ON THE BOUNDARY VALUES OF SOLUTIONS OF THE HEAT EQUATION

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1. Introduction. In a recent paper Hartman and Wintner [3] consider solutions of the heat equation

\[ u_{xx}(x, t) - u_t(x, t) = 0 \]

in a rectangle \( R: 0 < x < 1 \) \( (0 < t < k < \infty) \). There they obtain necessary and sufficient conditions for a solution of (1) in \( R \) to be representable in the form

\[ u(x, t) = \int_0^1 G(x, t; y, s) \, d A(y) + \int_0^t G_y(x, t; 0, s) \, d B(s) - \int_0^t G_y(x, t; 1, s) \, d C(s), \]

the Green's function \( G \) being defined by

\[ G(x, t; y, s) = \frac{1}{2} \left[ \partial_3 \left( \frac{x - y}{2}, t - s \right) - \partial_3 \left( \frac{x + y}{2}, t - s \right) \right] \]

where \( \partial_3 \) is the well known Jacobi theta function. (The first integral in (2) is an absolutely convergent improper Riemann-Stieltjes integral.) They proceed to show that the functions representable in the form (2) exhibit the following behavior at the boundary of \( R \):

\[ \lim_{t \to 0^+} u(x, t) = A'(x), \]

\[ \lim_{t \to 0^+} u(x, t) = B'(t), \quad \lim_{t \to 1^+} u(x, t) = C'(t) \]

wherever the derivatives in question exist.

In the present note we present an improvement of (5) first given in the author's thesis [2]. The admittedly slight mathematical improvement is physically significant. A solution of (1) which admits the representation (2) gives the

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temperature at time \( t \) and position \( x \) in an insulated rod of length unity and with a certain initial temperature distribution, given essentially by (4), and imposed end temperatures, given essentially by (5). We note that such solutions are not uniquely determined by (4) and (5).

As \( x \) approaches the boundary of \( R \) along a line \( t = t_0 \), it seems intuitively clear that the limit should be independent of values of \( B \) (or \( C \)) for \( t \geq t_0 \). Hence the expected result (for the left side of \( R \)) would be

\[
\lim_{x \to 0^+} u(x, t) = B'(t - 0) = \lim_{h \to 0^+} \frac{B(t - h) - B(t - 0)}{-h}
\]

wherever this derivative exists.

2. Theorem. For the above improvement it is sufficient to establish the following result.

**Theorem.** If \( B(s) \) is of bounded variation on every closed interval of \( 0 \leq s < k \leq \infty \), then

\[
\lim_{x \to 0^+} \int_0^t G_y(x, t; 0, s) dB(s) = B'(t - 0)
\]

wherever this derivative exists.

**Proof.** Let

\[
u(x, t) = \int_0^t G_y(x, t; 0, s) dB(s).
\]

Then since

\[
\theta_3 \left( \frac{x}{2}, t \right) = (\pi t)^{-1/2} \sum_{n = -\infty}^{\infty} \exp \left[ \frac{-(x + 2n)^2}{4t} \right]
\]

(see, for example, [1, p. 307]), we can write

\[
u(x, t) = \frac{1}{2} x \pi^{-1/2} \int_0^t (t - s)^{-3/2} \exp \left[ \frac{-x^2}{4(t - s)} \right] d B(s)
\]

\[
+ \frac{1}{2} \pi^{-1/2} \int_0^t (t - s)^{-3/2} \sum_{n = -\infty}^{\infty} (x + 2n) \exp \left[ \frac{-(x + 2n)^2}{4(t - s)} \right] d B(s)
\]

Clearly the latter integral vanishes with \( x \). Then denoting the first integral on
the right by $I$ and by setting $z = x^2/4$ and $t - s = 1/v$, we get

$$I = \left( \frac{z}{\pi} \right)^{1/2} \int_0^\infty e^{-zv} v^{3/2} \, dB(t - 1/v).$$

If we define

$$\alpha(v) = \begin{cases} \int_a^v r^{3/2} \, dB(t - 1/r) & (v \geq 1/t), \\ \alpha(1/t) & (v < 1/t), \end{cases}$$

where $\alpha$ is a suitable constant, then we have

$$I = \left( \frac{z}{\pi} \right)^{1/2} \int_0^\infty e^{-zv} \, d\alpha(v).$$

To evaluate $\lim_{z \to \infty} I$ we apply [4, Theorem 1, p. 181], which states: If

$$f(s) = \int_0^\infty e^{-st} \, d\alpha(t),$$

then for any $\gamma \geq 0$ any constant $A$ we have

$$\lim_{s \to 0^+} |S^\gamma f(s) - A| \leq \lim_{t \to \infty} |\alpha(t) t^{-\gamma} \Gamma(\gamma + 1) - A|.$$

To this end we evaluate $\lim_{v \to \infty} v^{-1/2} \, \alpha(v)$. Now

$$v^{-1/2} \, \alpha(v) = v^{-1/2} \int_r^v r^{3/2} \, dB(t - 1/r)$$

$$= v^{-1/2} \int_r^v r^{3/2} \, d[B(t - 1/r) - B(t - 0)]$$

$$= r^{3/2} \, v^{-1/2} [B(t - 1/r) - B(t - 0)] \bigg|_a^v$$

$$+ \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)] \, r^{1/2} \, dr$$

$$= \frac{B(t - 1/v) - B(t - 0)}{1/v} - \frac{B(t - 1/a) - B(t - 0)}{v^{1/2}} a^{3/2}$$

$$+ \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)] \, r^{1/2} \, dr.$$
As \( v \to \infty \) the first expression on the right tends to \(-B'(t - 0)\), if this derivative exists, and the second vanishes. Now consider the integral term: given \( \epsilon > 0 \), choose \( T \) so large that

\[
\left| B'(t - 0) - \frac{B(t - 0) - B(t - 1/r)}{1/r} \right| < \epsilon \text{ if } r > T.
\]

Then

\[
\frac{3}{2} v^{-1/2} \int_a^v \left[ B(t - 0) - B(t - 1/r) \right] r^{1/2} \, dr
\]

\[
= \frac{3}{2} v^{-1/2} \int_a^T \left[ B(t - 0) - B(t - 1/r) \right] r^{1/2} \, dr
\]

\[
+ \frac{3}{2} v^{-1/2} \int_T^v \frac{B(t - 0) - B(t - 1/r)}{1/r} r^{-1/2} \, dr.
\]

The first integral on the right \( \to 0 \) as \( v \to \infty \), and

\[
\frac{3}{2} v^{-1/2} \int_T^v \frac{B(t - 0) - B(t - 1/r)}{1/r} r^{-1/2} \, dr
\]

\[
= 3 \left[ B'(t - 0) + \eta(T, v) \right] (v^{1/2} - T^{1/2}) v^{-1/2},
\]

where \( |\eta| < \epsilon \) for all values of \( v > T \). Let \( v \to \infty \), then let \( \epsilon \to 0 \); the right side of the above equation approaches \( 3B'(t - 0) \). Consequently we now have

\[
\lim_{v \to \infty} v^{-1/2} a(v) = 2B'(t - 0).
\]

By applying the above-mentioned theorem with \( \gamma = 1/2, A = \pi^{1/2} B'(t - 0) \), we now obtain

\[
\lim_{z \to 0} \left| z^{1/2} \int_0^\infty e^{-zv} d\alpha(v) - \pi^{1/2} B'(t - 0) \right|
\]

\[
\leq \lim_{v \to \infty} \left| \frac{1}{2} \pi^{1/2} v^{-1/2} B(v) - \pi^{1/2} B'(t - 0) \right| = 0.
\]

Hence

\[
\lim_{x \to 0^+} u(x, t) = \lim_{z \to 0} l = B'(t - 0).
\]
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