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# TWO THEOREMS ON METRIC SPACES

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**1. Introduction.** Let $E$ be a metric space with distance function $d$. The space $E$ is called *two-point homogeneous* if given any four points $a$, $a'$, $b$, $b'$ with $d(a,a') = d(b,b')$, there exists an isometry of $E$ carrying $a$, $a'$ to $b$, $b'$, respectively. In a recent paper [7], the author has determined all the compact and connected two-point homogeneous spaces. It is the aim of the present note to discuss the noncompact case, and prove a conjecture of Busemann which can be regarded also as a sharpening of a theorem of Birkhoff [1]. The results concerning the noncompact two-point homogeneous spaces are not as satisfactory as the results for the compact case; we have to assume certain conditions on the metric.

By a segment in a metric space $E$, we shall mean an isometric image of a closed interval with the usual metric. A metric space will be said to have the property $(L)$ if given a point $p$, there exists a neighborhood $W$ of $p$ so that each point $x$ $(\neq p)$ of $W$ can be joined to $p$ by at most one segment in $E$. The following theorems will be proved:

THEOREM 1. *Let $E$ be a finite-dimensional, finitely compact, convex metric space with property $(L)$. If $E$ is two-point homogeneous, then $E$ is homeomorphic with a manifold.*

THEOREM 2. *Let $E$ be a metric space with all the properties mentioned in Theorem* 1. *If, moreover,* $\dim E$ *is odd, then $E$ is congruent either to the euclidean space, the hyperbolic space, the elliptic space, or the spherical space.*

Our Theorem 2 justifies the conjecture of Busemann [2, p. 233] that a two-point homogeneous three dimensional S.L. space [2, p. 78] is either elliptic, hyperbolic, or euclidean. It is to be noted that Theorem 2 no longer holds if $\dim E$ is even and greater than two. The complex elliptic spaces [7] and the hyperbolic Hermitian spaces[1] [2, p. 192] serve as counter examples.

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[1]These spaces were first introduced by H. Poincaré, and then discussed by G. Fubini and E. Study. Following E. Cartan, we call these spaces the hyperbolic Hermitian spaces.

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**2. Preliminary results.** Throughout this note, by a Busemann space [2, p. 11], we shall mean a finitely compact, convex metric space such that at each point $p$, there exists a neighborhood $W$ with the following property: given any two points $x$, $y$ of $W$ and any $\epsilon > 0$, we can find a positive number $\delta < \epsilon$ for which a unique point $z$ exists so that

$$d(x, y) + d(y, z) = d(x, z), \quad d(y, z) = \delta.$$

It can easily be verified that the class of all two-point homogeneous, finitely compact, convex metric space with the property $(L)$ coincides with the class of all two-point homogeneous Busemann spaces. In the statements of our Theorems, we use the property $(L)$ instead of Busemann's axioms merely because it is, geometrically, easier to visualize.

Let $E$ be a Busemann space. We shall first see that each $d$-sphere[1] of sufficiently small radius is locally connected. In fact, let $p$ be a point of $E$. We choose $\epsilon > 0$ so small that each point $x$ with $0 < d(p, x) \leq \epsilon$ can be joined to $p$ by one and only one segment. Let $K(p, \epsilon)$ be the $d$-sphere with center $p$ and radius $\epsilon$, and $R$ the totality of points $y$ with $0 < d(p, y) < \epsilon$. Then evidently $R$ is an open set of $E$. Since $E$ is convex, $E$ must be locally connected. It follows then that $R$ is locally connected.

For each point $y$ of $K(p, \epsilon)$, we denote by $P_y(s)$ $(0 \leq s \leq \epsilon)$ the isometric representation of the segment joining $p$ to $y$. Let $J$ be the open interval $0 < s < \epsilon$. By our choice of $\epsilon$, the mapping $h: K(p, \epsilon) \times J \longrightarrow R$ defined by $h(y, s) = P_y(s)$ is a one-to-one mapping of the topological product $K(p, \epsilon) \times J$ onto $R$. Moreover, from Busemann's results [2, I., §3] concerning the convergence of geodesics, we see immediately that $h$ is bicontinuous. This tells us that $K(p, \epsilon) \times J$ and $R$ are homeomorphic. Since $R$ is locally connected, $K(p, \epsilon) \times J$, and hence $K(p, \epsilon)$, is locally connected.

**3. Proof of Theorem 1.** Let $E$ be a metric space with all the properties mentioned in Theorem 1. From the above discussions, we know that for any point $p$ of $E$, the $d$-sphere $K(p, \epsilon)$ with sufficiently small radius $\epsilon$ is locally connected. Let $\Gamma$ be the group of all isometries of $E$, and $\Gamma_p$ the totality of all those isometries which leave $p$ invariant. In $\Gamma$, we introduce the topology as defined by van Dantzig and van der Waerden [4] (in fact, this is exactly the $g$-topology of R. Arens).

[1] By a $d$-sphere we mean the totality of points equidistant from a fixed point with respect to the metric $d$. This should be distinguished from the $(n-1)$-sphere which stands for the $(n-1)$-dimensional topological sphere.

Then $\Gamma_p$ forms a compact topological group [4]. Evidently, $\Gamma_p$ is a transformation group of $K(p, \epsilon)$ in the sense of Montgomery and Zippin. From the two-point homogeneity, $\Gamma_p$ is transitive on $K(p, \epsilon)$. Taking account of the finite dimensionality and local connectedness of $K(p, \epsilon)$ and the compactness of $\Gamma_p$, we can conclude [5] that $\Gamma_p$ is a Lie group, and hence $K(p, \epsilon)$ is locally euclidean (here as well as in what follows, locally euclidean is always used in the topological sense). The set $R$, being homeomorphic with the topological product of $K(p, \epsilon)$ and the open interval $J$, must be locally euclidean as well. Hence our space $E$ is locally euclidean at each point of $R$, and hence locally euclidean at all its points. Moreover, $E$ is obviously separable and connected. It follows then that $E$ is homeomorphic with a manifold.

4. **The structure of $d$-spheres.** Before proving Theorem 2, we find it convenient to establish some more properties of the $d$-spheres.

LEMMA. *Let $E$ be a metric space satisfying all the conditions in Theorem* 2. *Then each d-sphere with sufficient small radius is homeomorphic with the* $(n-1)$-*dimensional topological sphere where* $\dim E = n$.

*Proof.* If $\dim E$ is equal to one, this is trivial. Now we shall assume that $n > 1$. Let $p$ be a point of $E$, and $\epsilon$ so small that each point $x$ with $0 < d(p,x) \leq \epsilon$ can be joined to $p$ by one and only one segment. Set $K(p, \epsilon)$ to be the $d$-sphere with center p and radius $\epsilon$, and

$$U = \{x \mid d(p, x) < \epsilon\} .$$

We shall show first that $U$ is contractible to a point. Given each point $y$ of $K(p,\epsilon)$, let us denote by $P_y(s)$ the isometric representation of the segment joining $p$ to $y$. Then the pair $(y, s)$, where $y \in K(p, \epsilon)$ and $0 \leq s < \epsilon$, can be regarded as polar coordinates of points in $U$. For any real number $t$ with $0 \leq t \leq 1$, we define

$$\phi[t, P_y(s)] = P_y(ts).$$

We see immediately that $\phi$ is a well-defined mapping of the product $I \times U$, and

$$\phi[1, P_y(s)] = P_y(s), \quad \phi(t, p) = p, \quad \phi[0, P_y(s)] = p,$$

where $I$ denotes the closed interval $\{t \mid 0 \leq t \leq 1\}$. The continuity of $\phi$ can easily be verified. Thus $\phi$ gives a contraction of $U$ into the point $p$, and thus the homotopy group $\pi_i(U)$ vanishes for each $i$.

Now let us consider the set $R = U - p$. Since $U$ is an $n$-dimensional open manifold and $n > 1$, the set $R$ is connected and has the same homotopy group $\pi_i$ as $U$ for all dimensions $i$ less than $n - 1$. Thus $\pi_i(R) = 0$, $i = 1, 2, \cdots, n-2$. On the other hand, we have shown in §1 that $R$ is homeomorphic with the topological product $K(p, \epsilon) \times J$, where $J$ denotes an open interval. It follows then that $K(p, \epsilon)$ is connected and

$$(1) \qquad \pi_i[K(p, \epsilon)] = 0, \qquad i = 1, 2, \cdots, n-2.$$

From the proof of Theorem 1, we know that $K(p, \epsilon)$ is a homogeneous space of a compact Lie group. Its connectedness and its simply-connectedness imply that it is an orientable manifold.

Since both $K(p, \epsilon)$ and $J$ are manifolds, we have

$$\dim K(p, \epsilon) + \dim J = \dim R = \dim E = n,$$

and hence $\dim K(p, \epsilon) = n - 1$. It follows immediately from (1) that $K(p, \epsilon)$ is a simply-connected homology sphere of even dimension $n - 1$. Therefore [6] $K(p, \epsilon)$ is a topological sphere. The lemma is proved.

5. **Proof of Theorem 2.** Suppose $E$ to be a metric space with all the properties mentioned in Theorem 2. If $E$ is compact, then our Theorem 2 follows as a direct consequence of [7, Theorem VI]. Thus we can assume from now on that $E$ is not compact. We shall first show that $E$ is an open S. L. space in the sense of Busemann [2, p. 78]. To show this, it suffices [3, p.173] to establish that each geodesic is congruent to a euclidean line; for this, it suffices to demonstrate that given any two distinct points $x$, $y$ and any $k > 0$, there exists a point $z$ so that

$$d(x, y) + d(y, z) = d(x, z), \qquad d(y, z) = k.$$

In fact, since $E$ is finitely compact and noncompact, $E$ cannot be bounded. There exists then a sequence of points $p_0, p_1, p_2, \cdots$ with $d(p_0, p_i)$ tending to infinity. Thus we can choose $i$ so large that $d(p_0, p_i) \geq d(x, y) + k$. Let $\tau$ be a segment joining $p_0$ to $p_i$. Evidently there exist three points $x'$, $y'$, $z'$ in $\tau$ such that

$$d(x', y') + d(y', z') = d(x', z'), \qquad d(x', y') = d(x, y), \qquad d(y', z') = k.$$

From the two-point homogeneity of $E$, there is an isometry $f$ of $E$ carrying $x'$, $y'$ to $x, y$ respectively. Then we can see immediately that the point $z = f(z')$ has all the required properties. Thus $E$ is an open S. L. space.

Let $K(p, \epsilon)$ be the $d$-sphere with center $p$ and radius $\epsilon$, and $\Gamma_p$ the group of all isometries of $E$ which leave the point $p$ invariant. From the above lemma, we know that $K(p, \epsilon)$ is an $(n-1)$-sphere and $\Gamma_p$ a compact and transitive transformation group of $K(p, \epsilon)$. Moreover, it can easily be seen that $\Gamma_p$ is effective on $K(p, \epsilon)$. In our further discussions, we shall rule out the trivial case where $\dim E = n = 1$. Thus $K(p, \epsilon)$ is connected, and the identity component $\Gamma_p^0$ of $\Gamma_p$ forms a connected, compact, transitive, and effective transformation group of $K(p, \epsilon)$. Since $n-1$ is even, it follows [6] that $\Gamma_p^0$ is either isomorphic with the rotation group $R_{n-1}$ or Cartan's exceptional group $G_2$. We shall discuss these two cases separately.

*Case A*. Suppose $\Gamma_p^0$ to be isomorphic with the group $R_{n-1}$ of all rotations of the $(n-1)$-sphere. Let us represent $K(p, \epsilon)$ by the unit sphere in a certain $n$-dimensional euclidean space, and consider $R_{n-1}$ not only as a topological group but also as a transformation group of $K(p, \epsilon)$ in the usual sense. It is well known that $\Gamma_p^0$ and $R_{n-1}$ have the same topological type, that is, there exists a homeomorphism $\phi$ of $K(p, \epsilon)$ onto itself so that

$$R_{n-1} = \phi \Gamma_p^0 \phi^{-1} = \{\phi f \phi^{-1} \mid f \in \Gamma_p^0\}.$$

Since $n$ is odd, given any point $q$ of $K(p, \epsilon)$, there exists a rotation of period two which leaves fixed *only* $q$ and its diametrically opposite point. It follows then that for each point $q$ of $K(p, \epsilon)$, we can find a transformation $f$ in $\Gamma_p^0$ such that (a) $f$ is of period two, (b) $f$ leaves $q$ fixed, and (c) $f$ has only two fixed points on $K(p, \epsilon)$. Now let $g$ be any geodesic through $p$ in $E$. It intersects $K(p, \epsilon)$ at two points, say $q$ and $q'$. We consider the transformation $f$ in $\Gamma_p^0$ having the above three properties (a), (b), and (c). Since $f$ is an isometry leaving fixed $p$ and $q$, it leaves the geodesic $g$ pointwise invariant. Moreover, this isometry $f$ cannot have any other fixed point, for otherwise $f$ would have some other fixed points on $K(p, \epsilon)$ besides $q$ and $q'$. Thus $f$ is a reflection of $E$ about $g$. Since $p$ is an arbitrary point and $g$ an arbitrary geodesic through $p$, there exists a reflection of $E$ about each geodesic. From Schur's Theorem [2, p. 181], it follows that $E$ is either hyperbolic or euclidean.

*Case B*. Suppose $\Gamma_p^0$ to be isomorphic with the exceptional group $G_2$. To discuss this case, we have to digress into a few properties of Cayley numbers. Let $1, e_i$ $(i = 1, 2, \cdots, 7)$ be the units of Cayley algebra. The multiplication rule is given by

$$e_i e_i = -1, \quad e_i e_j = -e_j e_i, \quad e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_1 e_6 = e_7,$$
$$e_2 e_5 = e_7, \quad e_2 e_4 = -e_6, \quad e_3 e_4 = e_7, \quad e_3 e_5 = e_6,$$

together with the equalities obtained by cyclic permutation of the indices. Let

$$\Theta = \left\{ \sum_{i=1}^{7} x_i e_i \mid x_i = \text{real number}, \quad \sum_{i=1}^{7} (x_i)^2 = 1 \right\}$$

be the totality of all the Cayley numbers with vanishing real part and with norm equal to unity. Evidently, $\Theta$ forms a 6-sphere, and each automorphism of the Cayley algebra carries $\Theta$ into itself. We can regard therefore the group $H$ of all automorphisms of Cayley algebra as a transformation group of $\Theta$ (the topology over $H$ is defined in the usual manner). Now $H$ acts effectively and transitively on $\Theta$. Moreover, it is known that $H$ is isomorphic with the exceptional group $G_2$.

For each $x = \sum_{i=1}^{7} x_i e_i$ in $\Theta$, we shall denote the Cayley number $x_1 - \sum_{i=2}^{7} x_i e_i$ by $x^*$, and call it the *symmetric image* of $x$ with respect to $e_1$. It is evident that

$$(1) \qquad (x^*)^* = x, \quad x^* \begin{cases} = x, & \text{if } x = \pm e_1, \\ \neq x, & \text{otherwise.} \end{cases} \qquad x \in \Theta$$

Moreover, by a direct calculation, we can show that given any two Cayley numbers $y$, $z$ in $\Theta$, there exists an automorphism $f$ in $H$ such that

$$f(e_1) = e_1, \quad f(y) = y^*, \quad f(z) = z^*.$$

It is to be noted that this $f$ depends on $y$ and $z$. There is no automorphism of Cayley algebra which carries each $x$ in $\Theta$ into its symmetric image $x^*$.

Now we can proceed to the proof of Theorem 2. Since $\Gamma_p^0$ is isomorphic with the exceptional group $G_2$, $K(p, \epsilon)$ must be six-dimensional [6]. It is known that each transitive transformation group of the 6-sphere which is isomorphic with the exceptional group $G_2$ has the same topological type as $H$.[1] Thus we can identify $\Theta$ and $K(p, \epsilon)$ in such a manner that $\Gamma_p^0$ and $H$ coincide. Let $x$ be a point of $K(p,\epsilon)$. It determines a ray $\underrightarrow{px}$, that is, the totality of points $u$ of $E$ for which either $d(x, u) + d(u, p) = d(x, p)$ or $d(u, x) + d(x, p) = d(u, p)$ [2, p. 76]. For each nonnegative number $s$, we denote by $P_x(s)$ the point $u$ on the ray $\underrightarrow{px}$ with the property that

[1] This follows as a direct consequence of [6, Lemma 6].

$d(p, u) = s$. Since $E$ is an open S. L. space, each point of $E$ other than $p$ can be represented in a unique way as $P_x(s)$, where $x \in K(p, \epsilon)$ and $s > 0$. Let $y$, $z$ be any two points of $K(p, \epsilon)$, and let $y^*$, $z^*$ be, respectively, their symmetric images with respect to $e_1$ [note that we have identified $\Theta$ with $K(p, \epsilon)$]. Then there exists a transformation $f$ in $\Gamma_p^0$ such that $f(e_1) = e_1$, $f(y) = y^*$, $f(z) = z^*$. Since $f$ is an isometry of $E$ and leaves $p$ fixed, we have, for any $s$, $s' \geq 0$, the relations

$$f[P_y(s)] = P_{y^*}(s), \qquad f[P_z(s')] = P_{z^*}(s').$$

This tells us that

$$d[P_y(s),\ P_z(s')] = d[P_{y^*}(s),\ P_{z^*}(s')] \qquad (s,\ s' \geq 0). \tag{2}$$

Now let us consider the mapping $h: E \longrightarrow E$ defined by $h[P_x(s)] = P_{x^*}(s)$, where $x \in K(p, \epsilon)$ and $s \geq 0$. Equality (2) tells us that this mapping $h$ is an isometry of $E$. Moreover, from (1) we can see that $h$ is of period two and that $h$ has only two fixed points $e_1$ and $-e_1$ on $K(p, \epsilon)$. It follows then that $h$ is a reflection of $E$ about the geodesic joining $p$ and $e_1$. However, our space $E$ is two-point homogeneous so that there exists a reflection about every geodesic of $E$. From Schur's Theorem, we can conclude that $E$ is either hyperbolic or euclidean. Theorem 2 is hereby proved.

**6. Remarks.** In all the arguments, we use only the weaker two-point homogeneity; that is, there exists a number $\delta > 0$ such that, for any four points $x$, $x'$, $y$, $y'$ with $d(x, x') = d(y, y') < \delta$. there exists an isometry of $E$ carrying $x$, $x'$ to $y$, $y'$ respectively.

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