with $U \subset f^{-1}(V)$ and set $\pi(p_U) = p_V$. It is easy to verify that π is simplicial. Extending linearly, we have $\pi: N(U) \longrightarrow N(V)$. Again it is simple to verify that $g\pi(x)$ and $g'(x)$ are in a common set W for every $x \in N(U)$, and hence are homotopic.

Thus we see that f is homotopic to $g\pi K$, so that, with $\pi K = \mu$, the theorem is proved.

The property is not known to be equivalent with ANR. The theorem also holds for LC^n spaces, if $\dim X \leq n$; the polytope P can be chosen so that $\dim P \leq n$ in this case. We have the trivial consequence:

7.6 COROLLARY. *If Y is an ANR, and P is the polytope of the theorem, then the continuous homology groups of Y are direct summands of the corresponding groups of P.*

Proof. By taking $X = Y$ and $i: Y \longrightarrow Y$ the identity map, we have i homotopic to $g\mu$; hence, for each n, the homorphism $H_n(Y) \longrightarrow H_n(Y)$ induced by $g\mu$ is the identity automorphism. The result now follows from the trivial group theoretic result:

7.7 THEOREM. *If A, B are two abelian groups and $\mu: A \longrightarrow B$, $g: B \longrightarrow A$ homomorphisms such that $g\mu(a) = a$ for each $a \in A$, then A is isomorphic to a direct summand of B.*

Proof. Since $g\mu(a) = a$ for every $a \in A$, it follows at once that $\mu A \longrightarrow B$ is an isomorphism into. Furthermore, $\mu(A)$ is a retract of B. In fact, defining $r = \mu g$ we see that $r: B \longrightarrow \mu(A)$; further, for each $b = \mu(a)$, we have $r(b) = \mu g \mu(a) = \mu(a) = b$. Since $\mu(A)$ is a retract of B, it is a direct summand of B, and $B = \mu(A) \oplus \text{Kernel } \mu g$.

In the case that Y is a compactum, all coverings involved can be chosen finite, and 7.6 yields known results (Lefschetz [11; p. 109]). If the Y is a separable metric ANR, the coverings can be so chosen (Kaplan [7]) so that the polytope P is a locally finite one.

It should further be remarked that the method of proof used in Theorem 6.2 is a completely general procedure to prove that an ANR which is connected in all dimensions is in fact an absolute retract.

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THE POLARIZATION OF A LENS

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1. Introduction. In a previous paper [3], the author obtained inequalities comparing the capacity of a lens with various geometric quantities of a lens. A lens may be described simply as a solid of revolution determined by the intersection of two spheres. More precisely, if $c > 0$, the solid of revolution generated by revolving about the imaginary axis the area in the complex z-plane defined by the inequalities

$$\theta_1 \leq \arg \frac{z - c}{z + c} \leq \theta_2$$

is called a *lens*. We may suppose $0 < \theta_1 \leq \theta_2 < 2\pi$. It is, however, more convenient to characterize a lens in terms of its exterior angles. Accordingly we denote by α and β the exterior angles which the two portions of the boundary of the generating area make with the real axis. It is easily seen that $\beta = \theta_1$, $\alpha = 2\pi - \theta_2$. We shall assume, as we may without loss of generality, that $\alpha \leq \beta$. The sum of these angles, $\alpha + \beta$, is called the *dielectric angle* of the lens. Clearly we have $\alpha + \beta \leq 2\pi$, and hence we need consider only values of α not exceeding π. Sometimes it is convenient to introduce the radii a and b of the intersecting spheres; these are given by

$$c = a \sin \alpha = b \left|\sin \beta\right| .$$

It is clear that when $\alpha + \beta = \pi$ the lens becomes a sphere; and when $\alpha + \beta \geq \pi$, $\beta \leq \pi$, the lens is convex. When $\beta \neq 0$ and $\alpha \longrightarrow 0$, with a fixed, the lens becomes a sphere of radius a. When $\alpha, \beta \longrightarrow 0$ in such a manner that $\beta = k\alpha$, and a is kept fixed, the lens becomes two tangent spheres of radii a and a/k. When $\alpha, \beta \longrightarrow \pi$, with c fixed, the lens becomes a circular disk of radius c.

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In the present paper we consider the polarization of a lens, obtaining inequalities comparing the average polarization with the capacity and volume of the lens. This investigation is one phase of a general study of relationships between various physical and geometric quantities which has been carried on at Stanford University during the past four years under the direction of Professors Pólya and Szegö [5].

We now explain the concept of polarization as it has been defined by Schiffer and Szegö [6]. (Cf. also Pólya and Szegö [5].) Consider an infinite electric field whose direction is determined by the unit vector $\mathbf{h}$. When a conducting solid is placed in this field, the uniform field will be disturbed; the disturbance is equivalent to superimposing another field on the original one. If the electric potential of the superimposed field is denoted by ψ, then its energy is given, apart from trivial factors, by

$$P = \iiint |\operatorname{grad} \psi|^2 d\tau,$$

the integral being extended over the whole space exterior to the solid.

We note that the function ψ is harmonic and behaves like a dipole at infinity. Also ψ satisfies on the surface of the given solid the boundary condition

$$\psi = \mathbf{h} \cdot \mathbf{r} + \text{constant} ,$$

where $\mathbf{r}$ is the radius vector. (The additive constant must be chosen properly.)

We call the quantity P the *polarization* in the $\mathbf{h}$-direction. It is easily verified that P is a quadratic form in the components of $\mathbf{h}$:

$$P = \sum_{i,k=1}^{3} P_{i,k} h_i h_k .$$

The coefficients of this form depend naturally on the coordinate system used; however, the invariants of this form are independent of the coordinate system. The simplest of these invariants, and the one with which we shall be concerned in this paper, is the *average polarization* P_m, defined by

$$P_{1,1} + P_{2,2} + P_{3,3} = 3P_m . \tag{1}$$

The study of P is facilitated by introducing the expansion of the potential ψ at infinity, where, as has already been observed, it behaves like a dipole. The strength component of this dipole in the direction $\mathbf{h}$ can be represented in terms of the leading coefficient (that of r^{-2}) of the potential; it is a quadratic form in h_i , say

$$E = \sum_{\iota, k=1}^{3} e_{\iota,k} h_\iota h_k .$$

We call it the *dipole form* associated with the polarization. By use of Green's formula, it is easy to establish the elegant relation

$$P = 4\pi E - V ,$$

where V is the volume of the solid. It has been found that E shows a more regular behavior than P. We shall frequently find it convenient to consider $P_m + V$ in the present paper.

It is known that for a sphere we have $P = P_m = 2V$. It is conjectured that $P_m > 2V$ for other solids. Now for arbitrary solids it is well known that the capacity C is not less than the radius of a sphere having the same volume as the given solid. (See e.g. Pólya and Szegö [4].) Thus we have $V \leq (4\pi/3)C^3$. Hence the inequality $P_m \geq (8\pi/3)C^3$ is stronger than $P_m \geq 2V$. An even stronger inequality is

$$P_m + V \geq 4\pi C^3 . \tag{2}$$

Since, as has already been pointed out, E shows a more regular behavior than P, it is not surprising that this last inequality (2) is the easiest to investigate. It can be studied readily in the case of a lens by means of explicit expressions which Schiffer and Szegö [6] have given for e_x, e_y, e_z $(e_{1,1}, e_{2,2}, e_{3,3})$, where the z-axis is in the direction of the axis of the lens. From these we can write at once the expressions for P_x, P_y, P_z, the polarizations in the x-, y-, and z-directions. These formulas with others are collected together for convenience in §2. We then prove in §3 the strongest inequality (2) for the case of the spherical bowl (lens with $\alpha + \beta = 2\pi$). The same inequality is proven in §4 for the so-called Kelvin case (lens with $\alpha + \beta = \pi/2$), in §5 for the case of two tangent spheres, and in §6 for the symmetric lens. More detailed information concerning the behavior of the corresponding ratio is obtained for some of these cases.

2. Basic formulas. In this section we collect for convenience several formulas which will be useful in the later sections. Those for the polarization of the lens in the general case and in the several special cases are obtained from the paper of Schiffer and Szegö [6]. Those for the capacity are taken from a paper by Szegö

[7], which also gives references to the original literature; they are also collected together in the author's previous paper [3], and in the paper of Schiffer and Szegö [6].

For the polarization of the lens, we have

$$P_{1,1} = P_{2,2} = P_x = P_y = 4\pi e_x - V, \quad P_{3,3} = P_z = 4\pi e_z - V, \tag{3}$$

where

$$e_x = e_y = 2c^3 \int_{-\infty}^{\infty} (q^2 + \tfrac{1}{4}) \frac{\operatorname{sh} \pi q \operatorname{ch} (\alpha - \beta)q + \operatorname{sh} (\alpha + \beta - \pi)q}{\operatorname{sh} (\alpha + \beta)q \operatorname{ch} \pi q} dq, \tag{4}$$

$$e_z = 4c^3 \int_{-\infty}^{\infty} q^2 \frac{\operatorname{sh} \pi q \operatorname{ch} (\alpha - \beta)q + \operatorname{sh} (\pi - \alpha - \beta)q}{\operatorname{sh} (\alpha + \beta)q \operatorname{ch} \pi q} dq \tag{5}$$

$$- \frac{4c^3 \left[\int_{-\infty}^{\infty} q \operatorname{th} \pi q \dfrac{\operatorname{sh} (\alpha - \beta)q}{\operatorname{sh} (\alpha + \beta)q} dq \right]^2}{\int_{-\infty}^{\infty} \dfrac{\operatorname{sh} \pi q \operatorname{ch} (\alpha - \beta)q + \operatorname{sh} (\alpha + \beta - \pi)q}{\operatorname{sh} (\alpha + \beta)q \operatorname{ch} \pi q} dq}.$$

For the electrostatic capacity of the lens, we have

$$C = c \int_{-\infty}^{\infty} \frac{\operatorname{sh} \pi q \operatorname{ch} (\alpha - \beta)q + \operatorname{sh} (\alpha + \beta - \pi)q}{\operatorname{sh} (\alpha + \beta)q \operatorname{ch} \pi q} dq. \tag{6}$$

For the case of the spherical bowl, in which $\alpha + \beta = 2\pi$, these formulas yield

$$e_x = e_y = \tfrac{1}{2} c^3 [f''(\alpha) + f''(\pi) + f(\alpha)], \tag{7}$$

$$e_z = c^3 \{ f''(\alpha) - f''(\pi) - [f'(\alpha)]^2 / f(\alpha) \}, \tag{8}$$

and

$$C = cf(\alpha), \tag{9}$$

where

$$f(\alpha) = \int_{-\infty}^{\infty} \left[\frac{\operatorname{ch} (\pi - \alpha)q}{\operatorname{ch} \pi q} \right]^2 dq = \frac{1}{\pi} \left(1 + \frac{\pi - \alpha}{\sin \alpha} \right). \tag{10}$$

We note that $f''(\pi) = (3\pi)^{-1}$.

For the case of the lens with $\alpha + \beta = \pi/2$ (Kelvin case), we have

$$e_x = e_y = \tfrac{1}{2}c^3[k(\alpha) + k(\beta) - k(0) + k''(\alpha) + k''(\beta) - k''(0)]\,, \tag{11}$$

$$e_z = c^3\{k''(\alpha) + k''(\beta) + k''(0) - [k'(\alpha) - k'(\beta)]^2/[k(\alpha) + k(\beta) - k(0)]\}\,, \tag{12}$$

and

$$C = c[k(\alpha) + k(\beta) - k(0)] = a + b - c\,, \tag{13}$$

where

$$k(\alpha) = \int_{-\infty}^{\infty} \frac{\operatorname{ch} 2\alpha q}{\operatorname{ch} \pi q}\,dq = \sec \alpha\,. \tag{14}$$

[See formula (A-1) in Appendix A.].

For the limiting case of two tangent spheres of radii a and b, we have

$$e_x = e_y = \tfrac{1}{2}\left(\frac{ab}{a+b}\right)^3\left[2\psi''(1) - \psi''\left(\frac{a}{a+b}\right) - \psi''\left(\frac{b}{a+b}\right)\right], \tag{15}$$

$$e_z = \left(\frac{ab}{a+b}\right)^3\left\{-2\psi''(1) - \psi''\left(\frac{a}{a+b}\right) - \psi''\left(\frac{b}{a+b}\right) - \frac{\left[\psi'\left(\frac{a}{a+b}\right) - \psi'\left(\frac{b}{a+b}\right)\right]^2}{-\psi\left(\frac{a}{a+b}\right) - \psi\left(\frac{b}{a+b}\right) - 2\gamma}\right\}, \tag{16}$$

and

$$C = \frac{ab}{a+b}\left[-\psi\left(\frac{a}{a+b}\right) - \psi\left(\frac{b}{a+b}\right) - 2\gamma\right], \tag{17}$$

where $\psi(u) = \Gamma'(u)/\Gamma(u)$, $\Gamma(u)$ being Euler's gamma function, and where γ is Euler's constant.

Finally, for the case of the symmetric lens we obtain

$$e_x = e_y = 2c^3 \int_{-\infty}^{\infty} (q^2 + \tfrac{1}{4})(1 - \operatorname{th} \pi q \ \operatorname{th} \alpha q)\, dq \,, \tag{18}$$

$$e_z = 4c^3 \int_{-\infty}^{\infty} q^2 (\operatorname{th} \pi q \ \operatorname{cth} \alpha q - 1)\, dq \,, \tag{19}$$

and

$$C = c \int_{-\infty}^{\infty} (1 - \operatorname{th} \pi q \ \operatorname{th} \alpha q)\, dq \,. \tag{20}$$

An elementary calculation shows that the volume of the lens is given by

$$V = \frac{\pi c^3}{6} \left[(2 - \cos \alpha) \cot \frac{\alpha}{2} \csc^2 \frac{\alpha}{2} + (2 - \cos \beta) \cot \frac{\beta}{2} \csc^2 \frac{\beta}{2} \right]. \tag{21}$$

3. Spherical bowl, $\alpha + \beta = 2\pi$. The volume V of the spherical bowl is clearly zero, so that the inequality (2) becomes $P_m \geq 4\pi C^3$. In this section we consider the ratio $P_m / 4\pi C^3$.

From (1), (3), (7), (8), and (9) we obtain

$$\frac{P_m}{4\pi C^3} = \frac{2f''(\alpha) f(\alpha) + [f(\alpha)]^2 - [f'(\alpha)]^2}{3[f(\alpha)]^4} \,. \tag{22}$$

If we make use of equation (10), which gives $f(\alpha)$ explicitly, and substitute δ for $\pi - \alpha$, we easily obtain

$$\frac{P_m}{4\pi C^3} = \frac{\pi^2}{3(\delta + \sin \delta)^4} \left[3\delta^2 + 4\delta \sin \delta - \delta \sin 2\delta - \tfrac{1}{4} \sin^2 2\delta - 2 \sin \delta \sin 2\delta \right]. \tag{23}$$

We differentiate (23) with respect to δ and find

$$\frac{d}{d\delta} \left(\frac{P_m}{4\pi C^3} \right) = -\frac{\pi^2 (1 + \cos \delta)^2}{3(\delta + \sin \delta)^5} H(2\delta) \,, \tag{24}$$

where

$$H(\delta) = \delta^2 + \delta \sin \delta - 4(1 - \cos \delta) \,. \tag{25}$$

We now proceed to show that $H(\delta) \geq 0$ for $0 \leq \delta \leq 2\pi$. Clearly $H(0) = 0$. Also,

$$H'(\delta) = 2\delta + \delta\cos\delta - 3\sin\delta\,,$$

$$H''(\delta) = 2\sin\delta(\tan\tfrac{1}{2}\delta - \tfrac{1}{2}\delta)\,.$$

If $0 \leq \delta \leq \pi$, then $\sin\delta \geq 0$ and $\tan(\delta/2) \geq \delta/2$, so that $H''(\delta) \geq 0$. But if $\pi \leq \delta \leq 2\pi$, then $\sin\delta \leq 0$ and $\tan(\delta/2) \leq 0$, so that again $H''(\delta) \geq 0$. Thus $H'(\delta)$ increases monotonely as δ increases from 0 to 2π. But $H'(0) = 0$. Thus $H'(\delta) \geq 0$. Since $H(0) = 0$, it follows that $H(\delta) \geq 0$ for $0 \leq \delta \leq 2\pi$. From (24) it follows that $P_m/4\pi C^3$ decreases monotonely as δ increases from 0 to π. But from (23) we easily find that $P_m/4\pi C^3$ has the value $\pi^2/9$ for $\delta = 0$ and the value 1 for $\delta = \pi$. It follows that $P_m/4\pi C^3$ increases monotonely from 1 to $\pi^2/9 \cong 1.097$ as α increases from 0 (sphere) to π (circular disk). Thus for the spherical bowl the inequality $P_m \geq 4\pi C^3$ is proven.

4. **Kelvin case,** $\alpha + \beta = \pi/2$. We now consider the case of a lens of dielectric angle $\pi/2$ formed by the intersection of orthogonal spheres. The polarization and capacity can again be expressed in terms of elementary functions, so that the study of the ratio $(P_m + V)/4\pi C^3$ is not difficult. For this case we use equations (1), (3), (11), (12), and (13) to obtain

$$(26)\quad \frac{P_m + V}{4\pi C^3} = \frac{2[k''(\alpha) + k''(\beta)][k(\alpha) + k(\beta) - k(0)]}{3[k(\alpha) + k(\beta) - k(0)]^4} + \frac{[k(\alpha) + k(\beta) - k(0)]^2 - [k'(\alpha) - k'(\beta)]^2}{3[k(\alpha) + k(\beta) - k(0)]^4}\,,$$

where $k(\alpha)$ is given by (14), and, as throughout this section, $\beta = \pi/2 - \alpha$. We note that $k(\alpha) + k(\beta) - k(0)$ becomes infinite when α tends to zero or $\pi/2$; in order to obtain a fraction whose numerator and denominator remain finite, it is convenient to multiply the numerator and denominator in (26) by $\sin^4\alpha\cos^4\alpha$. If we subtract 1 from both sides of (26), we obtain

$$(27)\qquad \frac{P_m + V}{4\pi C^3} - 1 = \frac{k^*(\alpha)}{3\sin^4\alpha\cos^4\alpha\,[k(\alpha) + k(\beta) - k(0)]^4}\,,$$

where

$$k^*(\alpha) = \sin^4\alpha\cos^4\alpha\{2[k''(\alpha)+k''(\beta)][k(\alpha)+k(\beta)-k(0)] + [k(\alpha)+k(\beta)-k(0)]^2 - [k'(\alpha)-k'(\beta)]^2 - 3[k(\alpha)+k(\beta)-k(0)]^4\}. \tag{28}$$

We note that $k^*(\alpha)$ is always finite. In order to prove the inequality (2), it suffices to prove that $k^*(\alpha) \geq 0$ for $0 \leq \alpha \leq \pi/4$ since, as was pointed out earlier, we can always suppose $\alpha \leq \beta$. We make use of (14) to obtain the following necessary expressions:

$$\sin\alpha\cos\alpha[k(\alpha)+k(\beta)-k(0)] = \sin\alpha + \cos\alpha(1-\sin\alpha), \tag{29}$$

$$\sin^2\alpha\cos^2\alpha[k'(\alpha)-k'(\beta)] = \sin^3\alpha - \cos^3\alpha, \tag{30}$$

$$\sin^3\alpha\cos^3\alpha[k''(\alpha)+k''(\beta)] = \sin^3\alpha + \sin^5\alpha + \cos^3\alpha(2-\sin^2\alpha). \tag{31}$$

If we substitute (29), (30), and (31) in (28) we obtain, after some simplification, $k^*(\alpha) = 2\sin\alpha\cos\alpha(1-\cos\alpha)^2(1-\sin\alpha)^2[\cos\alpha(4-\sin\alpha) + 2(1+2\sin\alpha)]$. It is clear that each factor in this product is nonnegative, and hence $k^*(\alpha) \geq 0$ for $0 \leq \alpha \leq \pi/4$ and indeed for $0 \leq \alpha \leq \pi/2$. As previously noted, this is sufficient to prove the inequality (2) for this case.

5. Two tangent spheres. We now consider two tangent spheres of radii a and b. (We assume without loss of generality that $b \leq a$.) We write $b/(a+b) = z$ (z should not be confused with the z-coordinate), and make use of (1), (3), (15), (16), and (17), obtaining

$$\frac{P_m+V}{4\pi C^3} = \frac{2[-\psi''(z)-\psi''(1-z)][-\psi(z)-\psi(1-z)-2\gamma]}{3[-\psi(z)-\psi(1-z)-2\gamma]^4} - \frac{[\psi'(z)-\psi'(1-z)]^2}{3[-\psi(z)-\psi(1-z)-2\gamma]^4}. \tag{32}$$

Recalling that

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right),$$

we obtain

$$(33)\quad -\psi(z) - \psi(1-z) - 2\gamma = \frac{1}{z(1-z)} + \sum_{n=1}^{\infty}\left[\frac{2n+1}{n^2+n+z(1-z)} - \frac{2}{n}\right].$$

We note that this expression is a function of $z(1-z)$, and make the substitution $z(1-z) = y$. It is clear that z lies between 0 and 1/2, and hence y lies between 0 and 1/4. For such values of y, z is a single-valued function of y given by $2z = 1 - (1-4y)^{1/2}$. We have also

$$(34)\qquad \frac{dy}{dz} = 1 - 2z = (1-4y)^{1/2}.$$

It follows that both $\psi'(z) - \psi'(1-z)$ and $-\psi''(z) - \psi''(1-z)$ are single-valued functions of y for $0 \le y \le 1/4$. Thus, by (32), the same is true of $(P_m + V)/4\pi C^3$. We denote this function by $t(y)$. We shall show that $t(y)$ increases as y increases. We therefore consider $t'(y)$. A simple calculation gives

$$(35)\qquad t'(y) = \frac{M}{3(1-4y)^{1/2}[-\psi(z) - \psi(1-z) - 2\gamma]^5},$$

where

$$M = 4\{\psi'(z) - \psi'(1-z)\}\{2[-\psi''(z) - \psi''(1-z)][-\psi(z) - \psi(1-z) - 2\gamma] - [\psi'(z) - \psi'(1-z)]^2\} - 2\{\psi'''(z) - \psi'''(1-z\}\{-\psi(z) - \psi(1-z) - 2\gamma\}^2.$$

If we make the substitution $z(1-z) = y$ in (33), and let $1/[n(n+1)] = a_n$, we obtain

$$y[-\psi(z) - \psi(1-z) - 2\gamma] = 1 + y\sum_{n=1}^{\infty}\left[\frac{(2n+1)a_n}{1+a_n y} - \frac{2}{n}\right].$$

But

$$\frac{(2n+1)a_n}{1+a_n y} - \frac{2}{n} = -a_n - (2n+1)a_n^2 y + (2n+1)a_n^3 y^2 - \cdots + (-1)^{m-1}(2n+1)a_n^m y^{m-1} + \cdots.$$

It is easily verified that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2n+1)a_n^2 = 1 . \tag{36}$$

For convenience we let

$$b_m = \sum_{n=1}^{\infty} (2n+1)a_n^m \qquad (m = 2, 3, 4, \cdots) . \tag{37}$$

By (36), we have $b_2 = 1$. It follows that

$$y[-\psi(z) - \psi(1-z) - 2\gamma] = 1 - y - y^2 + \sum_{m=3}^{\infty} (-1)^{m-1} b_m \, y^m . \tag{38}$$

Now

$$b_m = \sum_{n=1}^{\infty} (2n+1)a_n^{m-1} \frac{1}{n(n+1)} \leq \tfrac{1}{2} b_{m-1} \qquad (m = 3, 4, 5, \cdots) . \tag{39}$$

Repeated application of this inequality shows that

$$b_m \leq \frac{1}{2^{m-2}} b_2 = \frac{1}{2^{m-2}} . \tag{40}$$

Thus the series in equation (38) certainly converges uniformly and absolutely for $0 \leq y \leq 1/4$.

If we divide equation (38) by y, differentiate with respect to z, and make use of (34), we obtain

$$\begin{aligned} y^2[\psi'(z) - \psi'(1-z)] \\ = (1-4y)^{1/2} \left[1 + y^2 + \sum_{m=3}^{\infty} (-1)^m (m-1) b_m \, y^m \right]. \end{aligned} \tag{41}$$

Similarly we find that

(42) $y^3[-\psi''(z)-\psi''(1-z)]$

$$= 2\left\{1-3y+\sum_{m=3}^{\infty}(-1)^{m-1}(m-2)\cdot\left[\frac{m-1}{2}b_m+(2m-5)b_{m-1}\right]y^m\right\},$$

and

(43) $$y^4[\psi'''(z)-\psi'''(1-z)] = 6(1-4y)^{1/2}Q,$$

where

$$Q = 1-2y+\sum_{m=4}^{\infty}(-1)^m\frac{(m-2)(m-3)}{3}\cdot\left[\frac{m-1}{2}b_m+(2m-5)b_{m-1}\right]y^m .$$

By means of (40) it is easily seen that the series in (41), (42), and (43) are uniformly and absolutely convergent for $0 \le y \le 1/4$. Moreover, the terms of these series as well as those of the series in (38) alternate in sign after the first few terms. If we make use of (39) and (40), we easily verify that the terms in each of these series decrease in absolute value for $0 \le y \le 1/4$. Consequently, each of these series may be conveniently estimated by taking a finite number of its terms. In order to simplify the estimates we need a better estimate for b_m than is given by (40). We easily find that

(44) $$\frac{1}{2^{m-1}} < \frac{3}{2^m} < b_m < \frac{3}{2^m}+\frac{1}{2^{2m-3}} < \frac{1}{2^{m-2}} \qquad (m = 3, 4, 5, \cdots).$$

The following estimates are then obtained:

(45) $$y[-\psi(z)-\psi(1-z)-2\gamma] \le 1-y-y^2+b_3y^3-b_4y^4+\tfrac{1}{8}y^5,$$

(46) $y[-\psi(z)-\psi(1-z)-2\gamma]$

$$\ge 1-y-y^2+b_3y^3-b_4y^4+\tfrac{1}{16}y^5-\tfrac{1}{16}y^6,$$

(47) $y^2[\psi'(z)-\psi'(1-z)]$

$$\le (1-4y)^{1/2}[1+y^2-2b_3y^3+3b_4y^4-\tfrac{1}{4}y^5+\tfrac{1}{4}y^6],$$

(48) $y^2[\psi'(z)-\psi'(1-z)]$

$$\ge (1-4y)^{1/2}[1+y^2-2b_3y^3+3b_4y^4-\tfrac{1}{2}y^5],$$

(49) $$y^3[-\psi''(z)-\psi''(1-z)]$$
$$\geq 2[1-3y+(b_3+1)y^3-3(b_4+2b_3)y^4+3y^5-4y^6],$$

(50) $$y^4[\psi'''(z)-\psi'''(1-z)]$$
$$\leq 6(1-4y)^{1/2}[1-2y+(b_4+2b_3)y^4-2y^5+4y^6].$$

All of these estimates are valid for $0 \leq y \leq 1/4$.

Before substituting these estimates in (35), we find it convenient to estimate certain combinations which appear there. From (49), (46), and (47) we obtain

(51) $$y^4\{2[-\psi''(z)-\psi''(1-z)][-\psi(z)-\psi(1-z)-2\gamma]-[\psi'(z)-\psi'(1-z)]^2\}$$
$$\geq 3-12y+6y^2+12(b_3+2)y^3-(22b_4+56b_3+5)y^4$$
$$+(48b_4+24b_3+\tfrac{51}{4})y^5+(6b_4+12b_3-\tfrac{63}{2})y^6$$
$$+(20b_4-4b_3b_4-8b_3^2+\tfrac{29}{4})y^7+(\tfrac{45}{4}b_3-24b_3b_4+3b_4^2+\tfrac{55}{4})y^8$$
$$-(\tfrac{45}{4}b_4+\tfrac{51}{4}b_3-36b_4^2-\tfrac{7}{4})y^9+(\tfrac{37}{4}b_4-\tfrac{5}{2}b_3+\tfrac{11}{16})y^{10}$$
$$-(\tfrac{11}{8}-6b_4)y^{11}+\tfrac{7}{16}y^{12}+\tfrac{1}{4}y^{13}$$
$$\geq 3-12y+6y^2+12(b_3+2)y^3-(22b_4+56b_3+5)y^4+24y^5-28y^6.$$

In passing to the last inequality we have made use of the inequalities (44) to estimate the coefficients of y^5 and y^6 and to prove that the sum of the last seven terms is nonnegative for $0 \leq y \leq 1/4$. From (45) we find in a similar way that

(52) $$y^2[-\psi(z)-\psi(1-z)-2\gamma]^2$$
$$\leq 1-2y-y^2+2(b_3+1)y^3-(2b_4+2b_3-1)y^4+\tfrac{1}{2}y^6.$$

We proceed in a similar manner using (48) and (51) to obtain

(53) $$4y^6\{\psi'(z)-\psi'(1-z)\}\{2[-\psi''(z)-\psi''(1-z)][-\psi(z)-\psi(1-z)-2\gamma]$$
$$-[\psi'(z)-\psi'(1-z)]^2\}$$
$$\geq (1-4y)^{1/2}[12-48y+36y^2+24(b_3+2)y^3$$
$$-(52b_4+128b_3-4)y^4+150y^5-344y^6].$$

The last combination which we shall need is obtained from (50) and (52); it is

$$\begin{aligned}(54)\qquad & 2y^6[\psi'''(z) - \psi'''(1-z)][-\psi(z) - \psi(1-z) - 2\gamma]^2 \\ & \leq (1-4y)^{1/2}[12 - 48y + 36y^2 + 24(b_3+2)y^3 \\ & \quad - 12(b_4 + 4b_3 + 3)y^4 - 42y^5 + 95y^6].\end{aligned}$$

If we substitute from (53) and (54) into (35), and use the result that

$$b_4 + 2b_3 = \sum_{n=1}^{\infty} (2n+1)(a_n^4 + 2a_n^3) = \sum_{n=1}^{\infty} \left[\frac{1}{n^4} - \frac{1}{(n+1)^4}\right] = 1,$$

we obtain

$$(55)\qquad t'(y) \geq \frac{192 - 439y}{3y[-\psi(z) - \psi(1-z) - 2\gamma]^5}.$$

Now for $0 \leq y \leq 1/4$ it is clear from (38) that $y[-\psi(z) - \psi(1-z) - 2\gamma]$ is finite and positive. Hence by (55) we see that $t'(y) > 0$ for $0 < y \leq 1/4$. It is easily verified from (35) that $t'(0) = 0$. Thus $t(y)$ increases monotonely as y increases from 0 to 1/4. This means that the ratio $(P_m + V)/4\pi C^3$ increases monotonely as b increases from 0 to a, where a and b are the radii of the tangent spheres.

Now we see that $b \longrightarrow 0$ implies $z \longrightarrow 0$ and hence $y \longrightarrow 0$. If we multiply the numerator and denominator of (32) by y^4 and make use of (38), (41), and (42), we find that $b \longrightarrow 0$ implies

$$\frac{P_m + V}{4\pi C^3} \longrightarrow \frac{2(2)(1) - 1}{3} = 1.$$

Also we see that $z = 1/2$ when $a = b$, and hence in this case (32) yields

$$\frac{P_m + V}{4\pi C^3} = \frac{-4\psi''(\frac{1}{2})}{3[-2\psi(\frac{1}{2}) - 2\gamma]^3} = \frac{-\psi''(\frac{1}{2})}{48 \log^3 2} = \frac{7\zeta(3)}{24 \log^3 2} \cong 1.053,$$

where $\zeta(z)$ denotes the Riemann zeta-function. (To obtain the values of $\psi''(1/2)$ and $\psi(1/2)$ see, for example, Copson [2, p. 229].)

Thus as b increases from 0 (one sphere) to a (equal spheres) we see that the ratio $(P_m + V)/4\pi C^3$ increases monotonely from 1 to $7\zeta(3)/(24 \log^3 2) \cong 1.053$. Thus we have proved the inequality (2) for the case of tangent spheres.

Of course the weaker inequalities $P_m \geq (8\pi/3)C^3$ and $P_m \geq 2V$ follow immediately from (2). However, it is instructive to consider the behavior of the corresponding ratios for this case of tangent spheres. This behavior can be deduced from the results just obtained if we first study the behavior of $4\pi C^3/3V$. It has already been pointed out that this ratio is never less than unity [4].

The volume may be obtained from (21) by setting $\beta = (a/b)\alpha$ and letting $\alpha \to 0$, or more simply by direct calculation. It is found to be

$$(56)\qquad V = \frac{4\pi}{3}(a^3 + b^3) = \frac{4\pi}{3}(a+b)(a^2 - ab + b^2)$$

$$= \frac{4\pi}{3}(a+b)^3\left(1 - \frac{3ab}{(a+b)^2}\right) = \frac{4\pi}{3}(a+b)^3(1-3y),$$

since $y = z(1-z)$, $z = b/(a+b)$.

If we now make use of (17) and (56) we find that

$$(57)\qquad \left(\frac{4\pi C^3}{3V}\right)^{1/3} = \frac{y[-\psi(z) - \psi(1-z) - 2\gamma]}{(1-3y)^{1/3}} .$$

This is a function of y and we could differentiate it with respect to y and prove the derivative nonnegative by a method similar to that used in treating $t'(y)$ above. But the following method seems to be more elegant. We have

$$(58)\qquad (1-3y)^{-1/3} = 1 + \sum_{m=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot \cdots \cdot (3m-2)}{m!}\, y^m .$$

If we substitute (58) and (38) into (57), we obtain

$$\left(\frac{4\pi C^3}{3V}\right)^{1/3} = 1 + \sum_{m=3}^{\infty} h_m y^m ,$$

where

$$h_m = \frac{1\cdot 4\cdot 7\cdot \cdots \cdot (3m-2)}{m!} - \frac{1\cdot 4\cdot 7\cdot \cdots \cdot (3m-5)}{(m-1)!} - \frac{1\cdot 4\cdot 7\cdot \cdots \cdot (3m-8)}{(m-2)!}$$

$$+ \sum_{\mu=3}^{m-1} (-1)^{\mu-1} \frac{1\cdot 4\cdot 7\cdot \cdots \cdot(3m-3\mu-2)}{(m-\mu)!} b_\mu + (-1)^{m-1} b_m$$

$$= \frac{1\cdot 4\cdot 7\cdot \cdots \cdot(3m-8)(5)(m-2)(m-1)}{m!}$$

$$+ \sum_{\mu=3}^{m-1} (-1)^{\mu-1} \frac{1\cdot 4\cdot 7\cdot \cdots \cdot(3m-3\mu-2)}{(m-\mu)!} b_\mu + (-1)^{m-1} b_m .$$

The first term in this last expression for h_m is positive, and the rest of the terms are alternately positive and negative and decrease in absolute value. It follows that $h_m > 0$. Thus $(4\pi C^3/3V)^{1/3}$ increases as y increases. The same is therefore true of $4\pi C^3/3V$. Now (57) shows that when $b \longrightarrow 0$, that is, when $y \longrightarrow 0$, this ratio tends to one. When $b = a$, we have $y = 1/4$, and (57) shows that the ratio $4\pi C^3/3V$ is $4 \log^3 2 \cong 1.332$. Thus as b increases from 0 to a we see that the ratio $4\pi C^3/3V$ increases monotonely from 1 to $4 \log^3 2$.

Combining with our previous result we conclude that the ratio $(P_m + V)/3V$ increases monotonely from 1 to $(7/6)\zeta(3) \cong 1.402$ as b increases from 0 to a. Now since

$$\frac{P_m}{2V} = \frac{3}{2}\frac{P_m + V}{3V} - \frac{1}{2},$$

it is clear that $P_m/2V$ increases monotonely from 1 to $(7/4)\zeta(3) - 1/2 \cong 1.604$ as b increases from 0 to a. Finally since

$$\frac{P_m}{(8\pi/3)C^3} = \frac{3}{2}\frac{P_m + V}{4\pi C^3} - \frac{1}{2}\frac{3V}{4\pi C^3},$$

we see that $P_m/[(8\pi/3)C^3]$ increases monotonely from one to the quantity $[7\zeta(3) - 2]/(16 \log^3 2) \cong 1.204$ as b increases from zero to a.

6. Symmetric lens. In this section we prove the inequality (2) for the case of the symmetric lens. If we make use of (1), (3), (18), (19), and (20) we find that

$$\frac{P_m + V}{4\pi C^3} = \frac{\int_{-\infty}^{\infty} (1 - \operatorname{th}\pi q \ \operatorname{th}\alpha q)\, dq + 8\int_{-\infty}^{\infty} \frac{q^2 \ \operatorname{th}\pi q}{\operatorname{sh} 2\alpha q}\, dq}{3\left[\int_{-\infty}^{\infty} (1 - \operatorname{th}\pi q \ \operatorname{th}\alpha q)\, dq\right]^3} . \tag{59}$$

We denote this ratio by $S(\alpha)$. We note that $S(0) = 7\zeta(3)/(24\log^3 2) \cong 1.053$ (two equal tangent spheres), $S(\pi/2) = 1$ (sphere), and $S(\pi) = \pi^2/9 \cong 1.097$ (circular disk). We wish to prove that $S(\alpha) \geq 1$ for $0 \leq \alpha \leq \pi$. We write

$$S(\alpha) = \frac{\alpha^2 g(\alpha) + G(\alpha)}{3g^3(\alpha)}, \tag{60}$$

where

$$g(\alpha) = \alpha \int_{-\infty}^{\infty} (1 - \operatorname{th} \pi q \ \operatorname{th} \alpha q)\, dq = \int_{-\infty}^{\infty} (1 - \operatorname{th} \pi q/\alpha \ \operatorname{th} q)\, dq \tag{61}$$

and

$$G(\alpha) = 8\alpha^3 \int_{-\infty}^{\infty} \frac{q^2 \ \operatorname{th} \pi q}{\operatorname{sh} 2\alpha q}\, dq = 8 \int_{-\infty}^{\infty} \frac{q^2 \ \operatorname{th} \pi q/\alpha}{\operatorname{sh} 2q}\, dq\ . \tag{62}$$

We note that $g(\alpha)$ in (61) is the same function $g(\alpha)$ that was used in [3]. Next we let

$$d(\alpha) = \alpha^2 g(\alpha) + G(\alpha), \tag{63}$$

so that

$$S(\alpha) - 1 = \frac{d(\alpha) - 3g^3(\alpha)}{3g^3(\alpha)}\ . \tag{64}$$

Since $g(\alpha) > 0$ for $0 < \alpha \leq \pi$, it suffices to prove $d(\alpha) - 3g^3(\alpha) \geq 0$ in order to establish the inequality (2).

We note that $d(\alpha) - 3g^3(\alpha)$ has the value zero when $\alpha = \pi/2$, because $S(\pi/2) = 1$. Its value when $\alpha = \pi/4$ or π can also be calculated as we shall see. In proving the desired inequality we shall find it convenient to estimate $d(\alpha)$ and $g(\alpha)$ by means of Taylor's series expansions of these functions in the neighborhoods of the points $\alpha = \pi/4, \pi/2$ and π. We shall therefore need to compute some of the derivatives of $g(\alpha)$, $G(\alpha)$, and $d(\alpha)$, and to study their behavior.

From [3] we find that

$$g'(\alpha) = \pi \int_{-\infty}^{\infty} \frac{q \ \operatorname{th} \alpha q}{\operatorname{ch}^2 \pi q}\, dq\ , \tag{65}$$

$$g''(\alpha) = \pi \int_{-\infty}^{\infty} \frac{q^2}{\text{ch}^2\, \pi q\ \text{ch}^2\, \alpha q}\, dq\ , \tag{66}$$

$$g'''(\alpha) = -2\pi \int_{-\infty}^{\infty} \frac{q^2\ \text{sh}\, \alpha q}{\text{ch}^2\, \pi q\ \text{ch}^3\, \alpha q}\, dq\ . \tag{67}$$

It is clear that $g'(\alpha) \geq 0$, $g''(\alpha) \geq 0$ and $g'''(\alpha) \leq 0$, so that $g(\alpha)$ and $g'(\alpha)$ are monotone increasing functions and $g''(\alpha)$ is a monotone decreasing function. Turning to the consideration of the derivatives of $G(\alpha)$, we have

$$G'(\alpha) = -8\pi\alpha^{-2} \int_{-\infty}^{\infty} \frac{q^3}{\text{sh}\, 2q\ \text{ch}^2\, \pi q/\alpha}\, dq = -8\pi\alpha^2 \int_{-\infty}^{\infty} \frac{q^3}{\text{ch}^2\, \pi q\ \ \text{sh}\, 2\alpha q}\, dq\,, \tag{68}$$

$$G''(\alpha) = 2\alpha^{-1} G'(\alpha) + 16\pi\alpha^2 \int_{-\infty}^{\infty} \frac{q^4\ \text{ch}\, 2\alpha q}{\text{ch}^2\, \pi q\ \ \text{sh}^2\, 2\alpha q}\, dq\,, \tag{69}$$

$$\begin{aligned} G'''(\alpha) = {} & -16\pi \int_{-\infty}^{\infty} \frac{q^3}{\text{ch}^2\, \pi q\ \ \text{sh}\, 2\alpha q}\, dq + 64\pi\alpha \int_{-\infty}^{\infty} \frac{q^4\ \text{ch}\, 2\alpha q}{\text{ch}^2\, \pi q\ \text{sh}^2\, 2\alpha q}\, dq \\ & -32\pi\alpha^2 \int_{-\infty}^{\infty} \frac{q^5\,(2 + \text{sh}^2\, 2\alpha q)}{\text{ch}^2\, \pi q\ \ \text{sh}^3\, 2\alpha q}\, dq. \end{aligned} \tag{70}$$

For the derivatives of $d(\alpha)$, we have

$$d'(\alpha) = \alpha^2 g'(\alpha) + 2\alpha g(\alpha) + G'(\alpha)\,, \tag{71}$$

$$d''(\alpha) = \alpha^2 g''(\alpha) + 4\alpha g'(\alpha) + 2g(\alpha) + G''(\alpha)\,, \tag{72}$$

$$d'''(\alpha) = \alpha^2 g'''(\alpha) + 6\alpha g''(\alpha) + 6g'(\alpha) + G'''(\alpha)\,. \tag{73}$$

We first consider the interval $\pi/2 \leq \alpha \leq \pi$. If we let $\delta = \alpha - \pi/2$, we have

$$g(\alpha) \leq g(\pi/2) + \delta g'(\pi/2) + \frac{\delta^2}{2}\, g''(\pi/2)\,, \quad \pi/2 \leq \alpha \leq \pi\,, \tag{74}$$

and

$$d(\alpha) \geq d(\pi/2) + \delta d'(\pi/2) + \frac{\delta^2}{2}\, d''(\pi/2), \quad \pi/2 \leq \alpha \leq \pi\,, \tag{75}$$

since $g'''(\alpha) \leq 0$ by (67), and $d'''(\alpha) > 0$ for $\pi/2 \leq \alpha \leq \pi$, as we now shall show.

From (73) we find at once that

$$(76)\quad d'''(\alpha) \geq \alpha^2 g'''(\alpha) + 6\alpha\, g''(\pi) + 6g'(\pi/2) + G'''(\alpha)\,, \quad \pi/2 \leq \alpha \leq \pi\,.$$

We must now find estimates for $g'''(\alpha)$ and $G'''(\alpha)$. From (67) we see that, for $\pi/2 \leq \alpha \leq \pi$, we have

$$0 \leq -g'''(\alpha) \leq 2\pi \int_{-\infty}^{\infty} \frac{q^3}{\mathrm{ch}^2\, \pi q} \cdot \frac{\mathrm{sh}\, \pi q}{\mathrm{ch}\, \pi q} \cdot \frac{1}{\mathrm{ch}^2\, \pi q/2}\, dq$$

$$= 32\pi \int_{-\infty}^{\infty} \frac{q^3\ \mathrm{sh}^2\, \pi q(\mathrm{ch}\, \pi q - 1)}{\mathrm{sh}^3\, 2\pi q}\, dq$$

$$= \pi \int_{-\infty}^{\infty} \frac{q^3\ \mathrm{sh}\, \pi q/2}{\mathrm{sh}^2\, \pi q}\, dq - \pi \int_{-\infty}^{\infty} \frac{q^3\,(\mathrm{ch}\, \pi q - 1)}{\mathrm{sh}^3\, \pi q}\, dq\,.$$

If we make use of formulas (A-16), (A-32), and (A-31) in Appendix A to evaluate these integrals, we find that

$$(77)\qquad 0 \geq g'''(\alpha) \geq -1/\pi + 3 - 7\pi/8, \quad \pi/2 \leq \alpha \leq \pi\,.$$

Equation (70) shows that $G'''(\alpha)$ is the sum of three integrals each of which may be estimated by methods similar to that used above in the estimation of $g'''(\alpha)$; it is convenient to observe that the function $q/\mathrm{sh}\, q$ decreases monotonely for $q > 0$ and is an even function of q. The necessary formulas from Appendix A are (A-16), (A-18), (A-20), (A-26), and (A-30). We find that

$$(78)\quad G'''(\alpha) \geq [\pi/(2\alpha)](12 - 44\pi - 25\pi^2 + 12\pi^3)$$
$$+ 34\pi/45 - 16/(3\pi) + \pi\alpha(25 - 8\pi)\,, \ \pi/2 \leq \alpha \leq \pi\,.$$

The values of $g''(\pi)$ and $g'(\pi/2)$ are given in equations (B-3) and (B-5) of Appendix B. If we substitute these values as well as values from (77) and (78) into (76), we find that, for $\pi/2 \leq \alpha \leq \pi$, $\alpha d'''(\alpha)$ is not less than a certain polynomial of third degree in α. It is easily verified that this polynomial has three real zeros, none of which lies between $\pi/2$ and π, and that it is positive for $\alpha = \pi/2$ and $\alpha = \pi$. Consequently, it is positive for $\pi/2 \leq \alpha \leq \pi$. It follows that $d'''(\alpha) > 0$ in the same interval.

If we substitute into (74) and (75) the necessary values from Appendix B, we find that, for $\pi/2 \leq \alpha \leq \pi$, $\delta^{-2}[d(\alpha) - 3g^3(\alpha)]$ is not less than a certain fourth degree polynomial in δ which is readily shown to decrease monotonely as

δ increases from 0. Moreover, this polynomial is positive for $\delta = 4/3$. It follows that

$$d(\alpha) - 3g^3(\alpha) \geq 0\,, \quad \pi/2 \leq \alpha \leq \pi/2 + \tfrac{4}{3}\,. \tag{79}$$

Since the desired inequality has not yet been proven for $\alpha > \pi/2 + 4/3$, we consider further the interval $\pi/2 \leq \alpha \leq \pi$ and let $\epsilon = \pi - \alpha$. We first recall that $g''(\alpha)$ decreases monotonely. Also it has been shown that $d'''(\alpha) > 0$ for $\pi/2 \leq \alpha \leq \pi$; it follows that $d''(\alpha)$ increases monotonely in this interval. We thus obtain

$$g(\alpha) \leq g(\pi) - \epsilon g'(\pi) + \frac{\epsilon^2}{2}\, g''(\pi/2)\,, \quad \pi/2 \leq \alpha \leq \pi\,, \tag{80}$$

and

$$d(\alpha) \geq d(\pi) - \epsilon d'(\pi) + \frac{\epsilon^2}{2}\, d''(\pi/2)\,, \quad \pi/2 \leq \alpha \leq \pi\,. \tag{81}$$

If we substitute into (80) and (81) the necessary values from Appendix B, we find that, for $\pi/2 \leq \alpha \leq \pi$, $d(\alpha) - 3g^3(\alpha)$ is not less than a certain sixth degree polynomial in ϵ which is readily shown to be positive for $0 \leq \epsilon \leq 1/2$. It follows that

$$d(\alpha) - 3g^3(\alpha) > 0\,, \quad \pi - \tfrac{1}{2} \leq \alpha \leq \pi\,. \tag{82}$$

If we combine (79) and (82) the desired inequality is proven for the interval $\pi/2 \leq \alpha \leq \pi$.

Next we turn our attention to the interval $\pi/4 \leq \alpha \leq \pi/2$. We first need to obtain estimates for $g'''(\alpha)$ and $d'''(\alpha)$ in this interval. If we make use of (67) and employ (A-16) and (A-28) in Appendix A to evaluate the integrals which arise, we find that

$$0 \geq g'''(\alpha) \geq -\tfrac{3}{2} + 7\pi/16\,, \quad 0 \leq \alpha \leq \pi/2\,. \tag{83}$$

Before we can estimate $d'''(\alpha)$ we need to estimate $G'''(\alpha)$. We proceed as we did for the interval $\pi/2 \leq \alpha \leq \pi$, and we find that two of the integrals that have to be evaluated are the same as before although the inequalities are reversed. However, the third integral is different; it may be estimated by making use of (A-8), (A-10), and (A-17). We find that

$$(84)\quad G'''(\alpha) \leq [\pi/(2\alpha)](12 - 44\pi - 25\pi^2 + 12\pi^3) + 512\pi + 176\pi(2)^{1/2} - 171\pi^2(2)^{1/2} + \pi\alpha(25 - 8\pi), \quad \pi/4 \leq \alpha \leq \pi/2 .$$

If we recall that $g'(\alpha)$ is an increasing function, $g''(\alpha)$ a decreasing function, and $g'''(\alpha)$ a nonpositive function, we find from (73) that

$$d'''(\alpha) \leq 0 + 6\alpha g''(0) + 6g'(\pi/2) + G'''(\alpha), \quad \pi/4 \leq \alpha \leq \pi/2 .$$

If we insert the values of $g''(0)$ and $g'(\pi/2)$ from Appendix B, and make use of (84), we obtain

$$\begin{aligned}(85)\quad d'''(\alpha) &\leq [\pi/(2\alpha)](12 - 44\pi - 25\pi^2 + 12\pi^3) + 6 + 1021\pi/2 \\ &\quad + 176\pi(2)^{1/2} - 171\pi^2(2)^{1/2} + \alpha(1 + 25\pi - 8\pi^2) \\ &\leq 18 + 467\pi + 176\pi(2)^{1/2} - 25\pi^2/2 \\ &\quad - 171\pi^2(2)^{1/2} + 8\pi^3, \quad \pi/4 \leq \alpha \leq \pi/2 .\end{aligned}$$

In passing to the last inequality we have replaced α by $\pi/2$ because the first parenthesis is negative and the last one is positive. We also point out that the last member of (85) is positive.

If we now let $\zeta = \pi/2 - \alpha$, we find from the Taylor's series expansions of $g(\alpha)$ and $d(\alpha)$, on using (83) and (85), that

$$(86)\quad g(\alpha) \leq g(\pi/2) - \zeta g'(\pi/2) + \frac{\zeta^2}{2} g''(\pi/2) + \zeta^3\left(\tfrac{1}{4} - 7\pi/96\right), \quad \pi/4 \leq \alpha \leq \pi/2$$

and

$$(87)\quad d(\alpha) \geq d(\pi/2) - \zeta d'(\pi/2) + \frac{\zeta^2}{2} d''(\pi/2) - \frac{\zeta^3}{6} d_1, \quad \pi/4 \leq \alpha < \pi/2 ,$$

where d_1 denotes the last member of (85).

If we substitute into (86) and (87) the necessary values from Appendix B, we find that, for $\pi/4 \leq \alpha \leq \pi/2$, $\zeta^{-2}[d(\alpha) - 3g^3(\alpha)]$ is not less than a certain seventh degree polynomial in ζ which is easily shown to be positive for $0 \leq \zeta \leq 1/2$. It follows that

$$(88)\quad d(\alpha) - 3g^3(\alpha) \geq 0, \quad \pi/2 - \tfrac{1}{2} \leq \alpha \leq \pi/2 .$$

Since the desired inequality has not yet been proven for $\alpha < \pi/2 - 1/2$, we consider further the interval $\pi/4 \le \alpha \le \pi/2$ and let $\eta = \alpha - \pi/4$. We need another estimate for $d'''(\alpha)$, but of the opposite sense to that given by (85). This in turn requires a new estimate for $G'''(\alpha)$. The necessary integrals may be evaluated by using (A-17), (A-23), and (A-24). We find that

$$(89) \qquad G'''(\alpha) \ge -8/(3\alpha) + 32\pi - 10\pi^2 - 14\alpha/15\,, \quad 0 < \alpha \le \pi/2\,.$$

From (73) we find at once that

$$d'''(\alpha) \ge \alpha^2 g'''(\alpha) + 6\alpha g''(\pi/2) + 6g'(\pi/4) + G'''(\alpha)\,, \quad \pi/4 \le \alpha \le \pi/2\,.$$

If we insert the necessary values from Appendix B, and make use of (83) and (89), we obtain

$$(90) \quad d'''(\alpha) \ge \frac{1}{\alpha}\{-8/3 + [12(2)^{1/2} - 6 + 3\pi(2)^{1/2} + 49\pi/2 - 10\pi^2]\alpha$$
$$+ (136/15 - 3\pi)\alpha^2 + (7\pi/16 - 3/2)\alpha^3\}\,, \quad \pi/4 \le \alpha \le \pi/2\,.$$

But it is easily shown that the polynomial in the braces increases monotonely when α increases from 0 to $\pi/2$. Moreover, it is negative if $\alpha = \pi/4$. Hence we may replace α by $\pi/4$ in the right-hand member of (90). If we denote the resulting value by d_2, we see that $d'''(\alpha) \ge d_2$ for $\pi/4 \le \alpha \le \pi/2$. Using this fact and recalling that $g'''(\alpha) \le 0$, we have

$$(91) \qquad g(\alpha) \le g(\pi/4) + \eta g'(\pi/4) + \frac{\eta^2}{2} g''(\pi/4)\,, \quad \pi/4 \le \alpha \le \pi/2\,,$$

and

$$(92) \quad d(\alpha) \ge d(\pi/4) + \eta d'(\pi/4) + \frac{\eta^2}{2} d''(\pi/4) + \frac{\eta^3}{6} d_2\,, \quad \pi/4 \le \alpha \le \pi/2\,.$$

If we substitute into (91) and (92) the necessary values from Appendix B, we find that, for $\pi/4 \le \alpha \le \pi/2$, $d(\alpha) - 3g^3(\alpha)$ is not less than a certain sixth degree polynomial in η which is easily shown to be positive for $0 \le \eta \le 0.4$. It follows that

$$(93) \qquad d(\alpha) - 3g^3(\alpha) > 0\,, \quad \pi/4 \le \alpha \le \pi/4 + 0.4\,.$$

If we combine (88) and (93), the desired inequality is proven for the interval $\pi/4 \leq \alpha \leq \pi/2$.

Finally we consider the interval $0 \leq \alpha \leq \pi/4$. We first need to obtain estimates for $d''(\alpha)$ and $G''(\alpha)$ in this interval. If we make use of (68) and (69), and employ (A-22), (A-23), (A-24), and (A-30) to evaluate the integrals which arise, we find that

$$G''(\alpha) \geq -4/3 + 121\pi^2/60 - 5\pi^3/8\,, \quad 0 \leq \alpha \leq \pi/4\,. \tag{94}$$

From (72) we find at once that

$$d''(\alpha) \geq 2g(0) + G''(\alpha)\,, \quad 0 \leq \alpha \leq \pi/4\,,$$

since $g''(\alpha)$ and $g'(\alpha)$ are both nonnegative. If we take the value of $g(0)$ from Appendix B, and make use of (94), we find that

$$d''(\alpha) \geq 4\, \log 2 - 4/3 + 121\pi^2/60 - 5\pi^3/8\,, \quad 0 \leq \alpha \leq \pi/4\,. \tag{95}$$

If we now let $\kappa = \pi/4 - \alpha$, recall that $g''(\alpha)$ decreases monotonely when α increases, and make use of (95), we find that

$$g(\alpha) \leq g(\pi/4) - \kappa g'(\pi/4) + \frac{\kappa^2}{2} g''(0)\,, \quad 0 \leq \alpha \leq \pi/4\,, \tag{96}$$

and

$$d(\alpha) \geq d(\pi/4) - \kappa d'(\pi/4) + \frac{\kappa^2}{2} d_3\,, \quad 0 \leq \alpha \leq \pi/4\,, \tag{97}$$

where d_3 denotes the right-hand member of (95).

If we substitute into (96) and (97) the necessary values from Appendix B, we find that, for $0 \leq \alpha \leq \pi/4$, $d(\alpha) - 3g^3(\alpha)$ is not less than a certain sixth degree polynomial in κ which is positive for $0 \leq \kappa \leq \pi/4$. It follows that

$$d(\alpha) - 3g^3(\alpha) > 0\,, \quad 0 \leq \alpha \leq \pi/4\,. \tag{98}$$

Combining this with our previous results, we see that the desired inequality has now been established for the whole interval $0 \leq \alpha \leq \pi$. As previously observed, this proves the inequality (2) for the symmetric lens.

7. Appendix A. In this appendix we give a table of integrals which includes all the integrals needed in the proof in §6 and in the calculations in Appendix B. Some of these integrals can be deduced easily from formulas given in the integral tables of Bierens de Haan [1]. When this is the case the formula is followed by two numbers in parenthesis giving first the table number and second the formula number of the necessary formula in the tables of Bierens de Haan.

Since not all of our formulas can be deduced from these tables we indicate alternative methods of proof. Formulas (A-1) to (A-6) can be derived by standard methods of contour integration. In connection with (A-5), we mention that it is necessary to integrate both $z\ \mathrm{sh}\,\alpha z/\mathrm{sh}^2\pi z$ and $\mathrm{ch}\,\alpha z/\mathrm{sh}^2\pi z$ around an indented rectangular contour; and in (A-6) it is necessary to integrate $z^2\ \mathrm{sh}\,\alpha z/\mathrm{sh}^3\pi z$, $z\ \mathrm{ch}\,\alpha z/\mathrm{sh}^3\pi z$, and $\mathrm{sh}\,\alpha z/\mathrm{sh}^3\pi z$ around the same contour. Formulas (A-7) to (A-20) can be derived by differentiation of the formulas (A-1) to (A-6). Finally, formulas (A-21) to (A-32) are all special or limiting cases of formulas (A-7) to (A-20). It may be noted that (A-32) may be derived most easily by using an integration by parts and (A-29).

(A-1) $$\int_{-\infty}^{\infty} \frac{\mathrm{ch}\,\alpha q}{\mathrm{ch}\,\pi q}\,dq = \sec\frac{\alpha}{2}, \quad -\pi < \alpha < \pi \qquad (27,\ 4)$$

(A-2) $$\int_{-\infty}^{\infty} \frac{\mathrm{ch}\,\alpha q}{\mathrm{ch}^2\,\pi q}\,dq = \frac{\alpha}{\pi}\csc\frac{\alpha}{2}, \quad -2\pi < \alpha < 2\pi \qquad (27,\ 18)$$

(A-3) $$\int_{-\infty}^{\infty} \frac{\mathrm{ch}\,\alpha q}{\mathrm{ch}^4\,\pi q}\,dq = \frac{\alpha}{6\pi^3}\left(4\pi^2 - \alpha^2\right)\csc\frac{\alpha}{2}, \quad -4\pi < \alpha < 4\pi \qquad (27,\ 18)$$

(A-4) $$\int_{-\infty}^{\infty} \frac{\mathrm{sh}\,\alpha q}{\mathrm{sh}\,\pi q}\,dq = \tan\frac{\alpha}{2}, \quad -\pi < \alpha < \pi \qquad (27,\ 10)$$

(A-5) $$\int_{-\infty}^{\infty} \frac{q\ \mathrm{sh}\,\alpha q}{\mathrm{sh}^2\,\pi q}\,dq = \frac{1}{2\pi}\left(\alpha\csc^2\frac{\alpha}{2} - 2\cot\frac{\alpha}{2}\right), \quad -2\pi < \alpha < 2\pi$$

(A-6) $$\int_{-\infty}^{\infty} \frac{q^2\ \mathrm{sh}\,\alpha q}{\mathrm{sh}^3\,\pi q}\,dq = \frac{1}{4\pi^2}\left[4\tan\frac{\alpha}{2} + 4\alpha\sec^2\frac{\alpha}{2}\right.$$
$$\left. + \left(\alpha^2 - \pi^2\right)\sec^2\frac{\alpha}{2}\tan\frac{\alpha}{2}\right], \quad -3\pi < \alpha < 3\pi$$

(A-7) $$\int_{-\infty}^{\infty} \frac{q^2 \operatorname{ch} \alpha q}{\operatorname{ch} \pi q}\, dq = \tfrac{1}{4} \sec \frac{\alpha}{2}\left(1 + 2 \tan^2 \frac{\alpha}{2}\right), \quad -\pi < \alpha < \pi \qquad \mathbf{(84,\ 17)}$$

(A-8) $$\int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \alpha q}{\operatorname{ch} \pi q}\, dq = \frac{1}{16} \sec \frac{\alpha}{2}\left(5 + 28 \tan^2 \frac{\alpha}{2} + 24 \tan^4 \frac{\alpha}{2}\right),$$
$$-\pi < \alpha < \pi \qquad \mathbf{(82,\ 16)}$$

(A-9) $$\int_{-\infty}^{\infty} \frac{q^2 \operatorname{ch} \alpha q}{\operatorname{ch}^2 \pi q}\, dq = \frac{1}{4\pi} \csc \frac{\alpha}{2}\left[-4 \cot \frac{\alpha}{2}\right.$$
$$\left. + \alpha\left(1 + 2 \cot^2 \frac{\alpha}{2}\right)\right], \quad -2\pi < \alpha < 2\pi$$

(A-10) $$\int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \alpha q}{\operatorname{ch}^2 \pi q}\, dq = \frac{1}{16\pi} \csc \frac{\alpha}{2}\left[-8 \cot \frac{\alpha}{2}\left(5 + 6 \cot^2 \frac{\alpha}{2}\right)\right.$$
$$\left. + \alpha\left(5 + 28 \cot^2 \frac{\alpha}{2} + 24 \cot^4 \frac{\alpha}{2}\right)\right], \quad -2\pi < \alpha < 2\pi$$

(A-11) $$\int_{-\infty}^{\infty} \frac{q^2 \operatorname{ch} \alpha q}{\operatorname{ch}^4 \pi q}\, dq = \frac{1}{24\pi^3} \csc \frac{\alpha}{2}\left[-24\alpha + 4(3\alpha^2 - 4\pi^2) \cot \frac{\alpha}{2}\right.$$
$$\left. + \alpha(4\pi^2 - \alpha^2)\left(1 + 2 \cot^2 \frac{\alpha}{2}\right)\right], \quad -4\pi < \alpha < 4\pi$$

(A-12) $$\int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \alpha q}{\operatorname{ch}^4 \pi q}\, dq = \frac{1}{96\pi^3} \csc \frac{\alpha}{2}\left[192 \cot \frac{\alpha}{2} - 144\alpha\left(1 + 2 \cot^2 \frac{\alpha}{2}\right)\right.$$
$$+ 8(3\alpha^2 - 4\pi^2) \cot \frac{\alpha}{2}\left(5 + 6 \cot^2 \frac{\alpha}{2}\right)$$
$$+ \alpha(4\pi^2 - \alpha^2)\left(5 + 28 \cot^2 \frac{\alpha}{2}\right.$$
$$\left.\left. + 24 \cot^4 \frac{\alpha}{2}\right)\right], \quad -4\pi < \alpha < 4\pi$$

(A-13) $$\int_{-\infty}^{\infty} \frac{q \operatorname{ch} \alpha q}{\operatorname{sh} \pi q}\, dq = \tfrac{1}{2} \sec^2 \frac{\alpha}{2}, \quad -\pi < \alpha < \pi \qquad \mathbf{(84,\ 16)}$$

(A-14) $$\int_{-\infty}^{\infty} \frac{q^3 \operatorname{ch} \alpha q}{\operatorname{sh} \pi q}\, dq = \tfrac{1}{4} \sec^2 \frac{\alpha}{2} \left(1 + 3 \tan^2 \frac{\alpha}{2}\right), \quad -\pi < \alpha < \pi \quad \mathbf{(82, 15)}$$

(A-15) $$\int_{-\infty}^{\infty} \frac{q^2 \operatorname{ch} \alpha q}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{2\pi} \csc^2 \frac{\alpha}{2} \left(2 - \alpha \cot \frac{\alpha}{2}\right), \quad -2\pi < \alpha < 2\pi$$

(A-16) $$\int_{-\infty}^{\infty} \frac{q^3 \operatorname{sh} \alpha q}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{4\pi} \csc^2 \frac{\alpha}{2} \left[-6 \cot \frac{\alpha}{2} + \alpha\left(1 + 3 \cot^2 \frac{\alpha}{2}\right)\right], \quad -2\pi < \alpha < 2\pi$$

(A-17) $$\int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \alpha q}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{2\pi} \csc^2 \frac{\alpha}{2} \left[2\left(1 + 3 \cot^2 \frac{\alpha}{2}\right) - \alpha \cot \frac{\alpha}{2}\left(2 + 3 \cot^2 \frac{\alpha}{2}\right)\right], \quad -2\pi < \alpha < 2\pi$$

(A-18) $$\int_{-\infty}^{\infty} \frac{q^5 \operatorname{sh} \alpha q}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{4\pi} \csc^2 \frac{\alpha}{2} \left[-10 \cot \frac{\alpha}{2}\left(2 + 3 \cot^2 \frac{\alpha}{2}\right) + \alpha\left(2 + 15 \cot^2 \frac{\alpha}{2} + 15 \cot^4 \frac{\alpha}{2}\right)\right], \quad -2\pi < \alpha < 2\pi$$

(A-19) $$\int_{-\infty}^{\infty} \frac{q^3 \operatorname{ch} \alpha q}{\operatorname{sh}^3 \pi q}\, dq = \frac{1}{8\pi^2} \sec^2 \frac{\alpha}{2} \left[12 + 12\alpha \tan \frac{\alpha}{2} + (\alpha^2 - \pi^2)\left(1 + 3 \tan^2 \frac{\alpha}{2}\right)\right], \quad -3\pi < \alpha < 3\pi$$

(A-20) $$\int_{-\infty}^{\infty} \frac{q^5 \operatorname{ch} \alpha q}{\operatorname{sh}^3 \pi q}\, dq = \frac{1}{8\pi^2} \sec^2 \frac{\alpha}{2} \left[20\left(1 + 3 \tan^2 \frac{\alpha}{2}\right) + 20\alpha \tan \frac{\alpha}{2}\left(2 + 3 \tan^2 \frac{\alpha}{2}\right) + (\alpha^2 - \pi^2)\left(2 + 15 \tan^2 \frac{\alpha}{2} + 15 \tan^4 \frac{\alpha}{2}\right)\right], \quad -3\pi < \alpha < 3\pi$$

(A-21) $$\int_{-\infty}^{\infty} \frac{q^2}{\operatorname{ch} \pi q}\, dq = \tfrac{1}{4} \qquad (84,\ 3)$$

(A-22) $$\int_{-\infty}^{\infty} \frac{q^4}{\operatorname{ch} \pi q}\, dq = \frac{5}{16} \qquad (84,\ 7)$$

(A-23) $$\int_{-\infty}^{\infty} \frac{q^2}{\operatorname{ch}^2 \pi q}\, dq = \frac{1}{6\pi} \qquad (86,\ 2)$$

(A-24) $$\int_{-\infty}^{\infty} \frac{q^4}{\operatorname{ch}^2 \pi q}\, dq = \frac{7}{120\pi} \qquad (86,\ 2)$$

(A-25) $$\int_{-\infty}^{\infty} \frac{q^2}{\operatorname{ch}^4 \pi q}\, dq = \frac{1}{9\pi} - \frac{2}{3\pi^3}$$

(A-26) $$\int_{-\infty}^{\infty} \frac{q^4}{\operatorname{ch}^4 \pi q}\, dq = \frac{7}{180\pi} - \frac{1}{3\pi^3}$$

(A-27) $$\int_{-\infty}^{\infty} \frac{q}{\operatorname{sh} \pi q}\, dq = \tfrac{1}{2} \qquad (84,\ 2)$$

(A-28) $$\int_{-\infty}^{\infty} \frac{q^3}{\operatorname{sh} \pi q}\, dq = \tfrac{1}{4} \qquad (84,\ 5)$$

(A-29) $$\int_{-\infty}^{\infty} \frac{q^2}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{3\pi} \qquad (86,\ 5)$$

(A-30) $$\int_{-\infty}^{\infty} \frac{q^4}{\operatorname{sh}^2 \pi q}\, dq = \frac{1}{15\pi} \qquad (86,\ 5)$$

(A-31) $$\int_{-\infty}^{\infty} \frac{q^3}{\operatorname{sh}^3 \pi q}\, dq = \frac{3}{2\pi^2} - \tfrac{1}{8}$$

(A-32) $$\int_{-\infty}^{\infty} \frac{q^3 \operatorname{ch} \pi q}{\operatorname{sh}^3 \pi q}\, dq = \frac{1}{2\pi^2}$$

8. Appendix B. In the proof given in §6 we had to use the values of $g(\alpha)$ and $d(\alpha)$ for certain values of α. The necessary values are listed in this appendix;

the method of calculation of each is also indicated. Following is the list of values used:

(B-1, 2, 3) $g(\pi) = 2$, $g'(\pi) = 1/\pi$, $g''(\pi) = 1/9 - 2/(3\pi^2)$

(B-4, 5, 6) $g(\pi/2) = \pi/2$, $g'(\pi/2) = 1 - \pi/4$, $g''(\pi/2) = 5/3 - \pi/2$

(B-7) $g(\pi/4) = \pi(2)^{1/2}/2 - \pi/4$

(B-8) $g'(\pi/4) = -1 + 2(2)^{1/2} - 5\pi/4 + \pi(2)^{1/2}/2$

(B-9) $g''(\pi/4) = 34/3 + 4(2)^{1/2} + \pi - 9\pi(2)^{1/2}/2$

(B-10, 11) $g(0) = 2 \log 2$, $g''(0) = 1/6$

(B-12, 13) $d(\pi) = 8\pi^2/3$, $d'(\pi) = 6\pi - \pi^3/8$

(B-14, 15) $d(\pi/2) = 3\pi^3/8$, $d'(\pi/2) = 9\pi^2/4 - 9\pi^3/16$

(B-16) $d''(\pi/2) = 9\pi + 23\pi^2/12 - 11\pi^3/8$

(B-17) $d(\pi/4) = 7\pi^3(2)^{1/2}/32 - \pi^3/64$

(B-18) $d'(\pi/4) = -3\pi^2/16 + 21\pi^2(2)^{1/2}/8 - 133\pi^3/64 + 23\pi^3(2)^{1/2}/32$

(B-19) $d''(\pi/4) = -3\pi/2 + 21\pi(2)^{1/2} + 371\pi^2/24 + 69\pi^2(2)^{1/2}/4$
$+ \pi^3/16 - 351\pi^3(2)^{1/2}/32$

Formulas (B-4), (B-5), (B-6), (B-10), and (B-11) will be found in [3]; (B-1) and (B-2) can be proven by starting with (61) and (65) and using (A-2), an integration by parts being first needed in the case of (B-2). (B-3) follows at once from (66) and (A-25). In order to prove (B-7) we observe that

$$g(\pi/4) = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{\operatorname{ch} 3\pi q/4}{\operatorname{ch}\pi q \, \operatorname{ch}\pi q/4}\, dq = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{2\operatorname{ch}\pi q/2 - 1}{\operatorname{ch}\pi q}\, dq,$$

and the result follows from (A-1). For (B-8) we have

$$g'(\pi/4) = \pi \int_{-\infty}^{\infty} \frac{q \operatorname{sh}\pi q/4}{\operatorname{ch}^2 \pi q \, \operatorname{ch}\pi q/4}\, dq = \pi \int_{-\infty}^{\infty} \frac{q(\operatorname{ch}\pi q + 1 - 2\operatorname{ch}\pi q/2)}{\operatorname{ch}^2 \pi q \, \operatorname{sh}\pi q}\, dq$$

$$= 2\pi \int_{-\infty}^{\infty} q \left[\frac{1}{\text{sh}\, 2\pi q} + \frac{2(\text{sh}\, \pi q - \text{sh}\, 3\pi q/2 - \text{sh}\, \pi q/2)}{\text{sh}^2\, 2\pi q} \right] dq \,,$$

and the result follows from (A-27) and (A-5). For (B-9) we start with (66) and have

$$g''(\pi/4) = 4\pi \int_{-\infty}^{\infty} \frac{q^2\ \text{sh}^2\, \pi q/4}{\text{ch}^2\, \pi q\ \text{sh}^2\, \pi q/2}\, dq$$

$$= 16\pi \int_{-\infty}^{\infty} \frac{q^2(\text{ch}\, \pi q/2 - 1)(\text{ch}\, \pi q + 1)}{\text{sh}^2\, 2\pi q}\, dq$$

$$= \pi \int_{-\infty}^{\infty} \frac{q^2(\text{ch}\, 3\pi q/4 - 2\ \text{ch}\, \pi q/2 + 3\ \text{ch}\, \pi q/4 - 2)}{\text{sh}^2\, \pi q}\, dq \,,$$

and the result follows from (A-15) and (A-29).

Formulas (B-12) to (B-19) follow immediately from (63), (71), and (72) if we show how to calculate the following:

(B-20, 21) $\quad G(\pi) = 2\pi^2/3\,, \qquad G'(\pi) = \pi - \pi^3/8$

(B-22, 23) $\quad G(\pi/2) = \pi^3/4\,, \qquad G'(\pi/2) = 3\pi^3/2 - \pi^4/2$

(B-24) $\quad G''(\pi/2) = 6\pi + 2\pi^2 - 5\pi^3/4$

(B-25) $\quad G(\pi/4) = 3\pi^3(2)^{1/2}/16$

(B-26) $\quad G'(\pi/4) = 9\pi^2(2)^{1/2}/4 - 2\pi^3 + 11\pi^3(2)^{1/2}/16$

(B-27) $\quad G''(\pi/4) = 18\pi(2)^{1/2} + 16\pi^2 + 33\pi^2(2)^{1/2}/2 - 171\pi^3(2)^{1/2}/16$

Formula (B-20) follows easily from (62) and (A-23), (B-22) from (62) and (A-21), (B-23) from (68) and (A-16), (B-24) from (69) and (A-17), and (B-25) from (62) and (A-7). For (B-21) we use (68) and find that

$$G'(\pi) = -8\pi^3 \int_{-\infty}^{\infty} \frac{q^3}{\text{ch}^2\, \pi q\ \text{sh}\, 2\pi q}\, dq = -16\pi^3 \int_{-\infty}^{\infty} \frac{q^3(\text{ch}\, 2\pi q - 1)}{\text{sh}^3\, 2\pi q}\, dq \,,$$

whence the result follows from (A-32) and (A-31). For (B-26) we obtain, from (68),

$$G'(\pi/4) = -\pi^3 \int_{-\infty}^{\infty} \frac{q^3\ \text{ch}\, \pi q/2}{\text{ch}^2\, \pi q\ \text{sh}\, \pi q}\, dq = -2\pi^3 \int_{-\infty}^{\infty} \frac{q^3(\text{sh}\, 3\pi q/2 + \text{sh}\, \pi q/2)}{\text{sh}^2\, 2\pi q}\, dq \,,$$

and the result follows from (A-16). Finally for (B-27) we obtain, from (69),

$$G''(\pi/4) = (8/\pi)G'(\pi/4) + \pi^3 \int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \pi q/2}{\operatorname{ch}^2 \pi q \operatorname{sh}^2 \pi q/2} dq$$

$$= (8/\pi)G'(\pi/4) + 2\pi^3 \int_{-\infty}^{\infty} \frac{q^4 \operatorname{ch} \pi q/2}{\operatorname{ch}^2 \pi q(\operatorname{ch} \pi q - 1)} dq$$

$$= (8/\pi)G'(\pi/4)$$

$$+ 2\pi^3 \int_{-\infty}^{\infty} q^4 \operatorname{ch} \pi q/2 \left(\frac{1}{2 \operatorname{sh}^2 \pi q/2} - \frac{1}{\operatorname{ch} \pi q} - \frac{1}{\operatorname{ch}^2 \pi q} \right) dq ,$$

and the result follows from (A-17), (A-8), and (A-10).

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THE BOREL PROPERTY OF SUMMABILITY METHODS

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1. Introduction. Let T denote the method of summability corresponding to the real matrix $(a_{n,k})$, for the moment arbitrary, by means of which a sequence $\{s_k\}$ is said to be summable-T to s if each of the series in

$$(1.1) \qquad t_n = \sum_{k=1}^{\infty} a_{n,k}\, s_k \qquad (n = 1, 2, 3, \cdots),$$

is convergent and if $t_n \longrightarrow s$.

We shall be concerned here exclusively with the class $\mathfrak{X}$ of all sequences $x = \{\alpha_k\}$ where the α_k are 0 or 1 with infinitely many 1's. A biunique mapping of the class $\mathfrak{X}$ into the real interval $\mathfrak{Y} \equiv (0 < y \leq 1)$ is obtained by defining y as the dyadic fraction $0.\alpha_1\alpha_2\alpha_3 \cdots$ corresponding to $x = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, and conversely. This enables us to employ the phrase, "almost all sequences of 0's and 1's," by which is meant a subset of $\mathfrak{X}$ for which the corresponding subset of $\mathfrak{Y}$ has Lebesgue measure one.

A classical result of Borel [2] may be interpreted as asserting that almost all sequences of 0's and 1's are summable-$(C, 1)$, Cesàro of order one, to the value $1/2$. If the corresponding statement is true for the method T defined by (1.1) we shall say that T has the Borel property, or more briefly, that $T \in (BP)$.

A study of the Borel property for regular methods T was undertaken recently by the author [5]. In the present paper we dispense with the assumption of regularity, and in §2 we investigate the consequences of assuming merely that $T \in (BP)$. Two independent necessary conditions, (2.2) and (2.5), are obtained.

In §3 it is shown by means of an example due essentially to Erdös that these conditions are not sufficient in order that $T \in (BP)$ even if condition (2.10) is added. By virtue of a lemma of Khintchine we are able to state in Theorem (3.5) a new sufficient condition considerably weaker than that given in Theorem (2.14) of [5]. For comparison the latter result is repeated here in Theorem (3.3). In Theorem

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(3.11) we deal with a conjecture of Erdös and prove incidentally that in general the Borel property does not depend on the rate at which $\Sigma_{k=1}^{\infty} a_{n,k}^2$ approaches zero.

At the present time it appears unlikely that the Borel property can be characterized in any reasonably simple manner, at least if no restrictions are imposed on the matrix $(a_{n,k})$ at the outset. This aspect of the problem remains to be considered.

2. Necessary conditions. We shall establish the following result.

(2.1) THEOREM. *In order that $T \in (BP)$ the following conditions are necessary*:

(2.2) $$\sum_{k=1}^{\infty} a_{n,k} \text{ converges for each } n \text{ and tends to } 1 \text{ as } n \longrightarrow \infty;$$

(2.3) $$A_n \equiv \sum_{k=1}^{\infty} a_{n,k}^2 < \infty \text{ for each } n;$$

(2.4) $$\lim_{n \to \infty} a_{n,k} = 0 \text{ for each } k;$$

(2.5) $$\lim_{n \to \infty} A_n = 0.$$

Proof. If $T \in (BP)$, there exists a subset $\mathfrak{Y}^*$ of $\mathfrak{Y} \equiv (0 < y \leq 1)$ of measure one such that

$$t_n(y) \equiv \sum_{k=1}^{\infty} a_{n,k}\alpha_k(y)$$

is defined for each n and each $y = 0.\alpha_1 \alpha_2 \alpha_3 \cdots \in \mathfrak{Y}^*$ and such that $t_n(y) \longrightarrow 1/2$. Since $\mathfrak{Y}^*$ is of measure one it contains a subset $\mathfrak{Y}^{**}$ of measure one such that if $y \in \mathfrak{Y}^{**}$ then also $1 - y \in \mathfrak{Y}^{**}$. Choosing any such y we may write $y = 0.\alpha_1 \alpha_2 \alpha_3 \cdots$ and $1 - y = 0.\beta_1 \beta_2 \beta_3 \cdots$, where $\alpha_k + \beta_k = 1$ for all k. Then (2.2) follows from the fact that

$$\sum_{k=1}^{\infty} a_{n,k}\alpha_k + \sum_{k=1}^{\infty} a_{n,k}\beta_k = \sum_{k=1}^{\infty} a_{n,k}.$$

To verify (2.3) we introduce the Rademacher functions $R_k(y)$ defined for each

k and each $y = 0.\alpha_1 \alpha_2 \alpha_3 \cdots \in \mathfrak{Y}$ as $1 - 2\alpha_k(y)$. Then

$$t_n(y) \equiv \sum_{k=1}^{\infty} a_{n,k} \alpha_k(y) = \frac{1}{2} \sum_{k=1}^{\infty} a_{n,k} - \frac{1}{2} \sum_{k=1}^{\infty} a_{n,k} R_k(y), \tag{2.6}$$

must exist almost everywhere in $\mathfrak{Y}$ for each n. In view of (2.2) the necessity of (2.3) follows from a well-known result of Kolmogoroff [6, p. 126].

To establish (2.4) let k be fixed and denote by $\mathfrak{Y}_1^*$ and $\mathfrak{Y}_2^*$ the subsets of $\mathfrak{Y}^*$ (defined above) of measure 2^{-k} which lie, respectively, in the intervals $0 < y < 2^{-k}$ and $2^{-k} < y < 2^{-k+1}$. It is evident that there exist subsets $\mathfrak{Y}_1^{**}$ of $\mathfrak{Y}_1^*$ and $\mathfrak{Y}_2^{**}$ of $\mathfrak{Y}_2^*$, of measure 2^{-k}, such that if $y \in \mathfrak{Y}_1^{**}$ then $y + 2^{-k} \in \mathfrak{Y}_2^{**}$. For such a value of y we have $y = 0.00 \cdots 0\alpha_{k+1}\alpha_{k+2} \cdots$ ($k+1$ zeros) and $y + 2^{-k} = 0.00 \cdots 01\alpha_{k+1}\alpha_{k+2} \cdots$ (k zeros). Consequently, $t_n(y + 2^{-k}) - t_n(y) = a_{n,k} \longrightarrow 0$ as $n \longrightarrow \infty$.

The proof of the necessity of (2.5) is more involved. Since (2.3) implies the convergence almost everywhere in $\mathfrak{Y}$ of the series $\sum_{k=1}^{\infty} a_{n,k} R_k(y)$ for each n, it follows from Egoroff's theorem that there exists for each n a subset I_n of $\mathfrak{Y}$ of measure $|I_n| > 1 - 2^{-n-1}$, and an index $\phi_1(n)$, increasing to infinity with n, such that

$$\left| \sum_{k=m+1}^{\infty} a_{n,k} R_k(y) \right| < \frac{1}{n} \quad \text{for all } m \geq \phi_1(n) \text{ and all } y \in I_n. \tag{2.7}$$

Setting $I = \prod_{n=1}^{\infty} I_n$ and using $\mathfrak{C}E$ to denote the complement of E with respect to $\mathfrak{Y}$, we have

$$|\mathfrak{C}I| \leq \sum_{n=1}^{\infty} |\mathfrak{C}I_n| < \frac{1}{2}.$$

Consequently we have $|I| > 1/2$, and (2.7) holds in I. We need also the fact that (2.3) insures for each n the existence of an index $\phi(n) \geq \phi_1(n)$ for which

$$\sum_{k > \phi(n)} a_{n,k}^2 < \frac{1}{n}. \tag{2.8}$$

Now it follows from (2.2), (2.3), and (2.6) that T will have the Borel property if and only if $\tau_n(y) \equiv \sum_{k=1}^{\infty} a_{n,k} R_k(y)$ approaches zero almost everywhere in $\mathfrak{Y}$ as

$n \longrightarrow \infty$. Writing $\tau_n(y)$ in the form

$$\sum_{k=1}^{\phi(n)} a_{n,k} R_k(y) + \sum_{k>\phi(n)} a_{n,k} R_k(y)$$

and using (2.7), we see that $T \in (BP)$ implies the approach to zero almost everywhere in I of

$$\sigma_n(y) \equiv \sum_{k=1}^{\phi(n)} a_{n,k} R_k(y) .$$

Let E be a subset of I with $|E| > 0$ on which $\sigma_n(y)$ approaches zero uniformly, and let

$$\sigma_{n,m}(y) \equiv \sum_{k=m}^{\phi(n)} a_{n,k} R_k(y) .$$

We can now follow an argument due to Kolmogoroff (for the details see [6, pp.127-128] or [4]) and arrive at the inequality

$$\int_E \sigma_{n,\mu}^2(y)\, dy \geq \frac{1}{2} |E| \sum_{k=\mu}^{\phi(n)} a_{n,k}^2 , \tag{2.9}$$

for a certain fixed μ and all n sufficiently large. From (2.4) it follows that

$$\sigma_{n,\mu}(y) = \sigma_n(y) - \sum_{k=1}^{\mu-1} a_{n,k} R_k(y)$$

tends to zero uniformly in E together with $\sigma_n(y)$. Then (2.9) yields

$$\sum_{k=\mu}^{\phi(n)} a_{n,k}^2 = o(1)$$

as $n \longrightarrow \infty$. Finally from (2.4) and (2.8) we conclude that

$$A_n = \sum_{k=1}^{\mu-1} a_{n,k}^2 + \sum_{k=\mu}^{\phi(n)} a_{n,k}^2 + \sum_{k>\phi(n)} a_{n,k}^2 = o(1)$$

as $n \longrightarrow \infty$. This completes the proof of Theorem (2.1).

It will be noticed incidentally that conditions (2.2) and (2.4) are among the familiar Silverman-Toeplitz conditions for the regularity of T. The remaining condition for regularity, namely,

$$\sum_{k=1}^{\infty} |a_{n,k}| = O(1) \qquad (n \longrightarrow \infty), \tag{2.10}$$

is not necessary in order that T have the Borel property. This is shown by the example of the following matrix which appears in [1]:

$$\left\| \begin{matrix} 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & \cdots \\ & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \\ \frac{1}{n}\ \ \frac{1}{n} & \cdots & \frac{1}{n} & \frac{(-1)^n}{n+1} & \frac{(-1)^{n+1}}{n+2} & \cdots \\ & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \end{matrix} \right\|$$

This matrix violates condition (2.10) but satisfies the sufficient conditions of Theorem (3.3) below. It has been proved in [1], however, that T is necessarily regular if it evaluates to 1/2 all sequences of 0's and 1's which are summable-$(C, 1)$ to 1/2.

3. Sufficient conditions. We first raise the obvious question of whether the conditions (2.2) and (2.5), which imply (2.3) and (2.4), are sufficient in order that $T \in (BP)$. Before showing that the answer is in the negative, even with the addition of (2.10), we make a few preliminary remarks. Using the notations of §2, and appealing to the Riesz-Fisher theorem, we are led at once to the Parseval relation $\int_0^1 \tau_n^2(y)\,dy = A_n$. The condition $A_n \longrightarrow 0$ is therefore equivalent to the convergence of $\{\tau_n\}$ to zero in the space L^2, and this assures the existence of a sequence of indices $\{n_i\}$ such that $\tau_{n_i}(y) \longrightarrow 0$ almost everywhere in $\mathfrak{Y}$. In other words, if (2.2) and (2.5) are satisfied, the matrix $(a_{n,k})$ contains a row-submatrix $(a_{n_i,k})$ defining a method T^* (not weaker than T) having the Borel property; this fact was obtained in [5] with the aid of (2.10). We proceed now to the construction of an example which shows that in the absence of further conditions nothing more can be said.

We need the following result due to Borel [3, pp. 37-47]. The form stated here is less general than the original, in that the groups of consecutive α's are not permitted to overlap, but it is sufficient for our purposes.

(3.1) LEMMA (Borel). *Let $\{\lambda_n\}$ be a sequence of positive integers, and let the positive integers $\{n_j\}$ be such that $n_j \geq n_{j-1} + \lambda_{j-1}$ $(j = 2, 3, 4, \cdots)$. Then in order that almost all dyadic fractions $y = 0.\alpha_1\alpha_2\alpha_3\cdots$ have the property that for infinitely many j, α_{n_j} is followed by λ_j zeros, and for infinitely many j, by λ_j ones, it is necessary and sufficient that $\sum_{n=1}^{\infty} 2^{-\lambda_n} = \infty$.*

We can now construct the example of Erdös which was outlined in a letter to the author. The details have been modified to render the matrix triangular but the idea otherwise remains essentially as communicated. We use the notation $a(n, k)$ as alternative to $a_{n,k}$, and define a matrix as follows, wherein, as usual, $[\log m]$ means the greatest integer in $\log m$. Let

$$a\{(m^2 + i - 1),\ (\overline{m-1}^2 + j - 2)\} \equiv [\log m]^{-1}$$

for $j = i+1,\ i+2, \cdots,\ i + [\log m]$; $i = 1, 2, \cdots, 2m+1$; $m = 3, 4, 5, \cdots$;

and let $a_{n,k} \equiv 0$ otherwise. This matrix of nonnegative terms is evidently triangular, regular, and such that (2.5) is satisfied. On the other hand we have

$$t_{m^2+2m}(y) = [\log m]^{-1} \sum_{\nu=1}^{[\log m]} \alpha_{m^2+\nu}(y) \qquad (m = 3, 4, 5, \cdots), \tag{3.2}$$

and since $\sum 2^{-[\log m]} = \infty$ it follows from Lemma (3.1) that for almost all $y = 0.\alpha_1\alpha_2\alpha_3\cdots$ there are infinitely many values of m for which α_{m^2} is followed by $[\log m]$ zeros, and also infinitely many m for which α_{m^2} is followed by $[\log m]$ ones. Hence we see from (3.2) that for almost all y the sequence $\{t_n(y)\}$ contains both infinitely many zeros and infinitely many ones. Consequently the matrix $(a_{n,k})$ fails to have the Borel property.

The search for conditions which are necessary as well as sufficient has so far yielded no results. However, the sufficient conditions set forth in the following theorems appear to be of interest.

(3.3) THEOREM. *In order that* $T \in (BP)$, *the conditions* (2.2) *and*

$$\sum_{n=1}^{\infty} A_n^q < \infty \qquad (\textit{for some } q > 0), \tag{3.4}$$

are sufficient [5].

Proof. The proof of this theorem given in [5] remains valid under the present weaker conditions. A new criterion involving, as we show later, a condition considerably weaker than (3.4) is contained in the following theorem.

(3.5) THEOREM. *In order that* $T \in (BP)$ *the conditions* (2.2) *and*

$$\sum_{n=1}^{\infty} \exp(-\delta^2/2A_n) < \infty \qquad (\textit{for each } \delta > 0), \tag{3.6}$$

are sufficient.

For the proof it is convenient to have the following lemma.[1]

(3.7) LEMMA. *In order that a sequence* $\{f_n(y)\}$ *of measurable functions on* $\mathfrak{Y}$ *converge to zero almost everywhere it is necessary and sufficient that given* $\delta > 0$ *and* $\epsilon > 0$ *there should exist an index* $\nu = \nu(\epsilon, \delta)$ *such that*

$$\left| \prod_{n=\nu}^{\infty} E_n(\delta) \right| > 1 - \epsilon, \tag{3.8}$$

where $E_n(\delta) \equiv E\{|f_n(y)| \leq \delta\}$.

Proof of Lemma (3.7). Inasmuch as we make no use of the necessity we give only the proof of the sufficiency. Let $\lambda(y) = \overline{\lim}_{n\to\infty} |f_n(y)|$, and set $H = E\{\lambda(y) > 0\}$. For $m = 1, 2, 3, \cdots$, we set $H_m = E\{\lambda(y) > 1/m\}$ so that

$$H = \sum_{m=1}^{\infty} H_m.$$

[1] Added in proof: see P. R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950, p. 91, Theorem A.

If $|H| > 0$, contrary to the statement of the lemma, then there is an index μ such that $|H_\mu| > 0$. For $\delta = 1/\mu$ and $\epsilon = (1/2)|H_\mu|$ the condition (3.8) becomes

$$\left| \prod_{n=\nu}^{\infty} E_n\left(\frac{1}{\mu}\right) \right| > 1 - \frac{1}{2}\,|H_\mu|$$

for an index $\nu = \nu(\mu)$. Consequently

$$\left| H_\mu \cdot \prod_{n=\nu}^{\infty} E_n\left(\frac{1}{\mu}\right) \right| > \frac{1}{2}\,|H_\mu| > 0\,.$$

For any point

$$y_0 \in H_\mu \cdot \prod_{n=\nu}^{\infty} E_n\left(\frac{1}{\mu}\right)$$

we have $\lambda(y_0) > 1/\mu$ since $y_0 \in H_\mu$. On the other hand, since

$$y_0 \in \prod_{n=\nu}^{\infty} E_n\left(\frac{1}{\mu}\right),$$

we have $|f_n(y_0)| \leq 1/\mu$ for all $n \geq \nu$, and this yields $\lambda(y_0) \leq 1/\mu$. With this contradiction the proof is complete

Proof of (3.5). Proceeding as we did in proving the necessity of (2.5), we first determine an index $\phi(n)$, approaching infinity with n, and a set I of positive measure such that

$$(3.9) \qquad \left| \sum_{k>\phi(n)} a_{n,k} R_k(y) \right| < \frac{1}{n} \text{ for all } y \in I \text{ and } n = 1, 2, 3, \cdots.$$

If we set

$$B_n = \sum_{k=1}^{\phi(n)} a_{n,k}^2\,,$$

then it follows from (3.6) that

$$\sum_{n=1}^{\infty} \exp(-\delta^2/2B_n) < \infty \qquad \text{(for each } \delta > 0) , \tag{3.10}$$

since $B_n \leq A_n$. Now

$$\tau_n(y) = \sum_{k=1}^{\phi(n)} a_{n,k} R_k(y) + \sum_{k>\phi(n)} a_{n,k} R_k(y) \equiv \sigma_n(y) + \rho_n(y) ,$$

where $\rho_n(y) \longrightarrow 0$ for all $y \in I$, by (3.9). Let $I^* \supset I$ denote the entire subset of $\mathfrak{Y}$ on which $\rho_n(y) \longrightarrow 0$, so that $|I^*| > 0$. If $0.\alpha_1 \alpha_2 \cdots \alpha_n \cdots$ is any point of I^*, it is clear from the definition of $\rho_n(y)$ that every point of the form $0.\beta_1 \beta_2 \cdots \beta_p \alpha_{p+1} \alpha_{p+2} \cdots$ is likewise in I^*. Hence I^* is a *homogeneous set* of positive measure, and therefore of measure one (see [9] and [4]). Since $\rho_n(y) \longrightarrow 0$ almost everywhere, we complete the proof by showing that (3.10) implies that $\sigma_n(y) \longrightarrow 0$ almost everywhere. For this purpose let $E_n(\delta) = E\{|\sigma_n(y)| \leq \delta\}$ for $\delta > 0$. By a lemma of Khintchine [7] we have

$$|\mathfrak{C} E_n(\delta)| < M \exp(-\delta^2/2B_n)$$

for $n = 1, 2, 3, \cdots$, where M is an absolute constant. Let $\delta > 0$ and $\epsilon > 0$ be given. Then from (3.10) there exists an index $\nu = \nu(\epsilon, \delta)$ such that

$$M \sum_{n=\nu}^{\infty} \exp(-\delta^2/2B_n) < \epsilon .$$

Consequently

$$\left| \prod_{n=\nu}^{\infty} E_n(\delta) \right| \geq 1 - \sum_{n=\nu}^{\infty} |\mathfrak{C} E_n(\delta)| > 1 - \epsilon .$$

It now follows from Lemma (3.7) that $\sigma_n(y) \longrightarrow 0$ almost everywhere.

As a partial consequence of Theorem (3.5) we are able to decide a conjecture of Erdös (made in a letter to the author) to the effect that (2.2) and $A_n \log n = o(1)$ are necessary and sufficient in order that $T \in (BP)$.

(3.11) THEOREM. *In order that T have the Borel property, the conditions* (2.2) *and*

$$A_n \log n = o(1) \qquad (n \longrightarrow \infty), \tag{3.12}$$

are sufficient; but neither (3.12) *nor* (3.6) *is necessary.*

Proof. To prove the sufficiency it is enough to show that (3.12) implies (3.6). For this purpose, let $\delta > 0$ be given and fix $\epsilon > 0$ so that $\delta^2/2\epsilon > 1$. By (3.12) there exists an index n_0 such that $A_n < \epsilon/(\log n)$ for all $n \geq n_0$. Then

$$\exp(-\delta^2/2A_n) < n^{-\delta^2/2\epsilon}$$

for $n \geq n_0$ with $\delta^2/2\epsilon > 1$, and (3.6) follows.

To complete the proof we show somewhat more, namely, that *no condition of the form* $A_n \psi(n) = o(1)$, *with* $\psi(n) \longrightarrow \infty$, *is necessary.* Consequently the Borel property can not be characterized in terms of the rate at which A_n approaches zero. For let $0 < \theta(n) < 1$, $\theta(n) \longrightarrow 0$, with $\theta(n)$ arbitrary otherwise. Let $x_n = [1 - \theta(n)]/[1 + \theta(n)]$, so that $\theta(n) = (1 - x_n)/(1 + x_n)$, $0 < x_n < 1$, and $x_n \longrightarrow 1$. Since the Abel method has the Borel property [5], the same is true of the "discrete" Abel method defined by the matrix

$$a_{n,k} = (1 - x_n)\, x_n^{k-1} \qquad (k, n = 1, 2, 3, \cdots).$$

For this matrix we find that

$$A_n = \sum_{k=1}^{\infty} a_{n,k}^2 = \theta(n),$$

where $\theta(n)$ may tend to zero in any preassigned manner. Thus, for example, if

$$\theta(n) = \frac{\log \log(n+2)}{\log(n+2)},$$

we have $A_n \log n \longrightarrow \infty$. Finally, if we take $\theta(n)$ as $1/\log\log(n + p)$, for p sufficiently large, the series in (3.6) diverges for every $\delta > 0$.

We now wish to show, as mentioned earlier, that condition (3.4) of Theorem (3.3) implies condition (3.6) of Theorem (3.5), but not conversely. If (3.4) holds for some $q > 0$, we have

$$0 < z_n \equiv 2A_n/\delta^2 \longrightarrow 0$$

for each $\delta > 0$. Since

$$\exp(-1/z_n) = o(z_n^q) \quad \text{or} \quad \exp(-\delta^2/2A_n) = o(A_n^q)$$

as $n \longrightarrow \infty$, it follows that (3.6) is satisfied. On the other hand, for the logarithmic method of regular Riesz means defined by

$$a_{n,k} = 1/k \log(n+1) \quad \text{for } k = 1, 2, \cdots, n; \quad n = 1, 2, 3, \cdots,$$

we have

$$A_n \cong \pi^2/6 \log^2 n .$$

Hence for every $q > 0$ the series in (3.4) diverges, but $A_n \log n = o(1)$, so that (3.6) holds by the proof of Theorem (3.11).

As a simple application of Theorem (3.11), we call attention to the existence of a regular method having the Borel property and which is weaker than (C, α) for every $\alpha > 0$. It suffices to consider the harmonic method N_h of regular Nörlund means defined by

$$a_{n,k} = 1/(n-k+1) \log(n+1) \quad \text{for} \quad k = 1, 2, \cdots, n; n = 1, 2, 3, \cdots.$$

It is known [8] that $N_h \subset (C, \alpha)$ for all $\alpha > 0$, and we have here again

$$A_n \cong \pi^2/6 \log^2 n .$$

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ON THE THEORY OF SPACES Λ

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1. Introduction. In this paper we discuss properties of the spaces $\Lambda(\phi, p)$, which were defined for the special case $\phi(x) = \alpha x^{\alpha-1}$, $0 < \alpha \leq 1$, in our previous paper [8]. A function $f(x)$, measurable on the interval $(0, l)$, $l < +\infty$ belongs to the class $\Lambda(\phi, p)$ provided the norm $\|f\|$, defined by

$$\|f\| \equiv \left\{\int_0^l \phi(x)\, f^*(x)^p dx\right\}^{1/p}, \tag{1.1}$$

is finite. Here $\phi(x)$ is a given nonnegative integrable function on $(0, l)$, not identically 0, and $f^*(x)$ is the decreasing rearrangement of $|f(x)|$, that is, the decreasing function on $(0, l)$, equimeasurable with $|f(x)|$. (For the properties of decreasing rearrangements see [5, 12, 7, and 8].) We write also $\Lambda(\alpha, p)$ instead of $\Lambda(\phi, p)$ with $\phi(x) = \alpha x^{\alpha-1}$, and $\Lambda(\phi)$ instead of $\Lambda(\phi, 1)$. We shall also consider spaces $\Lambda(\phi, p)$ for the infinite interval $(0, +\infty)$. In §2 we give some simple properties of the spaces Λ, and show in particular that $\Lambda(\phi, p)$ has the triangle property if and only if $\phi(x)$ is decreasing. In §3 we discuss the conjugate spaces $\Lambda^*(\phi, p)$, and show that the spaces $\Lambda(\phi, p)$ are reflexive. In §4 we give a generalization of the spaces $\Lambda(\phi, p)$, and characterize the conjugate spaces in case $p = 1$. In §5 we give applications; we prove that the Hardy-Littlewood majorants $\theta(x, f)$ of a function $f \in \Lambda(\phi, p)$ or $f \in \Lambda^*(\phi, p)$ also belong to the same class. We give sufficient conditions for an integral transformation to be a linear operation from one of these spaces into itself, and apply them to solve the moment problem for the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$.

2. Properties of spaces $\Lambda(\phi, p)$. We shall establish the following result.

THEOREM 1. *The norm $\|f\|$ defined by* (1.1) *has the triangle property if and only if $\phi(x)$ is equivalent to a decreasing function; in this case $f, g \in \Lambda(\phi, p)$*

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implies $f + g \in \Lambda(\phi, p)$.

Proof. (a) Suppose that $\|f\|$ has the triangle property. Let $\delta > 0$, $h > 0$, $a > 0$, and $a + 2h \leq l$. Set

$$f(x) = \begin{cases} 1 + \delta & \text{on } (0, a+h) \\ 1 & \text{on } (a+h, a+2h) \\ 0 & \text{on } (a+2h, l), \end{cases} \qquad g(x) = \begin{cases} 1 & \text{on } (0, h) \\ 1 + \delta & \text{on } (h, a+2h) \\ 0 & \text{on } (a+2h, l); \end{cases}$$

then

$$(f+g)^*(x) = \begin{cases} 2 + 2\delta & \text{on } (0, a) \\ 2 + \delta & \text{on } (a, a+2h) \\ 0 & \text{on } (a+2h, l). \end{cases}$$

We have $\|f\| = \|g\|$; hence the inequality $\|f+g\| \leq \|f\| + \|g\|$ is equivalent to

$$\left\{(2+2\delta)^p \int_0^a \phi(x)\,dx + (2+\delta)^p \int_a^{a+2h} \phi(x)\,dx\right\}^{1/p}$$

$$\leq 2\left\{(1+\delta)^p \int_0^{a+h} \phi(x)\,dx + \int_{a+h}^{a+2h} \phi(x)\,dx\right\}^{1/p},$$

or to

$$(2+\delta)^p \int_a^{a+2h} \phi(x)\,dx \leq (2+2\delta)^p \int_a^{a+h} \phi(x)\,dx + 2^p \int_{a+h}^{a+2h} \phi(x)\,dx,$$

and thus to

$$\text{(2.1)} \qquad \frac{(1+\delta)^p - (1+\frac{1}{2}\delta)^p}{(1+\frac{1}{2}\delta)^p - 1} \int_a^{a+h} \phi(x)\,dx \geq \int_{a+h}^{a+2h} \phi(x)\,dx.$$

If $\Phi(x)$ is the integral of ϕ over $(0, x)$, we obtain from (2.1), making $\delta \longrightarrow 0$,

$$\Phi(a+h) \geq \tfrac{1}{2}\left[\Phi(a) + \Phi(a+2h)\right];$$

that is, $\Phi(x)$ is concave, and thus $\phi(x)$ is equivalent to a decreasing function.

(b) Suppose that ϕ is decreasing. Instead of (2.1) we can now write

$$\text{(2.2)} \qquad \|f\| = \sup_{\phi_r} \left\{\int_0^l \phi_r |f|^p\,dx\right\}^{1/p},$$

the supremum being taken over all possible rearrangements ϕ_r of ϕ. It follows from (2.2) that $f, g \in \Lambda(\phi, p)$ implies $f+g \in \Lambda(\phi, p)$ and $\|f+g\| \leq \|f\| + \|g\|$.

It is now easy to see that, for $\phi(x)$ decreasing, $\Lambda(\phi, p)$ is a Banach space; the completeness may be proved by usual methods (compare [8]). In general, $\Lambda(\phi, p)$ is not uniformly convex. Suppose, for instance, that there is a sequence $\delta_n \longrightarrow 0$ such that

$$\Phi(2\delta_n)/\Phi(\delta_n) \longrightarrow 1 . \tag{2.3}$$

This condition is satisfied, for example, if $\phi(x) = x^{-1} |\log x|^{-p}$, $p > 1$. We take $f_n(x) = h_n$ on $(0, 2\delta_n)$, $f_n(x) = 0$ on $(2\delta_n, l)$; we take $g_n(x) = h_n$ on $(0, \delta_n)$, $g_n(x) = -h_n$ on $(\delta_n, 2\delta_n)$, and $g_n(x) = 0$ on $(2\delta_n, l)$; and we choose h_n so that

$$\|f_n\|^p = \|g_n\|^p = h_n^p \, \Phi(2\delta_n) = 1 .$$

Then we have

$$\tfrac{1}{2}\{f_n(x) + g_n(x)\} = \begin{cases} h_n & \text{on} \quad (0, \delta_n) , \\ 0 \text{ elsewhere} , \end{cases}$$

and $(1/2)(f_n - g_n)^*(x)$ is the same function. Therefore

$$\left\| \frac{f_n + g_n}{2} \right\|^p = \left\| \frac{f_n - g_n}{2} \right\|^p = h_n^p \, \Phi(\delta_n) \longrightarrow 1 ,$$

and so $\Lambda(\phi, p)$ is not uniformly convex. In case of the spaces $\Lambda(\alpha, p)$, the problem remains open.

The remarks made above apply also to the spaces $\Lambda(\phi, p)$ in case of the infinite interval $(0, +\infty)$. We assume in this case that $\int_0^l \phi(x)\, dx < +\infty$ for any $l < +\infty$; the additional hypothesis on $f \in \Lambda(\phi, p)$ is that the rearrangement $f^*(x)$ exists, which is the case if and only if any set $[\,|f(x)| \geq \epsilon]$, $\epsilon > 0$, has finite measure. The completeness of $\Lambda(\phi, p)$ in this case follows from the fact that the set of such f is a closed linear subset of the Banach space of all f for which (2.2) is finite. If

$$\int_0^{+\infty} \phi(x)\, dx = +\infty , \tag{2.4}$$

this subspace coincides with the whole space. Condition (2.4) is in particular satisfied if $\phi(x) = \alpha x^{\alpha - 1}$.

3. Reflexivity of the spaces $\Lambda(\phi, p)$. We shall first give some definitions and lemmas which will be useful in the sequel. If $g(x)$, $g_1(x)$ are two positive functions defined on $(0, l)$, $0 < l \leq +\infty$, we write $g \prec g_1$, if for all finite $0 \leq x \leq l$ we have

$$\int_0^x g(t)\, dt \leq \int_0^x g_1(t)\, dt.$$

Integration by parts readily yields:

LEMMA 1. *If $g \prec g_1$, and f is positive and decreasing on $(0, l)$, then*

$$\int_0^l g f\, dx \leq \int_0^l g_1 f\, dx. \tag{3.1}$$

LEMMA 2. *If $g \prec g_1$, and g, g_1 are positive and decreasing, then also $\psi(g) \prec \psi(g_1)$ for any convex increasing positive function, in particular for $\psi(u) = u^p$, $p \geq 1$.*

For the proof, let $f(x) = \{\psi(g_1(x)) - \psi(g(x))\}/\{g_1(x) - g(x)\}$ if $g(x) \neq g_1(x)$, and let $f(x)$ be equal to one of the derivates of $\psi(u)$ at $u = g(x)$ if $g(x) = g_1(x)$. Then $f(x)$ is the slope of the chord of the curve $v = \psi(u)$ on the interval (u, u_1), $u = g(x)$, $u_1 = g_1(x)$. The slope decreases as both u, u_1 decrease. Therefore $f(x)$ is decreasing and positive. Applying Lemma 1, we obtain

$$\int_0^l f(x)[g(x) - g_1(x)]\, dx \leq 0,$$

which proves our assertion.

THEOREM 2. *Suppose that $f(x)$, $g(x)$ are positive and decreasing on $(0, l)$, and $f \in \Lambda(\phi, p)$, $p > 1$. Then*

$$\int_0^l f g\, dx \leq \|f\|_\Lambda \inf_{\phi D \succ g} \left\{ \int_0^l \phi D^q\, dx \right\}^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.2}$$

where infimum is taken for all decreasing positive $D(x)$ for which $\phi D \succ g$. Moreover, this infimum is equal to the supremum of $\int_0^l fg\, dx$ for all positive decreasing f with $\|f\| \leq 1$, if there is a function D with $\phi D \succ g$ and $\int \phi D^q\, dx < +\infty$, and is to $+\infty$ if there is no such D.

This theorem is due to I. Halperin. For the proofs, see a paper of Halperin appearing in the Canadian Journal of Mathematics and, for a simpler proof, [10].

Inequality (3.2) is a combination of (3.1) and the usual Hölder inequality. For if $g_1 = \phi D \succ g$, then

$$\int_0^l fg\,dx \le \int_0^l fg_1\,dx = \int_0^l \phi^{1/p} f \phi^{1/q} D\,dx \tag{3.3}$$

$$\le \|f\| \left\{ \int_0^l \phi D^q\,dx \right\}^{1/q}.$$

Here and in the next section, the following theorem will be useful:

THEOREM 3. *Suppose that X is a normed linear space of measurable functions $f(x)$ on $(0, l)$, $0 < l < +\infty$, with the properties*: (i) *X contains all constants*; (ii) *if f_1 is measurable and $|f_1(x)| \le |f(x)|$, $f \in X$, then $f_1 \in X$ and $\|f_1\| \le \|f\|$*; (iii) *if $f \in X$ and f_e denotes the characteristic function of the set e, then $\|f f_e\| \longrightarrow 0$ as* meas $e \longrightarrow 0$.

Let Y consist of all measurable functions g for which $\int_0^l fg\,dx$ exists for all $f \in X$. Then

$$F(f) = \int_0^l fg\,dx, \qquad g \in Y, \tag{3.4}$$

is the general form of a linear functional on X, and its norm is equal to

$$\|g\| \equiv \sup_{\|f\| \le 1} \int_0^l fg\,dx < +\infty.$$

Proof. (a) Let $g \in Y$; then $\int_0^l f|g|\,dx$ exists for all $f \in X$, and $\|g\| = \sup \int_0^l f|g|\,dx$, where f runs through all *positive* $f \in X$ with $\|f\| \le 1$. If $\|g\| = +\infty$, there is a sequence $f_n \ge 0$, $\|f_n\| \le 1$ such that $\int f_n|g|\,dx > n^3$. Then $f = \sum n^{-2} f_n \in X$, and therefore $\int_0^l f|g|\,dx$ must exist. However $\int f|g|\,dx \ge n^{-2} \int f_n|g|\,dx \ge n$, which is a contradiction. Hence $\|g\| < +\infty$ for $g \in Y$. We see now that for $g \in Y$, $\int fg\,dx$ is a linear functional with norm $\|g\|$.

(b) Suppose that $F(f)$ is a given linear functional on X. By (i) and (ii), any characteristic function $f_e(x)$ belongs to X. Define $G(e) = F(f_e)$; since $|G(e) \le \|F\|\,\|f_e\| \longrightarrow 0$ as meas $e \longrightarrow 0$, there is an integrable $g(x)$ with $G(e) = \int_e g\,dx$. This means that (3.4) holds for $f = f_e$, and therefore also for all step-functions $\bar{f}$ (which are linear combinations of the f_e). For a bounded f, there is a sequence $\bar{f}_n(x) \longrightarrow f(x)$ uniformly. As $\|\bar{f}_n - f\| \longrightarrow 0$, this establishes (3.4) for all bounded f. Now suppose $f \in X$ is such that $fg = |f|\,|g|$. Let $f_n(x) = f(x)$ if $|f(x)| \le n$,

$f_n(x) = 0$ otherwise; then $\|f - f_n\| \longrightarrow 0$ by (iii), and hence $\int_0^l f_n g\, dx = F(f_n)$ has a finite limit. This shows that $\int |f|\,|g|\,dx < +\infty$; therefore $g \in Y$. Repeating the last part of this argument for an arbitrary $f \in A$, we obtain (3.4).

REMARKS. (A) Let X have the additional property: (iv) $f_n(x) \longrightarrow f(x)$ almost everywhere, $f_n \in X$, *and* $\|f_n\| \leq M$ *imply* $f \in X$. Then the existence of $\int fg\, dx$ for all $g \in Y$ implies $f \in X$.

For taking the subsequence $f_n(x) \longrightarrow f(x)$ of (b), we see that $F_n(g) = \int f_n g\, dx$ is a sequence of linear functionals convergent toward $\int fg\, dx$ for any $g \in Y$. Then the norms $\|F_n\| = \|f_n\|$ are uniformly bounded, and using (iv) we obtain $f \in X$.

(B) Since Y is the conjugate space to X, Y is a Banach space, and Y clearly satisfies (ii). Suppose now that X satisfies (i)–(iv) and that Y satisfies (i) and (iii). Then Remark A and Theorem 3 together imply that X is the conjugate space of Y, in other words that any linear functional $F(g)$ in Y is of the form $F(g) = \int fg\, dx$, $f \in X$ and $\|F\| = \|f\|$.

(C) The above results hold for the interval $(0, +\infty)$ if the conditions (i)–(iii) [and eventually (iv)] are true for functions vanishing outside of a finite interval, and also (v) *for any* $f \in X$, $\|f - f^l\| \longrightarrow 0$ *as* $l \longrightarrow \infty$, *where* f^l *is defined by* $f^l(x) = f(x)$ *on* $(0, l)$ *and* $f^l(x) = 0$ *on* $(l, +\infty)$.

Applying these general results to the space $\Lambda(\phi, p)$ in case of a finite interval, we see that (i) and (ii) are satisfied. Condition (iii) follows from

$$\|h_e\|^p \leq \int_0^{\text{meas } e} \phi f^{*p}\, dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0,$$

[$h_e(x)$ is the function $f(x) f_e(x)$], and (iv) from (2.2) and Fatou's theorem. We obtain the result that *the space* $\Lambda^*(\phi, p)$ *conjugate to* $\Lambda(\phi, p)$ *consists of all measurable functions* g *such that there is a decreasing positive* D *with* $\phi D \succ g^*$ *and* $\int_0^l \phi D^q\, dx < +\infty$; *further,*

$$\|g\|_{\Lambda^*} = \inf_{\phi D \succ g^*} \left\{ \int_0^l \phi D^q\, dx \right\}^{1/q}. \tag{3.5}$$

For it follows from Theorem 2 that

$$\left| \int_0^l f g\, dx \right| \leq \int_0^l f^* g^*\, dx \leq \|f\|_\Lambda \, \|g\|_{\Lambda^*},$$

and that $\|g\|_{\Lambda^*}$ is the supremum of the integral $\int fg\,dx$ for all $\|f\| \le 1$.

Now if $g(x) = C > 0$ is a constant, we take an $l_1 > 0$ with $\phi(l_1) > 0$ and $C_1 = Cl[l_1\phi(l_1)]^{-1}$. Then $\int_0^{l_1} C_1\phi(x)\,dx \ge Cl$; and if $D(x) = C_1$ on $(0, l_1)$, $D(x) = 0$ on (l_1, l), then $\phi D \succ g$. Therefore Λ^* satisfies (i). Also (iii) holds, for if $h_e(x) = g(x)f_e(x)$, $g \in \Lambda^*$, $g^* \prec \phi D$, then $h_e^* \prec \phi D_1$, where $D_1(x) = D(x)$ on $(0, \text{meas } e)$, $D_1(x) = 0$ on $(\text{meas } e, l)$, and

$$\|h_e\|_{\Lambda^*}^q \le \int_0^l \phi D_1^q\,dx = \int_0^{\text{meas } e} \phi D^q\,dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0.$$

We have proved the theorem:

THEOREM 4. *The space $\Lambda(\phi, p)$, $p > 1$, is reflexive. Its conjugate is defined by* (3.5).

We now consider the case of an infinite interval and assume $\int_0^\infty \phi\,dx = +\infty$. Then $f \in \Lambda(\phi, p)$ implies $f^*(x) \longrightarrow 0$ for $x \longrightarrow \infty$. If $a > 0$ is fixed and l sufficiently large, then the function $|f^l(x)|$ of (v) will take values $\ge f^*(a)$ only on a set of arbitrarily small measure. In view of (iii), condition (v) will follow for $\Lambda(\phi, p)$, if we can show that the norm of the function $f^*(a + x)$, $0 \le x < +\infty$, tends to 0 as $a \longrightarrow \infty$, or even if this is true for some sequence $a \longrightarrow \infty$. This norm does not exceed

$$\left\{\int_0^\infty \phi(x) f^*(a + x)^p\,dx\right\}^{1/p} = \left\{\int_0^\infty \phi(x) f^*(x)^p \left[\frac{f^*(x + a)}{f^*(x)}\right]^p dx\right\}^{1/p} \longrightarrow 0,$$

as the integrand has the majorant ϕf^{*p}, and $f^*(x + a)/f^*(x) \longrightarrow 0$ for $a \longrightarrow \infty$.

To prove (v) for $\Lambda^*(\phi, p)$, we need a result going beyond Lemma 1, namely that if g and D are decreasing and positive, and $\phi D \succ g$, then there is another such function D_0 for which $\phi D \succ \phi D_0 \succ g$, and that except for certain open intervals I where D_0 is constant, $\int_0^x \phi D_0\,dt = \int_0^x g\,dt$. (This fact is proved in the paper of Halperin, mentioned at the beginning of this section and in [10]). As before, we have to prove that if $g \in \Lambda^*(\phi, p)$ is positive and decreasing, then the norm of the function $h(x) = g(x + a)$, $x \ge 0$, tends to 0 as $a \longrightarrow \infty$ for certain values of a. There is a D with $\phi D \succ g$ and $\int_0^\infty \phi D^q\,dx < +\infty$; and, by Lemma 2, $\int_0^\infty \phi D_0^q\,dx < +\infty$. As $\int_0^\infty \phi\,dx = +\infty$, we deduce that $D_0(x) \longrightarrow 0$ for $x \longrightarrow \infty$. Therefore

$$\int_0^x \phi D_0\,dx = o[\Phi(x)].$$

On intervals I, $\int_0^x \phi D_0\,dt$ is of the form $C\Phi(x) + C_1$, where $\Phi(x) = \int_0^x \phi\,dt$. If an I extends to $+\infty$, we have $C = 0$, that is $\int_0^x \phi D_0\,dt = C_1$ for all large x. and $D_0(x)$ is necessarily 0 for all such x. In this case also $g(x) = 0$ for all large x, and our assertion is trivial. If, on the other hand, there are arbitrarily large values a which do not belong to any I, then we have for these a,

$$\int_0^a \phi D_0\,dt = \int_0^a g\,dt .$$

It follows that $\int_0^x \phi D_0\,dt \geq \int_0^x g\,dt$, $x \geq a$, or $\phi(x+a)D_0(x+a) \succ g(x+a)$, and this implies $\phi(x)D_0(x+a) > g(x+a)$. Therefore,

$$\|h\|^q \leq \int_0^\infty \phi(x)\,D_0(x+a)^q\,dx = \int_0^\infty \phi(x)\,D_0(x)^q \left[\frac{D_0(x+a)}{D_0(x)}\right]^q dx \longrightarrow 0$$

for $a \longrightarrow \infty$. We obtain in this way:

THEOREM 5. *The space* $\Lambda(\phi, p)$, $p > 1$, $l = \infty$ *is reflexive; its conjugate is given by* (3.5).

4. A generalization. There is an obvious generalization of the spaces $\Lambda(\phi, p)$. Consider a class C of functions $\phi(x) \geq 0$ integrable over $(0, l)$, and let $X(C, p)$ consist of all those functions $f(x)$ for which

$$\|f\| = \sup_{\phi \in C} \left\{\int_0^l \phi\,|f|^p\,dx\right\}^{1/p} < +\infty. \tag{4.1}$$

A special type of these spaces is obtained if C is chosen to consist of all integrable positive functions $\phi(x)$ whose integrals $\phi_1(e)$ satisfy the condition

$$\phi_1(e) \leq \Phi(e)\,, \tag{4.2}$$

where $\Phi(e)$ is a given positive finite set function of measurable sets $e \subset (0, l)$. We may then assume that

$$\Phi(e) = \sup_{\phi_1} \phi_1(e)\,. \tag{4.3}$$

(A full characterization of set functions $\Phi(e)$ which may be represented in form (4.3) by means of a class of positive additive ϕ_1 will be given by the author elsewhere [9].) In particular, let $\phi_0(x)$ be a fixed decreasing positive function, and let $\Phi(e) = \int_0^{\text{meas}\,e} \phi_0\,dx$; then condition (4.2) is equivalent to the condition

$$\phi^*(x) \leq \phi_0(x) .$$

Therefore, in this case the norm (4.1) is equal to (1.1), and so $X(\Phi, p) = \Lambda(\phi_0, p)$.

For the space $X(\Phi, p)$, the condition $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere if and only if $\Phi(e) > 0$ for any set e of positive measure. Suppose now that $\Phi(e)$, defined by (4.3), vanishes on certain sets e with meas $e > 0$. There is then [2, p. 80, Theorem 15] a least measurable set e_0 which contains any such set e up to a null set; and e_0 is a union of a properly chosen denumerable set of these sets e. Hence $\phi_1(e_0) = 0$, and $\Phi(e_0) = 0$. It is easy to see that in this case $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere on $(0, l) - e_0$, and that the values of $f(x)$ on e_0 have no significance whatsoever for $\|f\|$. Omitting e_0 from $(0, l)$, we do not change the space $X(\phi, p)$, and we obtain a $\Phi(e)$ satisfying the above condition. In the sequel, ϕ is assumed to have this property.

The spaces $X(\Phi, p)$ are normed linear spaces. Their completeness may be proved by usual methods, if for instance $F(e)$ has the property that meas $e \longrightarrow 0$ implies $\Phi(e) \longrightarrow 0$ and if $l < +\infty$.

The spaces $X(C, p)$ satisfy the conditions (i), (ii), and (iv) of 3 [(iv) follows easily by Fatou's theorem]. Condition (iii) is not fulfilled in general. We can however enforce (iii) by defining the spaces $\Lambda(C, p)$ and $\Lambda(\Phi, p)$ to consist of all those functions $f \in X(C, p)$ or $f \in X(\Phi, p)$, respectively, for which $\|f f_e\| \longrightarrow 0$ with meas $e \longrightarrow 0$ in X. Then the conjugate space $\Lambda^*(C, p)$ and all linear functionals in $\Lambda(C, p)$ are given by Theorem 3. We conclude this section by describing the spaces $\Lambda^*(\Phi, 1)$ more precisely:

THEOREM 6. *If $f \in \Lambda(\Phi, 1)$, then*

$$\left| \int_0^l f g \, dx \right| \leq \|f\| \sup_{\Phi(e) > 0} \frac{1}{\Phi(e)} \int_e |g| \, dx , \tag{4.4}$$

and the left integral exists provided the right side is finite; moreover, the supremum $M(g)$ in the right side is equal to the supremum of $\int_0^l fg \, dx$ for all $f \in \Lambda(\Phi, 1)$ with $\|f\| \leq 1$.

Proof. Consider the function $\phi_0(x) = M(g)^{-1} \, |g(x)|$; then

$$\int_0^l |f| \, |g| \, dx = M(g) \int_0^l \phi_0 \, |f| \, dx \leq M(g) \, \|f\|_\Lambda ,$$

since

$$\int_e \phi_0(x)\,dx = M(g)^{-1}\int_e g(x)\,dx \le \Phi(e)\,, \qquad e \subset (0,\,l)\,.$$

This proves (4.4). On the other hand, if e is an arbitrary subset of $(0,l)$ with $\Phi(e) > 0$, then the function $f(x) = \Phi(e)^{-1} f_e(x)$ sign $g(x)$ has norm 1 in $\Lambda(\phi,1)$, and

$$\int_0^l f g\,dx = \Phi(e)^{-1}\int_e |g|\,dx\,.$$

Therefore the integral $\int_0^l fg\,dx$ takes values arbitrarily close to $M(g)$.

From Theorems 3 and 6 we deduce that the space $\mathrm{M}(\Phi,1) = \Lambda^*(\Phi,1)$ consists of all $g(x)$ for which

$$\|g\| = \sup_e \left\{\Phi(e)^{-1}\int_e |g(x)|\,dx\right\} < +\infty. \tag{4.5}$$

In particular, the space $\mathrm{M}(\phi)$, conjugate to $\Lambda(\phi)$, is given by

$$\|g\|_{\mathrm{M}(\phi)} = \sup_e \left\{\phi_1(e)^{-1}\int_e |g|\,dx\right\}. \tag{4.6}$$

It is easy to see that the expression (4.6) is the limit, for $p \longrightarrow 1$, of the norm of g in the space $\Lambda^*(\phi,p)$, $p > 1$.

5. Applications. We shall make three applications.

5.1. *Hardy-Littlewood majorants.* We take in this section $l = 1$. We write

$$\theta(x,f) = \sup_{0 \le y \le 1} \frac{1}{y-x}\int_x^y |f(t)|\,dt, \tag{5.1}$$

and denote by $\theta_1(x,f)$ and $\theta_2(x,f)$ the supremum of the same expression for $0 \le y < x$ or $x < y \le 1$, respectively. Then

$$\theta(x,f) \le \max\,\{\theta_1(x,f),\ \theta_2(x,f)\}\,. \tag{5.2}$$

On the other hand, it is well known [5, p. 291] that

$$\theta_1^*(x,f) \le \theta(x,f^*) = \frac{1}{x}\int_0^x f^*(t)\,dt\,, \tag{5.3}$$

and this is also true with θ_2 in place of θ_1. From (5.2) we derive, for any $p \ge 1$,

$$\theta^p(x, f) \le \theta_1^p(x, f) + \theta_2^p(x, f) .$$

It follows that

$$\theta^*(x, f)^p \le (\theta_1^p + \theta_2^p)^* < (\theta_1^p)^* + (\theta_2^p)^* = \theta_1^{*p} + \theta_2^{*p} \le 2\theta(x, f^*)^p ;$$

that is,

$$\theta^*(x, f)^p < 2\theta(x, f^*)^p . \tag{5.4}$$

We shall make repeated use of the inequality of Hardy [12, p. 72]:

$$\int_0^l x^{s-p} F(x)^p \, dx \le \left(\frac{p}{p-s-1}\right)^p \int_0^l x^s f(x)^p \, dx , \tag{5.5}$$

where $p > 1$, $s < p - 1$, $0 < l \le +\infty$, and $F(x)$ is the integral of the positive function $f(x)$.

In our present situation it follows from (5.3) and (5.5), if $p > 1$, that

$$\int_0^x \theta(t, f^*)^p \, dt \le \left(\frac{p}{p-1}\right)^p \int_0^x f^*(t)^p \, dt ;$$

and, by Lemma 1,

$$\int_0^1 \phi(x)\theta^*(x, f)^p \, dx \le 2\left(\frac{p}{p-1}\right)^p \int_0^1 \phi(x) f^*(x)^p \, dx . \tag{5.6}$$

This is case (i) of the following theorem:

THEOREM 7. (i) *If $f \in \Lambda(\phi, p)$ and $p > 1$, then also $\theta(x, f) \in \Lambda(\phi, p)$;* (ii) *if $f^*(x) \log (1/x) \in \Lambda(\phi)$, then $\theta(x, f) \in \Lambda(\phi)$;* (iii) *if $f \in \Lambda(\phi)$, and $\phi(x)$ is decreasing with respect to $x^{-\delta}$ for some $\delta > 0$, then $\theta(x, f) \in \Lambda(\phi)$.*

To prove (ii) we observe that (5.4) with $p = 1$ and Lemma 1 imply

$$\|\theta\|_{\Lambda(\phi)} = \int_0^1 \phi(x)\theta^*(x, f) \, dx \le 2\int_0^1 \phi(x) \frac{1}{x} dx \int_0^x f^*(t) \, dt$$

$$= 2\int_0^1 f^*(t) \, dt \int_t^1 \frac{\phi(x)}{x} dx \le 2\int_0^1 \phi(t) f^*(t) \log \frac{1}{t} dt < +\infty .$$

Finally, if the hypothesis of (iii) holds, that is if $\phi(x) = x^{-\delta} D(x)$ with a decreasing positive D, then the preceeding inequality gives

$$\|\theta\| \leq 2 \int_0^1 f^*(t) D(t) \int_t^1 x^{-\delta-1}\, dx \leq 2\,\delta^{-1} \int_0^1 \phi(t) f^*(t)\, dt\,.$$

THEOREM 8. (i) *If* $f^*(x) \log (1/x) \in \Lambda^*(\phi, p)$, $p \geq 1$, *then* $\theta(x, f) \in \Lambda^*(\phi, p)$; (ii) *if* $f \in \Lambda^*(\alpha, p)$, $p > 1$, *then* $\theta(f) \in \Lambda^*(\alpha, p)$.

Proof. (i) Let $p > 1$ [the case $p = 1$, $\Lambda^*(\phi, p) = M(\phi)$ is simpler]. By (5.4), and since $\theta(x, f^*)$ decreases, we have

$$\|\theta(f)\|^q \leq 2^q \|\theta(f^*)\|^q = 2^q \inf_{\phi D \succ \theta(f^*)} \int_0^1 \phi(x) D(x)^q\, dx\,.$$

But by (5.3), we have

$$\int_0^x \theta(u, f^*)\, du = \int_0^x f^*(t)\, dt \int_t^x \frac{du}{u} \leq \int_0^x f^*(t) \log \frac{1}{t}\, dt\,,$$

which means that $\theta(x, f^*) \prec f^*(x) \log (1/x) = h(x)$; hence

$$\|\theta(f)\|^q \leq 2^q \inf_{\phi D \succ h} \int_0^1 \phi D^q\, dx = 2^q \|h\|^q < +\infty\,.$$

(ii) Let $f \in \Lambda^*(\alpha, p)$; because of (5.4) we may assume that $f = f^*$, that is, that f is positive and decreasing. Suppose $f \prec \phi D$ and $\int_0^1 \phi D^q\, dx < +\infty$ with $\phi(x) = \alpha x^{\alpha-1}$. Then by (5.3) we have

$$\theta(x, f) = \frac{1}{x} \int_0^x f(t)\, dt \leq \frac{\alpha}{x} \int_0^x t^{\alpha-1} D(t)\, dt$$

$$= \alpha x^{\alpha-1} \frac{1}{x^\alpha} \int_0^x t^{\alpha-1} D(t)\, dt = \phi(x) D_1(x)\,,$$

say. The function $D_1(x)$ is positive and decreasing, as

$$D_1'(x) = -\alpha x^{-\alpha-1} \int_0^x t^{\alpha-1} D\, dt + x^{-1} D(x)$$

$$\leq -\alpha x^{-\alpha-1} D(x) \int_0^x t^{\alpha-1}\, dt + x^{-1} D(x) = 0\,.$$

Therefore, by Hardy's inequality, we have

$$\|\theta(f)\|^q \leq \alpha \int_0^1 x^{\alpha-1} D_1^q\, dx = \alpha \int_0^1 x^{(1-\alpha)(q-1)} \left\{ \frac{1}{x} \int_0^x t^{\alpha-1} D\, dt \right\}^q dx$$

$$\leq C \int_0^1 x^{(1-\alpha)(q-1)+(\alpha-1)q} D(x)^q \, dx = C \int_0^1 x^{\alpha-1} D^q \, dx$$

with some constant C. Thus $\theta(f) \in \Lambda^*$, which proves (ii).

It should be remarked that $f^* \log (1/x)$ behaves very much like $f^* \log^+ f^*$:

(a) *If* $f^* \log (1/x)$ *belongs to* $\Lambda^*(\phi, p)$, $p \geq 1$, *then* $f \log^+ |f|$ *belongs to* $\Lambda^*(\phi, p)$. For if $p > 1$ [the case $p = 1$ is similar but simpler], there is a $D(x)$ with $f^* \log (1/x) \prec \phi D$ and $\int_0^1 \phi D \, dx < +\infty$. Then also $f^*(\delta) \log (1/x) \prec \phi D$ on $(0, \delta)$; in particular,

$$f^*(\delta) \int_0^\delta \log \frac{1}{x} \, dx \leq \int_0^\delta \phi D \, dx \leq 1$$

if δ is small. Therefore $f^*(\delta) \leq \delta^{-1}$ for all small δ, which shows that

$$f \log^+ |f| \in \Lambda^*(\phi, p).$$

(b) Now suppose $\phi(x)$ is such that, for some $\delta > 0$, we have $\int_0^1 \phi(x) x^{-\delta} dx < +\infty$. *If* $f \log^+ |f|$ *belongs to* $\Lambda^*(\phi, p)$, $p \geq 1$, *then* $f^* \log (1/x)$ *also does*. In fact, by Young's inequality [5, p. 111; or 11, p. 64], for the pair of inverse functions $\phi(u) = \log^+ u$, $\psi(v) = e^v$, we obtain $ab \leq a \log^+ a + e^b$ $(a, b \geq 0)$ and therefore

$$f^* \log \frac{1}{x} \leq \delta^{-1} f^* \log^+ (\delta^{-1} f^*) + x^{-\delta} \leq \delta^{-1} f^* \log^+ \frac{1}{\delta} + \delta^{-1} f^* \log^+ f^* + x^{-\delta}$$

$$\leq A f^* \log^+ f^* + B + x^{-\delta}$$

for some constants A, B.

It follows from these remarks, that Theorem 7 (ii) may be regarded as a generalization of the theorem of Hardy-Littlewood [12, p. 245] that $f \log^+ |f| \in L$ implies $\theta(f) \in L$.

Theorems 7 and 8 have many applications which may be derived in the same way as the corresponding results for the spaces L^p (see [12, p. 246]). As an example, we give the following result. Let $k > 0$, and let $\sigma_n^{(k)}(x, f)$ denote the Cesàro sum of order k of the Fourier series of a function $f(x)$. If $\theta(x, f)$ is taken for the interval $(0, 4\pi)$, we have: *if* $f(x)$ *satisfies one of the hypotheses of Theorems 7 or 8, then* $|\sigma_n^{(k)}(x, f)| \leq C_k \theta(x, f)$, $n = 0, 1, \cdots$. We may give another formulation of this result. In the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$ we introduce a partial ordering, writing $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ almost everywhere. With this ordering, Λ and Λ^* become Banach lattices for which the order convergence $f_n \longrightarrow f$ is identical with the convergence $f_n(x) \longrightarrow f(x)$ almost everywhere and the existence of a function $h(x)$ of the lattice such that $|f_n(x)| \leq h(x)$ almost everywhere. This is an immediate consequence of the fact that the lattices Λ, Λ^* satisfy the condition (ii) of Theorem 3 (see [6, pp. 154-156]). Then the above result implies that $\sigma_n^{(k)} \longrightarrow f$ in order in the corresponding space. Theorems of this section may also be used to obtain analogues of theorems of Hardy [3] and Bellman [1] for spaces Λ and Λ^*; see Petersen [11].

5.2. *Integral transformations.* Let $K(x, t)$ be measurable on the square $0 \leq x \leq 1$, $0 \leq t \leq 1$, and let

$$F(x) = \int_0^1 K(x, t) f(t)\, dt. \tag{5.7}$$

THEOREM 9. *Suppose that there is a constant M such that*

(i) $$\int_0^1 |K(x, t)|\, dt \leq M \quad \textit{almost everywhere};$$

(ii) *for any rearrangement $\phi_r(x)$ of $\phi(x)$, the function $h_r(t) = \int_0^1 \phi_r(x) K(x, t)\, dx$ belongs to $\mathrm{M}(\phi)$ and has a norm not exceeding M. Then* (5.7) *is a linear operator of norm $\leq M$ mapping $\Lambda(\phi, p)$ into itself. Condition* (ii) *may also be replaced by*

(iii) $$\int_0^1 |K(x, t)|\, dx \leq M \quad \textit{almost everywhere}.$$

Proof. Condition (ii) is equivalent to

$$h_r^*(t) \prec M\phi(t). \tag{5.8}$$

Assuming $f \in \Lambda(\phi, p)$, $p > 1$, we have

$$\begin{aligned}\int_0^1 \phi_r(x)\, |F(x)|^p\, dx &\leq \int_0^1 \phi_r\, dx \left\{\int_0^1 |K|\, |f(t)|\, dt\right\}^p \\ &\leq \int_0^1 \phi_r\, dx \int_0^1 |K|\, |f|^p\, dt \left\{\int_0^1 |K|\, dt\right\}^{p/q} \\ &\leq M^{p/q} \int_0^1 |f(t)|^p\, dt \int_0^1 \phi_r(x)\, |K(x, t)|\, dx \\ &\leq M^{p/q} \int_0^1 h_r^*(t) f^*(t)^p\, dt;\end{aligned}$$

by (5.8) and Lemma 1, this is

$$\leq M^{1+p/q} \int_0^1 \phi(t) f^*(t)^p \, dt = M^p \|f\|^p ,$$

which proves the first part of the theorem. Suppose now that (i) and (iii) hold. Let $\delta > 0$, e an arbitrary set of measure δ, and e_1 a set of measure δ such that $\phi_r(x) \geq \phi(\delta)$ on e_1 and $\phi_r(x) \leq \phi(\delta)$ on the complement Ce_1 of e_1. Then we have

$$\int_e |h_r(t)| \, dt \leq \int_e dt \int_{e_1} |\phi_r(x)| \, |K| \, dx + \int_e \int_{Ce_1}$$

$$\leq M \int_{e_1} |\phi_r(x)| \, dx + \phi(\delta) \int_e dt \int_0^1 |K(x, t)| \, dx$$

$$\leq M \Phi(\delta) + M \delta \phi(\delta) \leq 2M \Phi(\delta) .$$

This shows that the norm of $h_r(t)$ in $M(\phi)$ does not exceed $2M$, and proves (ii).

REMARK. If the conditions of Theorem 9 are satisfied, then

$$G(t) = \int_0^1 K(x, t) g(x) \, dx \tag{5.9}$$

is a linear operator of norm $\leq 2M$ mapping $\Lambda^*(\phi, p)$ into itself.

We have in fact, for $g \in \Lambda^*(\phi, p)$ and $f \in \Lambda(\phi, p)$,

$$\int_0^1 G(t) f(t) \, dt = \int_0^1 g(x) \, dx \int_0^1 K(x, t) f(t) \, dt = \int_0^1 g(x) F(x) \, dx$$

$$\leq \|g\|_{\Lambda^*} \|F\|_\Lambda \leq M \|f\|_\Lambda \|g\|_{\Lambda^*} ,$$

(the integrals evidently exist), and this shows that $G \in \Lambda^*$ and that $\|G\| \leq M \|g\|$.

Theorem 9 is akin to the "convexity theorem" of M. Riesz [12, p.198]. We mention for completeness that there is a generalization of this theorem, in which the different spaces L^p involved are replaced by the spaces $\Lambda(\phi, p)$ *with the same* ϕ. The proof, which follows closely the proof of M. Riesz's theorem in [12], is omitted.

5.3. *Moment problems.* We give an application of Theorem 9 to moment problems of the form

$$\mu_n = \int_0^1 x^n f(x) \, dx, \qquad n = 0, 1, 2, \cdots . \tag{5.10}$$

We shall write

$$\Phi_{n\nu} = \Phi\left(\frac{\nu+1}{n+1}\right) - \Phi\left(\frac{\nu}{n+1}\right), \qquad \Phi(x) = \int_0^x \phi\, dt,$$

$$\mu_{n\nu} = \binom{n}{\nu}\Delta^{n-\nu}\mu_\nu = \int_0^1 f(x)p_{n\nu}(x)\, dx,$$

$$p_{n\nu} = \binom{n}{\nu} x^\nu(1-x)^{n-\nu}, \qquad \nu = 0, 1, \cdots, n,$$

and $\mu_{n\nu}^*$ for the decreasing rearrangement of the $|\mu_{n\nu}|$, $\nu = 0, 1, \cdots, n$. Moreover, we set

$$(5.11) \qquad f_n(x) = (n+1)\mu_{n\nu} \qquad \text{for } \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1},$$

and obtain

$$(5.12) \qquad f_n(x) = \int_0^1 K_n(x, t) f(t)\, dt,$$

$$K_n(x,t) = (n+1)p_{n\nu}(t), \qquad \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1},$$

For the special case $\phi(x) = \alpha x^{\alpha-1}$ and $p = 1$, the following theorem (with another proof) has been given in [8].

THEOREM 10. *The sequence of real numbers μ_n is a moment sequence of a function of the space $\Lambda(\phi, p)$ or of $\Lambda^*(\phi, p)$ [for the case $\Lambda(\phi, 1)$, we assume $\phi(x) \longrightarrow \infty$ for $x \longrightarrow 0$] if and only if the norms of the functions* (5.11) *are uniformly bounded in this space.*

For the space $\Lambda(\phi, p)$, the condition is

$$(5.13) \qquad \sum_{\nu=0}^{n} \Phi_{n\nu}\mu_{n\nu}^{*p} \le M(n+1)^{-p},$$

and for $\Lambda^*(\phi, p)$, $p > 1$,

$$(5.14) \qquad \mu_{n\nu}^* \prec \Phi_{n\nu}D_{n\nu}, \qquad \sum_{\nu=0}^{n} \Phi_{n\nu}D_{n\nu}^q \le M^q,$$

with some positive decreasing $D_{n\nu}$, $\nu = 0, 1, \cdots, n$.

Proof. If $f \in \Lambda(\phi, p)$, then condition (5.13) is satisfied by Theorem 9, because the kernel (5.12) satisfies (i) and (iii) with $M = 1$.

Conversely, let $\|f_n\|_\Lambda \le M$. Since

$$\int_e |f_n(x)|\, dx \le \phi(\delta)^{-1} \int_0^\delta \phi(x)\, |f_n(x)|\, dx \le M\phi(\delta)^{-1}, \qquad \text{meas } e = \delta,$$

it follows in case $p = 1$ that the integrals $\int_e |f_n|\, dx$ are uniformly absolutely continuous and uniformly bounded. In case $p > 1$, this follows by Hölder's inequality. We deduce that for a certain subsequence $f_{n_k}(x)$, the integrals $\int_e f_{n_k}(x)\, dx$ converge for any $e = (0, x)$ with x rational; hence they converge for any measurable set $e \subset (0,1)$. We then have

$$\lim_{k\to\infty} \int_e f_{n_k}(x)\, dx = \int_e f(x)\, dx, \tag{5.15}$$

with some $f \in L$. Then also

$$\int_0^1 f_{n_k}\psi\, dx \longrightarrow \int_0^1 f\psi\, dx \tag{5.16}$$

for any bounded ψ. For any such ψ we have, by (3.2),

$$\left|\int_0^1 f\psi\, dx\right| \le \lim \left|\int_0^1 f_{n_k}\psi\, dx\right| \le M\|\psi\|_{\Lambda^*};$$

hence this must be true for any ψ in Λ^*. Thus by §3, it follows that $f \in \Lambda(\phi, p)$.

We remark also that it follows easily from (5.16) that we have

$$\int_0^1 f_{n_k}\psi_k\, dx \longrightarrow \int_0^1 f\psi\, dx, \tag{5.17}$$

if the sequence $\psi_k(x)$ is uniformly convergent towards a bounded function $\psi(x)$.

Now let P be the vector space of all polynomials

$$\psi(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with usual addition and scalar multiplication. On P we define an additive and homogeneous functional F by

$$F(\psi) = a_0\mu_0 + a_1\mu_1 + \cdots + a_m\mu_m .$$

Let

$$B_n^\psi(x) = \sum_{\nu=0}^{n} \psi\left(\frac{\nu}{n}\right) p_{n\nu}(x)$$

be the Bernstein polynomial of order n of $\psi(x)$; then it is known [10] that

$$B_n^\psi(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_m^{(n)}x^m ,$$

and that $a_i^{(n)} \longrightarrow a_i$ for $n \longrightarrow \infty$. Hence $F(B_n^\psi) \longrightarrow F(\psi)$. In particular, let $\psi(x) = x^m$. We have

$$F(B_n^\psi) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m F(p_{n\nu}) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m \mu_{n\nu} \tag{5.18}$$

$$= \int_0^1 f_n(x) g_n(x)\, dx ,$$

where $\psi_n(x)$ is equal to $(\nu/n)^m$ in the interval $[\nu/(n+1), (\nu+1)/(n+1)]$. As $\psi_n(x) \longrightarrow \psi(x)$ uniformly, we deduce from (5.18) and (5.17) that

$$\int_0^1 f(x)x^m\, dx = \lim F(B_n) = F(\psi) = \mu_m , \qquad m = 0, 1, \cdots .$$

Since $f \in \Lambda(\phi, p)$, this proves that the condition is sufficient in case of the space Λ. The proof for the space $\Lambda^*(\phi, p)$, which is similar, is omitted.

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ON A CERTAIN NONLINEAR INTEGRAL EQUATION OF THE VOLTERRA TYPE

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1. Introduction. In an earlier paper by Mann and Wolf [1], the following problem of heat transfer between a gas at constant unit temperature and the semi-infinite solid was considered:

GAS	SOLID
Temperature 1°	Temperature $U(x,t)$
	$U(x, 0) = 0$ for $x \geq 0$
	$\longrightarrow x$

$$U_t(x, t) = U_{x,x}(x, t) , \tag{1.1}$$

$$U(x, 0) = 0 , \tag{1.2}$$

$$|U(x, t)| < M , \qquad x > 0 , \quad t > 0 , \tag{1.3}$$

$$U_x(0, t) = \frac{-[1 - U(0, t)]}{K} f[1 - U(0, t)] = -G[U(0, t)] . \tag{1.4}$$

It will be noted that, in boundary condition (1.4), Newton's Law of Cooling has been replaced by the weaker, more realistic hypothesis that the net rate of heat exchange from the gas to the solid, $-KU_x(0,t)$, is some function, $KG[U(0,t)]$, of the surface temperature. In every heat transfer problem of physical significance, the following conditions must be satisfied by $G[U]$:

$$G[U] \text{ is continuous,} \tag{1.5}$$

$$G[1] = 0, \tag{1.6}$$

$$G[U] \text{ is strictly decreasing.} \tag{1.7}$$

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Pacific J. Math. 1 (1951), 431-445.

By Duhamel's Principle, the solution $U(x, t)$ of the above boundary value problem is easily constructed once we know the surface temperature, $U(0,t)$, which it can be shown must satisfy the nonlinear integral equation,

$$U(0, t) = \int_0^t \frac{G[U(0,\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{1.8}$$

Equation (1.8) was shown in [1] to have at least one solution for all G satisfying (1.5), (1.6), and (1.7). Under the additional *ad hoc* assumption that G satisfy a Lipschitz condition on the unit interval, the solution of (1.8) was proved to be unique and nondecreasing.

It is the purpose of the present paper to show that conditions (1.5), (1.6), and (1.7) alone are sufficient to imply that $U(0,t)$ is not only unique but also strictly increasing. Besides being a stronger result than that previously obtained, it has the advantage of requiring only those conditions imposed upon G by the most elementary physical consideration.

2. The theorems. More general results are obtained without increasing the complexity of the proofs if instead of the function $[\pi(t-\tau)]^{-1/2}$ we write $K(t - \tau)$, or $K(z)$ where $t - \tau = z$, subject to specified conditions, namely:

(2.1) $K(z)$ is positive, continuous, and strictly decreasing for $z > 0$;

(2.2) $\int_0^t K(z)\,dz$ is finite for each $t > 0$;

(2.3) $K(z + \alpha)/K(z)$ is strictly increasing in z for each fixed α greater than zero;

(2.4) $\int_0^t K(z)\,dz \longrightarrow \infty$ as $t \longrightarrow \infty$.

It is easily verified, for example, that $[\pi(t - \tau)]^{-p}$ satisfies the above conditions for $0 < p < 1$.

THEOREM 1. *The equation*

$$y(t) = \int_0^t G[y(\tau)]K(t - \tau)\, d\tau \tag{2.5}$$

can have at most one bounded solution, given that $G[y]$ *satisfies* (1.5), (1.6), *and* (1.7), *and that* $K(z)$ *satisfies* (2.1) *and* (2.2).

THEOREM 2. *In addition to the hypotheses of Theorem 1, assume that* K *satisfies* (2.3). *If* $y(t)$ *is a bounded solution of* (2.5), *then* $y(t)$ *is strictly increasing in* t. *If, in addition,* K *satisfies* (2.4), *then* $y(t) \longrightarrow 1$ *as* $t \longrightarrow \infty$.

3. On Theorem 1. In this section we arrive at a proof of Theorem 1.

LEMMA 3.1. *Suppose that* $f(\tau)$ *is continuous for* $a \le \tau \le b$, *and that* $\int_a^t f(\tau)\,d\tau$ *is positive for some* t *on* $[a, b]$. *Let* t_1 *be the smallest value of* t *on* $[a, b]$ *for which* $\int_a^t f(\tau)\,d\tau$ *is a maximum. Then either* $f(t_1) = 0$ *or* $t_1 = b$. *Suppose that* $K(\tau)$ *is positive and strictly increasing on* $a \le \tau < t_1$, *and that* $\int_a^{t_1} K(\tau)\,d\tau$ *exists. Then* $\int_a^{t_1} f(\tau)\,K(\tau)\,d\tau > 0$.

Proof. Set $\int_a^{t_1} f(\tau)\,d\tau = M > 0$. Divide f into its positive and negative parts by writing $f_1(\tau) = \max\,[f(\tau), 0]$ and $f_2(\tau) = -\min\,[f(\tau), 0]$, so that $f(\tau) = f_1(\tau) - f_2(\tau)$. Let $c_0 = a$, and define c_1 to be the smallest number $c\,(c > c_0)$ such that $\int_a^c f_1(\tau)\,d\tau = M$. Then $c_1 \le t_1$. In general, choose c_{n+1} as the smallest number greater than c_n for which

$$(3.2)\qquad \int_{c_n}^{c_{n+1}} f_1(\tau)\,d\tau = \int_{c_{n-1}}^{c_n} f_2(\tau)\,d\tau.$$

Since $\int_{c_1}^{t_1} f_1(\tau)\,d\tau = \int_{c_0}^{t_1} f_2(\tau)\,d\tau$, it follows that for each n we have $c_n \le t_1$. Let c be the number to which the sequence $c_0, c_1, c_2, \cdots$ converges. Then $c \le t_1$ and

$$\int_{c_0}^{c} f(\tau)\,d\tau = \int_{c_0}^{c_1} f_1(\tau)\,d\tau + \sum_{n=1}^{\infty} \left[\int_{c_n}^{c_{n+1}} f_1(\tau)\,d\tau - \int_{c_{n-1}}^{c_n} f_2(\tau)\,d\tau\right] = M,$$

since each summand of the infinite series is zero. Thus $c = t_1$.

We have

$$\begin{aligned}(3.3)\quad &\int_{c_0}^{t_1} f(\tau) K(\tau)\,d\tau \\ &= \int_{c_0}^{t_1} [f_1(\tau) - f_2(\tau)] K(\tau)\,d\tau \\ &= \int_{c_0}^{c_1} f_1(\tau) K(\tau)\,d\tau + \sum_{n=1}^{\infty} \left[\int_{c_n}^{c_{n+1}} f_1(\tau) K(\tau)\,d\tau - \int_{c_{n-1}}^{c_n} f_2(\tau) K(\tau)\,d\tau\right].\end{aligned}$$

Now for $n \geq 1$ we have

$$\int_{c_n}^{c_{n+1}} f_1(\tau) K(\tau)\, d\tau \geq K(c_n) \int_{c_n}^{c_{n+1}} f_1(\tau)\, d\tau,$$

since $K(\tau)$ is strictly increasing; and

$$\int_{c_{n-1}}^{c_n} f_2(\tau) K(\tau)\, d\tau \leq K(c_n) \int_{c_{n-1}}^{c_n} f_2(\tau)\, d\tau .$$

Thus, by (3.2), each summand in the expansion (3.3) is positive or zero, and the first one is positive. Hence $\int_a^{t_1} f(\tau)\, K(\tau) d\tau > 0$.

The first assertion of the lemma, namely that either $f(t_1) = 0$ or $t_1 = b$, is obvious.

LEMMA 3.4. *Assume that* $f(\tau)$ *is continuous on* $0 \leq \tau \leq T$ *and that* $K(z)$ *satisfies* (2.1) *and* (2.2). *Suppose furthermore that* $F(t) f(t) \leq 0$ *for* $0 \leq t \leq T$, *where* $F(t) = \int_0^t f(\tau)\, K(t-\tau) d\tau$. *Then* $f(\tau) = 0$ *for* $0 \leq \tau \leq T$.[1]

Proof. Assume the lemma to be false. Then for some t we have $\int_0^t f(\tau) d\tau \neq 0$. There is no loss of generality in assuming $\int_0^t f(\tau) d\tau > 0$, since replacing f by $-f$ results in replacing F by $-F$, so that the inequality $F(t) f(t) \leq 0$ persists. Clearly $f(\tau)$ must change signs, so there exists a number b, $0 < b < T$, such that $f(b) = 0$ and, for some $t < b$, $\int_0^t f(\tau) d\tau > 0$. Let t_1 be the smallest value of t $(0 < t \leq b)$ for which $\int_0^t f(\tau) d\tau$ is a maximum and apply Lemma 3.1 using $K(t_1 - \tau)$ in place of $K(\tau)$. We have

$$F(t_1) = \int_0^{t_1} f(\tau) K(t_1 - \tau)\, d\tau > 0 .$$

Then we have $F(t) > 0$ over the segment $(t_1 - \delta, t_1)$ for some $\delta > 0$; and since $\int_{t_1-\delta}^{t_1} f(\tau) d\tau > 0$ there is some t between $t_1 - \delta$ and t_1 for which $f(t) > 0$. But for this t we have $F(t)\, f(t) > 0$, violating our hypothesis. Thus $f(t)$ is identically zero on $[0, T]$. This completes the proof of Lemma 3.4 and we are now ready to prove the uniqueness theorem.

Proof of Theorem 1. Suppose $y_1(t)$ and $y_2(t)$ are bounded solutions of (2.5). Obviously both are continuous. Letting $F(t) = y_1(t) - y_2(t)$, and $f(\tau) = G[y_1(\tau)] - G[y_2(\tau)]$, we have $F(t) = \int_0^t f(\tau)\, K(t - \tau) d\tau$. If $f(\tau) < 0$ then, since G is

[1] In place of assuming continuity we may assume that $f(\tau)$ has a Lebesgue integral over $[0, T]$ and that the condition $F(t) f(t) \leq 0$ holds except for a set of measure zero. Then we may conclude that $f(\tau) = 0$ over $[0, T]$ except at points of a set of measure zero.

strictly decreasing, we have $y_2(\tau) < y_1(\tau)$, whence $F(\tau) > 0$ and $F(\tau)f(\tau) < 0$. Similarly, if $f(\tau) > 0$ it follows that $F(\tau)f(\tau) < 0$. Thus the hypotheses of Lemma 3.4 are satisfied and we can infer that $f(t)$, and hence $F(t)$, is identically zero for $t > 0$. This means that $y_1(t) \equiv y_2(t)$.

4. The function $K(z)$. In preparation for the proof of Theorem 2, we give the following lemma concerning $K(z)$.

LEMMA 4.1. *If $K(z)$ satisfies* (2.1) *and* (2.3), *then*:

(4.1) *For $\alpha > 0$ and $z > 0$, we have*

$$[K(z+\alpha) - K(z+2\alpha)]/[K(z) - K(z+\alpha)] < K(z+\alpha)/K(z)\,;$$

(4.2) *$K(z) - K(z+\alpha)$ is strictly decreasing in z for all fixed $\alpha > 0$;*

(4.3) *$K(z)$ is a convex function;*

(4.4) *For each interval $[0, b]$, there exists a number $R > 0$ such that*

$$K(z) - K(z+\alpha) > R\alpha \quad for\ 0 < z < z+\alpha < b.$$

Proof. By (2.3) we know that $K(z+\alpha)/K(z) < K(z+2\alpha)/K(z+\alpha)$. Subtracting 1 from both sides of this inequality and performing a simple rearrangement of terms, we easily arrive at conclusion (4.1) above.

To prove (4.2) we observe that, by (2.3), $[K(z+\alpha)/K(z)] - 1$ is strictly increasing, so that $[K(z) - K(z+\alpha)]/K(z)$ is strictly decreasing. But by (2.1), both the numerator and the denominator are positive and the denominator is decreasing. Hence, the numerator must also be decreasing.

That $K(z)$ is convex follows readily from (4.2), in view of the hypotheses that $K(z)$ is positive, decreasing, and continuous.

From (4.2) and (4.3) it follows that $K(z)$ has a right-hand derivative at each $z > 0$, and this derivative is negative and strictly increasing. The R of (4.4) can be taken as the negative of this derivative at $z = b$.

5. The function $y(t)$. Sections 5 through 10 are devoted to the proof of Theorem 2. Throughout, $y(t)$ will denote the bounded solution of (2.5), where $K(z)$ satisfies (2.1), (2.2), and (2.3). In §10 we assume in addition that $K(z)$ satisfies (2.4).

LEMMA 5.1. *If $y(t) < 1$ for $0 \leq t < T$, then $y(t)$ is nondecreasing on $[0, T]$.*

Proof. Assume the lemma is false. Then for some subinterval, $[0, b]$, $y(t)$

attains its maximum M at an interior point, a, and we set $y(a) - y(b) = 3\epsilon > 0$. We shall assume that a is the smallest number $(0 < a < b)$ such that $y(a) = M$. Choose $\delta_1 > 0$ so small that

$$\delta_1 \int_a^b K(b-\tau)\,d\tau < \epsilon . \tag{5.2}$$

Set $G[y(a)] = c$ and choose p_1 $(0 < p_1 < a)$ so near to a that (see Fig. 1)

$$G[y(t)] < c + \delta_1 \qquad \text{for } p_1 < t < a . \tag{5.3}$$

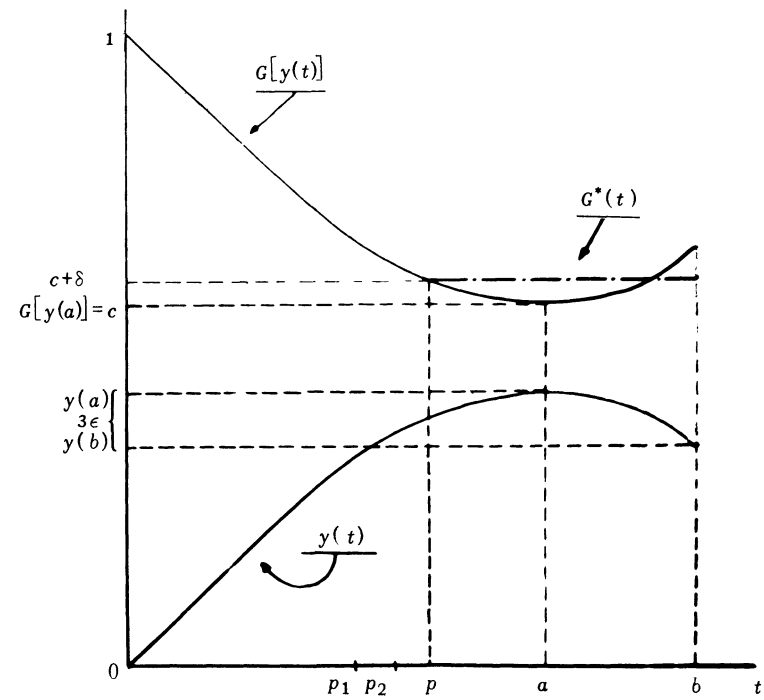

Fig. 1

Next, choose p_2 $(p_1 < p_2 < a)$ so close to a that

$$(c+\delta_1)\int_{p_2}^a K(a-\tau)\,d\tau < \epsilon , \tag{5.4}$$

and

$$(c + \delta_1) \int_{p_2}^{a} K(b - \tau)\, d\tau < \epsilon . \tag{5.5}$$

Define δ so that $0 < \delta < \delta_1$ and $c + \delta \leq G[y(t)]$ for $0 \leq t \leq p_2$. Let p be the largest value of t such that $G[y(t)] \geq c + \delta$ for $t \leq p$. Then $p_2 \leq p < a$. Define

$$G^*[t] = \begin{cases} G[y(t)] & \text{if } t \leq p , \\ c + \delta & \text{if } t > p . \end{cases} \tag{5.6}$$

Now since y attains its maximum on $[0, b]$ at $t = a$, and G is strictly decreasing, we have

$$G - G^* \geq -\delta \qquad \text{for } a \leq t \leq b .$$

We shall show (Lemma 7.1) that $\int_0^t G^*[\tau]\, K(t - \tau) d\tau$ is strictly increasing as t increases from a to b, and therefore $Y(b) > Y(a)$, where we use the following definition:

$$Y(t) = \int_0^t G^*[\tau] K(t - \tau)\, d\tau . \tag{5.7}$$

By (5.4) we have

$$|y(a) - Y(a)| = \left| \int_p^a \{G[y(\tau)] - G^*[\tau]\} K(a - \tau)\, d\tau \right| \tag{5.8}$$

$$\leq \delta_1 \int_p^a K(a - \tau)\, d\tau < \epsilon .$$

Similarly, we obtain

$$y(b) - Y(b) = \int_p^a \{G[y(\tau)] - G^*[\tau]\} K(b - \tau)\, d\tau \tag{5.9}$$

$$+ \int_a^b \{G[y(\tau)] - G^*[\tau]\} K(b - \tau)\, d\tau$$

$$= \alpha + \beta, \text{ say} .$$

By (5.5), we have $|\alpha| < \epsilon$. As for β, the integrand for any τ is either positive or numerically less than $\delta K(b - \tau)$. Hence, by (5.2), it follows that $\beta > -\epsilon$. From (5.8) and (5.9) we therefore have $y(a) < Y(a) + \epsilon$ and $y(b) > Y(b) - 2\epsilon$.

Subtracting, we get $y(b) - y(a) > Y(b) - Y(a) - 3\epsilon > -3\epsilon$, since $Y(b) - Y(a) > 0$ by Lemma 7.1. This contradicts the definition of ϵ, and thus the proof will be complete when Lemma 7.1 has been established.

6. The function $Y(t)$ for $t \le a$. We shall establish the following result.

LEMMA 6.1. *With the notation of* §5, *there exist numbers r and s $(p \le r < s \le a)$ such that $Y(s) > Y(r)$.*

Proof. Define $f(\tau)$ to be $G^*(\tau) - G[y(\tau)]$.

Case 1: for some q $(p < q \le a)$, we have $\int_p^q f(\tau)\,d\tau > 0$. In this case, set $r = p$ and let s be the smallest value of q on $[p, a]$ such that $\int_p^q f(\tau)\,d\tau$ is a maximum. Using $K(s - \tau)$ in place of $K(\tau)$, and p and s, respectively, in place of a and t_1, we see from Lemma 3.1 that $\int_p^s f(\tau)\,K(s-\tau)\,d\tau > 0$. This implies that

$$\int_p^s G^*[\tau]K(s-\tau)\,d\tau > \int_p^s G[y(\tau)]K(s-\tau)\,d\tau. \tag{6.2}$$

Now if $s < a$ then $f(s) = 0$, by Lemma 3.1. That is, $G[y(s)] = c + \delta$, so that $y(s) = y(p)$. If $s = a$, then obviously $y(s) > y(p)$. Since $G^*[\tau] = G[y(\tau)]$ for $\tau \le p$, we get immediately from (6.2) the result that

$$\begin{aligned}\int_0^s G^*[\tau]K(s-\tau)\,d\tau &> \int_0^s G[y(\tau)]K(s-\tau)\,d\tau \\ &= y(s) \ge y(p) \\ &= \int_0^p G^*[\tau]K(p-\tau)\,d\tau;\end{aligned}$$

that is,

$$Y(s) > Y(p).$$

Case 2: for every q $(p < q \le a)$, we have $\int_p^q f(\tau)\,d\tau \le 0$. Now $f(\tau)$ is not identically zero on $[p, a]$ since $f(a) = \delta$. Let r be the smallest number q on $[p, a]$ such that $\int_p^q f(\tau)\,d\tau$ is a minimum.

Then $\int_p^r f(\tau)\,d\tau = M < 0$ and $\int_r^t f(\tau)\,d\tau \ge 0$ $(r \le t \le a)$ by the minimum property for r. Let s be the smallest value of t $(r < t \le a)$ such that $\int_r^t f(\tau)\,d\tau$ is a maximum. We now apply Lemma 3.1 to the interval $[p, r]$, using $K(r - \tau) - K(s - \tau)$ as the function $K(\tau)$ [note that this function is increasing in τ by (4.2)]. We also use $-f(\tau)$ in place of $f(\tau)$. This gives

$$\int_p^r [-f(\tau)][K(r-\tau)-K(s-\tau)]\,d\tau > 0\,. \tag{6.3}$$

Similarly, applying Lemma 3.1 to the interval $[r, s]$ and using $K(s-\tau)$ as the function $K(\tau)$, we get

$$\int_r^s f(\tau)K(s-\tau)\,d\tau > 0\,. \tag{6.4}$$

We are now in a position to show that $Y(s)-Y(r)>0$. For we have

$$\begin{aligned} Y(s)-Y(r) = & \int_0^p G^*[\tau][K(s-\tau)-K(r-\tau)]\,d\tau \\ & + \int_p^r G^*[\tau][K(s-\tau)-K(r-\tau)]\,d\tau \\ & + \int_r^s G^*[\tau]K(s-\tau)\,d\tau\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} y(s)-y(r) = & \int_0^p G[y(\tau)][K(s-\tau)-K(r-\tau)]\,d\tau \\ & + \int_p^r G[y(\tau)][K(s-\tau)-K(r-\tau)]\,d\tau \\ & + \int_r^s G[y(\tau)]K(s-\tau)\,d\tau. \end{aligned}$$

We therefore get

$$\begin{aligned} & [Y(s)-Y(r)]-[y(s)-y(r)] \\ & \quad = \int_p^r f(\tau)[K(s-\tau)-K(r-\tau)]\,d\tau + \int_r^s f(\tau)K(s-\tau)\,d\tau \\ & \quad = \int_p^r [-f(\tau)][K(r-\tau)-K(s-\tau)]\,d\tau + \int_r^s f(\tau)K(s-\tau)\,d\tau > 0\,, \end{aligned}$$

by (6.3) and (6.4). But $f(r)=0$, so that $y(r)=y(p)$. Also either $f(s)=0$ or $s=a$. In either case we have $y(s)\geq y(p)$. Thus $y(s)-y(r)\geq 0$ and $Y(s)>Y(r)$.

7. The function $Y(t)$ for $t \geq a$. For $t \geq a$ we have the following stronger result.

LEMMA 7.1. *The function $Y(t)$ is strictly increasing for $t \geq a$.*

Proof. Suppose that $e \geq p$, $\alpha > 0$, and $Y(e+\alpha) \geq Y(e)$. We prove first that $Y(e+2\alpha) > Y(e+\alpha)$. Replacing $c+\delta$ by k, we may write

$$Y(e+\alpha) - Y(e) = \int_0^{e+\alpha} G^*(\tau) K(e+\alpha-\tau)\, d\tau - \int_0^{e+\alpha} G^*(\tau) K(e-\tau)\, d\tau$$

$$= \left\{ \int_0^{e+\alpha} kK(e+\alpha-\tau)\, d\tau - \int_0^{e} kK(e-\tau)\, d\tau \right\}$$

$$- \left\{ \int_0^{e} [G^*(\tau) - k][K(e-\tau) - K(e+\alpha-\tau)]\, d\tau \right\}$$

$$= A_1 - B_1, \text{ say}.$$

(We have used the fact that $G^*(\tau) - k = 0$ for $\tau \geq e$.) Similarly, we have

$$Y(e+2\alpha) - Y(e+\alpha) = \left\{ \int_0^{e+2\alpha} kK(e+2\alpha-\tau)\, d\tau - \int_0^{e+\alpha} K(e+\alpha-\tau)\, d\tau \right\}$$

$$- \left\{ \int_0^{e} [G^*(\tau) - k][K(e+\alpha-\tau) - K(e+2\alpha-\tau)]\, d\tau \right\}$$

$$= A_2 - B_2, \text{ say}.$$

Now $A_1 - B_1 \geq 0$ by hypothesis, and we wish to show that $A_2 - B_2 > 0$.

By simple changes of variable we get

$$\int_0^{e} K(e-\tau)\, d\tau = \int_0^{e} K(z)\, dz, \quad \int_0^{e+\alpha} K(e+\alpha-\tau)\, d\tau = \int_0^{e+\alpha} K(z)\, dz,$$

and

$$\int_0^{e+2\alpha} K(e+2\alpha-\tau)\, d\tau = \int_0^{e+2\alpha} K(z)\, dz.$$

Then we have the following:

$$A_1 = k \int_e^{e+\alpha} K(z)\, dz,$$

$$B_1 = \int_0^{e} [G^*(e-z) - k][K(z) - K(z+\alpha)]\, dz,$$

$$A_2 = k \int_{e+\alpha}^{e+2\alpha} K(z)\, dz\,,$$

$$B_2 = \int_0^e [G^*(e-z) - k][K(z+\alpha) - K(z+2\alpha)]\, dz\,.$$

Another change of variable gives

$$A_2 = k \int_e^{e+\alpha} K(z+\alpha)\, dz\,.$$

Now over the interval $e \le z \le e+\alpha$ we have, by (2.3),

$$K(z+\alpha) = K(z)[K(z+\alpha)/K(z)] \ge K(z)[K(e+\alpha)/K(e)]\,.$$

Furthermore, the strict inequality holds except for $z = e$. It follows that $A_2 > [K(e+\alpha)/K(e)]\, A_1$.

To obtain an inequality for B_2/B_1, we note first that $G^*(e-z) - k$ is positive or zero for $0 \le z \le e$. Over this range for z, we have

$$[K(z+\alpha) - K(z+2\alpha)]/[K(z) - K(z+\alpha)]$$
$$< K(z+\alpha)/K(z) \le K(e+\alpha)/K(e)\,,$$

by (4.1) and (2.3). Thus it follows that $B_2 < [K(e+\alpha)/K(e)]\, B_1$. Then

$$A_2 - B_2 > [K(e+\alpha)/K(e)][A_1 - B_1] \ge 0\,.$$

Thus we have seen that if $e \ge p$, $\alpha > 0$, and $Y(e+\alpha) \ge Y(e)$ then $Y(e+2\alpha) > Y(e+\alpha)$. But then it follows that $Y(e+3\alpha) > Y(e+2\alpha)$; $Y(e+4\alpha) > Y(e+3\alpha)$, and so on. Now if $e = r$, and $\alpha = s - r$, we have $Y(e+\alpha) > Y(e)$ by Lemma 6.1. Divide the interval $[r, s]$ into n equal subintervals by the points $x_0 = r, x_1, x_2, \cdots, x_n = s$. It follows that for some i we have $Y(x_{i+1}) > Y(x_i)$. But $x_{i+1} = x_i + \alpha n^{-1}$, so that $Y(x_i + \alpha n^{-1}) > Y(x_i)$. Thus we see that $Y(t)$ is strictly increasing over the points of an arbitrarily fine mesh. Hence, by continuity, it is always increasing for $t \ge s$, therefore *a fortiori* for $t \ge a$. This completes the proof of Lemma 7.1, and thereby establishes Lemma 5.1.

8. A stronger result concerning $y(t)$. We now prove:

LEMMA 8.1. *Under the hypothesis of Lemma* 5.1, *$y(t)$ is strictly increasing on the interval* $[0, T]$.

Proof. If the lemma is false then there exist points p and a $(0 < p < a)$ such that $y(\tau) < y(p)$ if $\tau < p$, and $y(\tau) = y(p)$ if $p \leq \tau \leq a$. Define $G^*(\tau) = G[y(\tau)]$ for $\tau \leq p$, and $G^*(\tau) = G[y(p)]$ for $\tau > p$. Then we have the situation of §7, and $Y(t)$ is strictly increasing for $t \geq p$. But over $[p, a]$, we have $Y(t) = y(t)$.

9. Another result concerning $y(t)$. Our last lemma is the following:

LEMMA 9.1. *For every t $(t \geq 0)$, we have $y(t) < 1$.*

Proof. Assume the lemma is false, and let b be the smallest number such that $y(b) = 1$. Then by (1.5), (1.6), and (1.7) it follows that $G[y(t)]$ strictly decreases from 1 to 0 as t increases from 0 to b. (See Fig. 2.)

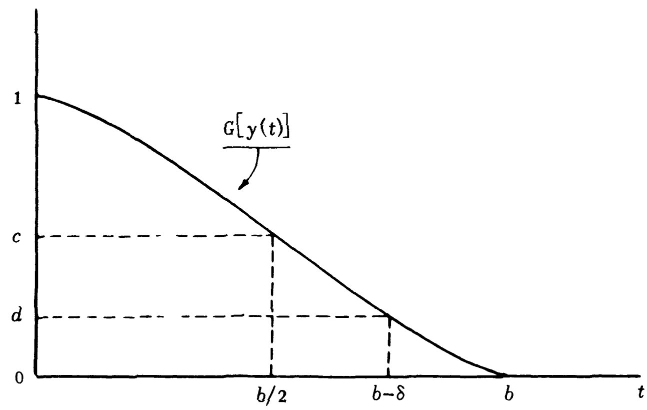

Fig. 2

By (4.4), there exists an $R > 0$ such that for every δ $(0 < \delta < b/2)$ we have

$$K[(b/2) - \delta] - K(b/2) > R\delta \,. \tag{9.1}$$

Set $c = G[y(b/2)]$ and $d = G[y(b - \delta)]$. Then c is fixed and d is a function of δ such that $d \longrightarrow 0$ as $\delta \longrightarrow 0$. Also $K(b) > 0$ and K is continuous. Therefore it is clear that we can fix δ so that

$$(b/2)(c - d)R > 2dK(b) \,, \tag{9.2}$$

$$K(b - \delta) < 2K(b) \,. \tag{9.3}$$

We shall show that for this choice of δ we have $y(b) < y(b - \delta)$, which is a contradiction. Now

$$y(b-\delta) = d\int_0^{b-\delta} K(b-\delta-\tau)\,d\tau + \int_0^{b/2} (G[y(\tau)]-d)K(b-\delta-\tau)\,d\tau$$

$$+ \int_{b/2}^{b-\delta} (G[y(\tau)]-d)K(b-\delta-\tau)\,d\tau$$

$$= \alpha+\beta+\gamma, \text{ say .}$$

Similarly, we have

$$y(b) < d\int_0^{b} K(b-\tau)\,d\tau + \int_0^{b/2} (G[y(\tau)]-d)K(b-\tau)\,d\tau$$

$$+ \int_{b/2}^{b-\delta} (G[y(\tau)]-d)K(b-\tau)\,d\tau = \lambda+\mu+\nu, \text{ say,}$$

where the inequality arises from replacing $G[y(\tau)]$ by the greater quantity d, for $b-\delta<\tau\le b$. Then

(9.4) $$y(b)-y(b-\delta) < (\lambda-\alpha)-(\beta-\mu)+(\nu-\gamma)\,.$$

By (2.1) we have

(9.5) $$\nu-\gamma<0\,.$$

Furthermore,

$$\lambda-\alpha = d\left[\int_0^{b} K(b-\tau)\,d\tau - \int_0^{b-\delta} K(b-\delta-\tau)\,d\tau\right] = d\int_0^{\delta} K(b-\tau)\,d\tau,$$

since

$$\int_0^{b} K(b-\tau)\,d\tau = \int_0^{\delta} K(b-\tau)\,d\tau + \int_{\delta}^{b} K(b-\tau)\,d\tau\,,$$

and since replacing τ by $z+\delta$ gives $\int_0^{b-\delta} K(b-\delta-z)\,dz$ for the second integral. But by (2.1) it follows that

$$\int_0^{\delta} K(b-\tau)\,d\tau < \delta[K(b-\delta)]\,,$$

so that

(9.6) $$\lambda-\alpha < d\delta[K(b-\delta)] < 2d\delta K(b)\,.$$

Similarly, by (4.2),

$$\beta - \mu = \int_0^{b/2} (G[y(\tau)] - d)[K(b - \delta - \tau) - K(b - \tau)]\, d\tau$$

$$> (c - d) \int_0^{b/2} [K(b - \delta - \tau) - K(b - \tau)]\, d\tau$$

$$> (c - d)[K(b/2 - \delta) - K(b/2)](b/2)\,.$$

Thus using (9.1) and (9.3) we have

$$\beta - \mu > (c - d) R \delta\, (b/2)\,. \tag{9.7}$$

In view of (9.2), (9.6), and (9.7), it is clear that $\beta - \mu > \lambda - \alpha$. Hence, from (9.4) and (9.5) we have $y(b) - y(b - \delta) < 0$, a contradiction.

10. Proof of Theorem 2. To complete the proof of Theorem 2, we now assume in addition that $K(z)$ satisfies (2.4).

We know that $y(t)$ is a strictly increasing function of t, $y(0) = 0$, and $y(t) < 1$ for all t. We must show that $y(t) \longrightarrow 1$ as $t \longrightarrow \infty$. Assume on the contrary that $y(t) \longrightarrow k$ as $t \longrightarrow \infty$, where $0 < k < 1$. Then $G[y(t)] > G(k) > 0$ for all t. By (2.5) we have

$$y(t) = \int_0^t G[y(\tau)]K(t - \tau)\, d\tau > \int_0^t G[k]K(t - \tau)\, d\tau$$

$$= G(k) \int_0^t K(t - \tau)\, d\tau = G(k) \int_0^t K(z)\, dz\,;$$

but, by (2.4), the last integral increases indefinitely as $t \longrightarrow \infty$, so that we have a contradiction.

11. Conclusion. In conclusion it will be shown that if hypothesis (2.3) on $K(z)$ is replaced by the stipulation that $K(z)$ be convex, then $y(t)$ is not necessarily monotonic increasing.

Let $G(y) = 1 - y$ and $K_1(z) = 1 - z$ $(0 \leq z \leq 1)$. Then if $y(t)$ denotes the bounded solution of the equation

$$y(t) = \int_0^t G[y(\tau)]K_1(t - \tau)\, d\tau,$$

it is readily shown that $y(t)$ is actually decreasing over a small segment, $1 - \delta < t < 1$.

To get a similar example where $\int_0^t K(z)dz \longrightarrow \infty$ as $t \longrightarrow \infty$, we select a fixed c, $1 - \delta < c < 1$, and write $K(z) = K_1(z)$ for $z \leq c$, $K(z) = dz^{-1/2}$ for $z > c$, where d is chosen so that the functions $1 - z$ and $dz^{-1/2}$ have the same value at $z = c$; that is, $d = c^{1/2}(1 - c)$.

Reference

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A NOTE ON UNRESTRICTED REGULAR TRANSFORMATIONS

W. R. Utz

1. Introduction. Let W be the class of real continuous functions defined on the nonnegative reals and such that for each $g(t) \in W$ the following conditions hold:

(a) $g(0) = 0$ and $g(t) > 0$ when $t > 0$,

(b) for each triple $t_1, t_2, t_3, \geq 0$, the inequality $t_1 + t_2 \geq t_3$ implies $g(t_1) + g(t_2) \geq g(t_3)$.

Let M be a metric space wherein $[p, q]$ denotes the distance between $p, q \in M$. A transformation $T(M) = N$ is called *unrestricted regular* by W. A. Wilson [2] if there exists a $g(t) \in W$ such that for each pair $p, q \in M$ we have $[T(p), T(q)] = g[p, q] \equiv g([p, q])$. The function g (not always unique) is called a *scale function* for T.

It is easily seen that every member of the class W is monotone increasing and that each unrestricted regular transformation is continuous and one-to-one. Thus an unrestricted regular transformation on a compact metric space is a homeomorphism. Wilson shows [2, p.65] that if M is dense and metric and T is unrestricted regular, then T is a homeomorphism.

In §2 of this note we examine the graphs of scale functions and show how the graph of the scale function of an unrestricted regular transformation determines the behavior of points under the transformation. Section 3 is devoted to a question involving a class of transformations investigated by E. J. Mickle [1].

2. The graphs of scale functions. We shall establish the following result.

THEOREM 1. *If M is a metric space and $T(M) = M$ is unrestricted regular with scale function $g(t)$, then for each $n = 1, 2, 3, \cdots$, the transformation $T^n(M) = M$ is unrestricted regular with scale function $g^n(t)$ (that is, g iterated n times).*

Proof. Obviously $g^n(t)$ is real and continuous, $g^n(0) = 0$, and $g^n(t) > 0$ when $t > 0$. Suppose $T^{n-1}(M) = M$ is unrestricted regular with scale function $g^{n-1}(t)$.

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Let $t_1 + t_2 \geq t_3$, where $t_1, t_2, t_3 \geq 0$. Then

$$g^{n-1}(t_1) + g^{n-1}(t_2) \geq g^{n-1}(t_3)\,,$$

and hence

$$g^n(t_1) + g^n(t_2) = g[g^{n-1}(t_1)] + g[g^{n-1}(t_2)] \geq g[g^{n-1}(t_3)] = g^n(t_3)\,.$$

Thus $g^n(t) \in W$. Also we have

$$[T^n(p),\, T^n(q)] = [T\{T^{n-1}(p)\},\, T\{T^{n-1}(q)\}]$$
$$= g[T^{n-1}(p),\, T^{n-1}(q)] = g(g^{n-1}[p,q]) = g^n[p,q]\,,$$

for each pair $p, q \in M$. Thus, since T is unrestricted regular with scale function $g(t)$, we have proved by induction that $T^n(M) = M$ is an unrestricted regular transformation with scale function $g^n(t)$.

If M is a metric space of at least two points, $p \in M$, and $T(M) = M$ is unrestricted regular, then we shall call the set $\sum_{n=0}^{\infty} T^n(p) \subset M$ the *orbit* of p under T.

Let $g(t)$ be a scale function for T. We distinguish three cases.

CASE I. *If $g(t) < t$ for all $t > 0$, then each pair of points of M will determine asymptotic orbits.* That is, given $p, q \in M$ and $\epsilon > 0$, there exists an integer N such that $[T^n(p), T^n(q)] < \epsilon$ for all $n > N$.

Proof. Let p and q be points of M. Since $g(t) < t$, we see that $[T^n(p), T^n(q)] = g^n[p,q]$ decreases monotonically as n increases. Suppose that the monotone decreasing sequence of real numbers $[p,q]$, $g[p,q]$, $g^2[p,q]$, $\cdots$, has $u \neq 0$ as limit point. Choose δ such that $0 < \delta < u$, and let s be the greatest lower bound of $t - g(t)$ on the interval $u - \delta \leq t \leq u + \delta$. Since u is the limit point of the sequence, there exists an integer n for which $g^n[p,q] - u < \min(s, \delta)$. Since $g^n[p,q]$ is in the interval $u - \delta \leq t \leq u + \delta$, it follows that $g^n[p,q] - g^{n+1}[p,q] \geq s$ and $u - g^{n+1}[p,q] > 0$. Thus for all $i > n$, the elements $g^i[p,q]$ of the sequence are smaller than u; this contradicts the assumption that $u \neq 0$ is the limit point of the sequence.

In Case I, T has equicontinuous powers.

CASE II. *If $g(t) > t$ for all $t > 0$, then T is unstable.* That is, there exists a $\delta > 0$ (in this case any positive number will serve) such that if $p, q \in M$, then there is an integer N for which $n > N$ implies $[T^n(p), T^n(q)] > \delta$.

CASE III. (1) *If* $g(t) \equiv t$, *then all orbits are parallel.* That is, T is an isometry.

If $g(t) \not\equiv t$, there are these possibilities:

(2a) When $g[p,q] = [p,q]$, the orbits of p and q are parallel (as in Case III).

(2b) If $g[p,q] > [p,q]$, and if there is a zero of $g(t) - t$ greater than $[p,q]$, then the orbits of p and q approach a distance apart equal to the first zero of $g(t) - t$ that is greater than $[p,q]$. If no zero of $g(t) - t$ is greater than $[p,q]$, the orbits of p and q separate as in Case II.

(2c) If $g[p,q] < [p,q]$, and if no positive zero of $g(t) - t$ is smaller than $[p,q]$, then p and g have asymptotic orbits as in Case I. If $g(t) - t$ has a positive zero smaller than $[p,q]$, then the orbits of p and q approach a distance apart equal to the first zero of $g(t) - t$ less than $[p,q]$.

The proofs of these cases are similar to the proof of Case I.

THEOREM 2. *If* M *is a bounded metric space, then Case* I *and Case* II *are not possible.*

Proof. That Case II cannot occur is obvious.

Suppose $g(t) < t$ (Case I). Let δ be the least upper bound of $[p,q]$ for all $p,q \in M$. Let $\sigma > 0$ be the greatest lower bound for $t - g(t)$ on the interval $\delta/2 \le t \le \delta$. Select $p,q \in M$ such that $[p,q] > \max(\delta - \sigma, \delta/2)$. Since $T^{-1}(p)$, $T^{-1}(q)$ are elements of M, and since

$$[T^{-1}(p),\ T^{-1}(q)] > g[T^{-1}(p),\ T^{-1}(q)] = [p,q]\ ,$$

it follows that

$$(1/2)\,\delta \le [T^{-1}(p),\ T^{-1}(q)] \le \delta\ .$$

Thus,

$$[p,q] = g[T^{-1}(p),\ T^{-1}(q)] < [T^{-1}(p),\ T^{-1}(q)] - \sigma \le \delta - \sigma\ ;$$

this contradicts $[p,q] > \delta - \sigma$ and completes the proof of the theorem.

LEMMA 1. *If* $g(t) \in W$, *then there exists a real number* s *such that, on* $0 < t \le s$, *either* (i) $g(t) \equiv t$, *or* (ii) $g(t) > t$, *or* (iii) $g(t) < t$.

Proof. Suppose that $g(t) \not\equiv t$ on every interval $0 < t \le s$. If $t = 0$ is not a limit point of the positive zeros of $g(t) - t$, then obviously on some interval $0 < t \le s$ we have $g(t) < t$ or $g(t) > t$. Suppose that $t = 0$ is a limit point of the zeros of

$g(t) - t$ and suppose that in every interval $0 < t \leq s$ there are values of t for which $g(t) < t$ and $g(t) > t$. Select u_1 and u_2 such that $g(u_1) = u_1$ and $g(u_2) = u_2$, and such that $g(t) > t$ on the interval $u_1 < t < u_2$. Select $u_3 > 0$ such that $g(u_3) < u_3$ and $u_3 < u_2 - u_1$. Define $u_4 = u_1 + u_3$. Since $u_1 < u_4 < u_2$, we have $g(u_4) > u_4$. Since $u_1 + u_3 \geq u_4$, we must have $g(u_1) + g(u_3) \geq g(u_4)$. This is not the case since $g(u_1) + g(u_3) = u_1 + g(u_3) < u_1 + u_3 = u_4 < g(u_4)$. Thus on some interval $0 < t \leq s$, either $g(t) \leq t$ or $g(t) \geq t$.

We must now eliminate the possibility of the equalities. Suppose $g(t) \leq t$ on $0 < t \leq s$ but there is no subinterval $0 < t \leq s_1$ on which $g(t) < t$ or $g(t) \equiv t$. Let $u \leq s$ be such that $g(u) = u$. Select $v < u$ such that $g(v) < v$. Now, $v + (u - v) = u$; but

$$g(v) + g(u-v) \leq g(v) + (u-v) < v + u - v = u = g(u),$$

and property (b) of $g(t)$ is violated. Thus $g(t) < t$.

If $g(t) \geq t$ on $0 < t \leq s$, but there is no subinterval $0 < t \leq s_1$ on which $g(t) > t$ or $g(t) \equiv t$, then choose $0 < u_1 < s$ and $0 < u_2 \leq s$ such that $g(u_1) = u_1$ and $g(u_2) = u_2$, and such that on the t-interval $0 < u_1 < t < u_2 \leq s$ we have $g(t) > t$. Select $0 < u_3 < u_2 - u_1$ such that $g(u_3) = u_3$ and define $u_4 = u_3 + u_1$. Then

$$g(u_3) + g(u_1) = u_3 + u_1 = u_4 < g(u_4),$$

since $u_1 < u_4 < u_2$. Thus $g(t)$ fails to have property (b). We conclude that $g(t) > t$. This proves the lemma.

LEMMA 2. *If* (i) *of Lemma* 1 *occurs, then either* $g(t) = t$ *for all* $t > 0$ *or there exists an* $r > 0$ *such that* $g(t) = t$ *for* $0 \leq t \leq r$ *and* $g(t) < t$ *for all* $t > r$. *If* (iii) *of Lemma* 1 *occurs, then* $g(t) < t$ *for all* $t > 0$.

Proof. Suppose that (i) of Lemma 1 occurs. Let r be the largest value of s for which $g(t) = t$ on $0 \leq t \leq s$ (if r does not exist, then $g(t) = t$ for all $t > 0$). Let t be any real number greater than r. Suppose $g(t) > t$. Then $t = mr + q$, where m is a positive integer and $0 \leq q < r$. Since $g(r) = r$, we have $g(mr) \leq mg(r) = mr$; and since $0 \leq q < r$, we have $g(q) = q$. Hence

$$g(mr) + g(q) \leq mr + q = t < g(t),$$

in violation of property (b) of $g(t)$. Thus $g(t) \leq t$ for all $t > 0$. Suppose $t > r$ and $g(t) = t$. Then there exists a nonnegative integer m, and real numbers u and q such that $mr + q + u = t$, and such that $g(u) < u$. However,

$$g(mr) + g(q) + g(u) \leq mr + q + g(u) < mr + q + u = t = g(t) ,$$

and condition (b) of $g(t)$ is violated. Thus $g(t) < t$ for $t > r$, and the first part of the lemma is proved.

Suppose that (iii) of Lemma 1 occurs. To show that $g(t) < t$ for all real values of t, we shall show that for no $t > 0$ is $g(t) = t$. If $g(t) = t$ for some $t > 0$, then there exists a smallest value u of t such that $g(u) = u$. Now, $g(u/2) < u/2$ since u is the smallest value of t for which $g(t) = t$. Hence $g(u/2) + g(u/2) < u$, contrary to property (b) of $g(t)$. This completes the proof of the lemma.

THEOREM 3. *If M is a bounded metric space and $T(M) = M$ is unrestricted regular and has equicontinuous powers, then T is an isometry.*

Proof. Since T has equicontinuous powers, given $\epsilon > 0$ there exists $\delta > 0$ such that when $[p,q] < \delta$ we have $[T^n(p), T^n(q)] < \epsilon$ for $n = 1,2,3,\cdots$. From this it follows that (ii) of Lemma 1 cannot occur. For if ϵ is taken as $s/2$ in Lemma 1, then regardless of the size of $[p,q]$, we have $[T^n(p), T^n(q)] > s/2$ for n sufficiently large (cf. 2b of Case III).

Further, (iii) of Lemma 1 cannot occur since by Lemma 2 this implies Case I, which is impossible since M is bounded.

Since (i) of Lemma 1 must occur, either $g(t) \equiv t$ for $t > 0$, or there exists an $r > 0$ such that $g(t) = t$ for all $0 \leq t \leq r$ and $g(t) < t$ for all $t > r$. If $g(t) \not\equiv t$, then we can show by the argument of Theorem 2 that distances in M are bounded by r. Hence we always have $[T(p), T(q)] = g[p,q]$ for each pair $p, q \in M$, and T is an isometry.

REMARK. Suppose that (ii) of Lemma 1 occurs and suppose that $g(t) - t$ has a positive zero. We can show easily that either there exist arbitrarily large zeros of $g(t) - t$ or there exists a real number $w > 0$ such that $t > w$ implies $g(t) < t$. If r is the smallest positive zero of $g(t) - t$, and N is the length of any interval of the t-axis on which $g(t) > t$, then $N \leq r$.

The following theorem relates periodicity to unrestricted regularity. Other theorems of this nature are possible.

THEOREM 4. *Let M be a metric space. If $T(M) = M$ is pointwise periodic and unrestricted regular then T is an isometry.*

Proof. Let p and q be arbitrary points of M. Since p and q are individually periodic (possibly having different periods), there exists an integer n (in particular, the products of the periods of p and q will serve) such that $T^n(p) = p$ and $T^n(q) = q$. Thus p and q are fixed under T^n. If $g(t)$ is the scale function of T, then $g^n(t)$ is the scale function of T^n. Since p and q are fixed under T^n, we have

$$g^n[T^n(p), T^n(q)] = [T^n(p), T^n(q)] = [p, q].$$

Thus we have $g^n[p, q] = [p, q]$. This implies that $g[p, q] = [p, q]$; and since g is the scale function for T, we have $[T(p), T(q)] = g[p, q] = [p, q]$, and the theorem is proved.

3. A class of transformations. Given a metric space M, Mickle [1] defines the associated class $P(M)$ of real continuous functions on the nonnegative reals as those functions $g(t)$ satisfying these conditions:

(a) $g(0) = 0$ and $g(t) > 0$ when $t > 0$,

(b) for any $m + 1$ points $p_0, p_1, p_2, \cdots, p_m$ in M the real quadratic form

$$\sum_{i,j=1}^{m} \{g[p_0, p_i]^2 + g[p_0, p_j]^2 - g[p_i, p_j]^2\} \xi_i \xi_j$$

is positive definite.

For example, let M be any set with metric $[p, q] = 1$ for $p \neq q$, $[p, q] = 0$ for $p = q$. Let $g(t)$ be any real continuous function that satisfies condition (a). If $p_0, p_1, \cdots, p_m$ are any set of $m + 1$ distinct points of M, then $g[p_i, p_j] = g(1) = a > 0$ for $i \neq j$. The elements of the matrix $\|a_{i,j}\|$ of the quadratic form of condition (b) are $2a^2$ if $i = j$ and a^2 if $i \neq j$. From this, and from well-known theorems concerning quadratic forms, it follows that condition (b) is always satisfied. Hence, in this case, $P(M)$ consists of all real continuous functions for which (a) holds.

Let $T(M) = N$ be a continuous transformation. Then T is said by Mickle to satisfy the *condition* $C(g)$, $g(t) \in P(M)$, if for each pair $p, q \in M$ we have $[T(p), T(q)] \leq g[p, q]$.

A transformation may satisfy the condition $C(g)$ for some $g(t) \in P(M)$, yet not be unrestricted regular. Let M be the interval $0 \leq x \leq 1$ with the metric described in the second paragraph of this section. Let N be the same interval with the Euclidean metric. Let $T(M) = N$ be the identity on the point set. That is, if $p \in M$ has coordinate x, then $T(p) \in N$ has coordinate x. If $g(t) \in P(M)$ and $g(1) \geq 1$, then for each distinct pair $p, q \in M$, we have $[T(p), T(q)] \leq 1 \leq g[p, q]$, and T satisfies $C(g)$. However, T is not unrestricted regular.

QUESTION. Suppose that $T(M) = N$ is an unrestricted regular transformation. When does there exist an element $g(t) \in P(M)$ such that T satisfies the condition $C(g)$?

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REMARKS ON THE SPACE H^p

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1. Introduction. The space H^p is the collection of all single-valued complex functions f which are regular on the interior of the unit circle in the complex plane, and for which

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty .$$

In [6] it was shown that H^p, $0 < p < 1$, is a linear topological space in which the metric is $\|f - g\|^p$, where we define

$$\|f\| = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} .$$

It was moreover shown that $(H^p)^*$, the conjugate of H^p, has sufficiently many elements (linear functionals on H^p) so as to distinguish elements in H^p, in the sense that if $f \neq 0$ is in H^p, then there is a $\gamma \in (H^p)^*$ such that $\gamma(f) \neq 0$.

In the present paper it will be shown that *if γ is in $(H^p)^*$, $0 < p < 1$, then there exists a unique function G which is regular in the open unit circle, continuous on the closed circle,*[1] *and such that*

$$\gamma(f) = \lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) G\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta , \qquad r < \rho < 1 ,$$

for every f in H^p. It is further shown that the following is true of G:

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[1] The author wishes to acknowledge the fact that the referee suggested the plausibility that G is continuous on the closed unit circle.

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(a) *if* $0 < p < 1/n$, $n = 2, 3, \cdots$, *then* $[d^{n-1} G(z)]/dz^{n-1}$ *is continuous on the closure of the unit circle*;

(b) *if* $0 < p < 1/2n$, $n = 1, 2, \cdots$, *then* $G(e^{it})$ *has a continuous nth derivative with respect to t*; *and*

(c) *if* $0 < p < 1/2$, *then the power series for G converges absolutely on the boundary of the unit circle.*

It is moreover shown that *if G is regular on the open unit circle and is such that*

$$\lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) G\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta, \qquad r < \rho < 1,$$

exists for every f in H^p, *then the functional so defined is in* $(H^p)^*$. Thus $(H^p)^*$ is equivalent to a subspace of the functions which are regular on the open unit circle and continuous on the closed unit circle when $0 < p < 1$; and indeed, as p tends toward zero, the spaces $(H^p)^*$ are equivalent to subspaces of spaces whose members have far stronger properties than merely the property of being continuous on the closure of the unit circle.

A generalization of a theorem by Khintchine and Ostrowski [1, p.157], which is a sort of generalization of Vitali's theorem, will also be presented; namely, it will be shown that *a bounded sequence in* H^p, $0 < p < \infty$, *whose boundary values converge on a set of positive measure, converges uniformly on all compact subsets of the unit circle.* Khintchine and Ostrowski proved this theorem in the case that the sequence consists of uniformly bounded elements.

It is worth remarking that under the present "norm" $\|\cdot\|$, H^p, $0 < p < 1$, is definitely *not* a normed linear space, this being due to the complete failure of Minkowski's inequality for index smaller than unity. As a result, it is conjectured by the author that H^p, $0 < p < 1$, is not a normed linear space at all (and hence contains no bounded convex neighborhood). If this conjecture is true, then H^p, $0 < p < 1$, offers an interesting example of a linear topological space which is not locally convex (since H^p is clearly locally bounded) and whose conjugate space has sufficiently many members so as to distinguish the elements in H^p.

2. Representation of linear functionals on H^p, $0 < p < 1$. In this section we shall suppose always that $0 < p < 1$. We let Δ be the set of all z such that $|z| < 1$, and $\mathfrak{A}$ the class of all single-valued complex functions which are regular on Δ. We shall first make some definitions and prove several lemmas before proving the representation theorem.

For many of the topological terms used in the ensuing, see [3]. By a *complete* linear topological space, we shall mean a space in which $f_n - f_m \longrightarrow 0$ implies $\lim_{n=\infty} f_n$ exists in the space. *Locally bounded linear topological space* and *normed linear space* will be abbreviated LBLTS and NLS respectively. By F^*, where F is a linear topological space, we shall mean the conjugate of F, that is, the space of linear functionals on F.

If F is a LBLTS, it is easy to show that F^* is a complete NLS (Banach space) in which

$$\|\gamma\| = \sup_{f \in U} |\gamma(f)| ,$$

where $\gamma \in F^*$, and U is a fixed bounded neighborhood of the origin. Moreover, the topology so introduced into F^* is independent of U. With respect to H^p, we let U be the unit sphere, so that

$$\|\gamma\| = \sup_{\|f\|=1} |\gamma(f)| .$$

It is then simple to prove the following theorem, merely by modeling the proof exactly after that given in the theory of NLS's.

LEMMA 1. *If F is a complete* LBLTS, *and Γ is a subset of F^* having the property that, for each fixed f in F, $\gamma(f)$ is bounded as γ varies over Γ, then Γ is a bounded set.*

We remind ourselves that H^p is locally bounded, and is moreover complete by [6]. We make the following definitions, where f and g are any elements in $\mathfrak{A}$:

(i) $$\gamma_n(f) = f^{(n)}(0)/N\,! , \qquad n = 0, 1, \cdots ,$$

(ii) $$T_w f : T_w f(z) = f(wz) , \qquad w \in \Delta , \ z \in \Delta ,$$

(iii) $$u_n : u_n(z) = z^n , \qquad z \in \Delta , \ n = 0, 1, \cdots ,$$

(iv) $$B(f, g; z) = \sum_{n=0}^{\infty} \gamma_n(f)\, \gamma_n(g)\, z^n , \qquad z \in \Delta .$$

It is easily verified that

$$B(f, g; z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta})\, g(z_2 e^{-i\theta})\, d\theta\,,$$

where $z_1 z_2 = z$, and z_1 and z_2 are in Δ. The proof is made by expansion of the integrand above in a Taylor series about the origin and then term-by-term integration. In particular,

$$B(f, g; r) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta})\, g\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta\,, \qquad r < \rho < 1.$$

LEMMA 2. *If f is in H^p, then $T_w f$ is in H^p, and moreover*

$$T_w f = \sum_{n=0}^{\infty} \gamma_n(f)\, w^n\, u_n.$$

Proof. Let $g = \sum_{n=0}^{\infty} \gamma_n(f) w^n u_n$. We first show that this series converges. Note that $\|u_n\| = 1$, and

$$|\gamma_n(f)| \le \left(\frac{pn+1}{pn}\right) (pn+1)^{1/p} \cdot \|f\|\,.$$

The last inequality appears in [6, Theorem 6]. Thus

$$\left\| \sum_{n=l}^{m} \gamma_n(f)\, w^n\, u_n \right\|^p \le \sum_{n=l}^{m} \|\gamma_n(f)\, w^n\, u_n\|^p \longrightarrow 0 \quad \text{as} \quad l, m \longrightarrow \infty\,,$$

whence $\sum_{n=0}^{\infty} \gamma_n(f)\, w^n\, u_n$ converges, by the completeness of H^p. Then, noting [6, Theorem 8], which tells us that a convergent sequence in H^p converges pointwise to its limit, we have

$$g(z) = \sum_{n=0}^{\infty} \gamma_n(f)\, w^n\, u_n(z) = \sum_{n=0}^{\infty} \gamma_n(f)(wz)^n$$

But $T_w f(z) = \sum_{n=0}^{\infty} \gamma_n(f)(wz)^n$. This completes the proof.

We note that it was obvious that $T_w f$ was in H^p in the first place, merely from the definition of H^p; but the form for $T_w f$, which was obtained above, will be needed later.

THEOREM 1. *If $G \in \mathfrak{A}$ such that $\lim_{r=1} B(f, G; r) = \gamma(f)$ (that is, we assume that this limit exists) for all f in H^p, then γ is in $(H^p)^*$. Conversely, if γ is in $(H^p)^*$, then there exists a unique G in $\mathfrak{A}$ such that $\gamma(f) = \lim_{r=1} B(f, G; r)$ for all f in H^p.*

Proof. To prove the first part of our theorem, let $\gamma_r(f) = B(f, G; r)$. Clearly $\gamma_r(f)$ is distributive in f. Suppose $\|f\| = 1$ and $r < \rho < 1$. Then

$$|\gamma_r(f)| \leq \sum_{n=0}^{\infty} |\gamma_n(f)| \cdot |\gamma_n(G)|\, r^n \leq \sum_{n=0}^{\infty} |\gamma_n(G)| \left(\frac{pn+1}{pn}\right)(pn+1)^{1/p} r^n .$$

Thus, $\gamma_r(f)$ is bounded in f for $\|f\| = 1$, r being fixed. It is then clear that γ_r is in $(H^p)^*$. Since $\lim_{r=1} \gamma_r(f)$ exists, it follows that $\gamma_r(f)$ is continuous on $0 \leq r \leq 1$ for each fixed f in H^p. Thus $\{\gamma_r(f)\}$ is bounded for $0 \leq r < 1$. As a result of Lemma 1, we may conclude that $\{\|\gamma_r\|\}$ is bounded for $0 \leq r < 1$; that is, there exists an M such that $\|\gamma_r\| \leq M$ for $0 \leq r < 1$. Let $\|f\| = 1$. Then $|\gamma_r(f)| \leq M$, whence $|\gamma(f)| \leq M$. Thus γ is necessarily in $(H^p)^*$ since it is bounded on the unit sphere in H^p.

We now prove the second part of Theorem 1. We note that if $\lim_{r=1} B(f, G; r) = \gamma(f)$ for some G and all f, then

$$\gamma(u_n) = \lim_{r=1} B(u_n, G; r) = \lim_{r=1} \gamma_n(G)\, r^n = \gamma_n(G) ;$$

that is, $\gamma_n(G) = \gamma(u_n)$ for all n, or merely $G(z) = \sum_{n=0}^{\infty} \gamma(u_n) z^n$. We note that $\sum_{n=0}^{\infty} \gamma(u_n) z^n$ converges, for $|\gamma(u_n)| \leq \|\gamma\| \cdot \|u_n\| = \|\gamma\|$. Let us now verify that G, as defined, has the desired property. We see that

$$B(f, G; r) = \sum_{n=0}^{\infty} \gamma_n(f)\, \gamma(u_n)\, r^n = \gamma\left\{\sum_{n=0}^{\infty} \gamma_n(f)\, r^n u_n\right\} = \gamma(T_r f) .$$

But $\|T_r f - f\| \longrightarrow 0$; see [5] for this result; note that

$$\|T_r f - f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta\right)^{1/p},$$

where $f(e^{i\theta})$ is the boundary function for $f(z)$. Thus $\gamma(T_r f) \longrightarrow \gamma(f)$, or $B(f, G; r) \longrightarrow \gamma(f)$. Our proof is thus complete.

THEOREM 2. *The function G in Theorem 1 is continuous on the closure of* Δ.

Proof. We first verify that $f_t(z) = (1 - ze^{it})^{-1}$ is in H^p for every real t. It suffices to show that f_0 is in H^p. We see that

$$|1 - re^{i\theta}|^{-2} = [(1 - re^{i\theta})(1 - re^{-i\theta})]^{-1} = (1 - 2r \cos\theta + r^2)^{-1},$$

whence

$$|1 - re^{i\theta}|^{-p} = (1 - 2r\cos\theta + r^2)^{-p/2} .$$

From the character of $(1 - 2r\cos\theta + r^2)$, we see that it suffices to show that $\int_0^\delta (1 - 2r\cos\pi + r^2)^{-p/2}\, d\theta$ is bounded in r, where δ is any positive number. We note that the following is true for $0 \leq \theta \leq \delta$ (where δ is some sufficiently small positive number) and for all r such that $1/2 \leq r < 1$:

$$1 - 2r\cos\theta + r^2 \geq 1 - 2r\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) + r^2 = (1 - 2r + r^2) + r\theta^2\left(1 - \frac{\theta^2}{12}\right)$$

$$= (1 - r)^2 + r\theta^2\left(1 - \frac{\theta^2}{12}\right)$$

$$\geq \frac{r\theta^2}{2} \geq \frac{\theta^2}{4} .$$

Thus, $(1 - 2r\cos\theta + r^2)^{-p/2} \leq 4^{p/2}\ \theta^{-p}$. Since θ^{-p} is integrable on $[0, \delta]$, our statement is proved.

We remind ourselves that we are trying to show that G is continuous on the assumption that

$$\gamma(f) = \lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta})\, G(\frac{r}{\rho} e^{-i\theta})\, d\theta , \qquad r < \rho < 1 ,$$

exists for each f in H^p. Let γ_r be defined as in the proof of Theorem 1. Then

$$\gamma_r(f_t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-\rho e^{i(\theta+t)}} \cdot G\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{G\left(\frac{r}{\rho} e^{i\theta}\right)}{1-\rho e^{i(t-\theta)}} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{G\left(\frac{r}{\rho} e^{i\theta}\right)\frac{r}{\rho} e^{i\theta}}{\frac{r}{\rho} e^{i\theta} - re^{it}} \, d\theta$$

$$= G(re^{it}), \qquad r < \rho < 1 .$$

The last equality is true by virtue of Cauchy's integral formula. We then have shown that $G(re^{it}) = \gamma_r(f_t)$. Consequently, since $\lim_{r=1} \gamma_r(f_t)$ exists by hypothesis, $\lim_{r=1} G(re^{it})$ exists for all t, and in fact

$$\gamma(f_t) = G(e^{it}),$$

where we define $G(e^{it})$ to be the boundary function $\lim_{r=1} G(re^{it})$.

We now show that $\lim_{t=t_0} f_t = f_{t_0}$ in the topology of H^p. Now, for *any* g in H^p, letting $g(e^{i\theta})$ be its boundary function, we know that

$$\sup_{0 \le r < 1} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta = \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta .$$

It is easily verified that (see, for example, [4, Theorem 7, p. 29])

$$\lim_{t=t_0} \int_0^{2\pi} |g(e^{i(\theta+t)}) - g(e^{i(\theta+t_0)})|^p \, d\theta = 0 .$$

Clearly $f_t(e^{i\theta}) = (1 - e^{i(\theta+t)})^{-1}$, whence $f_t(e^{i\theta}) = f_0(e^{i(\theta+t)})$. Thus $\lim_{t=t_0} f_t = f_{t_0}$, in the topology of H^p.

Now, by Theorem 1, γ is continuous, whence $\lim_{t=t_0}\gamma(f_t) = \gamma(f_{t_0})$; hence $\lim_{t=t_0} G(e^{it}) = G(e^{it_0})$. We have now shown that $G(e^{it})$ is continuous.

We remember that, in the course of proving Theorem 1, we showed that $\{\gamma_r\}$ is bounded in r as a subset of $(H^p)^*$. Obviously $\{f_t\}$ is a bounded subset of H^p, all of the elements having the same norm. Thus $\gamma_r(f_t)$ is bounded in both r and t. In other words, $G(re^{it})$ is bounded in r and t, or equivalently G is uniformly bounded on Δ. We then know that

$$G(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} G(e^{it})\, P_r(\theta - t)\, d\theta,$$

where $P_r(\theta)$ is the Poisson kernel. But, since $G(e^{it})$ is continuous, the right side above is necessarily a continuous function on the closed unit circle. Our proof is now complete.

It will now be shown that even more can be said of G when $0 < p < 1/2$.

THEOREM 3. *If $0 < p < 1/2$, then $G(e^{it})$ satisfies the Lipschitz condition of order one.*

Proof. It suffices to show that

$$\|f_{t+h} - f_t\| = \|f_h - f_0\| \le A \cdot |1 - e^{ih}|$$

for some fixed constant A. We have

$$\|f_h - f_0\| = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{1 - e^{i(\theta+h)}} - \frac{1}{1 - e^{i\theta}} \right|^p d\theta \right)^{1/p}$$

$$= \frac{|1 - e^{ih}|}{(2\pi)^{1/p}} \left\{ \int_0^{2\pi} |(1 - e^{i(\theta+h)})(1 - e^{i\theta})|^{-p}\, d\theta \right\}^{1/p},$$

The proof will then be complete after we have shown that

$$\int_0^{2\pi} |(1 - e^{i(\theta+h)})(1 - e^{i\theta})|^{-p}\, d\theta$$

is bounded for all sufficiently small h. It is evident that

$$|(1 - e^{i\theta})(1 - e^{i(\theta+h)})|^2 = 4(1 - \cos\theta)[1 - \cos(\theta + h)],$$

and hence

$$|(1-e^{i\theta})(1-e^{i(\theta+h)}|^p = 4^{p/2}(1-\cos\theta)^{p/2}(1-\cos(\theta+h))^{p/2}.$$

We now must show that

$$\int_0^{2\pi} (1-\cos\theta)^{-p/2}(1-\cos(\theta+h))^{-p/2}\,d\theta$$

is bounded in h for all sufficiently small h. We note that the following is true for all sufficiently small θ and h:

$$1-\cos\theta \geq \frac{\theta^2}{2}\left(1-\frac{\theta^2}{12}\right) \geq \frac{\theta^2}{4},$$

$$1-\cos(\theta+h) \geq \frac{(\theta+h)^2}{4}.$$

Thus we have

$$(1-\cos\theta)^{-p/2}[1-\cos(\theta+h)]^{-p/2} \leq 4^p\,\theta^{-p}(\theta+h)^{-p}$$

for all sufficiently small θ and h. Since θ^{-2p} is integrable on the interval $[0, 2\pi]$, it is then rather easy to show that

$$\int_0^{2\pi} (1-\cos\theta)^{-p/2}[1-\cos(\theta+h)]^{-p/2}\,d\theta$$

is bounded in h for all sufficiently small h.

We now have the rather interesting result:

COROLLARY. *If* $0 < p < 1/2$, *then* $\sum_{n=0}^{\infty}|\gamma_n(G)| < \infty$.

Proof. Since $G(e^{it})$ is of bounded variation, it follows that $G(z)$ is a power series of bounded variation according to [7, §7.5]. Hence the conclusion is obtained by [7, (i), p. 158].

We shall now show that even more may be said of G when $0 < p < 1/2$.

THEOREM 4. *If* $0 < p < 1/2$, *then* $(d/dz)G(z)$ *is continuous on the closure of*

Δ, *and moreover* $(d/dt)G(e^{it})$ *is continuous on* $[0, 2\pi]$.

Proof. By Cauchy's integral formulas (where $(d/dz)G(z) = G'(z)$):

$$G'(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{G\left(\frac{r}{\rho}e^{i\theta}\right)\frac{r}{\rho}e^{i\theta}}{\left(\frac{r}{\rho}e^{i\theta} - re^{it}\right)^2}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{G\left(\frac{r}{\rho}e^{-i\theta}\right)\frac{\rho}{r}e^{i\theta}}{(1-\rho e^{i(\theta+t)})^2}\,d\theta$$

$$= \frac{1}{2\pi r}\int_0^{2\pi}[f_t^2(\rho e^{i\theta})\cdot\rho e^{i\theta}]\cdot G\left(\frac{r}{\rho}e^{-i\theta}\right)d\theta, \quad r < p < 1.$$

Thus $G'(re^{it}) = (1/r)\gamma_r(f_t^2 u_1)$. We note that since $0 < p < 1/2$, we have $f_t^2 \in H^p$, whence $f_t^2 \cdot u_1 \in H^p$, since u_1 is bounded. Thus we show exactly as in Theorem 2 that

$$\gamma(f_t^2 u_1) = G'(e^{it}),$$

$G'(e^{it})$ is continuous in t,

$G'(z)$ is uniformly bounded on Δ,

$$G'(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} G'(e^{i\theta})P_r(\theta - t)d\theta,$$

where we define $G'(e^{it})$ to be the boundary value of $G'(z)$. Let us now consider

$$F(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi}\left[-ie^{-i\theta}\frac{d}{d\theta}G(e^{i\theta})\right]P_r(\theta - t)\,d\theta.$$

We note that $G(e^{i\theta})$ is absolutely continuous by virtue of Theorem 2, whence $(d/d\theta)G(e^{i\theta})$ is integrable. We also note that

$$G(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\theta})P_r(\theta - t)\,d\theta = \sum_{n=0}^{\infty} C_n r^n e^{int},$$

where

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\theta})\, e^{-in\theta}\, d\theta .$$

Moreover, it is not at all difficult to verify that the real and imaginary parts of $-ie^{i\theta}(d/d\theta)G(e^{i\theta})$ are conjugate, whence

$$F(re^{it}) = \sum_{n=0}^{\infty} d_n r^n e^{int} ,$$

where

$$d_n = \frac{1}{2\pi} \int_0^{2\pi} \left[-ie^{-i\theta} \frac{d}{d\theta} G(e^{i\theta}) \right] e^{-in\theta} d\theta .$$

Integration by parts readily yields

$$d_n = (n+1) C_n ;$$

that is,

$$F(re^{it}) = \sum_{n=0}^{\infty} (n+1)\, C_{n+1}\, r^n e^{int} ,$$

and hence $F(z) = G'(z)$. Thus, we necessarily have

$$G'(e^{i\theta}) = -ie^{-i\theta} \frac{d}{d\theta} G(e^{i\theta})$$

almost everywhere. Since $G'(e^{i\theta})$ is continuous, it follows that $G(e^{i\theta})$ necessarily has a continuous derivative, and in fact

$$\frac{d}{d\theta} G(e^{i\theta}) = ie^{i\theta} G'(e^{i\theta}) .$$

This completes the proof of the theorem.

We sum up by presenting the following theorem, which is readily proved by induction, the proof being modeled after that given for Theorem 4.

THEOREM 5. *If* $0 < p < 1/n$, $n = 2,3,\cdots$, *then* $(d^{n-1}/dz^{n-1})\, G(z)$ *is continuous on the closure of* Δ. *Moreover, if* $0 < p < 1/2n$, $n = 1,2,\cdots$, *then* $G(e^{it})$ *has a continuous nth derivative with respect to t.*

3. Generalization of Vitali's Theorem. In this section we assume merely that p is any positive real number. We here need the following:

LEMMA 3. *If* $\{f_n\}$ *is a bounded sequence in* H^p, *and if* $\lim_{n=\infty} f_n(z)$ *exists on a set having at least one limit point in* Δ, *then* $\lim_{n=\infty} f_n(z)$ *exists uniformly on all compact subsets of* Δ.

Proof. The proof is a simple consequence of the following inequalities:

$$|f(z)| \leq \frac{\|f\|}{(1-|z|)^{1/p}}, \qquad \text{when } 0 < p \leq 1,$$

and

$$|f(z)| \leq \frac{\|f\|}{1-|z|} \qquad \text{when } 1 \leq p < \infty .$$

The first of the above inequalities appears in [6, Theorem 2]. The second is easily obtained as follows. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i\theta})\, \rho e^{i\theta}}{\rho e^{i\theta} - z}\, d\theta ,$$

and hence, by Hölder's inequality,

$$|f(z)| \leq \frac{\rho}{\rho - |z|}\; \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|\, d\theta$$

$$\leq \frac{\rho}{\rho - |z|} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta \right)^{1/p},$$

whence

$$|f(z)| \leq \frac{\|f\|}{1-|z|}.$$

Let

$$N(r) = \begin{cases} \dfrac{1}{(1-r)^{1/p}}\,, & 0 < p \le 1\,. \\[2ex] \dfrac{1}{1-r}\,, & 1 \le p < \infty\,. \end{cases}$$

It is then clear that $|f_n(z)| \le N(r) \cdot M$ when $|z| \le r < 1$, where $\|f_n\| \le M$ for all n. We choose r so large that the set $|z| < r$ includes a set having a limit point in $|z| < r$ and such that $\lim_{n=\infty} f_n(z)$ exists on this set. Then, by Vitali's theorem, $\lim_{n=\infty} f_n(z)$ exists uniformly on all compact subsets of $|z| < r$, and hence on all compact subsets of Δ. This completes the proof.

THEOREM 6. *Suppose $\{f_n\}$ is a bounded sequence in H^p. Further, suppose $\lim_{n=\infty} f_n(e^{i\theta})$ exists on a set of positive measure in the interval $[0, 2\pi]$. Then $\lim_{n=\infty} f_n(z)$ exists uniformly on all compact subsets of Δ.*

Proof. It suffices, by the preceding lemma, to show that $\lim_{n=\infty} f_n(z)$ exists on some neighborhood of the origin. Thus, we shall show that this is the case whenever $|z| < 1/9$. Let $|z_0| < 1/9$, and suppose $\lim_{n=\infty} f_n(z_0)$ does *not* exist. Then we may find a positive number α and subsequences $\{f_{n_k}\}$ and $\{f_{m_k}\}$ of $\{f_n\}$ which have the property that $|f_{n_k}(z_0) - f_{m_k}(z_0)| > \alpha$ for all k. We then define $q_k = f_{n_k} - f_{m_k}$. It is clear that $\{q_k\}$ is a bounded sequence in H^p. We then write $q_k = g_k \cdot h_k$, by virtue of F. Riesz's decomposition theorem [5], where g_k and h_k are such that

(i) $\quad g_k \in H^p$ and $g_k(z) \neq 0$ for all z in Δ,

(ii) $\quad |h_k(z)| \le 1$ on Δ and $|h_k(e^{i\theta})| = 1$ almost everywhere,

(iii) $\quad \|g_k\| = \|q_k\|$.

We note that $l_k(z) = [g_k(z)]^{p/2}$ is in H^2, and in fact $\{l_k\}$ is a bounded sequence in H^2. Since $\lim_{k=\infty} [f_{n_k}(e^{i\theta}) - f_{m_k}(e^{i\theta})] = 0$ on a set of positive measure, it follows that $\lim_{k=\infty} l_k(e^{i\theta}) = 0$ on a set E of measure $\mu > 0$. We next shall show that $\lim_{k=\infty} l_k(z_0) = 0$, which will in turn imply that $\lim_{k=\infty} g_k(z_0) = 0$, and hence imply $\lim_{k=\infty} q_k(z_0) = 0$, a contradiction to $|q_k(z_0)| > \alpha$ for all k.

Let $A > 0$, and define

$$\phi(\theta) = \begin{cases} A/\mu & \text{on } E \\ A/(\mu - 2\pi) & \text{on } CE\,, \end{cases}$$

where CE is the set $[0, 2\pi] - E$. There is no loss in supposing that $\mu \leq \pi$. Define

$$u_0(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)\, P_r(\theta - t)\, dt\,,$$

where $P_r(\theta)$ is the Poisson kernel. Then u_0 is harmonic in Δ and $\lim_{r=1} u_0(re^{i\theta}) = \phi(\theta)$ a.e., by virtue of Fatou's theorem; see [7, § 3.442]. Let

$$u(re^{i\theta}) = u_0(re^{i\theta}) - u_0(z_0)\,.$$

We note that

$$u_0(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\,,$$

where $\{a_n, b_n\}$ are the Fourier coefficients of $\phi(\theta)$. Since $u_0(0) = 0$, this being due to the fact that $\int_0^{2\pi}\phi(t)\,dt = 0$ and $P_0(\theta - t) = 1$, we then have a_0 equal to zero, or

$$u_0(re^{i\theta}) = \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\,.$$

We note that $|a_n| \leq 2A/\pi$ as well as $|b_n| \leq 2A/\pi$, whence

$$|u_0(re^{i\theta})| \leq \frac{4A}{\pi}\sum_{n=1}^{\infty} r^n = \frac{4A}{\pi}\,\frac{r}{1-r} < \frac{A}{2\pi} \leq \frac{A}{2(2\pi - \mu)}$$

provided $0 \leq r < 1/9$.

Let $v(z)$ be the harmonic conjugate of $u(z)$ which vanishes at z_0, and define $g(z) = e^{u(z)+iv(z)}$. Then $g \in \mathfrak{A}$, and $g(z_0) = 1$. Moreover, since $|g(z)| = e^{u(z)}$, we have $\lim_{r=1}|g(re^{i\theta})| = e^{\phi(\theta)-u_0(z_0)}$. By Cauchy's integral formula we have

$$l_k(z_0) = \frac{1}{2\pi}\int_0^{2\pi} l_k(\rho e^{i\theta})\; g(\rho e^{i\theta})\; \frac{\rho e^{i\theta}}{\rho e^{i\theta} - z_0}\, d\theta\,, \quad |z_0| < \rho < 1\,.$$

This is true since $l_k(z_0) = l_k(z_0)\, g(z_0)$. We note that $u(z)$ is bounded in Δ, and hence so is $g(z)$. Since

$$\lim_{\rho=1} \int_0^{2\pi} | l_k(\rho e^{i\theta}) - l_k(e^{i\theta})|^2\, d\theta = 0\,,$$

and since $g(z)$ is bounded on Δ, it is then evident that

$$\lim_{\rho=1} \int_0^{2\pi} | l_k(\rho e^{i\theta})\; g(\rho e^{i\theta})|\, d\theta = \int_0^{2\pi} | l_k(e^{i\theta})\; g(e^{i\theta})|\, d\theta\,.$$

Hence

$$| l_k(z_0)| \le \frac{1}{2\pi}\;\frac{1}{1-|z_0|}\int_0^{2\pi} | l_k(e^{i\theta})|\; e^{\phi(\theta)-u_0(z_0)}\, d\theta\,.$$

Consequently

$$| l_k(z_0)| \le e^{A/\mu - u_0(z_0)}\;\frac{1}{2\pi}\left(\frac{1}{1-|z_0|}\right)\int_E | l_k(e^{i\theta})|\, d\theta$$

$$+\; e^{A/(\mu-2\pi) - u_0(z_0)}\left(\frac{1}{2\pi}\right)\left(\frac{1}{1-|z_0|}\right)\int_{CE} | l_k(e^{i\theta})|\, d\theta\,.$$

Since

$$\frac{1}{2\pi}\int_{CE} | l_k(e^{i\theta})|\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} | l_k(e^{i\theta})|\, d\theta \le \left(\frac{1}{2\pi}\int_0^{2\pi} | l_k(e^{i\theta})|^2 d\theta\right)^{1/2}$$

and since $\{l_k\}$ is a bounded subset of H^2, we see that

$$\frac{1}{2\pi}\int_{CE} |l_k(e^{i\theta})|\, d\theta$$

is bounded with respect to k. Moreover

$$\frac{A}{\mu - 2\pi} - u_0(z_0) \leq \frac{A}{\mu - 2\pi} + |u_0(z_0)| \leq \frac{A}{\mu - 2\pi} + \frac{4A}{\pi}\left(\frac{|z_0|}{1 - |z_0|}\right)$$

$$\leq \frac{A}{\mu - 2\pi} + \frac{A}{2(2\pi - \mu)} = \frac{A}{2(\mu - 2\pi)} \ .$$

By virtue of Schwarz's inequality, where ξ is an arbitrary measurable subset of $[0, 2\pi]$, we have

$$\int_{\xi} |\, l_k(e^{i\theta})\,| \, d\theta \leq [m(\xi)]^{1/2} \left(\int_{\xi} |\, l_k(e^{i\theta})\,| \, d\theta\right)^{1/2} .$$

Hence, by a convergence theorem of Lebesque (see [2, p. 190], we have

$$\lim_{k=\infty} \int_E |\, l_k(e^{i\theta})| \, d\theta = 0,$$

since $\lim_{k=\infty} l_k(e^{i\theta}) = 0$ on E. Now, for arbitrary $\epsilon > 0$, we choose A so large that

$$e^{A/[2(\mu - 2\pi)]} \cdot \left(\frac{1}{2\pi}\right)\left(\frac{1}{1 - |z_0|}\right) \int_{CE} |\, l_k(e^{i\theta})\,| \, d\theta < \epsilon/2 \ .$$

and hence we obtain, from the foregoing,

$$e^{A/[\mu - 2\pi] - u_0(z_0)}\left(\frac{1}{2\pi}\right)\left(\frac{1}{1 - |z_0|}\right) \int_{CE} |\, l_k(e^{i\theta})\,| \, d\theta < \epsilon/2 \ .$$

Having so chosen A, choose K so large that $k > K$ implies

$$e^{A/\mu - u_0(z_0)}\left(\frac{1}{2\pi}\right)\left(\frac{1}{1 - |z_0|}\right) \int_E |\, l_k(e^{i\theta})\,| \, d\theta < \epsilon/2 \, .$$

Hence, $k > K$ implies $|\, l_k(z_0)\,| < \epsilon/2 + \epsilon/2 = \epsilon$. This completes the proof of the theorem.

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TWO THEOREMS ON METRIC SPACES

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1. Introduction. Let E be a metric space with distance function d. The space E is called *two-point homogeneous* if given any four points a, a', b, b' with $d(a, a') = d(b, b')$, there exists an isometry of E carrying a, a' to b, b', respectively. In a recent paper [7], the author has determined all the compact and connected two-point homogeneous spaces. It is the aim of the present note to discuss the noncompact case, and prove a conjecture of Busemann which can be regarded also as a sharpening of a theorem of Birkhoff [1]. The results concerning the noncompact two-point homogeneous spaces are not as satisfactory as the results for the compact case; we have to assume certain conditions on the metric.

By a segment in a metric space E, we shall mean an isometric image of a closed interval with the usual metric. A metric space will be said to have the property (L) if given a point p, there exists a neighborhood W of p so that each point x $(\neq p)$ of W can be joined to p by at most one segment in E. The following theorems will be proved:

THEOREM 1. *Let E be a finite-dimensional, finitely compact, convex metric space with property (L). If E is two-point homogeneous, then E is homeomorphic with a manifold.*

THEOREM 2. *Let E be a metric space with all the properties mentioned in Theorem* 1. *If, moreover,* $\dim E$ *is odd, then E is congruent either to the euclidean space, the hyperbolic space, the elliptic space, or the spherical space.*

Our Theorem 2 justifies the conjecture of Busemann [2, p. 233] that a two-point homogeneous three dimensional S.L. space [2, p. 78] is either elliptic, hyperbolic, or euclidean. It is to be noted that Theorem 2 no longer holds if $\dim E$ is even and greater than two. The complex elliptic spaces [7] and the hyperbolic Hermitian spaces[1] [2, p. 192] serve as counter examples.

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[1]These spaces were first introduced by H. Poincaré, and then discussed by G. Fubini and E. Study. Following E. Cartan, we call these spaces the hyperbolic Hermitian spaces.

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2. Preliminary results. Throughout this note, by a Busemann space [2, p. 11], we shall mean a finitely compact, convex metric space such that at each point p, there exists a neighborhood W with the following property: given any two points x, y of W and any $\epsilon > 0$, we can find a positive number $\delta < \epsilon$ for which a unique point z exists so that

$$d(x, y) + d(y, z) = d(x, z), \quad d(y, z) = \delta .$$

It can easily be verified that the class of all two-point homogeneous, finitely compact, convex metric space with the property (L) coincides with the class of all two-point homogeneous Busemann spaces. In the statements of our Theorems, we use the property (L) instead of Busemann's axioms merely because it is, geometrically, easier to visualize.

Let E be a Busemann space. We shall first see that each d-sphere[1] of sufficiently small radius is locally connected. In fact, let p be a point of E. We choose $\epsilon > 0$ so small that each point x with $0 < d(p, x) \leq \epsilon$ can be joined to p by one and only one segment. Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and R the totality of points y with $0 < d(p, y) < \epsilon$. Then evidently R is an open set of E. Since E is convex, E must be locally connected. It follows then that R is locally connected.

For each point y of $K(p, \epsilon)$, we denote by $P_y(s)$ $(0 \leq s \leq \epsilon)$ the isometric representation of the segment joining p to y. Let J be the open interval $0 < s < \epsilon$. By our choice of ϵ, the mapping $h: K(p, \epsilon) \times J \longrightarrow R$ defined by $h(y, s) = P_y(s)$ is a one-to-one mapping of the topological product $K(p, \epsilon) \times J$ onto R. Moreover, from Busemann's results [2, I., §3] concerning the convergence of geodesics, we see immediately that h is bicontinuous. This tells us that $K(p, \epsilon) \times J$ and R are homeomorphic. Since R is locally connected, $K(p, \epsilon) \times J$, and hence $K(p, \epsilon)$, is locally connected.

3. Proof of Theorem 1. Let E be a metric space with all the properties mentioned in Theorem 1. From the above discussions, we know that for any point p of E, the d-sphere $K(p, \epsilon)$ with sufficiently small radius ϵ is locally connected. Let Γ be the group of all isometries of E, and Γ_p the totality of all those isometries which leave p invariant. In Γ, we introduce the topology as defined by van Dantzig and van der Waerden [4] (in fact, this is exactly the g-topology of R. Arens).

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[1] By a d-sphere we mean the totality of points equidistant from a fixed point with respect to the metric d. This should be distinguished from the $(n - 1)$-sphere which stands for the $(n - 1)$-dimensional topological sphere.

Then Γ_p forms a compact topological group [4]. Evidently, Γ_p is a transformation group of $K(p, \epsilon)$ in the sense of Montgomery and Zippin. From the two-point homogeneity, Γ_p is transitive on $K(p, \epsilon)$. Taking account of the finite dimensionality and local connectedness of $K(p, \epsilon)$ and the compactness of Γ_p, we can conclude [5] that Γ_p is a Lie group, and hence $K(p, \epsilon)$ is locally euclidean (here as well as in what follows, locally euclidean is always used in the topological sense). The set R, being homeomorphic with the topological product of $K(p, \epsilon)$ and the open interval J, must be locally euclidean as well. Hence our space E is locally euclidean at each point of R, and hence locally euclidean at all its points. Moreover, E is obviously separable and connected. It follows then that E is homeomorphic with a manifold.

4. **The structure of d-spheres.** Before proving Theorem 2, we find it convenient to establish some more properties of the d-spheres.

LEMMA. *Let E be a metric space satisfying all the conditions in Theorem 2. Then each d-sphere with sufficient small radius is homeomorphic with the $(n-1)$-dimensional topological sphere where* $\dim E = n$.

Proof. If $\dim E$ is equal to one, this is trivial. Now we shall assume that $n > 1$. Let p be a point of E, and ϵ so small that each point x with $0 < d(p,x) \leq \epsilon$ can be joined to p by one and only one segment. Set $K(p, \epsilon)$ to be the d-sphere with center p and radius ϵ, and

$$U = \{x \mid d(p, x) < \epsilon\} .$$

We shall show first that U is contractible to a point. Given each point y of $K(p,\epsilon)$, let us denote by $P_y(s)$ the isometric representation of the segment joining p to y. Then the pair (y, s), where $y \in K(p, \epsilon)$ and $0 \leq s < \epsilon$, can be regarded as polar coordinates of points in U. For any real number t with $0 \leq t \leq 1$, we define

$$\phi[t, P_y(s)] = P_y(ts).$$

We see immediately that ϕ is a well-defined mapping of the product $I \times U$, and

$$\phi[1, P_y(s)] = P_y(s), \quad \phi(t, p) = p, \quad \phi[0, P_y(s)] = p,$$

where I denotes the closed interval $\{t \mid 0 \leq t \leq 1\}$. The continuity of ϕ can easily be verified. Thus ϕ gives a contraction of U into the point p, and thus the homotopy group $\pi_i(U)$ vanishes for each i.

Now let us consider the set $R = U - $ p. Since U is an n-dimensional open manifold and $n > 1$, the set R is connected and has the same homotopy group π_i as U for all dimensions i less than $n-1$. Thus $\pi_i(R) = 0$, $i = 1, 2, \cdots, n-2$. On the other hand, we have shown in §1 that R is homeomorphic with the topological product $K(p, \epsilon) \times J$, where J denotes an open interval. It follows then that $K(p, \epsilon)$ is connected and

$$(1) \qquad \pi_i[K(p, \epsilon)] = 0, \qquad i = 1, 2, \cdots, n-2.$$

From the proof of Theorem 1, we know that $K(p, \epsilon)$ is a homogeneous space of a compact Lie group. Its connectedness and its simply-connectedness imply that it is an orientable manifold.

Since both $K(p, \epsilon)$ and J are manifolds, we have

$$\dim K(p, \epsilon) + \dim J = \dim R = \dim E = n,$$

and hence $\dim K(p, \epsilon) = n-1$. It follows immediately from (1) that $K(p, \epsilon)$ is a simply-connected homology sphere of even dimension $n-1$. Therefore [6] $K(p, \epsilon)$ is a topological sphere. The lemma is proved.

5. Proof of Theorem 2. Suppose E to be a metric space with all the properties mentioned in Theorem 2. If E is compact, then our Theorem 2 follows as a direct consequence of [7, Theorem VI]. Thus we can assume from now on that E is not compact. We shall first show that E is an open S. L. space in the sense of Busemann [2, p. 78]. To show this, it suffices [3, p. 173] to establish that each geodesic is congruent to a euclidean line; for this, it suffices to demonstrate that given any two distinct points x, y and any $k > 0$, there exists a point z so that

$$d(x, y) + d(y, z) = d(x, z), \qquad d(y, z) = k.$$

In fact, since E is finitely compact and noncompact, E cannot be bounded. There exists then a sequence of points $p_0, p_1, p_2, \cdots$ with $d(p_0, p_i)$ tending to infinity. Thus we can choose i so large that $d(p_0, p_i) \geq d(x, y) + k$. Let τ be a segment joining p_0 to p_i. Evidently there exist three points x', y', z' in τ such that

$$d(x', y') + d(y', z') = d(x', z'), \qquad d(x', y') = d(x, y), \qquad d(y', z') = k.$$

From the two-point homogeneity of E, there is an isometry f of E carrying x', y' to x, y respectively. Then we can see immediately that the point $z = f(z')$ has all the required properties. Thus E is an open S. L. space.

Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and Γ_p the group of all isometries of E which leave the point p invariant. From the above lemma, we know that $K(p, \epsilon)$ is an $(n-1)$-sphere and Γ_p a compact and transitive transformation group of $K(p, \epsilon)$. Moreover, it can easily be seen that Γ_p is effective on $K(p, \epsilon)$. In our further discussions, we shall rule out the trivial case where $\dim E = n = 1$. Thus $K(p, \epsilon)$ is connected, and the identity component Γ_p^0 of Γ_p forms a connected, compact, transitive, and effective transformation group of $K(p, \epsilon)$. Since $n-1$ is even, it follows [6] that Γ_p^0 is either isomorphic with the rotation group R_{n-1} or Cartan's exceptional group G_2. We shall discuss these two cases separately.

Case A. Suppose Γ_p^0 to be isomorphic with the group R_{n-1} of all rotations of the $(n-1)$-sphere. Let us represent $K(p, \epsilon)$ by the unit sphere in a certain n-dimensional euclidean space, and consider R_{n-1} not only as a topological group but also as a transformation group of $K(p, \epsilon)$ in the usual sense. It is well known that Γ_p^0 and R_{n-1} have the same topological type, that is, there exists a homeomorphism ϕ of $K(p, \epsilon)$ onto itself so that

$$R_{n-1} = \phi\Gamma_p^0\phi^{-1} = \{\phi f \phi^{-1} \mid f \in \Gamma_p^0\}.$$

Since n is odd, given any point q of $K(p, \epsilon)$, there exists a rotation of period two which leaves fixed *only* q and its diametrically opposite point. It follows then that for each point q of $K(p, \epsilon)$, we can find a transformation f in Γ_p^0 such that (a) f is of period two, (b) f leaves q fixed, and (c) f has only two fixed points on $K(p, \epsilon)$. Now let g be any geodesic through p in E. It intersects $K(p, \epsilon)$ at two points, say q and q'. We consider the transformation f in Γ_p^0 having the above three properties (a), (b), and (c). Since f is an isometry leaving fixed p and q, it leaves the geodesic g pointwise invariant. Moreover, this isometry f cannot have any other fixed point, for otherwise f would have some other fixed points on $K(p, \epsilon)$ besides q and q'. Thus f is a reflection of E about g. Since p is an arbitrary point and g an arbitrary geodesic through p, there exists a reflection of E about each geodesic. From Schur's Theorem [2, p.181], it follows that E is either hyperbolic or euclidean.

Case B. Suppose Γ_p^0 to be isomorphic with the exceptional group G_2. To discuss this case, we have to digress into a few properties of Cayley numbers. Let $1, e_i$ $(i = 1, 2, \cdots, 7)$ be the units of Cayley algebra. The multiplication rule is given by

$$e_i e_i = -1, \quad e_i e_j = -e_j e_i, \quad e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_1 e_6 = e_7,$$

$$e_2 e_5 = e_7, \quad e_2 e_4 = -e_6, \quad e_3 e_4 = e_7, \quad e_3 e_5 = e_6,$$

together with the equalities obtained by cyclic permutation of the indices. Let

$$\Theta = \left\{ \sum_{i=1}^{7} x_i e_i \mid x_i = \text{real number}, \quad \sum_{i=1}^{7} (x_i)^2 = 1 \right\}$$

be the totality of all the Cayley numbers with vanishing real part and with norm equal to unity. Evidently, Θ forms a 6-sphere, and each automorphism of the Cayley algebra carries Θ into itself. We can regard therefore the group H of all automorphisms of Cayley algebra as a transformation group of Θ (the topology over H is defined in the usual manner). Now H acts effectively and transitively on Θ. Moreover, it is known that H is isomorphic with the exceptional group G_2.

For each $x = \sum_{i=1}^{7} x_i e_i$ in Θ, we shall denote the Cayley number $x_1 - \sum_{i=2}^{7} x_i e_i$ by x^*, and call it the *symmetric image* of x with respect to e_1. It is evident that

$$(1) \qquad (x^*)^* = x, \qquad x^* \begin{cases} = x, & \text{if } x = \pm e_1, \\ \neq x, & \text{otherwise.} \end{cases} \qquad x \in \Theta$$

Moreover, by a direct calculation, we can show that given any two Cayley numbers y, z in Θ, there exists an automorphism f in H such that

$$f(e_1) = e_1, \quad f(y) = y^*, \quad f(z) = z^*.$$

It is to be noted that this f depends on y and z. There is no automorphism of Cayley algebra which carries each x in Θ into its symmetric image x^*.

Now we can proceed to the proof of Theorem 2. Since Γ_p^0 is isomorphic with the exceptional group G_2, $K(p, \epsilon)$ must be six-dimensional [6]. It is known that each transitive transformation group of the 6-sphere which is isomorphic with the exceptional group G_2 has the same topological type as H.[1] Thus we can identify Θ and $K(p, \epsilon)$ in such a manner that Γ_p^0 and H coincide. Let x be a point of $K(p,\epsilon)$. It determines a ray $\overrightarrow{px}$, that is, the totality of points u of E for which either $d(x, u) + d(u, p) = d(x, p)$ or $d(u, x) + d(x, p) = d(u, p)$ [2, p. 76]. For each nonnegative number s, we denote by $P_x(s)$ the point u on the ray $\overrightarrow{px}$ with the property that

[1] This follows as a direct consequence of [6, Lemma 6].

$d(p, u) = s$. Since E is an open S. L. space, each point of E other than p can be represented in a unique way as $P_x(s)$, where $x \in K(p, \epsilon)$ and $s > 0$. Let y, z be any two points of $K(p, \epsilon)$, and let y^*, z^* be, respectively, their symmetric images with respect to e_1 [note that we have identified Θ with $K(p, \epsilon)$]. Then there exists a transformation f in Γ_p^0 such that $f(e_1) = e_1$, $f(y) = y^*$, $f(z) = z^*$. Since f is an isometry of E and leaves p fixed, we have, for any s, $s' \geq 0$, the relations

$$f[P_y(s)] = P_{y^*}(s), \qquad f[P_z(s')] = P_{z^*}(s').$$

This tells us that

$$d[P_y(s),\ P_z(s')] = d[P_{y^*}(s),\ P_{z^*}(s')] \qquad (s,\ s' \geq 0). \tag{2}$$

Now let us consider the mapping $h: E \longrightarrow E$ defined by $h[P_x(s)] = P_{x^*}(s)$, where $x \in K(p, \epsilon)$ and $s \geq 0$. Equality (2) tells us that this mapping h is an isometry of E. Moreover, from (1) we can see that h is of period two and that h has only two fixed points e_1 and $-e_1$ on $K(p, \epsilon)$. It follows then that h is a reflection of E about the geodesic joining p and e_1. However, our space E is two-point homogeneous so that there exists a reflection about every geodesic of E. From Schur's Theorem, we can conclude that E is either hyperbolic or euclidean. Theorem 2 is hereby proved.

6. Remarks. In all the arguments, we use only the weaker two-point homogeneity; that is, there exists a number $\delta > 0$ such that, for any four points x, x', y, y' with $d(x, x') = d(y, y') < \delta$. there exists an isometry of E carrying x, x' to y, y' respectively.

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