LOOPS WITH TRANSITIVE AUTOMORPHISM GROUPS

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1. Introduction. Every automorphism of an additive loop $L$ maps 0 upon 0. The automorphism group $A(L)$ of $L$ will be called transitive if $A(L)$ is transitive on the nonzero elements of $L$. It is readily deduced from work of L. J. Paige [4] and P. T. Bateman [3] that, for every cardinal number $n$, there exists a loop $L$ of cardinal number $n$ with a transitive automorphism group. We shall demonstrate that (aside from the obvious exceptions) such a loop $L$ must be simple, that is, its only normal subloops must be 0 and $L$, if it satisfies the following ascending condition:

$$(C) \text{ If } N_1 \subset N_2 \subset N_3 \subset \cdots \text{ is an ascending chain of normal subloops of the loop } L, \text{ there exists an integer } i \text{ such that } N_i = N_{i+1}. $$

2. Theorem. We shall establish the following result.

Theorem 1. An additive loop $L$ which satisfies (C) and has a transitive automorphism group is either (i) a simple loop or (ii) a finite abelian $p$-group of type $(p, p, \cdots, p)$.

Proof. For each nonzero $a$ of $L$, denote by $M(a)$ the smallest normal subloop of $L$ which contains $a$.

(1) The subloop $M(a)$ has a transitive automorphism group and is a minimal normal subloop of $L$. If $b \neq 0$ is in $M(a)$, then there exists $\theta \in A(L)$ such that $a^\theta = b$. Since $\theta$ maps normal subloops upon normal subloops, we have $M(a) \theta = M(b)$. Since $b \in M(a)$, it follows that $M(b) \subset M(a)$. If $\phi = \theta^{-1}$, then $M(a) = M(b) \phi \subset M(a) \phi$, and, by induction, $M(a) \subset M(a) \phi \subset M(a) \phi^2 \subset \cdots$. In view of (C), we have $M(a) \phi^i = M(a) \phi^{i+1}$ for some integer $i$. Since $\phi$ is an automorphism of $L$, it follows that $M(a) = M(a) \phi^{-1} = M(a) \theta = M(b)$. Hence $\theta$ induces an automorphism of $M(a)$. This is enough to prove (1).

1 Readers unfamiliar with loop theory will get the sense of the paper if they read group in place of loop. The necessary loop theory will be found in Baer [1,2].

(2) If \( N \) is a normal subloop of \( L \), then \( N \cap M(a) = 0 \) or \( M(a) \). This follows from the minimality of \( M(a) \).

(3) The loop \( L \) is a direct sum of a finite number \( r \) of isomorphic simple subloops \( M_i \); that is, \( L = M_1 \oplus M_2 \oplus \cdots \oplus M_r \). If \( a, b \) are nonzero elements of \( L \), there exists \( \theta \in A(L) \) such that \( a\theta = b \). Then \( M(a)\theta = M(b) \), showing that all the minimal normal subloops \( M(a) \) are isomorphic. If \( a_1 \) is an arbitrary nonzero element of \( L \), define \( M_1 = M(a_1) \). Now suppose that \( M_i = M(a_i) \) has been defined, for \( i = 1, 2, \ldots, s \), such that the (normal) subloop \( N_s \) generated by the \( M_i \) is the direct sum \( N_s = M_1 \oplus \cdots \oplus M_s \). Write \( t = s + 1 \). If there exists a nonzero element \( a_t \) of \( L \) which is not in \( N_s \), define \( M_t = M(a_t) \). Then \( N_s \cap M_t = 0 \), by (2), and hence \( N_t = N_s \oplus M_t = M_1 \oplus \cdots \oplus M_t \). In view of (C), the strictly increasing chain \( N_1 \subseteq N_2 \subseteq \cdots \) must be finite. Therefore \( L = N_r \) for some integer \( r \). If \( M' \) is a normal subloop of \( M_1 \), then \( M' \) is normal in \( L \), by virtue of the direct decomposition. Hence, by (1), each \( M_i \) is simple. This proves (3).

The center \( Z(L) \) of a loop \( L \) is a characteristic subloop and an abelian group. In view of (1), either \( Z(M_i) = 0 \) or \( Z(M_i) = M_i \). Hence, by (3), either (i) \( Z(L) = 0 \) or (ii) \( L \) is a direct sum of isomorphic simple abelian groups. Since a simple abelian group is cyclic of prime order \( p \), (ii) implies (ii) of Theorem 1. (Conversely, every finite abelian \( p \)-group of type \( (p, p, \ldots, p) \) satisfies the hypotheses of the theorem.) In the case (i), assume \( r > 1 \) in (3). Since \( Z(L) = 0 \), the decomposition (3) is unique. However, the nonzero element \( c = a_1 + a_2 \) is in \( M_1 \oplus M_2 \) but not in any of the \( M_i \). Yet the proof of (3) shows that \( M(c) \) could be chosen as the first factor in the direct decomposition of \( L \), a contradiction. Therefore \( r = 1 \), and we have (i). This completes the proof of Theorem 1.

As the following (trivial) theorem shows, simple loops need not have transitive automorphism groups:

**Theorem 2.** A finite simple group \( G \neq 0 \) with a transitive automorphism group is necessarily cyclic of prime order.

**Proof.** Every nonzero element of \( G \) has the same order \( p \), necessarily prime. Thus \( G \) is a \( p \)-group, \( Z(G) \neq 0, Z(G) = G \), and \( G \) is cyclic of order \( p \).

3. **Remarks.** The author does not know whether finiteness is necessary for the conclusion of Theorem 2.
The following is the nonabelian loop $L$ of lowest order with a transitive automorphism group; it is readily verified that $A(L)$ is the (alternating) group of order 12 generated by $(12)(34)$ and $(123)$:

\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 4 \\
2 & 2 & 4 & 0 & 1 \\
3 & 3 & 2 & 4 & 0 \\
4 & 4 & 3 & 1 & 2 \\
\end{array}
\]

References

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