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**1. Introduction.** We define the Bernoulli numbers of order  $k$  by means of [3, Chapter 6]

$$\left( \frac{t}{e^t - 1} \right)^k = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m^{(k)} \quad (|t| < 2\pi);$$

in particular,  $B_m = B_m^{(1)}$  denotes the ordinary Bernoulli number. Not much seems to be known about divisibility properties of  $B_m^{(k)}$ . Using different notation, S. Wachs [4] proved a result which may be stated in the form

$$(1.1) \quad B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2},$$

where  $p$  is a prime  $\geq 3$ . In attempting to simplify Wachs' proof, the writer found the stronger result

$$(1.2) \quad B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3} \quad (p > 3).$$

We remark that  $B_5^{(4)} = -9$ .

The proof of (1.2) depends on some well-known properties of the Bernoulli numbers and factorial coefficients; in particular, we make use of some theorems of Glaisher and Nielsen. The necessary formulas are collected in §2; the proof of (1.2) is given in §3. In §4 we prove

$$(1.3) \quad B_p^{(p)} \equiv \frac{1}{2} p^2 \pmod{p^3} \quad (p \geq 3);$$

the proof of this result is somewhat simpler than that of (1.2). For the residue of  $B_p^{(p)} \pmod{p^4}$ , see (4.5) below.

In §5 we prove several formulas of a similar nature ( $p > 3$ ):

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$$(1.4) \quad B_{p+1}^{(p)} \equiv -p \frac{B_{p+1}}{p+1} + \frac{1}{24} p^2 \pmod{p^3},$$

$$(1.5) \quad B_{p+2}^{(p)} \equiv p^2 \frac{B_{p+1}}{p+1} \pmod{p^4},$$

$$(1.6) \quad B_{p+1}^{(p+1)} \equiv \frac{B_{p+1}}{p+1} - \frac{1}{24} p \pmod{p^2}.$$

In §6 we discuss the number  $B_m^{(p)}$  for arbitrary  $m$ ; this requires the consideration of a number of cases. In particular, we mention the following special results ( $p > 3$ ):

$$(1.7) \quad B_{p^r}^{(p)} \equiv -\frac{1}{2} p^{r+1} (p-1) B_{p^{r-1}} \pmod{p^{r+2}}$$

for  $r > 1$ ;

$$(1.8) \quad B_m^{(p)} \equiv \frac{1}{2} p(p-1) B_{m-1} \pmod{p^{r+2}}$$

for  $m \equiv 1 \pmod{p^r(p-1)}$ .

It also follows from the results of §6 that  $B_m^{(p)}$  is integral  $\pmod{p}$ ,  $p \geq 3$ , unless  $m \equiv 0 \pmod{p-1}$  and  $m \equiv 0$  or  $p-1 \pmod{p}$ , in which case  $pB_m^{(p)}$  is integral.

The number  $B_m^{(p+1)}$  requires a more detailed discussion than  $B_m^{(p)}$ ; this will be omitted from the present paper. However, we note the special formula

$$(1.9) \quad B_{p^r}^{(p+1)} \equiv p^r \left\{ \frac{1}{2} p(p+1) \frac{B_{p^{r-1}}}{p^r-1} + p! \frac{B_{p^{r-p}}}{p^r-p} \right\} \pmod{p^{r+2}}$$

for  $p > 3$ ,  $r > 1$ . The residue  $\pmod{p^{r+3}}$  can be specified.

**2. Some preliminary results.** We first state a number of formulas involving  $B_m^{(k)}$  which may be found in [3, Chapter 6].

$$(2.1) \quad B_m^{(k+1)} = (k+1) \binom{m}{k+1} \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} B_s^{(k+1)} \frac{B_{m-s}}{m-s},$$

$$(2.2) \quad (x-1)(x-2)\cdots(x-m) = \sum_{s=0}^m \binom{m}{s} B_s^{(m+1)} x^{m-s}.$$

We shall require the special values

$$(2.3) \quad B_1^{(k)} = -\frac{1}{2}k, \quad B_2^{(k)} = \frac{1}{12}k(3k-1), \quad B_3^{(k)} = -\frac{1}{8}k^2(k-1).$$

If we define the factorial coefficients by means of

$$(x+1) \cdots (x+m-1) = \sum_{s=0}^{m-1} C_s^{(m)} x^{m-1-s},$$

we see at once that

$$(2.4) \quad (-1)^s \binom{m}{s} B_s^{(m+1)} = C_s^{(m+1)}.$$

We have also the recurrence formula

$$(2.5) \quad C_s^{(m+1)} = C_s^{(m)} + m C_{s-1}^{(m)}.$$

In the next place [1, p. 325; 2, p. 328] for  $p$  a prime  $> 3$ ,

$$(2.6) \quad C_{2r}^{(p)} \equiv -p \frac{B_{2r}}{2r} \pmod{p^2} \quad (2 \leq 2r \leq p-3),$$

$$(2.7) \quad C_{2r+1}^{(p)} \equiv p^2 \frac{(2r+1)B_{2r}}{4r} \pmod{p^3} \quad (r \geq 1),$$

$$(2.8) \quad C_{p-1}^{(p)} = (p-1)! \equiv p(-1 + B_{p-1}) \pmod{p^2}.$$

It follows immediately from (2.8) and Wilson's theorem that

$$(2.9) \quad p(p+1)B_{p-1} \equiv (p-1)! \pmod{p^2}.$$

We shall require the following special case of Kummer's congruence [2, Chapter 14]:

$$(2.10) \quad \frac{B_{m+p-1}}{m+p-1} \equiv \frac{B_m}{m} \pmod{p} \quad (p-1 \nmid m);$$

also, the Staudt-Clausen theorem [3, 32] which we quote in the following form:

$$(2.11) \quad pB_m \equiv -1 \pmod{p} \quad (p-1 \mid m).$$

A formula of a different sort that will be used is [3, p.146, formula (83)]

$$B_m = -\frac{1}{m} \sum_{s=1}^m (-1)^s \binom{m}{s} B_s B_{m-s}.$$

In particular, replacing  $m$  by  $2m$ , this becomes

$$(2.12) \quad (2m+1)B_{2m} + \sum_{t=1}^{m-1} \binom{2m}{2t} B_{2t} B_{2m-2t} = 0$$

provided  $m > 1$ , a formula due to Euler. The formula [3, p.145]

$$(2.13) \quad B_m^{(k+1)} = \left(1 - \frac{m}{k}\right) B_m^{(k)} - m B_{m-1}^{(k)}$$

will also be employed. In particular, we note that

$$(2.14) \quad B_m^{(m+1)} = (-1)^m m!$$

**3. Proof of (1.2).** Let  $p$  be a prime  $> 3$ . In (2.1), taking  $k = p$ ,  $m = p + 2$ , we get

$$\begin{aligned} B_{p+2}^{(p+1)} &= (p+1)(p+2) \sum_{s=0}^p (-1)^{p-s} \binom{p}{s} \frac{B_{p+2-s}}{p+2-s} B_s^{(p+1)} \\ (3.1) \quad &= (p+1)(p+2) \sum_{t=0}^{(p-1)/2} \binom{p}{2t+1} \frac{B_{p+1-2t}}{p+1-2t} B_{2t+1}^{(p+1)} \\ &= (p+1)(p+2)A, \end{aligned}$$

say. We break the sum  $A$  into several parts:

$$(3.2) \quad A = u_0 + u_1 + \sum_{t=2}^{(p-3)/2} u_t + u_{(p-1)/2},$$

where

$$u_t = \binom{p}{2t+1} B_{2t+1}^{(p+1)} \frac{B_{p+1-2t}}{p+1-2t} \quad (0 \leq t \leq p-1).$$

Then by (2.2) and (2.3) we have

$$(3.3) \quad u_0 = p B_1^{(p+1)} \frac{B_{p+1}}{p+1} = -\frac{1}{2} p B_{p+1};$$

and

$$(3.4) \quad u_{(p-1)/2} = \frac{1}{2} B_2 B_p^{(p+1)} = -\frac{1}{12} p!$$

by (2.14). As for  $u_1$  we have, by (2.3),

$$\begin{aligned} \binom{p}{3} B_3^{(p+1)} &= -\frac{1}{48} p^2 (p+1)^2 (p-1)(p-2) \\ &\equiv -\frac{1}{48} (p^3 + 2p^2) \pmod{p^4}; \end{aligned}$$

thus, by (2.8),

$$\begin{aligned} (3.5) \quad u_1 &\equiv -\frac{1}{48} (p^2 + 2p) \frac{p B_{p-1}}{p-1} \equiv -\frac{1}{48} (p^2 + 2p) \frac{(p-1)!}{p^2-1} \\ &\equiv \frac{1}{48} (p^2 + 2p) (p-1)! \pmod{p^3}. \end{aligned}$$

In the next place, by (2.4) and (2.5),

$$\begin{aligned} \binom{p}{2t+1} B_{2t+1}^{(p+1)} &= -C_{2t+1}^{(p+1)} = -C_{2t+1}^{(p)} - p C_{2t}^{(p)} \\ &\equiv -p^2 \left( \frac{2t+1}{4t} - \frac{1}{2t} \right) B_{2t} \\ &\equiv -p^2 \frac{2t-1}{4t} B_{2t} \pmod{p^3} \end{aligned}$$

for  $2 \leq t \leq (p-3)/2$ . Hence

$$u_t \equiv p^2 \frac{B_{2t}}{4t} B_{p+1-2t} \pmod{p^3},$$

so that

$$(3.6) \quad \sum_{t=2}^{(p-3)/2} u_t \equiv p^2 \sum_{t=2}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p+1-2t} \pmod{p^3}.$$

On the other hand, by (2.12),

$$(p+2) B_{p+1} + \sum_{t=1}^{(p-1)/2} \binom{p+1}{2t} B_{2t} B_{p+1-2t} = 0,$$

which implies

$$(3.61) \quad \begin{aligned} (p+2) B_{p+1} + \frac{1}{6} p(p+1) B_{p-1} &\equiv p(p+1) \sum_2^{(p-3)/2} \frac{B_{2t}}{2t} \frac{B_{p+1-2t}}{p+1-2t} \\ &\equiv 2p \sum_2^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p+1-2t} \pmod{p^2}; \end{aligned}$$

the last congruence is a consequence of

$$\frac{1}{2t} + \frac{1}{p+1-2t} \equiv \frac{1}{2t(p+1-2t)} \pmod{p}.$$

Now using (3.6) we see that

$$(3.7) \quad \begin{aligned} \sum_2^{(p-3)/2} u_t &\equiv \frac{1}{4} p(p+2) B_{p+1} + \frac{1}{24} p^2 (p+1) B_{p-1} \\ &\equiv \frac{1}{4} p(p+2) B_{p+1} + \frac{1}{24} p(p-1)! \pmod{p^3} \end{aligned}$$

by (2.9). Collecting from (3.3), (3.4), (3.5), and (3.7) we get, after some simplification,

$$A \equiv \frac{1}{4} p^2 B_{p+1} + \frac{1}{48} p^2 (p-1)!$$

$$\begin{aligned} &\equiv \frac{1}{4} p^2 \left( \frac{B_{p+1}}{p+1} - \frac{B_2}{2} \right) + \frac{1}{48} p^2 + \frac{1}{48} p^2 (p-1)! \\ &\equiv 0 \pmod{p^3} \end{aligned}$$

by (2.10). Therefore, by (3.1),  $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}$ .

It would be of interest to determine the residue of  $B_{p+2}^{(p+1)} \pmod{p^4}$ . We have already noted that  $B_5^{(4)} \not\equiv 0 \pmod{3^3}$ ; for small  $p$  at least, it can be verified that  $B_{p+2}^{(p+1)} \not\equiv 0 \pmod{p^4}$ .

**4. Proof of (1.3).** We now take  $m = p > 3$ ,  $k = p - 1$  in (2.1), so that

$$\begin{aligned} (4.1) \quad B_p^{(p)} &= p \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} \frac{B_{p-s}}{p-s} B_s^{(p)} \\ &= p \left\{ \frac{1}{2} p B_{p-1} - \frac{1}{2} (p-1)! - \sum_{t=1}^{(p-3)/2} \binom{p-1}{2t+1} \frac{B_{p-1-2t}}{p-1-2t} B_{2t+1}^{(p)} \right\} \end{aligned}$$

by (2.3) and (2.4). Now, again using (2.4), we have

$$\begin{aligned} \binom{p-1}{2t+1} B_{2t+1}^{(p)} &= -C_{2t+1}^{(p)} \\ &\equiv -p^2 \frac{(2t+1) B_{2t+1}}{4t} \pmod{p^3}. \end{aligned}$$

Hence, the sum  $Q$  in the right member of (4.1) satisfies  $Q \equiv 0 \pmod{p^2}$ ; more precisely, we see that

$$(4.2) \quad Q \equiv p^2 \sum_1^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p-1-2t} \pmod{p^3},$$

to which we return presently. Thus, it is clear that (4.1) reduces to

$$B_p^{(p)} \equiv \frac{p}{2} (p B_{p-1} - (p-1)!) \pmod{p^3}.$$

But by (2.8) this implies

$$(4.3) \quad B_p^{(p)} \equiv \frac{1}{2} p^2 \pmod{p^3}.$$



Since  $B_3^{(3)} = -9/4 \equiv 9/2 \pmod{27}$ , (4.3) holds for  $p \geq 3$ .

To determine the residue of  $B_p^{(p)} \pmod{p^4}$  we make use of [2, p.366, formula (10)],

$$(4.4) \quad \sum_{t=1}^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p-1-2t} \equiv \frac{1}{p} (\mathbb{W}_p - K_p) - \mathbb{W}_p \pmod{p},$$

where  $\mathbb{W}_p, K_p$  are defined by

$$(p-1)! + 1 = p\mathbb{W}_p, \quad a^{p-1} - 1 = pk(a) \quad (p \nmid a),$$

$$K_p = k(1) + k(2) + \dots + k(p-1).$$

Then, by (4.1) and (4.2),

$$B_p^{(p)} \equiv \frac{1}{2} p \left\{ pB_{p-1} + 1 - pK_p - p^2\mathbb{W}_p \right\} \pmod{p^4};$$

since  $\mathbb{W}_p \equiv K_p \pmod{p}$ , this may also be put in the form

$$(4.5) \quad B_p^{(p)} \equiv \frac{1}{2} p^2 \left\{ B_{p-1} + \frac{1}{p} - (p+1)K_p \right\} \pmod{p^4}.$$

That (4.5) includes (4.3) is easily verified.

**5. Proof of (1.4), (1.5), (1.6).** In the remainder of the paper let  $p > 3$ .

In (2.13) take  $k = p, m = p + 2$ ; then

$$B_{p+2}^{(p+1)} = \left( 1 - \frac{p+2}{p} \right) B_{p+2}^{(p)} - (p+2) B_{p+1}^{(p)}.$$

Therefore, by (1.2),

$$(5.1) \quad \frac{2}{p} B_{p+2}^{(p)} + (p+2) B_{p+1}^{(p)} \equiv 0 \pmod{p^3}.$$

Now take  $k = p-1, m = p+2$  in (2.1), so that

$$B_{p+2}^{(p)} = p \binom{p+2}{p} \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} B_s^{(p)} \frac{B_{p+2-s}}{p+2-s}.$$

Clearly only odd values of  $s$  need be considered; we get, using (2.4),

$$\begin{aligned}
 B_{p+2}^{(p)} &= (p+2) \binom{p+1}{p-1} \sum_{t=0}^{(p-3)/2} C_{2t+1}^{(p)} \frac{B_{p+1-2t}}{p+1-2t} \\
 &\equiv (p+2) \binom{p+1}{p-1} \left( -\frac{1}{2} p \frac{B_{p+1}}{p+1} + \frac{1}{8} p^2 \frac{B_{p-1}}{p-1} \right) \pmod{p^3}
 \end{aligned}$$

by (2.3) and (2.7); next, by (2.9) and (2.10), we get

$$(5.2) \quad B_{p+2}^{(p)} \equiv \frac{1}{12} p^2 \pmod{p^3}.$$

In view of (5.1) we have also

$$(5.3) \quad B_{p+1}^{(p)} \equiv -\frac{1}{12} p \pmod{p^2}.$$

However, (5.2) and (5.3) do not imply (5.1) but only the weaker result with modulus  $p^2$ .

To improve these results we follow the method of §3. Thus

$$(5.4) \quad B_{p+1}^{(p)} = p(p+1) \sum_{s=0}^{(p-1)/2} C_{2s}^{(p)} \frac{B_{p+1-2s}}{p+1-2s} = p(p+1) A,$$

and

$$A = \frac{B_{p+1}}{p+1} + C_2^{(p)} \frac{B_{p-1}}{p-1} + \sum_{t=2}^{(p-3)/2} C_{2t}^{(p)} \frac{B_{p+1-2t}}{p+1-2t} + \frac{1}{12} (p-1)!.$$

But, by (3.61),

$$\begin{aligned}
 \sum_{t=2}^{(p-3)/2} C_{2t}^{(p)} \frac{B_{p+1-2t}}{p+1-2t} &\equiv -p \sum_2^{(p-3)/2} \frac{B_{2t}}{2t} \frac{B_{p+1-2t}}{p+1-2t} \\
 &\equiv -\frac{p+2}{p+1} B_{p+1} - \frac{1}{6} p B_{p-1} \pmod{p^2},
 \end{aligned}$$

so that after some simplification we get

$$A \equiv -\frac{B_{p+1}}{p+1} + \frac{1}{8} p \pmod{p^2},$$

and therefore, by (5.4) and (2.10),

$$(5.5) \quad B_{p+1}^{(p)} \equiv -p \frac{B_{p+1}}{p+1} + \frac{1}{24} p^2 \pmod{p^3}.$$

In view of (5.1) this implies

$$(5.6) \quad B_{p+2}^{(p)} \equiv p^2 \frac{B_{p+1}}{p+1} \pmod{p^4}.$$

That (5.5) and (5.6) include (5.3) and (5.2) is evident; also (5.5) and (5.6) imply (1.2).

We remark also that using (2.13), (5.5), and (1.3) we get

$$(5.7) \quad B_{p+1}^{(p+1)} \equiv \frac{B_{p+1}}{p+1} - \frac{1}{24} p \pmod{p^2}.$$

**6. Discussion of  $B_m^{(p)}$ .** Let first  $m > p$  be odd, so that (2.1) implies

$$(6.1) \quad B_m^{(p)} = p \binom{m}{p} \sum_{t=0}^{(p-3)/2} C_{2t+1}^{(p)} \frac{B_{m-1-2t}}{m-1-2t}.$$

Now let  $m \equiv a \pmod{p}$ ,  $0 \leq a < p$ ;  $m \equiv b \pmod{p-1}$ ,  $0 \leq b < p-1$ . Also, let  $p^r \mid m-a$ ,  $p^{r+1} \nmid m-a$ , so that the binomial coefficient  $\binom{m}{p}$  is divisible by exactly  $p^{r-1}$ . Clearly  $b$  is odd. Now by a well-known theorem [2, p. 252], if  $a \neq b$ , the quotient  $B_{m-a}/(m-a)$  is integral  $\pmod{p}$ . Thus, by (2.7), the right member of (6.1), except for the terms corresponding to  $t=0$ ,  $(b-1)/2$ , is a multiple of  $p^{r+2}$ . As for the exceptional terms

$$(6.2) \quad u_1 = p \binom{m}{p} C_1^{(p)} \frac{B_{m-1}}{m-1}, \quad u_b = p \binom{m}{p} C_b^{(p)} \frac{B_{m-b}}{m-b},$$

there are several possibilities.

(i) Suppose  $b=1$ , so that the two terms in (6.2) coincide. Then if  $a \neq 1$ , we see that the term in question is exactly divisible by  $p^r$ . On the other hand if  $a=1$ , the term is integral  $\pmod{p}$  but not divisible by  $p$ .

(ii) If  $b \neq 1$ ,  $u_1$  and  $u_b$  in (6.2) are distinct. There are several cases to consider. If  $a=b$ , then  $u_1$  is divisible by  $p^{r+1}$ , while  $u_b$  is divisible by exactly  $p^{r+1}$ . Thus, in this sub-case  $B_m^{(p)} \equiv 0 \pmod{p^{r+1}}$ ; for  $m=p+2$  this is less precise than (5.2).

In the next place, let  $m$  be even and define  $a$ ,  $b$ ,  $r$  as above so that  $b$  is now

even.. Then we have

$$(6.3) \quad B_m^{(p)} = p \binom{m}{p} \sum_{t=0}^{(p-1)/2} C_{2t}^{(p)} \frac{B_{m-2t}}{m-2t} = \sum_{t=0}^{(p-1)/2} u_{2t} .$$

Then by (2.6) the right member, except for the terms  $u_0, u_b, u_{p-1}$ , is a multiple of  $p^{r+1}$ . We consider a number of cases.

(iii) If  $b = 0$ , there are only two distinct terms  $u_0, u_{p-1}$ . If  $a = 0$ , we find that  $pu_0$  is integral (mod  $p$ ); indeed  $pu_0 \equiv -1 \pmod{p}$  by the Staudt-Clausen theorem (2.11). On the other hand,  $u_{p-1}$  is divisible by  $p^{r-1}$ ; indeed  $u_{p-1} \equiv m/(m-p+1) \pmod{p^r}$ . If  $a = p-1$ , then  $u_0 \equiv (m-p+1)/m \pmod{p^r}$  while  $pu_{p-1} \equiv 1 \pmod{p}$ . If  $a \neq 0$  or  $p-1$  then it can be verified that  $u_0 + u_{p-1}$  is divisible by  $p^r$ .

(iv) If  $b \neq 0$ , then all three terms  $u_0, u_b, u_{p-1}$  are distinct. By means of Kummer's congruence (2.10) we find that  $u_0 + u_{p-1} \equiv 0 \pmod{p^{r+1}}$ ; in other words;

$$(6.4) \quad B_m^{(p)} \equiv u_b \pmod{p^{r+1}} \quad (b \neq 0).$$

As for  $u_b$ , there are several possibilities. If  $a = b$ , it is easily seen that  $u_b$  is integral (mod  $p$ ); moreover, by (2.6),  $u_b \equiv 0 \pmod{p}$  if and only if  $B_b \equiv 0 \pmod{p}$ . If  $a \neq b$ , then  $u_b$  is divisible by  $p^r$  at least; indeed using (6.4) we get

$$(6.5) \quad B_m^{(p)} \equiv B_b \frac{m-a}{b(m-b)} \pmod{p^{r+1}} \quad (a \neq b, b \neq 0).$$

This result evidently includes (5.3) but not (5.5).

We remark that  $B_m^{(p)}$  is integral (mod  $p$ ) in cases (i), (ii), (iv). In case (iii), however, if  $a = 0$  or  $p-1$ , then  $B_m^{(p)}$  is no longer integral, but  $pB_m^{(p)}$  is integral; indeed it is easily verified that

$$pB_m^{(p)} \equiv \begin{cases} -1 & \pmod{p} & (a = 0), \\ +1 & \pmod{p} & (a = p-1). \end{cases}$$

**7. Some special cases.** Clearly  $m = p^r, r > 1$ , falls under (i) above with  $a = 0, b = 1$ . Thus,

$$(7.1) \quad B_{p^r}^{(p)} \equiv -\frac{1}{2} p^{r-1} (p-1) B_{p^{r-1}} \pmod{p^{r+2}},$$

and in particular,

$$B_{p^r}^{(p)} \equiv -\frac{1}{2} p^r \pmod{p^{r+1}}.$$

For  $m \equiv 1 \pmod{p^r(p-1)}$ , we have  $a = b = 1$  which also falls under (i); we now have

$$(7.3) \quad B_m^{(p)} = \frac{1}{2} (p-1) p B_{m-1} \pmod{p^{r+2}}.$$

For  $m = cp^r$ , where  $c$  is odd,  $p \nmid c$ , we have  $a = 0$ ,  $c \equiv b \pmod{p-1}$ , which evidently falls under (i) or (ii). Thus, we get ( $r \geq 1$ )

$$(7.4) \quad B_{cp^r}^{(p)} = \frac{1}{2} cp^{r+1} \frac{p-1}{cp^r-1} B_{cp^{r-1}} \pmod{p^{r+2}}$$

for  $c \equiv 1 \pmod{p-1}$ ;

$$(7.5) \quad B_{cp^r}^{(p)} \equiv -\frac{1}{2} cp^{r+1} \left( B_{cp^{r-b}} + \frac{1}{b-1} \right) \pmod{p^{r+2}}$$

for  $c \equiv b \pmod{p-1}$ ,  $b \neq 1$ .

Similarly, for  $m = cp^r$ ,  $c$  even,  $p \nmid c$ , we have  $a = 0$ ,  $c \equiv b \pmod{p-1}$ , which falls under (iii) or (iv). We consider only the case  $p-1 \nmid c$ ; that is,  $b \neq 0$ . Then, by (6.5), we have

$$(7.6) \quad B_{cp^r}^{(p)} \equiv -\frac{c}{b^2} p^r B_b \pmod{p^{r+1}}.$$

Again for  $m = cp^r + a$ ,  $c$  odd,  $a$  even, we find

$$(7.7) \quad B_m^{(p)} \equiv \frac{1}{2} cp^{r+1} (p-1) \frac{B_{m-1}}{m-1} \pmod{p^{r+2}}$$

for  $b = 1$ , while

$$(7.8) \quad B_m^{(p)} \equiv -\frac{1}{2} cp^{r+1} \left( \frac{B_{m-1}}{a-1} - \frac{b}{(b-1)(b-a)} \right) \pmod{p^{r+2}}$$

for  $b \neq 1$ ; in these two formulas we have  $0 < a < p-1$ ,  $b \equiv c+a \pmod{p-1}$ . For  $c$  even,  $a$  odd, (7.7) holds; but (7.8) requires modification. For  $c$  and  $a$  both odd or both even, there are several cases; in particular, by (6.5) we have

$$(7.9) \quad B_m^{(p)} \equiv \frac{c^p}{b(a-b)} \pmod{p^{r+1}}$$

for  $a \neq b$ ,  $b \neq 0$ .

For  $p = 2, 3$  it follows at once from (2.1) that

$$B_m^{(2)} = -m(m-1) \left( \frac{B_m}{m} + \frac{B_{m-1}}{m-1} \right),$$

$$B_m^{(3)} = \frac{1}{2} m(m-1)(m-2) \left( \frac{B_m}{m} + 3 \frac{B_{m-1}}{m-1} + 2 \frac{B_{m-2}}{m-2} \right),$$

by means of which numerous special formulas can easily be obtained, for example,

$$B_m^{(2)} = \begin{cases} -(m-1)B_m & (m \text{ even} > 2), \\ -mB_{m-1} & (m \text{ odd}), \end{cases}$$

$$B_m^{(3)} = \frac{3}{2} m(m-2)B_{m-1} \quad (m \text{ odd} > 1).$$

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