QUASI-CONVEXITY AND THE LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

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1. Introduction. We are concerned in this paper with integrals of the form

\begin{equation}
I(z,D) = \int_D f[x,z^i(x), z^i_{x\alpha}(x)] \, dx,
\end{equation}

where

\[x = (x^1, \ldots, x^\nu), \quad z = (z^1, \ldots, z^N), \quad p = p^i_{\alpha},\]

\[(i = 1, \ldots, N; \quad \alpha = 1, \ldots, \nu),\]

\(f(x,z,p)\) is continuous in its arguments, and \(D\) is a bounded domain.

The object of the paper is to discuss necessary and sufficient conditions on the function \(f\) for the integral \(I\) to be lower semicontinuous with respect to various types of convergence of the vector functions \(z\). Because of the success of the "direct methods" in the Calculus of Variations, many writers have shown that certain integrals are lower semicontinuous. However, the writer knows of no paper in which a necessary condition for lower semicontinuity was discussed, although such a condition is very easy to obtain (see Theorem 2.1).

In \$2\), a general condition called "quasi-convexity" (see Definition 2.2) on the behavior of \(f\) as a function of \(p\) is obtained which is both necessary and sufficient for the lower semicontinuity of \(I\) with respect to the type of convergence given in Definition 2.1. This condition is that any linear function furnish the absolute minimum to \(I(z,D)\) among all Lipschitzian (see below) functions which coincide with it on \(D^*, D\) being any bounded domain and \(D^*\) its boundary; here, of course, we consider \(f\) to be a function of \(p\) only. Section 3 discusses cases involving more general types of convergence and gives an existence theorem. In \$4\), it is shown that if \(f(p)\) is continuous and quasi-convex, then it satisfies a certain generalized Weierstrass condition which reduces to the ordinary one (for the case at hand) when \(f\) is of class \(C^1\); this is, in turn, seen to be equivalent to the Legendre-Hadamard condition (see (4.8)) (quasi-regularity in its general form) when \(f\) is of class \(C^1\). In \$5\), a general sufficient

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condition for quasi-convexity is proved and the necessary condition of §4 is seen to be sufficient when \( f \) is either a quadratic form in the \( p^i \) or is the integrand of a parametric problem with \( N = \nu + 1 \). The view of Terpstra’s negative result [5] that even the strong Legendre-Hadamard condition (\( > 0 \)) does not necessarily imply the existence of an alternating form \( C^{i\beta}_{ij} p^i p^\beta \) (\( C^{i\alpha}_{ij} = - C^{\alpha i}_{ij} \), and so on) such that

\[
(1.2) \quad f(p) + C^{i\beta}_{ij} p^i p^\beta = \left( a^{\alpha \beta}_{ij} + C^{\alpha i}_{ij} \right) p^i p^\beta
\]
is positive definite when \( \nu > 2 \), it would seem that there is still a wide gap in the general case between the necessary and sufficient conditions for quasi-convexity which the writer has obtained. In fact, after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves \( f \) and only a finite number of its derivatives, and which is both necessary and sufficient for quasi-convexity in the general case.

In (1.2), we have used the usual tensor summation convention, and will continue to use it throughout the paper; unless otherwise specified, the Greek letters will run from 1 to \( \nu \) and the Latin letters from 1 to \( N \).

We shall denote the sum and difference of vectors of the various sorts \((x, z, p, \text{ and so on})\) in the usual way. We shall define

\[
|x| = (x^a x^a)^{1/2}, \quad |z| = (z^i z^i)^{1/2}, \quad |p| = \left( p^i p^i \right)^{1/2}.
\]

If \( \zeta(x) \) is a vector function with derivatives, \( \pi(x) \) will denote the vector function \( \pi^i_a(x) = \zeta^i_{xa}(x) \); similar notations involving other letters will be introduced as the occasion demands.

All integrals are Lebesgue integrals, frequently of vector functions. It is sometimes desirable to consider the behavior of a function \( z(x) \) with respect to a particular variable \( x^a \) or to the \( \nu - 1 \) variables \((x^1, \ldots, x^{a-1}, x^{a+1}, \ldots, x^\nu)\). In such a case, we write \( x'_a \) for \((x^1, \ldots, x^{a-1}, x^{a+1}, \ldots, x^\nu)\), \((x'_a, x^a)\) for \( x \) and so on. It is also convenient to write the boundary integrals

\[
\int_{D^*} A_a(x) \, dx'_a,
\]

where each \( A_a(x) \) may be a vector \( A^i_a(x) \) and the boundary \( D^* \) of the domain is sufficiently regular; such an integral is to be regarded as a Lebesgue-Stieltjes integral with respect to the set function \( x'_a(e) \) on \( D^* \) chosen so that Green’s theorem

\[
\int_{D^*} \zeta^i \, dx'_a = \int_D \zeta^i_{xa} \, dx
\]
holds. The closure of a set $E$ will be denoted by $\overline{E}$.

Ordinary functions of class $\mathcal{C}_s$, $\mathcal{C}_s'$, $\mathcal{C}_s''$, and so on, $s \geq 1$, have been discussed at length in the papers [1] and [2]; the extension to vector functions is trivial. We define the integrals $\overline{D}_s(z, G)$ and $D_s(z, G)$ by

$$
\overline{D}_s(z, G) = \int_G |z(x)|^s \, dx + D_s(z, G),
$$

$$
D_s(z, G) = \int_G \left[ \sum_{\alpha} x_\alpha (x) \right]^{s/2} \, dx.
$$

Each function $z$ of class $\mathcal{C}_s$ is equivalent to a function $\overline{z}$ defined uniquely almost everywhere as that number such that the Lebesgue derivative of the set function

$$
\int_e |z(x) - \overline{z}(x_0)|^s \, dx
$$

is zero at $x_0$; $\overline{z}$ is supposed to be defined at every point $x_0$ where such a number exists; $\overline{z}$ is of class $\mathcal{C}_s'$ (see [1] and [2]) and is also of class $\mathcal{C}_s$ in any coordinate system related to the original by a regular Lipschitzian transformation (cf. [2], Theorem 6.3; the $\overline{z}$ there used has a slightly different definition from the present one but the present theorem has been proved for vectors $z$ with values in a Riemannian manifold in [4], Lemma 2.3 and Theorem 2.5).

A function $z$ is said to satisfy a (uniform) Lipschitz condition with coefficient $M$ on a set $S$ if and only if

$$
|z(x_1) - z(x_2)| \leq M |x_1 - x_2|, x_1, x_2 \in S.
$$

A function is Lipschitzian if it satisfies a Lipschitz condition.

If $g(y)$, $y = (y^1, \cdots, y^n)$, is summable on a domain $D$, we define the $h$-average function $g_h$ by

$$
g_h(y) = (2h)^{-n} \int_{y-h}^{y+h} g(\eta) \, d\eta,
$$

if $g$ is summable then $g_h$ is continuous where defined; if $g$ is continuous on $D$ then $g_h$ is of class $C'$ and $g_h$ tends uniformly to $g$ on each bounded closed set interior to $D$; if $g$ is of class $\mathcal{C}_s$ on $D$ then $g_h$ tends strongly in $\mathcal{C}_s$ to $g$ on each domain $G$ with $G \subset D$ (see [1], Lemma 5.1).

A form

$$
\sum_{\alpha_1, \cdots, \alpha_\mu} \pi^{a_1 \cdots a_\mu}_{i_1 \cdots i_\mu} \left( \mu \leq \nu, 1 \leq \alpha_\gamma \leq \nu, 1 \leq i_\gamma \leq N, \gamma = 1, \cdots, \mu \right),
$$

is called alternating if and only if the $C$'s satisfy the obvious symmetry requirements and also the antisymmetry condition that
\[ C_{i_1, \ldots, i_\mu}^{\beta_1, \ldots, \beta_\mu} = \pm C_{i_1, \ldots, i_\mu}^{\alpha_1, \ldots, \alpha_\mu} \]

according as \((\beta_1, \ldots, \beta_\mu)\) is an even or odd permutation of the indices \((\alpha_1, \ldots, \alpha_\mu)\); if \(\zeta(x)\) is a vector function, then

\[
\mu! \frac{C_{i_1, \ldots, i_\mu}^{\alpha_1, \ldots, \alpha_\mu}}{\pi^{i_1}_{\alpha_1} \cdots \pi^{i_\mu}_{\alpha_\mu}} (x) = \frac{\partial (\zeta^{i_1}, \ldots, \zeta^{i_\mu})}{\partial (x^{\alpha_1}, \ldots, x^{\alpha_\mu})},
\]

the fractions on the right denoting Jacobians.

2. A necessary and sufficient condition for lower-semicontinuity. We begin with some definitions.

**Definition 2.1.** For the purposes of this section, we say that the vector functions \(z_n\) tend to the vector function \(z\) on the domain \(D\) if and only if the \(z_n\) and \(z\) all satisfy a uniform Lipschitz condition on \(D\), independent of \(n\), and the \(z_n\) tend uniformly to \(z\) on \(D\). We shall write \(z_n \rightarrow z\) to denote this type of convergence.

**Definition 2.2.** A function \(f(p^i_a)\) is said to be quasi-convex if and only if

\[
\int_D f[p^i_a(x) + \pi^i_a(x)] \, dx \geq f(p) \cdot m(D),
\]

for each constant \(p\), each domain \(D\), and each vector function \(\zeta\) which satisfies a uniform Lipschitz condition on \(D\) and vanishes on \(D^*\).

We shall show in this section that the integral \(I(z, D)\) is lower semicontinuous with respect to the type of convergence specified in Definition 2.1 on each bounded domain \(D\) if and only if \(f(x, z, p)\) is quasi-convex in \(p\) for each fixed \((x, z)\).

**Theorem 2.1.** Suppose \(I(z, D)\) is lower semicontinuous with respect to the type of convergence indicated on every region \(D\). Then \(f\) is quasi-convex in \(p\) for each fixed \((x, z)\).

**Proof.** Let \(x_0\) be any point, \(R\) be the cell \(x_0 \leq x^i \leq x_0^i + h\), \(Q\) be the cell \(0 \leq x^i \leq 1\), and \(\zeta\) be any function of class \(C^r\) and periodic in each \(x^i\) with period 1. Let \(z_0\) be any function of class \(C^r\) on \(R\).

For each \(n\), define \(\zeta_n(x)\) on \(R\) by
\[ \zeta^i_n(x) = n^{-1} h \zeta^i \left[ nh^{-1} (x - x_0) \right]. \]

Then
\[ \zeta^i_{n \cdot x^a}(x) = \zeta^i_{x^a} \left[ nh^{-1} (x - x_0) \right] \]

and
\[
I(z_0 + \zeta_n, R) = \int_R f \left\{ x, z_0^i(x) + \zeta_n^i(x), p_0^i(x) + \pi_i \left[ nh^{-1} (x - x_0) \right] \right\} dx
\]

\[
= \sum_a \int_{R^a} \left( f \left\{ x, z_0^i(x) + \zeta_n^i(x), p_0^i(x) + \pi_i \left[ nh^{-1} (x - x_0) \right] \right\} - f \left\{ x_a, z_0^i(x_a), p_0^i(x_a) + \pi_i \left[ nh^{-1} (x - x_0) \right] \right\} \right) dx
\]

\[
+ \sum_a (n^{-1} h)^{\nu} \int_Q f \left\{ x_a, z_0^i(x_a), p_0^i(x_a) + \pi_i (\xi) \right\} d\xi
\]

where
\[
\alpha = (\alpha_1, \ldots, \alpha_\nu), R_\alpha = R_{\alpha_1}, \ldots, \alpha_\nu, n^{-1} (\alpha_\beta - 1) \leq x_\beta \leq n^{-1} \alpha_\beta,
\]
\[
x_\alpha = \left( x_\alpha^1, \ldots, x_\alpha^{\nu} \right), x_\alpha^1, \ldots, x_\alpha^{\nu} = n^{-1} (\alpha_\beta - 1), \beta = 1, \ldots, \nu.
\]

As \( n \to \infty \), we see, since \( f \) is uniformly continuous on any bounded part of space, \( \zeta_n(x) \) tends uniformly to zero, and the \( \pi_i \) are bounded, that
\[
\lim_{n \to \infty} I(z_0 + \zeta_n, R) = \int_R \left[ \int_Q f(x, z_0(x), p_0(x) + \pi (\xi)) d\xi \right] dx.
\]

From the lower semicontinuity of \( I \), we must have
\[
\int_R \left\{ \int_Q f(x, z_0(x), p_0(x) + \pi (\xi)) d\xi \right\} dx \geq \int_R f(x, z_0(x), p_0(x)) dx.
\]

Now, let \( x_0, z_0, \) and \( p_0 \) be any constant vectors. By letting
\[
z_0(x) = z_0 + p_0 \alpha (x^a - x_0^a),
\]

dividing by \( h^{\nu} \) and letting \( h \to 0 \), we obtain
\[
\int_Q f(x_0, z_0, p_0 + \pi (\xi)) d\xi \geq f(x_0, z_0, p_0).
\]
By approximations, we can extend this to all $\zeta$ which satisfy a uniform Lipschitz condition over the whole space and are periodic of period 1 in each $x^\alpha$.

Now, let $D$ be a bounded domain and suppose $\zeta$ satisfies a uniform Lipschitz condition on $D$ and vanishes on $D^*$. Let $R$ be a hypercube of edge $h$, with edges parallel to the axes which contains $D$. Extend $\zeta$ to the whole space by first defining it to be zero on $\overline{R} - D$ and then extending it to be periodic of period $h$ in each variable. Then a simple change of function and variable reduces $R$ to $Q$ and establishes the result.

**Lemma 2.1.** Suppose $R$ is a cell with edges $(2h^1), \ldots, (2h^\nu)$ and center $x_0$. Let $h$ be the smallest $h^\alpha$. Suppose also that $0 < k < h$, that $\zeta^*$ satisfies a uniform Lipschitz condition with coefficient $M \geq 1$ on $R^*$, and suppose

$$|\zeta^*(x)| \leq k, \ x \in R^*.$$  

Then there is a function $\zeta$ on $R$ which satisfies a Lipschitz condition with coefficient $M$ on $R$, coincides with $\zeta^*$ on $R^*$, and is zero except on a set of measure at most

$$m(R) \cdot [1 - (1 - h^{-1}k)^\nu].$$

**Proof.** Let $R_1$ be the cell with center at $x_0$ and edges $2(h^\alpha - k)$, $\alpha = 1, \ldots, \nu$. Then, since $h = \min h^\alpha$, we have

$$m(R_1) \geq m(R) \cdot (1 - h^{-1}k)^\nu.$$  

Define $\zeta_1 = 0$ on $\overline{R_1}$ and equal to $\zeta^*$ on $R^*$. Then

$$|\zeta_1(x_1) - \zeta_1(x_2)| \leq |x_1 - x_2| \quad \text{if } x_1 \subset \overline{R_1}, \ x_2 \subset R^*.$$  

Thus $\zeta_1$ satisfies a uniform Lipschitz condition with coefficient $M$ on $\overline{R_1} \cup R^*$. By a well known theorem, there exists an extension of $\zeta_1$ to $\overline{R}$ (the whole space in fact) which satisfies the same Lipschitz condition.

**Lemma 2.2.** Suppose the vectors $\zeta_n \to 0$ (in our sense) on $\overline{R}$ and suppose $f$ is quasi-convex in $p$. Then if $p_0$ is a constant vector we have

$$m(R) f(p_0) \leq \liminf_{n \to \infty} \int_R f[p_0 + \pi_n(x)] \, dx.$$  

**Proof.** For all sufficiently large $n$, we have $k_n < h$, and $k_n \to 0$, $k_n$ being the maximum of $|\zeta_n(x)|$ for $x \in R^*$. For each $n$ for which $k_n < h$, let $\eta_n$ be the
function of the preceding lemma which coincides on $R^*$ with $\zeta_n$, and let $\omega_n = \zeta_n - \eta_n$. Then if each $\zeta_n$ satisfies a uniform Lipschitz condition with coefficient $M \geq 1$ on $R$, then each $\eta_n$ and $\omega_n$ satisfies one with coefficient $M$ and $2M$, respectively. Moreover, each derivative $\eta_n^{i^*}$ is uniformly bounded and $\eta_n^{i^*} \to 0$ almost everywhere. Since $f$ is uniformly continuous on any bounded portion of $p$-space, we see that

$$\lim_{n \to \infty} \int_R |f(p_n^{i^*} + \eta_n^{i^*} + \omega_n^{i^*}) - f(p_0^{i^*} + \omega_n^{i^*})| \, dx = 0.$$ 

But the result then follows, since, for each $n$, we have

$$\int_R f(p_n^{i^*} + \omega_n^{i^*}) \, dx \geq m(R) f(p_0),$$

because of the quasi-convexity of the function $f$.

**Theorem 2.2.** Suppose $f$ is continuous in $(x,z,p)$ for all $(x,z,p)$ and is quasi-convex in $p$ for each $(x,z)$. Suppose also that $z_n \to z_0$ on the bounded domain $D$. Then

$$I(z_0,D) \leq \liminf_{n \to \infty} I(z_n,D).$$

**Proof.** Let $\epsilon$ be any positive number. For each positive integer $k$, let $D_k$ consist of all the hypercubes of edge $2^{-k}$ whose faces lie along hyperplanes $x^a = 2^{-k} i^a$ (each $i^a$ an integer) which lie in $D$. Since all the points $[x,z_0(x),p_0(x)]$ and $[x,z_n(x),p_n(x)]$ for $x \in D$ lie in a bounded portion of $(x,z,p)$ space, we may choose $k_1$ so large that

$$\int_{D-D_{k_1}} |f(x,z_n,p_n)| \, dx < \epsilon/5, \quad \int_{D-D_{k_1}} |f(x,z_0,p_0)| \, dx < \epsilon/5$$

for all $n$.

Let the hypercubes of $D_{k_1}$ be $R_1, \ldots, R_N$. For each $k \geq k_1$, let $R_{k_1}$, $i = 1, \ldots, N \cdot 2^{v(k-k_1)}$, be all the hypercubes of side $2^{-k}$ described above which lie in $D_{k_1}$. For each such $k$, define $z_k^*(x), x_k^*(x), p_k^*(x)$ on $D_{k_1}$ by

$$z_k^*(x) = [m(R_{k_1})]^{-1} \int_{R_{k_1}} x \, dx, \quad x_k^*(x) = [m(R_{k_1})]^{-1} \int_{R_{k_1}} z_0(x) \, dx,$$

$$p_k^*(x) = [m(R_{k_1})]^{-1} \int_{R_{k_1}} p_0(x) \, dx$$

and

$$r_k(x) = \left\{ |x_k^*(x) - x|^2 + |z_k^*(x) - z_0(x)|^2 + |p_k^*(x) - p_0(x)|^2 \right\}^{1/2},$$

(2.2)
where $x \in R_{k_i}$. Let $\zeta_n(x) = z_n(x) - z_0(x)$, $\pi_n(x) = p_n(x) - p_0(x)$. Then, on $D_{k_1}$,

$$
\begin{align*}
&f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_0(x)] \\
&= \left\{ f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)] \right\} \\
&\quad + \left\{ f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)] \right\} \\
&\quad - \left\{ f[x, z_0(x), p_0(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x)] \right\} \\
&\quad + \left\{ f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)] - f[x^*_k(x), p^*_k(x), p^*_k(x)] \right\}.
\end{align*}
$$

(2.3)

Now, all the arguments of $f$ occurring in (2.3) for $x \in D_{k_1}$ lie in a bounded closed cell in $(x, z, p)$-space over which $f$ is uniformly continuous. Let

$$
\varepsilon(p) = \max |f(x', z', p') - f(x'', z'', p'')|, \quad \rho \geq 0
$$

for all $(x', z', p')$ and $(x'', z'', p'')$ in this cell with

$$
|x' - x''|^2 + |z' - z''|^2 + |p' - p''|^2 \leq \rho^2.
$$

then $\varepsilon(p)$ is continuous for $\rho \geq 0$ with $\varepsilon(0) = 0$. Then, for each $n$ and each $k \geq k_1$, we have

$$
\begin{align*}
&|f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]| \leq \varepsilon(|z_n(x) - z_0(x)|), \\
&|f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)]| \leq \varepsilon[r_k(x)], \\
&|f[x, z_0(x), p_0(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x)]| \leq \varepsilon[r_k(x)].
\end{align*}
$$

Now, the $r_k(x)$ are uniformly bounded on $D_{k_1}$ and tend to zero almost everywhere on $D_{k_1}$. Hence we may choose a $k \geq k_1$ so large that

$$
\begin{align*}
&\int_{D_{k_1}} |f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)]| \, dx < \varepsilon/5, \\
&(2.4) \\
&\int_{D_{k_1}} |f[x, z_0(x), p_0(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x)]| \, dx < \varepsilon/5,
\end{align*}
$$

for all $n$. Since $z_n$ converges uniformly to $z_0$, there is an $n_1$ such that

$$
\begin{align*}
&(2.5) \\
&\int_{D_{k_1}} |f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]| \, dx < \varepsilon/5, \quad n > n_1.
\end{align*}
$$
Finally, since $x_k^*(x)$, and so on, are constant on each $R_{ki}$, and $f$ is quasi-convex, we conclude from the previous lemma that

$$\liminf_{n \to \infty} \int_{D_{ki}} \left\{ f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)] \right\} dx \geq 0.$$  

Using (2.3)-(2.5) and the above inequality, we see that

$$\liminf_{n \to \infty} I(z_n, D) \geq I(z_0, D) - \epsilon.$$  

Since $\epsilon$ is any positive number, the result follows.

3. Lower semicontinuity and weak convergence in $\frac{\mathbb{B}}{s}$ ($s \geq 1$). In this section, we discuss additional conditions which, with the quasi-convexity of $f$ in $p$, are sufficient to guarantee the lower semicontinuity of $I(z, D)$ with respect to weak convergence in $\frac{\mathbb{B}}{s}$ on $D$.

**Definition 3.1.** Suppose $\zeta$ is of class $\frac{\mathbb{B}}{s}$ on the bounded domain $D$ and suppose $R$ is a cell with $\overline{R} \subset D$. Then $\zeta$ is said to be strongly of class $\frac{\mathbb{B}}{s}$ on $R^*$ if and only if $\overline{R}$ is of class $\frac{\mathbb{B}}{s}$ in $x_\alpha$ on each face $x_\alpha = \text{const.}$ of $R^*$ and there is a sequence $\zeta_n$ of class $C^\alpha$ on $\overline{R}$ such that

$$D_s (\zeta_n - \zeta, R) \to 0, \quad D_s (\zeta_n - \zeta, R^*) \to 0.$$  

**Lemma 3.1.** Suppose $\zeta$ is of class $\frac{\mathbb{B}}{s}$ ($s \geq 1$) on the bounded domain $D$. For each $\alpha$, $1 \leq \alpha \leq \nu$, let $(a^\alpha, b^\alpha)$ be the open interval projection of $D$ on the $x_\alpha$-axis. Then there exist sets $Z^\alpha$ of measure zero such that if $R^n = [C, D]$ with $C^\alpha, D^\alpha$ rational and $R = (a^\alpha, b^\alpha)$ is any closed cell in $D$ with

$$c^\alpha \subseteq (a^\alpha, b^\alpha) - Z^\alpha, \quad d^\alpha \subseteq (a^\alpha, b^\alpha) - Z^\alpha \quad (\alpha = 1, \ldots, \nu),$$

then $\zeta$ is strongly of class $\frac{\mathbb{B}}{s}$ on $R^*$.

**Proof.** Let $R'$ be any rational cell in $D$ (that is, $R' = [C, D]$ with $C^\alpha, D^\alpha$ rational). In [1], Lemma 5.1, we have seen that if $\zeta$ is of class $\frac{\mathbb{B}}{s}$ on $D$, then

$$(3.1) \quad \lim_{h \to 0} D_s (\zeta_h - \zeta, R) = 0.$$  

For each $\alpha$, define

$$\phi^\alpha_h (x^\alpha, R') = \int_{C^\alpha}^{D^\alpha} \left\{ \left| \zeta_h - \zeta \right|^s + \left[ \sum_{\beta=1}^{\nu} \left| \zeta_{x^\beta} - \zeta_{x^\beta} \right|^2 \right]^{s/2} \right\} dx^\alpha.$$
Since $\zeta$ is obviously of class $\frac{\psi}{s}$ in $x_\alpha'$ for almost all $x_\alpha$ on $[C_\alpha, D_\alpha]$, $\phi_h^\alpha (x_\alpha, R')$ is defined for almost all $x_\alpha$ and
\[
\lim_{h \to 0} \int_{C_\alpha} D^\alpha \phi_h^\alpha (x_\alpha, R') \, dx_\alpha = 0.
\]

By arranging the rational cells $R'$ in some order and choosing successive subsequences, we may choose (on account of (3.1)) a final sequence $h_n \to 0$ such that $\phi_h^\alpha (x_\alpha, R') \to 0$ and $\zeta$ is of class $\frac{\psi}{s}$ in $x_\alpha'$ on $[C_\alpha, D_\alpha]$ for each $x_\alpha$ not in a set $Z^\alpha (R')$ of measure zero ($\alpha = 1, \ldots, \nu$). Now let
\[
Z^\alpha = U Z^\alpha (R');
\]
then
\[
m(Z^\alpha) = 0 \quad (\alpha = 1, \ldots, \nu).
\]

Now suppose $R$ is one of the cells described in the lemma. Then it lies in some rational cell $R'$ and we may take $\zeta_n = \zeta_{h_n}^\alpha$.

**Lemma 3.2.** Suppose $R$ is a cell with edges $(2h^1), \ldots, (2h^\nu)$ and center $x_0$. Let
\[
h = \min_{1 \leq \alpha \leq \nu} h_\alpha, \quad K = h^{-1} (h^a h^a)^{1/2},
\]
Suppose also that $0 < k < h$, that $\zeta^*$ is of class $\frac{\psi}{s}$ on an open domain containing $\overline{R}$ in its interior, and that $\zeta^*$ is strongly of class $\frac{s}{s}$ on $R^*$ with
\[
\int_{R^*} |\zeta^*|^s \, dS \leq k^s, \quad D_s (\zeta^*, R^*) \leq M^s \quad (s \geq 1).
\]
Then there is a function $\zeta$ of class $\frac{\psi}{s}$ on $R$ which coincides with $\zeta^*$ on $R^*$, is zero except on a set of measure
\[
m(R) \cdot [1 - (1 - h^{-1} k)^\nu],
\]
and satisfies
\[
D_s (\zeta, R) \leq \tau_s h^{-1} k (1 + K^s M^s), \quad \tau_s = \begin{cases} 2^{s/2} & (s \leq 2), \\ 2^s - 1 & (s \geq 2). \end{cases}
\]

**Proof.** For each $x \in \overline{R}$, $x \neq x_0$, let $x^*(x)$ be the intersection of the ray $\overrightarrow{x_0 x}$ with $\overline{R}^*$, and for each $x \in \overline{R}$ define
\[
r(x) = \begin{cases} 0 & (x = x_0), \\ \frac{1}{|x^*(x) - x_0|^{-1} \cdot |x - x_0|} & (x \neq x_0). \end{cases}
\]
Let $\Pi^+_a$ be the pyramid in $\overline{R}$ with vertex $x_0$ and base the face $F^+_a$ where
On the pyramid $\Pi^+_\nu$, introduce coordinates $\xi_1, \ldots, \xi^{\nu-1}, r$ by

$$x^\nu = x_0^\nu + rh^\nu, \quad x^\gamma = x_0^\gamma + r\xi^\gamma \quad (0 \leq r \leq 1, \gamma = 1, \ldots, \nu - 1).$$

Then, if $r$ and $\xi^\gamma$ are considered as functions of $x$, we have

$$r(x) = r, x^*(x) = [\xi^1(x) + x_0^1, \ldots, \xi^{\nu-1}(x) + x_0^{\nu-1}, h^\nu + x_0^\nu].$$

Similar coordinate systems may be set up on each of the other $\Pi^+_\alpha$.

Define

$$\phi(r) = \begin{cases} 0 & (0 \leq r \leq 1 - Kh^{-1}), \\ kh^{-1}(r - 1 + Kh^{-1}) & (1 - Kh^{-1} \leq r \leq 1). \end{cases}$$

Choose a sequence $\zeta^*_n$ satisfying the conditions of Definition 3.1; and for each $n$, define

$$\zeta_n(x) = \phi [r(x)] \cdot \zeta^*_n [x^*(x)].$$

Then each $\zeta_n(x)$ is of class $D'$ on $\overline{R}$.

We now compute the derivatives of $\zeta_n$ on each pyramid $\Pi^+_\alpha$ taking $\Pi^+_{\nu}$ as an example. Then

$$\xi_{nx}^\gamma = r^{-1} \phi'(r) \xi^*_n \xi^\gamma \quad (1 \leq \gamma \leq \nu - 1),$$

$$\xi_{nx}^\nu = (h^\nu)^{-1} \phi'(r) \xi^*_n - (h^\nu)^{-1} r \phi'(r) \xi^\nu \xi^*_n \xi^\gamma \quad (\gamma \text{ summed from 1 to } \nu - 1).$$

Then, since $r^{-1} \phi'(r) \leq 1$ and $\phi'(r) = k^{-1} h$ for $1 - Kh^{-1} \leq r \leq 1$,

$$|\pi_n(x)|^2 \leq \left(\xi^*_n \xi^\nu \xi^*_{n \xi^\gamma} + 2k^{-2} |\zeta_n^*|^2 + 2(h^\nu)^{-2} (\xi^\nu \xi^*_{n \xi^\gamma})(\xi^*_n \xi^\gamma \xi_{n \xi^i})\right)$$

$$\leq 2 \left[ k^{-2} |\zeta_n^*|^2 + K^2 (\xi^*_n \xi^\nu \xi^*_{n \xi^i}) \right] \quad (n \text{ not summed}).$$

Using the inequality

$$(a^2 + b^2)^{s/2} \leq \sigma_s (|a|^s + |b|^s), \quad \sigma_s \leq \begin{cases} 1 & (s \leq 2) \\ 2(s-2)/2 & (s > 2) \end{cases},$$

$$x^\alpha = x_0^\alpha + h^\alpha.$$
we obtain
\[
D_s (\zeta_n, \Pi_\nu^\perp) \leq \tau_s \int_{1-kh}^1 r^{\nu-1} \, dr \int_{F_\nu^+} \left[ k^{-s} |\zeta_n^*|^s + K^s \left( \zeta_n^* \zeta_n^* \right)^{s/2} \right] ds
\]
\[
\leq \tau_s h^{-1} k \left[ k^{-s} \int_{F_\nu^+} |\zeta_n^*|^s \, ds + K^s D_s (\zeta_n^*, F_\nu^+) \right].
\]

Also
\[
\int_{\Pi_\nu^+} |\zeta_n|^s \, dx = \int_{1-kh}^1 r^{\nu-1} \phi_s (r) \, dr \int_{F_\nu^+} |\zeta_n^*|^s \, dS
\]
\[
\leq h^{-1} k \int_{F_\nu^+} |\zeta_n^*|^s \, dS.
\]

Adding these results for all the \( \Pi_{\alpha}^\perp \), we obtain the result for each \( n \); and also \( D_s (\zeta_n^*, R) \) is uniformly bounded. Thus, we may extract a subsequence which tends weakly in \( \mathbb{V}_s \) to some function \( \zeta \) of class \( \mathbb{V}_s \) on \( R \). Since each \( \zeta_n = \zeta_n^* \) on \( R^* \), \( \zeta_n^* \) tends strongly in \( L_s \) to \( \zeta^* \) on \( R^* \), we see from [2], Theorem 8.5, that \( \zeta = \zeta^* \) on \( R^* \). From the lower semicontinuity of \( D_s \) (see [2], Theorem 8.2), the result follows.

**Lemma 3.3.** Suppose \( f \) is quasi-convex and of class \( C^* \) for all \( p \), and suppose for all \( p \) that
\[
\sum_{i, \alpha} \left( f_{, i} \right)^2 \leq K^2 \left( |p|^{s-1} + 1 \right)^2 \quad (s \geq 1).
\]

If \( p_0 \) is any constant vector, \( D \) is any bounded domain, and \( \zeta \) is of class \( \mathbb{V}_s \) on \( D \) and vanishes on \( D^* \), then \( f \left[ p_0 + \pi (x) \right] \) is summable over \( D \) and
\[
\int_D f \left[ p_0 + \pi (x) \right] \, dx \geq m (D) \cdot f (p_0).
\]

**Proof.** There exists a sequence of functions \( \zeta_n \), each of class \( C^* \) on \( D \) and vanishing on and near \( D^* \), such that \( D_s (\zeta_n - \zeta, D) \rightarrow 0 \) (see [2], Definition 9.1). For each \( n \) and almost all \( x \) on \( D \), we have
\[
|f \left[ p_0 + \pi_n (x) \right] - f \left[ p_0 + \pi (x) \right]| =
\]
\[
\left| \left[ \pi_n (x) - \pi (x) \right] \int_0^1 f_{, i} \left[ p_0 + (1 - t) \pi (x) + t \pi_n (x) \right] \, dt \right|
\]
\[ \leq |\pi_n(x) - \pi(x)| \cdot K \cdot \int_0^1 \left\{ |(1-t)p_0 + \pi(x) + tp_0 + \pi_n(x)|^{s-1} + 1 \right\} dt \]
(3.2)
\[ \leq K |\pi_n(x) - \pi(x)| \cdot \left\{ h_s |p_0 + \pi(x)|^{s-1} + h_s |p_0 + \pi_n(x)|^{s-1} + 1 \right\}, \]
where
\[ h_s = \begin{cases} s^{-1} & (1 \leq s \leq 2) \\ s^{-1} 2^{s-2} & (s > 2). \end{cases} \]

Using the Hölder inequality, and so on, and the strong convergence in \( \mathbb{B}_s \), we see that
\[
\lim_{n \to \infty} \int_D f[p_0 + \pi_n(x)] \, dx = \int_D f[p_0 + \pi(x)] \, dx.
\]
Since \( f \) is quasi-convex, the result follows.

**Lemma 3.4.** Suppose that \( f \) satisfies the hypotheses of Lemma 3.3. Suppose also that each \( \zeta_n \) is of class \( \mathbb{B}_s \) on a domain \( D \) and is strongly of class \( \mathbb{B}_s \) on \( R^* \), \( \overline{R} \subset D \), with
\[
\lim_{n \to \infty} \int_{R^*} |\zeta_n|^s \, dS = 0, \quad D_s(\zeta_n, R^*) \leq M^s, \quad D_s(\zeta_n, R) \leq M^s \quad (n = 1, 2, \ldots).
\]
Then for each \( p_0 \), \( f[p_0 + \pi_n(x)] \) is summable for all sufficiently large \( n \), and
\[
\liminf_{n \to \infty} \int_R f[p_0 + \pi_n(x)] \, dx \geq m(R) \cdot f(p_0), \quad \pi_{n,\alpha}^i(x) = \zeta_{n,\alpha}^i(x).
\]

**Proof.** For each \( n \), let
\[ k_n = \left[ \int_{R^*} |\zeta_n|^s \, dS \right]^{1/s}, \]
and let \( K \) and \( h \) be the quantities of Lemma 3.2 for \( R \). Since \( k_n \to 0 \), we have \( k_n < h \) for all \( n > \) some \( n_1 \). For each such \( n \), let \( \eta_n \) be the function of Lemma 3.2 which coincides on \( R^* \) with \( \zeta_n \), and let
\[ \chi_n = \zeta_n - \eta_n, \quad \kappa_{n,\alpha}^i = \eta_{n,\alpha}^i, \quad \omega_{n,\alpha}^i = \chi_{n,\alpha}^i. \]
Then, since \( \chi_n = 0 \) on \( R^* \), we have
\[
\int_R f[p_0 + \omega_n(x)] \, dx \geq m(R) \cdot f(p_0).
\]
As in (3.2), we see that, for each \( n \), and almost all \( x \) on \( D \),
\[
|f[p_0 + \omega_n(x) + \kappa_n(x)] - f[p_0 + \omega_n(x)]|
\leq K \cdot |\kappa_n(x)| \cdot (h_s |p_0 + \omega_n(x) + \kappa_n(x)|^{s-1} + h_s |p_0 + \omega_n(x)|^{s-1} + 1)
\leq K \cdot |\kappa_n(x)| \cdot [(1 + sh_s) h_s |p_0 + \pi_n(x)|^{s-1} + sh_s^2 |\kappa_n(x)|^{s-1} + 1].
\]

Using the Hölder inequality, and so on, we see that
\[
\lim_{n \to \infty} \int_R |f[p_0 + \pi_n(x)] - f[p_0 + \omega_n(x)]| \, dx = 0,
\]
from which the result follows.

**Theorem 3.1.** Suppose \( f \) is of class \( C^\infty \) in \((x,z,p)\) and quasi-convex in \( p \).
Suppose also that there are numbers \( k \) and \( K, K > 0 \), such that

(i) \( f(x,z,p) \geq k \),

(ii) \( f_{x^i} f_{p^i_a} \leq K^2 (|p|^s + 1)^2 \)

(iii) \( f_{x^\alpha} f_{x^{\alpha}} \leq K^2 (|p|^s + 1)^2 \)

(iv) \( f_{z^i} f_{z^i} \leq K^2 (|p|^s + 1)^2 \).

for all \((x,z,p)\).

Suppose also that \( z_n \to z_0 \) weakly in \( H_s \) on the bounded domain \( D \) and that either

(a) each \( z_n \) and \( z_0 \) are continuous on \( D \) and \( z_n \) converges uniformly to \( z_0 \) on each closed set interior to \( D \), or

(b) the set functions \( D_s(z_n,e) \) are uniformly absolutely continuous on each closed set interior to \( D \).

Then
\[
I(z_0,D) \leq \liminf_{n \to \infty} I(z_n,D).
\]

**Remark.** If \( s = 1 \), weak convergence in \( H_s \) implies the hypothesis (b).

**Proof.** We note first that hypothesis (ii) implies

\[
|f(x,z,p) - f(x,z,0)| = |p_a^i \int_0^1 f_{p^i_a}(x,z,tp^i_a) \, dt|
\leq |p| \cdot \int_0^1 [f_{p^i_a} f_{p^i_a}(x,z,tp^i_a)]^{1/2} \, dt
\]

(3.3)
\[ \leq |p| \cdot \int_0^1 K(t^{s-1} p^{s-1} + 1) \, dt \leq K(s^{-1} |p|^s + |p|). \]

Also, hypotheses (iii) and (iv) similarly imply

\[ |f(x, z, 0) - f(0, 0, 0)| \leq K(|x| + |z|). \]

Thus, for all \((x, z, p)\), we have

\[ |f(x, z, p)| \leq |f(0, 0, 0)| + K(|x| + |z| + s^{-1} |p|^s + |p|). \]

Therefore \(I(z_0, D)\) and the \(I(z_n, D)\) are uniformly bounded.

For each \(\alpha\) (\(1 \leq \alpha \leq \nu\)), let \((a^\alpha, b^\alpha)\) be the open interval projection of \(D\) on the \(x^\alpha\) axis and let \(Z^\alpha_0\) and \(Z^\alpha_n\) be the sets of Lemma 3.1 for \(z_0\) and \(z_n\). Also for each \(\alpha, n, k\), let \(E^\alpha_{n,k}\) be the set of \(x^\alpha\) in \((a^\alpha, b^\alpha) - Z^\alpha_n\), where

\[ D_s(z, D_{x^\alpha}) \leq k, \]

\(D_{x^\alpha}\) being the set of \(x^\alpha\) such that \((x^\alpha, x^\alpha) \in D\). Suppose that \(D_s(z, D) \leq M\), some uniform bound existing because of the weak convergence. Let

\[ Z^\alpha_{n,k} = (a^\alpha, b^\alpha) - E^\alpha_{n,k}. \]

Then

\[ m(Z^\alpha_{n,k}) \leq Mk^{-1}, \quad m(E^\alpha_{n,k}) \geq (b^\alpha - a^\alpha) - Mk^{-1}. \]

For each \(\alpha\), let

\[ E^\alpha = \bigcup_{k=1}^\infty \bigcap_{N=1}^N \bigcup_{n=N}^\infty E^\alpha_{n,k}, \quad Z^\alpha_0 = (a^\alpha, b^\alpha) - E^\alpha \cup Z^\alpha_0 \cup \bigcup_{n=1}^\infty Z^\alpha_n. \]

Then \(m(Z^\alpha_0) = 0\). For each \(\alpha\), each natural number \(n\), and each integer \(i\), define \(Z^\alpha_{n,i}\) as the set of all \(x^\alpha\) such that \(x^\alpha - i \cdot 2^{-n} \in Z^\alpha_0\), and define

\[ Z^\alpha = \bigcup_{n,i} Z^\alpha_{n,i}. \]

Then \(m(Z^\alpha) = 0\).

Now, choose a point \(x_0^\alpha\) such that \(x_0^\alpha\) is not in \(Z^\alpha\) (\(\alpha = 1, \ldots, \nu\)). For each natural number \(k\), let \(C^k\) be the totality of hypercubes of side \(2^{-k}\) bounded by hyperplanes of the form \(x^\alpha = x_0^\alpha + i \cdot 2^{-k}\). None of the numbers \(x_0^\alpha + i \cdot 2^{-k}\) is in \(Z^\alpha\) and, moreover, \(Z_0^\alpha\) and each \(Z_n^\alpha\) is strongly of class \(\frac{\beta}{\delta}\) on \(\mathbb{R}^*\) with
uniformly bounded for infinitely many values of \( n \), \( R \) being any hypercube of any \( Q_k \). Since the totality of these hypercubes is countable, we may choose a subsequence, still called \( z_n \), such that \( I(z_n, D) \) tends to the former

\[
\lim_{n \to \infty} \int_{R^*} |z_n - z_0|^s \, dS = 0
\]

for each such \( R \).

Now, we first consider the alternative (a). Let \( \epsilon \) be any positive number. For each \( k \), let \( D_k \) be the union of all the cells of \( Q_k \) which are interior to \( D \). Since \( f \) is bounded below and \( I(z_0, D) \) is finite, we first choose \( k_1 \) so large that

\[
I(z_n, D - D_{k_1}) > -\epsilon/5 \quad (n = 1, 2, \ldots).
\]

(3.6)

\[
I(z_0, D_{k_1}) > I(z_0, D) - \epsilon/5.
\]

For this \( k_1 \), let \( R_1, \ldots, R_q \) be the cells of \( D_{k_1} \) and for each \( k \geq k_1 \), let \( R_{ki} \) be the cells of \( Q_k \) in \( D_{k_1} \). For each \( k \), define \( x^*_k(x), z^*_k(x), p^*_k(x) \) on \( D_{k_1} \) by (2.9). Then, from (ii), (iii), and (iv), it follows that

\[
|f[x, z_0(x), p_0(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x)]| 
\leq K(\ |p_0(x)|^s + 1) \cdot (\ |x - x^*_k(x)| + |z_0(x) - z^*_k(x)|)
\]

\[
+ K(h_s \ |p_0(x)|^{s-1} + h_s \ |p^*_k(x)|^{s-1} + 1) \cdot |p_0(x) - p^*_k(x)|,
\]

(3.7)

where

\[
h_s = \begin{cases} 
s^{-1} & (1 \leq s \leq 2), \\
s^{-1} \cdot 2s^{-2} & (s \geq 2);
\end{cases}
\]

the method of proof is similar to that of (3.3). If we let

\[
\zeta_n = z_n - z_0, \quad \pi_n = p_n - p_0,
\]

we see similarly that
\[ |f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)]| \]

\[ \leq K(|p_n(x)|^s + 1) \left( |x - x^*_k(x)| + |z_0(x) - z^*_k(x)| \right) + K(h_s |p_n(x)|^{s-1} + h_s |p^*_k(x) + \pi_n(x)|^{s-1} + 1) \cdot |p_0(x) - p^*_k(x)|; \]

\[ |f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]| \]

\[ \leq K(|p_n(x)|^s + 1) \cdot |z_n(x) - z_0(x)|. \]

Now, by the Hölder inequality on each \( R_{k_i} \), we see that

\[ \int_{D_{k_1}} |p^*_k(x)|^s \, dx \leq \int_{D_{k_1}} |p_0(x)|^s \, dx. \]

By applying the Minkowski inequality, we see that the integrals

\[ \int_{D_{k_1}} |\pi_n(x)|^s \, dx, \int_{D_{k_1}} |p^*_k(x) + \pi_n(x)|^s \, dx \]

are uniformly bounded. Finally,

\[ \lim_{k \to \infty} \int_{D_{k_1}} |p_0(x) - p^*_k(x)|^s \, dx = 0. \]

Hence, using (3.7)-(3.12), we may choose a \( k \) so large that

\[ \int_{D_{k_1}} \left| f[x, z_0(x), p_0(x)] - f[x^*_k(x), z^*_k(x), p^*_k(x)] \right| \, dx < \epsilon/5, \]

\[ \int_{D_{k_1}} \left| f[x, z_0(x), p_n(x)] - f[x^*_k(x), z^*_k(x) p^*_k(x) + \pi_n(x)] \right| \, dx < \epsilon/5 \]

\( (n = 1, 2, \ldots) \),

and then choose \( n_1 \) so large that

\[ \int_{D_{k_1}} \left| f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)] \right| \, dx < \epsilon/5, \quad n > n_1. \]

Since \( x^*_k(x), z^*_k(x), p^*_k(x) \) are constant on each \( R_{k_i} \), it follows from Lemma 3.4 that

\[ \liminf_{n \to \infty} \int_{D_{k_1}} f[x^*_k(x), z^*_k(x), p^*_k(x) + \pi_n(x)] \, dx \]
Using (3.6) and (3.13)-(3.16), we see that
\[
\liminf_{n \to \infty} I(z_n, D) \geq I(z_0, D) - \epsilon.
\]
The result follows in this case.

We now consider the alternative (b). For each natural number \( q \), we define
\[
f_q(x, z, p) = \begin{cases} 
0 & (0 \leq R \leq q), \\
3(R - q)^2 - 2(R - q)^3 & (q \leq R \leq q + 1), \\
1 & (R \geq q + 1), 
\end{cases}
\]
and
\[
a_q(x, z) = \begin{cases} 
3(R - q)^2 - 2(R - q)^3 & (q \leq R \leq q + 1), \\
3(\Lambda - 2) - 2(R - q)^3 & (q < R < q + 1), \\
1 & (q < R < q + 1), 
\end{cases}
\]
Remembering (3.3)-(3.5), we see that each \( f_q \) satisfies hypotheses (i)-(iv) with the same \( k \) and some \( K_q \). Moreover \( f_q \) is independent of \( (x, z) \) for \( R \geq q + 1 \), and also
\[
f_q(x, z, p) \leq f_{q+1}(x, z, p), \quad \lim_{q \to \infty} f_q(x, z, p) = f(x, z, p).
\]
Thus it is sufficient to prove the lower semicontinuity for each \( q \).

For a fixed \( q \), we note that we may replace \( |z_0(x) - z_k^*(x)| \) by \( \phi_k(x) \) in (3.7) and (3.8) and \( |z_n(x) - z_0(x)| \) by \( \psi_n(x) \) in (3.9), where
\[
\phi_k(x) = \min \left( |z_0(x) - z_k^*(x)|, 2q + 2 \right),
\]
\[
\psi_n(x) = \min \left( |z_n(x) - z_0(x)|, 2q + 2 \right).
\]
From the uniform boundedness of the \( \phi_k \) and \( \psi_n \) (\( q \) fixed), the uniform absolute continuity of the set function \( D_s(z_n, e) \), and the facts that
\[
\lim_{k \to \infty} \phi_k(x) = 0, \quad \lim_{n \to \infty} \psi_n(x) = 0
\]
almost everywhere, it follows that the argument can be carried through as before for each fixed \( q \).

**Theorem 3.2.** Suppose \( s > \nu \) and suppose \( f \) satisfies the hypotheses of Theorem 3.1 with (i) replaced by
(i') \[ f(x,z,p) \geq m|p|^s + k \quad (m > 0). \]

If \( z^* \) is any function of class \( \mathbb{P}_s \) on the bounded domain \( D \), then there is a function \( z_0 \) of class \( \mathbb{P}_s \) which coincides with \( z^* \) on \( D^* \) and minimizes \( I(z,D) \) among all such functions.

**Proof.** Let \( z_n \) be a minimizing sequence. It follows from (i') that \( D_s(z_n,D) \) is uniformly bounded. From [2], Theorem 9.4, it follows that \( \overline{D}_s(z_n,D) \) is uniformly bounded. But then a subsequence, still called \( \{ z_n \} \), converges weakly in \( \mathbb{P}_s \) to some function \( z_0 \) of class \( \mathbb{P}_s \) which coincides with \( z^* \) on \( D^* \) by [2], Theorem 9.2. But, from [3], Chapter II, Theorem 2.1, it follows that the equivalent functions \( \overline{z}_n \) and \( \overline{z}_0 \) are equicontinuous on closed sets interior to \( D \). Hence \( z_n \) converges uniformly to \( \overline{z}_0 \) on each closed set interior to \( D \). Hence, from the preceding theorem, \( z_0 \) is a desired solution.

More general theorems involving variable boundary values, similar to those in [3], Chapter III, §5, with \( s > \nu \), can be proved.

### 4. Necessary conditions for quasi-convexity.

In the two preceding sections, we have established the connection between quasi-convexity and lower semicontinuity. In this section, we shall establish some necessary conditions for quasi-convexity. In the next section, we establish some sufficient conditions which are also necessary when \( f \) has certain interesting special forms. Unfortunately, the writer is unable to establish conditions which are both necessary and sufficient in the general case.

**Lemma 4.1.** Suppose \( f \) is continuous, \( Q \) is the cell

\[ |x^\alpha| \leq 1 \quad (\alpha = 1, \ldots, \nu), \delta > 0, \]

and suppose

\[ \int_Q f|p + \pi(x)| \, dx \geq f(p) \cdot m(Q) \quad (4.1) \]

for every function \( \zeta \) which satisfies a Lipschitz condition with coefficient \( \delta \) on \( \overline{Q} \) and vanishes on \( Q^* \). Then (4.1) also holds with \( Q \) replaced by any bounded domain \( D \).

**Proof.** Suppose \( \zeta \) satisfies the conditions on the bounded domain \( \overline{D} \). Let \( \overline{R} \) be a hypercube of side \( h \) which contains \( \overline{D} \), and extend \( \zeta \) to \( \overline{R} \):

\[ x_0^\alpha \leq x^\alpha \leq x_0^\alpha + h \]
by defining $\zeta = 0$ on $\bar{R} - \bar{D}$. Then $\zeta$ satisfies the conditions on $\bar{R}$, and

$$\zeta^*(x) = h^{-1} \zeta(x_0 + hx)$$

satisfies the conditions on $\bar{Q}$, and

$$\zeta^*_{\alpha}(x) = \zeta^i_{\alpha}(x_0 + hx).$$

**Definition 4.1.** The function $f$ is said to be weakly quasi-convex if with each $p$ is associated a $\delta_p > 0$ such that (4.1) holds for all $D$ and all $\zeta$ satisfying a Lipschitz condition with coefficient $< \delta_p$ and vanishing on $D^*$.

In other words, $f$ is weakly quasi-convex if and only if each linear function furnishes a weak relative minimum among all Lipschitzian functions coinciding with it on the boundary, whereas $f$ is quasi-convex if and only if any linear function furnishes the absolute minimum among all such functions. Thus we have the following result.

**Theorem 4.1.** If $f$ is continuous and quasi-convex, it is weakly quasi-convex.

We shall see that if $f$ is weakly quasi-convex and continuous, then $f$ satisfies a uniform Lipschitz condition on any bounded set in $p$-space and satisfies a generalized Weierstrass condition (see Theorem 4.3) which reduces to the ordinary Weierstrass condition if $f$ is of class $C'$ (see (4.7)) and is equivalent to the Legendre-Hadamard condition (see (4.8)) if $f$ is of class $C''$.

**Lemma 4.2.** Suppose $\phi$ is continuous, and suppose corresponding to any point $\lambda_0$ in $E_v$ there is a $\delta > 0$ such that for any unit vector $\mu$ we have

$$k\phi(\lambda_0 - h\mu) + h\phi(\lambda_0 + k\mu) \geq (h + k) \phi(\lambda_0) \quad (0 < h < \delta, \ 0 < k < \delta).$$

Then $\phi$ is convex in $\lambda$.

**Proof.** Let $\lambda_0$ be any point, and $\mu$ any point with $|\mu| = 1$. We shall show that

$$\psi(t) = \phi(\lambda_0 + \mu t)$$

is convex in $t$. From the hypothesis, it follows that for each $t_0$, there is a $\delta(t_0) > 0$ such that

$$k\psi(t_0 - h) + h\psi(t_0 + k) \geq (h + k) \psi(t_0) \quad (0 \leq h \leq \delta, \ 0 \leq k \leq \delta).$$
Now, suppose \( t_1 < t_2 \). Let
\[
\chi(t) = \psi(t) - \psi(t_1) - \frac{t - t_1}{t_2 - t_1} [\psi(t_2) - \psi(t_1)].
\]
Then \( \chi(t) \) satisfies (4.2) and \( \chi(t_1) = \chi(t_2) = 0 \). Suppose \( M = \max \chi(t) \) \( (t_1 \leq t \leq t_2) \), and suppose \( M > 0 \). Let \( t_0 \) be the smallest value of \( t \) such that \( \chi(t) = M \), and let the number \( \delta(t_0) \) be chosen as above. Clearly \( t_1 < t_0 < t_2 \).

Choose \( t_3 \) and \( t_4 \) with
\[
|t_3 - t_0| < \delta, \quad |t_4 - t_0| < \delta \quad (t_1 \leq t_3 < t_0 < t_4 \leq t_2).
\]
Then \( \chi(t_3) < M, \chi(t_4) \leq M \), so that
\[
(t_4 - t_0) \chi[t_0 - (t_0 - t_3)] + (t_0 - t_3) \chi[t_0 + (t_4 - t_0)] < (t_4 - t_3) \chi(t_0),
\]
which contradicts the hypothesis. Thus \( \chi(t) \leq 0 \), so that
\[
\psi(t) \leq \psi(t_1) + \frac{t - t_1}{t_2 - t_1} [\psi(t_2) - \psi(t_1)].
\]

Since \( t_1 \) and \( t_2 \) were arbitrary with \( t_1 < t_2 \), the function \( \psi \) is convex in \( t \). Thus \( \phi \) is convex in \( \lambda \).

**Theorem 4.2.** If \( f \) is weakly quasi-convex, then \( f(p^i + \lambda \xi^i) \) is convex in \( \lambda \) for each fixed \( p \) and \( \xi \).

**Proof.** Let \( p^i, \xi^i \) and \( \lambda_0^i \) be fixed and let \( \mu_\lambda \) be any unit vector, and suppose \( h > 0, k > 0 \). Choose \( \delta(p^i, \xi^i, \lambda_0^i) > 0 \) but so small that, for any bounded domain \( G \),
\[
(4.3) \quad \int_G f[p^i + \lambda_0^i \xi^i + \xi^i(x)] dx \geq m(G) f(p^i + \lambda_0^i \xi^i)
\]
for all \( \xi \) satisfying a Lipschitz condition of constant \( < \delta \) on \( G \) and vanishing on \( G^* \). Let \( (\mu_1, \cdots, \mu_\nu) \) be a normal orthogonal set of unit vectors. If \( \xi = 0 \), the result is obvious. If \( \xi \neq 0 \), choose \( h \) and \( k \) with \( 0 < h |\xi| < \delta, 0 < k |\xi| < \delta \), and let \( \rho \) be any number \( > |\xi|/\delta \). Let \( H = (1/\rho) k, K = (1/\rho) h \), and let \( R \) be the rectangular parallelepiped
\[
-\rho H \leq y^1 \leq \rho K, \quad |y^\beta| \leq \rho \quad (\beta = 2, \cdots, \nu)
\]
where

\[ \gamma^\beta = x, \mu_\beta. \]

Let \( F^- \) be the face \( y^1 = -\rho H \), \( F^+ \) be the face \( y^1 = \rho K \), \( F^- \) be the face \( y^\beta = -\rho \beta, F^- \) be the face \( y^\beta = \rho \), and let \( \Pi^- \) and \( \Pi^+ \) be the pyramids with vertex at the origin and base \( F^- \) and \( F^+ \) respectively. Let \( \zeta \) be defined on \( \overline{R} \) to be continuous on \( \overline{R^*} \), zero on \( R^* \), linear on each \( \Pi^- \) and \( \Pi^+ \), with \( \zeta(0) = \xi \). Then

\[
(4.4) \quad \zeta^i(x) = \begin{cases} 
(\rho H)^{-1} \mu_{1a} \xi^i = k \mu_{1a} \xi^i & \text{on } \Pi^-
\end{cases} 
\]

\[
(\rho K)^{-1} \mu_{1a} \xi^i = -h \mu_{1a} \xi^i & \text{on } \Pi^+
\]

\[
\rho^{-1} \mu_{\beta a} \xi^i & \text{on } \Pi^- \beta
\]

\[
- \rho^{-1} \mu_{\beta a} \xi^i & \text{on } \Pi^+ \beta
\]

Also

\[
m(\Pi^-) = \nu^{-1} 2^{\nu-1} \rho^\nu H, m(\Pi^+) = \nu^{-1} 2^{\nu-1} \rho^\nu K, m(R) = 2^{\nu-1} \rho^\nu (H + K)
\]

(4.5)

\[
m(\Pi^-) = m(\Pi^+) = \nu^{-2} 2^{\nu-2} \rho^\nu (H + K)
\]

Then, by applying (4.3), (4.4), and (4.5), we obtain

\[
\frac{1}{2^\nu} \left[ \frac{2k}{h+k} f[p_a^i + (\lambda_{0a} - h \mu_{1a}) \xi^i] + \frac{2h}{h+k} f[p_a^i + (\lambda_{0a} + k \mu_{1a}) \xi^i] \right]
\]

\[
+ \sum_{\beta=2}^\nu \left[ f[p_a^i + (\lambda_{0a} - \rho^{-1} \mu_{\beta a}) \xi^i] + f[p_a^i + (\lambda_{0a} + \rho^{-1} \mu_{\beta a}) \xi^i] \right] \geq f(p_a^i + \lambda_{0a} \xi^i).
\]

Letting \( \rho \to \infty \), we obtain

\[ kf[p_a^i + (\lambda_{0a} - h \mu_{1a}) \xi^i] + hf[p_a^i + (\lambda_{0a} + k \mu_{1a}) \xi^i] \geq (h + k) f(p_a^i + \lambda_{0a} \xi^i) \]

From the preceding lemma, it follows that \( f(p_a^i + \lambda_{0a} \xi^i) \) is convex in \( \lambda \) for each \( \xi \) and \( p \).
THEOREM 4.3. Suppose \( f \) is continuous and convex in \( \lambda \) for \( p \) and \( \xi \).

Then \( f \) satisfies a uniform Lipschitz condition on each bounded closed set, and for each fixed \( p \) there exists a set of constants \( A^\alpha_i \) such that

\[
(4.6) \quad f(p^i + \lambda^\alpha \xi^i) \geq f(p) + A^\alpha_i \lambda^\alpha \xi^i
\]

for all \( \lambda \) and \( \xi \). If \( f \) is of class \( C^\alpha \), \( (4.6) \) holds if and only if \( A^\alpha_i = f_{p^i} \), that is,

\[
(4.7) \quad f(p^i + \lambda^\alpha \xi^i) \geq f(p) + f_{p^i}(p) \lambda^\alpha \xi^i.
\]

If \( f \) is of class \( C^\beta \), \( (4.7) \) holds for all \( p, \lambda, \xi \) if and only if

\[
(4.8) \quad f_{p^i + \lambda^\beta \xi^j}(p) \lambda^\alpha \lambda^\beta \xi^i \xi^j \geq 0
\]

for all \( \lambda, \xi, p \).

Proof. Suppose, first, that \( f \) is of class \( C^\alpha \). Let \( p \) and \( \xi \) be fixed. Then \( (4.7) \) follows from the convexity in \( \lambda \). Moreover, since each unit vector \( e^i_a \) in the \( p \)-space is of the form \( \lambda^\alpha \xi^i \), we see from the convexity in \( \lambda \) that

\[
(4.9) \quad f(p) - f(p - e^i_a) \leq f_{p^i}(p) \leq f(p + e^i_a) - f(p)
\]

for all \( p \). Thus the derivatives of \( f \) are uniformly bounded by these differences in the values of \( f \) on any bounded part of space. Moreover, in this case, if constants \( A^\alpha_i \) satisfy \( (4.6) \), we must have

\[
A^\alpha_i = f_{p^i}(p).
\]

Now, if \( f \) is of class \( C^\beta \), equation \( (4.8) \) with \( p \) replaced by \( p^i + \lambda^\alpha \xi^i \) is equivalent to the condition that \( f \) is convex in \( \lambda \) for each fixed \( p \) and \( \xi \).

Finally, if \( f \) is continuous and has this stated convexity property, it is clear that the \( h \)-average function also does, and \( f_h \) is of class \( C^\alpha \). By letting \( h \to 0 \), we see that \( f \) satisfies a uniform Lipschitz condition on any bounded closed set. Now, choose \( h_n = n^{-1} \) and choose \( p \) fixed. From \( (4.9) \) and the uniform convergence of \( f_h \) to \( f \) on any bounded part of space, we conclude that the derivatives \( f_{h_n p^i}(p) \) are uniformly bounded. We may therefore choose a subsequence, still called \( h_n \), such that

\[
\lim_{n \to \infty} f_{h_n p^i}(p) = A^\alpha_i.
\]
Since (4.7) holds for all $\lambda$ and $\xi$ for each $n$, (4.6) holds in the limit.

5. Sufficient conditions for quasi-convexity. In this section we prove one general sufficient condition and then give conditions which are necessary and sufficient when $f$ has certain interesting special forms.

**Lemma 5.1.** Suppose $\zeta$ satisfies a uniform Lipschitz condition on the closure $\overline{D}$ of the bounded domain $D$ and suppose $\zeta = 0$ on $D^\ast$. If

$$1 \leq \mu \leq \nu, \quad 1 \leq i_1, \cdots, i_\mu \leq N, \quad 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\mu \leq \nu,$$

then

$$\int_D \frac{\partial (\zeta^{i_1}, \cdots, \zeta^{i_\mu})}{\partial (x^{\alpha_1}, \cdots, x^{\alpha_\mu})} \, dx = 0.$$

**Proof.** Choose a large cell $R$ containing $\overline{D}$ in its interior, and extend $\zeta$ by defining it to be zero outside $\overline{D}$. Then the second $h$-average function $\zeta_{hh}$ is of class $C''$ on $\overline{R}$ and vanishes on and near $R^\ast$. Since any integral of the above type formed for $\zeta_{hh}$ tends to that for $\zeta$ as $h \to 0$, we need prove the theorem only for functions $\zeta$ of class $C''$ on cells $\overline{R}$.

As an example, take $i_\beta = \alpha_\beta = \beta$, $\beta = 1, \cdots, \mu$, $D = R$. Then

$$\int_R \frac{\partial (\zeta^{i_1}, \cdots, \zeta^{i_\mu})}{\partial (x^{i_1}, \cdots, x^{i_\mu})} \, dx = \int_R \sum_{\alpha = 1}^{\mu} (-1)^{\mu + \alpha} \zeta^{i_\mu}_{x^\alpha} \, Q \, dx$$

$$= \int_{R^\ast} \zeta^{i_\mu} \sum_{\alpha = 1}^{\mu} (-1)^{\mu + \alpha} \, Q \, dx^\ast$$

$$- \int_R (-1)^{\mu} \zeta^{i_\mu} \sum_{\alpha = 1}^{\mu} (-1)^{\alpha} \frac{\partial}{\partial x^\alpha} Q \, dx,$$

where

$$Q = \frac{\partial (\zeta^{i_1}, \cdots, \zeta^{i_{\mu-1}}, \zeta^\alpha, \cdots, \zeta^{i_\mu})}{\partial (x^{i_1}, \cdots, x^{i_{\mu-1}}, x^\alpha + 1, \cdots, x^{i_\mu})},$$

the last equality holding by Green's theorem. But the boundary integral vanishes since $\zeta = 0$ on $R^\ast$, and the integrand in the second integral vanishes on $R$ (see [3], Chapter II, Lemma 1.1, for instance).
THEOREM 5.1. A sufficient condition that \( f \) be quasi-convex is that for each \( p \) there exist alternating forms

\[
A_i^a \pi^i_a, A_{ij}^{a\beta} \pi^i_a \pi^j_{\beta}, \ldots, A_{i_1,\ldots,i_{\nu}}^{a_1,\ldots,a_{\nu}} \pi^{i_1}_{a_1} \cdots \pi^{i_{\nu}}_{a_{\nu}}
\]

such that for all \( \pi \) we have

\[
f(p + \pi) \geq f(p) + A_i^a \pi^i_a + \cdots + A_{i_1,\ldots,i_{\nu}}^{a_1,\ldots,a_{\nu}} \pi^{i_1}_{a_1} \cdots \pi^{i_{\nu}}_{a_{\nu}}.
\]

Proof. This is an immediate consequence of the preceding lemma.

THEOREM 5.2. If the \( a_{ij}^{a\beta} \) are constants and

\[
f(p) = a_{ij}^{a\beta} p_i^a p_j^\beta,
\]

a necessary and sufficient condition that \( f \) be quasi-convex is that

\[
a_{ij}^{a\beta} \lambda_a \lambda_\beta \xi^i \xi^j \geq 0
\]

for all \( \lambda \) and \( \xi \).

Proof. If \( \zeta = 0 \) on \( D^* \), we see from Lemma 5.1 that

\[
\int_D f(p + \pi(x)) \, dx = f(p) \, m(D) + \int_D a_{ij}^{a\beta} \pi_i^a(x) \pi_j^\beta(x) \, dx.
\]

But Van Hove [6] has shown that the condition (5.2) is necessary (this also follows from Theorem 4.3) and sufficient for the second integral to be \( \geq 0 \) for all \( \zeta \) of class \( D' \) on \( D \) which vanish on \( D^* \) (hence this is true also for all \( \zeta \) of class \( \mathbb{D}'_2 \) on \( D \) and vanishing on \( D^* \)).

LEMMA 5.2. Suppose

\[
\sum_{i,j=1}^n a_{ij} x^i y^j = 0
\]

for all \( x \) and \( y \) for which

\[
\sum_{i,j=1}^n b_{ij} x^i y^j = 0.
\]

Then there is a constant \( K \) such that
\[ a_{ij} = Kb_{ij} \ (i, j = 1, \ldots, n). \]

\textit{Proof.} We may introduce new variables \( \xi \) and \( \eta \) by
\[ x = c \xi, \ y = d \eta, \]
c and \( d \) being \( n \times n \) nonsingular matrices. Let \( a \) and \( b \) be the matrices of the original forms and \( A \) and \( B \) those of the transformed forms. Then
\[ A = c'ad, \ B = c'bd \ (c'_{ij} = c_{ij}). \]

We shall show that there is a scalar \( K \) such that \( A = KB \). We may assume that
\[ B_{ii} = 1 \ (i = 1, \ldots, r); \ B_{ij} = 0 \text{ otherwise,} \]
unless \( B = 0 \) in which case \( A = 0 \) also and the theorem holds. By taking \( \eta^s = 1, \ \eta^j = 0 \ (j \neq s, s = 1, \ldots, n) \) in turn we see that
\[ A_{is} = 0 \ (i = 1, \ldots, n, s > r); \ A_{is} = 0 \ (i \neq s, s = 1, \ldots, r, i = 1, \ldots, n). \]

Then, by choosing \( 1 \leq s < t \leq r \) and setting \( \eta^s = \eta^t = 1, \ \eta^j = 0, j \neq s, j \neq t, \) we have
\[ (A_{is} + A_{it}) \xi^i = 0 \quad \text{for all } \xi \text{ with } \xi^s + \xi^t = 0. \]
Thus there exists a constant \( K(s, t) \) such that
\[ A_{ss} + A_{st} = K(s, t), \quad A_{ts} + A_{tt} = K(s, t). \]

Hence
\[ A_{11} = A_{22} = \cdots = A_{rr} = K, \]
so that \( A = KB \).

\textbf{Theorem 5.3.} Suppose that \( N = \nu + 1 \) and
\begin{equation}
(5.3) \quad f(p) = F(X_1, \ldots, X_{\nu+1}),
\end{equation}
where \( F \) is positively homogeneous of the first degree in the \( X_i \) and
\[ X_i = -\det M_{i} \ (i = 1, \ldots, \nu), \ X_{\nu+1} = \det M_{\nu+1}, \]
\[ M_{\nu+1} = \| p_1^i, \ldots, p_\nu^i \|, \ M_{i} = \| p_1^i, \ldots, p_{i-1}^i, p_{i+1}^\nu, p_{i+1}^i, \ldots, p_\nu^i \|
\]
\[ (i = 1, \ldots, \nu). \]
Then $f$ is quasi-convex in $p$ if and only if $F$ is convex in the $X_i$.

Proof. If $F$ is convex in the $X_i$, it follows from Theorem 5.1 that $f$ is quasi-convex in $p$.

Hence suppose $f$ is given by (5.3) and is quasi-convex in $p$. If

$$\Delta X_k = X_k(p_i^a + \lambda \xi^i) - X_k(p_i^a),$$

then

$$(5.4) \quad \Delta X_k = X_k p_i^a \lambda \xi^i.$$ 

Also, since

$$p_k^\beta X_k = 0 \quad (\beta = 1, \ldots, \nu),$$

we have

$$(5.5) \quad p_k^\beta X_k p_i^a = - \delta_{\beta}^a X_i^i.$$ 

Now, choose a set of $X_i$ not all zero and choose any $p$ such that

$$X_i(p) = X_i.$$ 

Since $f$ is quasi-convex and hence weakly so, there are constants $A_i^\alpha$ such that

$$f(p_i^a + \lambda \xi^i) \geq f(p) + A_i^\alpha \lambda \xi^i.$$ 

Since $f$ depends only on the $X_i$, we must have

$$(5.6) \quad A_i^\alpha \lambda \xi^i \leq 0 \quad \text{for all } \lambda, \xi \text{ with } X_k p_i^a \lambda \xi^i = 0 \quad (k = 1, \ldots, \nu + 1).$$

Obviously, then, the equality must hold in (5.6). Using (5.4) and (5.5), we see that

$$(5.7) \quad p_k^\beta \Delta X_k = - \lambda \beta (X_i \xi^i) \quad (\beta = 1, \ldots, \nu).$$ 

Hence, we must have

$$(5.8) \quad A_i^\alpha \lambda \xi^i = 0$$ 

for all $\lambda, \xi$ for which

$$(5.9) \quad X_i \xi^i = 0 \text{ and } D_i^a \lambda \xi^i = 0, \quad D_i^a = X_k X_k p_i^a.$$
Now, since not all the $X_i$ are zero, assume $X_k \neq 0$. Then

\[(5.10) \sum_{i \neq k} \left( A_i X_k - A_k X_i \right) \lambda \xi = 0 \]

for all $\lambda$, $\xi$ for which

\[(5.11) \sum_{i \neq k} \left( D_i X_k - D_k X_i \right) \lambda \xi = 0. \]

From the preceding lemma, it follows that there is a constant $K$ such that

\[(5.12) A_i X_k - A_k X_i = K \left( D_i X_k - D_k X_i \right). \]

Hence

\[(5.13) A_i^\alpha = KD_i^\alpha + L_i^\alpha X_i, \quad L_i^\alpha = X_k^{-1} \left( A_k^\alpha - KD_k^\alpha \right). \]

From (5.7) and (5.13) it follows that

\[(5.14) A_i^\alpha \lambda \xi = KD_i^\alpha \lambda \xi + L_i^\alpha \lambda X_i \xi = C_k \Delta X_k, \quad C_k = \left( KX_k - L_i^\alpha p_i^k \right). \]

Finally, if we are given any values of the $\Delta X$, the quantities

$$h_i = p_i^k \Delta X_k \quad (i = 1, \cdots, \nu) \quad \text{and} \quad h_{\nu+1} = X_k \Delta X_k$$

are determined and the $\Delta X_k$ are also uniquely determined by the $h_i$. Using (5.7), we may determine the $\lambda$ in terms of the $h_i$ ($i = 1, \cdots, \nu$), and substitute them into

$$h_{\nu+1} = X_k \Delta X_k = D_i^\alpha \lambda \xi,$$

and we merely have to choose the $\xi$ to satisfy the equation

$$\left( D_i^\alpha h_a + h_{\nu+1} X_i \right) \xi = 0 \quad \text{with} \quad X_i \xi \neq 0;$$

this is always possible unless all the $D_i^\alpha h_a = 0$. Thus, unless these linear relations in the $\Delta X$ hold, we have

\[(5.15) F(X + \Delta X) = f \left( p_i^\alpha + \lambda \xi \right) \geq f(p) + A_i^\alpha \lambda \xi = F(X) + C_k \Delta X_k. \]

The result follows in this case by continuity.
Finally, since $F$ is homogeneous of the first degree, we see by taking

$$\Delta X = hX, \ h > -1,$$

that

$$F[(1 + h)X] = (1 + h)F(X) \geq F(X) + hC^kX_k,$$

or

$$h\left[F(X) - C^kX_k\right] \geq 0, \ h > -1.$$ 

Hence $F(X) = C^kX_k$. Then by setting $X = hX_0, \ X_0 \neq 0$, choosing the $C^k$ for this $X_0$, and then letting $h \to 0$, we see that (5.15) holds for some $C^k$ even if $X = 0$.

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