SOME THEOREMS ON BERNOULLI NUMBERS OF HIGHER ORDER

L. CARLITZ
1. Introduction. We define the Bernoulli numbers of order \( k \) by means of [3, Chapter 6]

\[
\left( \frac{t}{e^t - 1} \right)^k = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m^{(k)} \quad (|t| < 2\pi);
\]

in particular, \( B_m = B_m^{(1)} \) denotes the ordinary Bernoulli number. Not much seems to be known about divisibility properties of \( B_m^{(k)} \). Using different notation, S. Wachs [4] proved a result which may be stated in the form

\[
B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2},
\]

where \( p \) is a prime \( \geq 3 \). In attempting to simplify Wachs' proof, the writer found the stronger result

\[
B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3} \quad (p > 3).
\]

We remark that \( B_5^{(4)} = -9 \).

The proof of (1.2) depends on some well-known properties of the Bernoulli numbers and factorial coefficients; in particular, we make use of some theorems of Glaisher and Nielsen. The necessary formulas are collected in §2; the proof of (1.2) is given in §3. In §4 we prove

\[
B_p^{(p)} = \frac{1}{2} p^2 \pmod{p^3} \quad (p \geq 3);
\]

the proof of this result is somewhat simpler than that of (1.2). For the residue of \( B_p^{(p)} \pmod{p^4} \), see (4.5) below.

In §5 we prove several formulas of a similar nature \( (p > 3) \):

Received August 14, 1951.

Pacific J. Math. 2 (1952), 127-139
\[(1.4)\quad B(p)_{p+1} = -p \frac{B_{p+1}}{p+1} + \frac{1}{24} p^2 \pmod{p^3},\]

\[(1.5)\quad B(p)_{p+2} = p^2 \frac{B_{p+1}}{p+1} \pmod{p^4},\]

\[(1.6)\quad B(p+1)_{p+1} = \frac{B_{p+1}}{p+1} - \frac{1}{24} p \pmod{p^2}.

In §6 we discuss the number \(B_m\) for arbitrary \(m\); this requires the consideration of a number of cases. In particular, we mention the following special results \((p > 3)\):

\[(1.7)\quad B(p)_{p^r} = -\frac{1}{2} p^{r+1} (p - 1) B_{p^r-1} \pmod{p^{r+2}}\]

for \(r > 1;\)

\[(1.8)\quad B(p)_{m} = \frac{1}{2} p (p - 1) B_{m-1} \pmod{p^{r+2}}\]

for \(m \equiv 1 \pmod{p^r(p - 1)}\).

It also follows from the results of §6 that \(B_m\) is integral \(\pmod{p}\), \(p \geq 3\), unless \(m \equiv 0 \pmod{p - 1}\) and \(m \equiv 0\) or \(p - 1 \pmod{p}\), in which case \(pB_m\) is integral.

The number \(B_{m+1}^{(p+1)}\) requires a more detailed discussion than \(B_m^{(p)}\); this will be omitted from the present paper. However, we note the special formula

\[(1.9)\quad B_{p^r}^{(p+1)} = p^r \left\{ \frac{1}{2} p(p+1) \frac{B_{p^r-1}}{p^r-1} + p! \frac{B_{p^r-p}}{p^r-p} \right\} \pmod{p^{r+2}}\]

for \(p > 3, r > 1;\) The residue \(\pmod{p^{r+3}}\) can be specified.

2. Some preliminary results. We first state a number of formulas involving \(B_m^{(k)}\) which may be found in [3, Chapter 6].

\[(2.1)\quad B_m^{(k+1)} = (k + 1) \binom{m}{k+1} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} B_s^{(k+1)} \frac{B_{m-s}}{m-s},\]

\[(2.2)\quad (x-1)(x-2)\cdots(x-m) = \sum_{s=0}^{m} \binom{m}{s} B_s^{(m+1)} x^{m-s}.\]
We shall require the special values

\[ B_1^{(k)} = -\frac{1}{2} k, \quad B_2^{(k)} = \frac{1}{12} k (3k - 1), \quad B_3^{(k)} = -\frac{1}{8} k^2 (k - 1). \]

If we define the factorial coefficients by means of

\[ (x + 1) \cdots (x + m - 1) = \sum_{s=0}^{m-1} C_s^{(m)} x^{m-1-s}, \]

we see at once that

\[ (-1)^s \binom{m}{s} B_s^{(m+1)} = C_s^{(m+1)}. \]

We have also the recurrence formula

\[ C_s^{(m+1)} = C_s^{(m)} + mC_{s-1}^{(m)}. \]

In the next place [1, p. 325; 2, p. 328] for \( p \) a prime > 3,

\[ C_{2r}^{(p)} = -p \frac{B_{2r}}{2r} \pmod{p^2} \quad (2 \leq 2r \leq p - 3), \]

\[ C_{2r+1}^{(p)} = p^2 \frac{(2r + 1) B_{2r}}{4r} \pmod{p^3} \quad (r \geq 1), \]

\[ C_{p-1}^{(p)} = (p - 1)! = p (-1 + B_{p-1}) \pmod{p^2}. \]

It follows immediately from (2.8) and Wilson’s theorem that

\[ p (p + 1) B_{p-1} = (p - 1)! \pmod{p^2}. \]

We shall require the following special case of Kummer’s congruence [2, Chapter 14]:

\[ \frac{B_{m+p-1}}{m+p-1} = \frac{B_m}{m} \pmod{p} \quad (p - 1 \nmid m); \]

also, the Staudt-Clausen theorem [3, 32] which we quote in the following form:
(2.11) \[ pB_m \equiv -1 \pmod{p} \quad \text{(} p - 1 \mid m \text{).} \]

A formula of a different sort that will be used is [3, p. 146, formula (83)]

\[ B_m = -\frac{1}{m} \sum_{s=1}^{m} (-1)^s \binom{m}{s} B_s B_{m-s}. \]

In particular, replacing \( m \) by \( 2m \), this becomes

(2.12) \[ (2m + 1) B_{2m} + \sum_{t=1}^{m-1} \binom{2m}{2t} B_{2t} B_{2m-2t} = 0 \]

provided \( m > 1 \), a formula due to Euler. The formula [3, p. 145]

(2.13) \[ B_{m+1}^{(k+1)} = \left(1 - \frac{m}{k}\right) B_m^{(k)} - mB_{m-1}^{(k)} \]

will also be employed. In particular, we note that

(2.14) \[ B_{(m+1)} = (-1)^m m! \]

3. Proof of (1.2). Let \( p \) be a prime > 3. In (2.1), taking \( k = p, m = p + 2 \), we get

\[
B_{p+2}^{(p+1)} = (p + 1) (p + 2) \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} \frac{B_{p+2-s}}{p + 2 - s} B_{s}^{(p+1)}
\]

(3.1) \[
= (p + 1) (p + 2) \sum_{t=0}^{(p-1)/2} \binom{p}{2t + 1} \frac{B_{p+1-2t}}{p + 1 - 2t} B_{2t+1}^{(p+1)}
\]

say. We break the sum \( A \) into several parts:

(3.2) \[ A = u_0 + u_1 + \sum_{t=2}^{(p-3)/2} u_t + u_{(p-1)/2}, \]

where
\[ u_t = \binom{p}{2t + 1} B_{2t+1}^{(p+1)} \frac{B_{p+1-2t}}{p + 1 - 2t} \quad (0 \leq t \leq p - 1). \]

Then by (2.2) and (2.3) we have

\[ u_0 = pB_1^{(p+1)} \frac{B_{p+1}}{p + 1} = -\frac{1}{2} pB_{p+1}; \]

and

\[ u_{(p-1)/2} = \frac{1}{2} B_2 B_{p}^{(p+1)} = -\frac{1}{12} p! \]

by (2.14). As for \( u_1 \) we have, by (2.3),

\[ \binom{p}{3} B_3^{(p+1)} = -\frac{1}{48} p^2 (p + 1)^2 (p - 1) (p - 2) \]

\[ = -\frac{1}{48} (p^3 + 2p^2) \pmod{p^4}; \]

thus, by (2.8),

\[ u_1 = -\frac{1}{48} (p^2 + 2p) \frac{pB_{p-1}}{p - 1} = -\frac{1}{48} (p^2 + 2p) \frac{(p - 1)!}{p^2 - 1} \]

\[ \equiv \frac{1}{48} (p^2 + 2p) (p - 1)! \pmod{p^3}. \]

In the next place, by (2.4) and (2.5),

\[ \binom{p}{2t + 1} B_{2t+1}^{(p+1)} = -C_{2t+1}^{(p+1)} = -C_{2t+1}^{(p)} - pC_{2t}^{(p)} \]

\[ = -p^2 \left( \frac{2t + 1}{4t} - \frac{1}{2t} \right) B_{2t} \]

\[ \equiv -p^2 \frac{2t - 1}{4t} B_{2t} \pmod{p^3} \]

for \( 2 \leq t \leq (p - 3)/2 \). Hence
\[
\sum_{t=2}^{(p-3)/2} u_t = p^2 \sum_{t=2}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p+1-2t} \pmod{p^3},
\]
so that

\[
(3.6) \quad \sum_{t=2}^{(p-3)/2} u_t = p^2 \sum_{t=2}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p+1-2t} \pmod{p^3}.
\]

On the other hand, by (2.12),

\[
(p + 2) B_{p+1} + \sum_{t=1}^{(p-1)/2} \left( p + 1 \right) \frac{1}{2t} B_{2t} B_{p+1-2t} = 0,
\]

which implies

\[
(3.61) \quad (p + 2) B_{p+1} + \frac{1}{6} p (p + 1) B_{p-1} = p (p + 1) \sum_{t=2}^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p+1-2t} \pmod{p^3};
\]

the last congruence is a consequence of

\[
\frac{1}{2t} + \frac{1}{p + 1 - 2t} = \frac{1}{2t(p + 1 - 2t)} \pmod{p}.
\]

Now using (3.6) we see that

\[
(3.7) \quad \sum_{t=2}^{(p-3)/2} u_t = \frac{1}{4} p (p + 2) B_{p+1} + \frac{1}{24} p^2 (p + 1) B_{p-1}
\]

by (2.9). Collecting from (3.3), (3.4), (3.5), and (3.7) we get, after some simplification,

\[
A = \frac{1}{4} p^2 B_{p+1} + \frac{1}{48} p^2 (p - 1)!
\]
SOME THEOREMS ON BERNOULLI NUMBERS OF HIGHER ORDER

\[
\frac{1}{4} p^2 \left( \frac{B_{p+1}}{p+1} - \frac{B_2}{2} \right) + \frac{1}{48} p^2 + \frac{1}{48} p^2 (p - 1)!
\]

\[\equiv 0 \pmod{p^3}\]

by (2.10). Therefore, by (3.1), \(B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}\).

It would be of interest to determine the residue of \(B_{p+2}^{(p+1)} \pmod{p^4}\). We have already noted that \(B_5^{(4)} \not\equiv 0 \pmod{3^3}\); for small \(p\) at least, it can be verified that \(B_{p+2}^{(p+1)} \not\equiv 0 \pmod{p^4}\).

4. Proof of (1.3). We now take \(m = p > 3, k = p - 1\) in (2.1), so that

\[
B_p^{(p)} = p \sum_{s=0}^{p-1} (-1)^s \binom{p - 1}{s} \frac{B_{p-s}}{p-s} B_s^{(p)}
\]

(4.1)

by (2.3) and (2.4). Now, again using (2.4), we have

\[
\binom{p - 1}{2t + 1} B_{2t+1}^{(p)} = - C_{2t+1}^{(p)}
\]

\[= - p^2 \frac{(2t + 1) \ B_{2t+1}}{4t} \pmod{p^3}.
\]

Hence, the sum \(Q\) in the right member of (4.1) satisfies \(Q \equiv 0 \pmod{p^2}\); more precisely, we see that

\[
Q \equiv p^2 \sum_{t=1}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p-1-2t} \pmod{p^3},
\]

(4.2)

to which we return presently. Thus, it is clear that (4.1) reduces to

\[
B_p^{(p)} \equiv \frac{p}{2} \ (pB_{p-1} - (p - 1)!) \pmod{p^3}.
\]

But by (2.8) this implies

\[
B_p^{(p)} \equiv \frac{1}{2} p^2 \pmod{p^3}.
\]
Since $B_{3}^{(3)} = -9/4 \equiv 9/2 \pmod{27}$, (4.3) holds for $p \geq 3$.

To determine the residue of $B_{p}^{(p)} \pmod{p^4}$ we make use of [2, p. 366, formula (10)],

$$\sum_{t=1}^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p-1-2t} = \frac{1}{p} (\#_p - K_p) - \#_p \pmod{p},$$

where $\#_p, K_p$ are defined by

$$(p - 1)! + 1 = p\#_p, \quad a^{p-1} - 1 = pk(a) \quad \quad (p \nmid a),$$

$$K_p = k(1) + k(2) + \cdots + k(p - 1).$$

Then, by (4.1) and (4.2),

$$B_{p}^{(p)} = \frac{1}{2} p \left\{ pR_{p-1} + 1 - pK_p - p^2\#_p \right\} \pmod{p^4};$$

since $\#_p \equiv K_p \pmod{p}$, this may also be put in the form

$$B_{p}^{(p)} = \frac{1}{2} p^2 \left\{ B_{p-1} + \frac{1}{p} - (p + 1)K_p \right\} \pmod{p^4}.$$

That (4.5) includes (4.3) is easily verified.

5. **Proof of (1.4), (1.5), (1.6).** In the remainder of the paper let $p > 3$.

In (2.13) take $k = p$, $m = p + 2$; then

$$B_{p+2}^{(p+1)} = \left( 1 - \frac{p + 2}{p} \right) B_{p+1}^{(p)} - (p + 2) B_{p+1}^{(p+1)}.$$

Therefore, by (1.2),

$$\frac{2}{p} B_{p+2}^{(p)} + (p + 2) B_{p+1}^{(p)} \equiv 0 \pmod{p^3}.$$

Now take $k = p - 1$, $m = p + 2$ in (2.1), so that

$$B_{p+2}^{(p)} = p \left( \frac{p + 2}{p} \right) \sum_{s=0}^{p-2} (-1)^s \left( \frac{p - 1}{s} \right) B_{s}^{(p)} \frac{B_{p+2-s}}{p + 2 - s}.$$

Clearly only odd values of $s$ need be considered; we get, using (2.4),
\[
B_{p+2}^{(p)} = (p + 2) \left( \frac{p + 1}{p - 1} \right)^{(p-3)/2} \sum_{t=0}^{(p-3)/2} \binom{p-1}{2t+1} \frac{B_{p+1-2t}}{p + 1 - 2t} \\
= (p + 2) \left( \frac{p + 1}{p - 1} \right) \left( -\frac{1}{2} \frac{B_{p+1}}{p + 1} + \frac{1}{8} p^2 \frac{B_{p-1}}{p - 1} \right) \pmod{p^3}
\]

by (2.3) and (2.7); next, by (2.9) and (2.10), we get

\[(5.2) \quad B_{p+2}^{(p)} \equiv \frac{1}{12} p^2 \pmod{p^3}.\]

In view of (5.1) we have also

\[(5.3) \quad B_{p+1}^{(p)} \equiv -\frac{1}{12} p \pmod{p^2}.\]

However, (5.2) and (5.3) do not imply (5.1) but only the weaker result with modulus \(p^2\).

To improve these results we follow the method of \(\S 3\). Thus

\[(5.4) \quad B_{p+1}^{(p)} = p(p+1) \sum_{s=0}^{(p-1)/2} \binom{p-1}{2s} \frac{B_{p+1-2s}}{p + 1 - 2s} = p(p+1) A,\]

and

\[A = \frac{B_{p+1}}{p + 1} + \binom{p}{2} \frac{B_{p-1}}{p - 1} + \sum_{t=2}^{(p-3)/2} \binom{p-3/2}{2t} \frac{B_{p+1-2t}}{p + 1 - 2t} + \frac{1}{12} (p - 1)!.\]

But, by (3.61),

\[
\sum_{t=2}^{(p-3)/2} \binom{p}{2t} \frac{B_{p+1-2t}}{p + 1 - 2t} \equiv -p \sum_{t=2}^{(p-3)/2} \frac{B_{2t}}{2t} \frac{B_{p+1-2t}}{p + 1 - 2t} \\
= -\frac{p + 2}{p + 1} B_{p+1} - \frac{1}{6} p B_{p-1} \pmod{p^2},
\]

so that after some simplification we get

\[A \equiv -\frac{B_{p+1}}{p + 1} + \frac{1}{8} p \pmod{p^2},\]
and therefore, by (5.4) and (2.10),

\[(5.5) \quad B_{p+1}^{(p)} = -p \frac{B_{p+1}}{p+1} + \frac{1}{24} p^2 \pmod{p^3}.\]

In view of (5.1) this implies

\[(5.6) \quad B_{p+2}^{(p)} = p^2 \frac{B_{p+1}}{p+1} \pmod{p^4}.\]

That (5.5) and (5.6) include (5.3) and (5.2) is evident; also (5.5) and (5.6) imply (1.2).

We remark also that using (2.13), (5.5), and (1.3) we get

\[(5.7) \quad B_{p+1}^{(p+1)} = \frac{B_{p+1}}{p+1} - \frac{1}{24} p \pmod{p^2}.\]

6. Discussion of $B_m^{(p)}$. Let first $m > p$ be odd, so that (2.1) implies

\[(6.1) \quad B_m^{(p)} = p \binom{m}{p} \sum_{t=0}^{(p-3)/2} C_{2t+1}^{(p)} \frac{B_{m-1-2t}}{m-1-2t}.\]

Now let $m \equiv a \pmod{p}$, $0 \leq a < p$; $m \equiv b \pmod{p-1}$, $0 < b < p-1$. Also, let $p^r \mid m - a$, $p^{r+1} \nmid m - a$, so that the binomial coefficient $\binom{m}{p}$ is divisible by exactly $p^{r-1}$. Clearly $b$ is odd. Now by a well-known theorem [2, p. 252], if $a \neq b$, the quotient $B_{m-a}/(m-a)$ is integral (mod $p$). Thus, by (2.7), the right member of (6.1), except for the terms corresponding to $t = 0, (b-1)/2$, is a multiple of $p^{r+2}$. As for the exceptional terms

\[(6.2) \quad u_1 = p \binom{m}{p} C_1^{(p)} \frac{B_{m-1}}{m-1}, \quad u_b = p \binom{m}{p} C_b^{(p)} \frac{B_{m-b}}{m-b},\]

there are several possibilities.

(i) Suppose $b = 1$, so that the two terms in (6.2) coincide. Then if $a \neq 1$, we see that the term in question is exactly divisible by $p^r$. On the other hand if $a = 1$, the term is integral (mod $p$) but not divisible by $p$.

(ii) If $b \neq 1$, $u_1$ and $u_b$ in (6.2) are distinct. There are several cases to consider. If $a = b$, then $u_1$ is divisible by $p^{r+1}$, while $u_b$ is divisible by exactly $p^{r+1}$. Thus, in this sub-case $B_m^{(p)} = 0 \pmod{p^{r+1}}$; for $m = p+2$ this is less precise than (5.2).

In the next place, let $m$ be even and define $a$, $b$, $r$ as above so that $b$ is now
Then we have

$$B_m^{(p)} = p \binom{m}{p} \frac{(p-1)/2}{\sum_{t=0}^{m-2t} C_{2t}^{(p)} \frac{B_{m-2t}}{m-2t}} = \sum_{t=0}^{(p-1)/2} u_{2t}. $$

Then by (2.6) the right member, except for the terms $u_0, u_b, u_{p-1}$, is a multiple of $p^{r+1}$. We consider a number of cases.

(iii) If $b = 0$, there are only two distinct terms $u_0, u_{p-1}$. If $a = 0$, we find that $pu_0$ is integral (mod $p$); indeed $pu_0 = -1$ (mod $p$) by the Staudt-Clausen theorem (2.11). On the other hand, $u_{p-1}$ is divisible by $p^{r-1}$; indeed $u_{p-1} = m/(m - p + 1)$ (mod $p^r$). If $a = p - 1$, then $u_0 \equiv (m - p + 1)/m$ (mod $p^r$) while $pu_{p-1} \equiv 1$ (mod $p$). If $a \neq 0$ or $p - 1$ then it can be verified that $u_0 + u_{p-1}$ is divisible by $p^r$.

(iv) If $b \neq 0$, then all three terms $u_0, u_b, u_{p-1}$ are distinct. By means of Kummer's congruence (2.10) we find that $u_0 + u_{p-1} \equiv 0$ (mod $p^{r+1}$); in other words;

$$B_m^{(p)} \equiv u_b \pmod{p^{r+1}} \quad (b \neq 0).$$

As for $u_b$, there are several possibilities. If $a = b$, it is easily seen that $u_b$ is integral (mod $p$); moreover, by (2.6), $u_b \equiv 0$ (mod $p$) if and only if $B_b \equiv 0$ (mod $p$). If $a \neq b$, then $u_b$ is divisible by $p^r$ at least; indeed using (6.4) we get

$$B_m^{(p)} \equiv B_b \frac{m-a}{b(m-b)} \pmod{p^{r+1}} \quad (a \neq b, b \neq 0).$$

This result evidently includes (5.3) but not (5.5).

We remark that $B_m^{(p)}$ is integral (mod $p$) in cases (i), (ii), (iv). In case (iii), however, if $a = 0$ or $p - 1$, then $B_m^{(p)}$ is no longer integral, but $pB_m^{(p)}$ is integral; indeed it is easily verified that

$$pB_m^{(p)} \equiv \begin{cases} -1 & \pmod{p} \quad (a = 0), \\ +1 & \pmod{p} \quad (a = p - 1). \end{cases}$$

7. Some special cases. Clearly $m = p^r$, $r > 1$, falls under (i) above with $a = 0, b = 1$. Thus,

$$B_m^{(p)} \equiv - \frac{1}{2} p^{r-1} (p - 1) B_{p^r-1} \pmod{p^{r+2}},$$

and in particular,
\[
B_{m}^{(p)} = \frac{1}{2} (p - 1) pB_{m-1} \quad (\text{mod } p^{r+2}).
\]

For \( m = 1 \pmod{p^{r}(p-1)} \), we have \( a = b = 1 \) which also falls under (i); we now have

(7.3)

\[
B_{m}^{(p)} = \frac{1}{2} (p - 1) pB_{m-1} \quad (\text{mod } p^{r+2}).
\]

For \( m = cp^{r} \), where \( c \) is odd, \( p \nmid c \), we have \( a = 0, c \equiv b \pmod{p-1} \), which evidently falls under (i) or (ii). Thus, we get \( r \geq 1 \)

(7.4)

\[
B_{c^{p^{r}}}^{(p)} = \frac{1}{2} c^{p^{r+1}} \frac{p - 1}{c^{p^{r}} - 1} B_{c^{p^{r-1}}} \quad (\text{mod } p^{r+2})
\]

for \( c \equiv 1 \pmod{p-1} \);

(7.5)

\[
B_{c^{p^{r}}}^{(p)} = -\frac{1}{2} c^{p^{r+1}} \left( B_{c^{p^{r-1}}} + \frac{1}{b-1} \right) \quad (\text{mod } p^{r+2})
\]

for \( c \equiv b \pmod{p-1}, b \neq 1 \).

Similarly, for \( m = cp^{r} \), \( c \) even, \( p \nmid c \), we have \( a = 0, c \equiv b \pmod{p-1} \), which falls under (iii) or (iv). We consider only the case \( p - 1 \nmid c \); that is, \( b \neq 0 \). Then, by (6.5), we have

(7.6)

\[
B_{c^{p^{r}}}^{(p)} = -\frac{c}{b^{2}} p^{r} B_{b} \quad (\text{mod } p^{r+1}).
\]

Again for \( m = cp^{r} + a, c \) odd, \( a \) even, we find

(7.7)

\[
B_{m}^{(p)} = \frac{1}{2} cp^{r+1} (p - 1) \frac{B_{m-1}}{m - 1} \quad (\text{mod } p^{r+2})
\]

for \( b = 1 \), while

(7.8)

\[
B_{m}^{(p)} = -\frac{1}{2} cp^{r+1} \left( \frac{B_{m-1}}{a - 1} - \frac{b}{(b - 1)(b - a)} \right) \quad (\text{mod } p^{r+2})
\]

for \( b \neq 1 \); in these two formulas we have \( 0 < a < p - 1, b \equiv c + a \pmod{p-1} \). For \( c \) even, \( a \) odd, (7.7) holds; but (7.8) requires modification. For \( c \) and \( a \) both odd or both even, there are several cases; in particular, by (6.5) we have
for \( a \neq b, b \neq 0. \)

For \( p = 2, 3 \) it follows at once from (2.1) that

\[
B_m^{(2)} = -m(m - 1) \left( \frac{B_m}{m} + \frac{B_{m-1}}{m-1} \right),
\]

\[
B_m^{(3)} = \frac{1}{2} m(m - 1)(m - 2) \left( \frac{B_m}{m} + 3 \frac{B_{m-1}}{m-1} + 2 \frac{B_{m-2}}{m-2} \right),
\]

by means of which numerous special formulas can easily be obtained, for example,

\[
B_m^{(2)} = \begin{cases} 
- (m - 1) B_m & \text{if } m \text{ even } > 2, \\
- mB_{m-1} & \text{if } m \text{ odd }, 
\end{cases}
\]

\[
B_m^{(3)} = \frac{3}{2} m(m - 2) B_{m-1} & \text{if } m \text{ odd } > 1.
\]

References

1. J. W. L. Glaisher, On the residues of the sums of products of the first \( p - 1 \) numbers, and their powers, to modulus \( p^2 \) or \( p^3 \), Quarterly Journal of Pure and Applied Mathematics, 31 (1900), 321-353.


L. Carlitz, *Some theorems on Bernoulli numbers of higher order* ............ 127
Watson Bryan Fulks, *On the boundary values of solutions of the heat equation* ................................................................. 141
John W. Green, *On the level surfaces of potentials of masses with fixed center of gravity* .......................................................... 147
Isidore Heller, *Contributions to the theory of divergent series* ............ 153
Melvin Henriksen, *On the ideal structure of the ring of entire functions* ... 179
James Richard Jackson, *Some theorems concerning absolute neighborhood retracts* ............................................................... 185
Everett H. Larguier, *Homology bases with applications to local connectedness* ................................................................. 191
Janet McDonald, *Davis’s canonical pencils of lines* .............................. 209
J. D. Niblett, *Some hypergeometric identities* ........................................ 219
Elmer Edwin Osborne, *On matrices having the same characteristic equation* ................................................................. 227
Robert Steinberg and Raymond Moos Redheffer, *Analytic proof of the Lindemann theorem* ........................................................ 231
Edward Silverman, *Set functions associated with Lebesgue area* .......... 243
James G. Wendel, *Left centralizers and isomorphisms of group algebras* ... 251
Kosaku Yosida, *On Brownian motion in a homogeneous Riemannian space* ................................................................. 263