

Pacific Journal of Mathematics

**ON THE BOUNDARY VALUES OF SOLUTIONS OF THE HEAT
EQUATION**

WATSON BRYAN FULKS

ON THE BOUNDARY VALUES OF SOLUTIONS OF THE HEAT EQUATION

W. FULKS

1. **Introduction.** In a recent paper Hartman and Wintner [3] consider solutions of the heat equation

$$(1) \quad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle $R: 0 < x < 1$ ($0 \leq t < k \leq \infty$). There they obtain necessary and sufficient conditions for a solution of (1) in R to be representable in the form

$$(2) \quad u(x, t) = \int_{0+}^{1-0} G(x, t; y, s) dA(y) \\ + \int_0^t G_y(x, t; 0, s) dB(s) - \int_0^t G_y(x, t; 1, s) dC(s),$$

the Green's function G being defined by

$$(3) \quad G(x, t; y, s) = \frac{1}{2} \left[\vartheta_3\left(\frac{x-y}{2}, t-s\right) - \vartheta_3\left(\frac{x+y}{2}, t-s\right) \right]$$

where ϑ_3 is the well known Jacobi theta function. (The first integral in (2) is an absolutely convergent improper Riemann-Stieltjes integral.) They proceed to show that the functions representable in the form (2) exhibit the following behavior at the boundary of R :

$$(4) \quad \lim_{t \rightarrow 0+} u(x, t) = A'(x),$$

$$(5) \quad \lim_{x \rightarrow 0+} u(x, t) = B'(t), \quad \lim_{x \rightarrow 1-0} u(x, t) = C'(t)$$

wherever the derivatives in question exist.

In the present note we present an improvement of (5) first given in the author's thesis [2]. The admittedly slight mathematical improvement is physically significant. A solution of (1) which admits the representation (2) gives the

Received July 6, 1951.

Pacific J. Math. 2 (1952), 141-145

temperature at time t and position x in an insulated rod of length unity and with a certain initial temperature distribution, given essentially by (4), and imposed end temperatures, given essentially by (5). We note that such solutions are not uniquely determined by (4) and (5).

As x approaches the boundary of R along a line $t = t_0$, it seems intuitively clear that the limit should be independent of values of B (or C) for $t \geq t_0$. Hence the expected result (for the left side of R) would be

$$\lim_{x \rightarrow 0^+} u(x, t) = B'(t - 0) = \lim_{h \rightarrow 0^+} \frac{B(t - h) - B(t - 0)}{-h}$$

wherever this derivative exists.

2. Theorem. For the above improvement it is sufficient to establish the following result.

THEOREM. *If $B(s)$ is of bounded variation on every closed interval of $0 \leq s < k \leq \infty$, then*

$$\lim_{x \rightarrow 0^+} \int_0^t G_y(x, t; 0, s) dB(s) = B'(t - 0)$$

wherever this derivative exists.

Proof. Let

$$u(x, t) = \int_0^t G_y(x, t; 0, s) dB(s).$$

Then since

$$\vartheta_3\left(\frac{x}{2}, t\right) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x + 2n)^2}{4t}\right]$$

(see, for example, [1, p. 307]), we can write

$$\begin{aligned} u(x, t) &= \frac{1}{2} x \pi^{-1/2} \int_0^t (t-s)^{-3/2} \exp\left[\frac{-x^2}{4(t-s)}\right] dB(s) \\ &\quad + \frac{1}{2} \pi^{-1/2} \int_0^t (t-s)^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (x + 2n) \exp\left[\frac{-(x + 2n)^2}{4(t-s)}\right] dB(s). \end{aligned}$$

Clearly the latter integral vanishes with x . Then denoting the first integral on

the right by I and by setting $z = x^2/4$ and $t - s = 1/v$, we get

$$I = \left(\frac{z}{\pi}\right)^{1/2} \int_{v=1/t}^{\infty} e^{-zv} v^{3/2} dB(t - 1/v).$$

If we define

$$\alpha(v) = \begin{cases} \int_{r=a}^v r^{3/2} dB(t - 1/r) & (v \geq 1/t), \\ \alpha(1/t) & (v < 1/t), \end{cases}$$

where a is a suitable constant, then we have

$$I = \left(\frac{z}{\pi}\right)^{1/2} \int_0^{\infty} e^{-zv} d\alpha(v).$$

To evaluate $\lim_{z \rightarrow \infty} I$ we apply [4, Theorem 1, p. 181], which states: If

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t),$$

then for any $\gamma \geq 0$ any constant A we have

$$\lim_{s \rightarrow 0+} |S^\gamma f(s) - A| \leq \lim_{t \rightarrow \infty} |\alpha(t) t^{-\gamma} \Gamma(\gamma + 1) - A|.$$

To this end we evaluate $\lim_{v \rightarrow \infty} v^{-1/2} \alpha(v)$. Now

$$\begin{aligned} v^{-1/2} \alpha(v) &= v^{-1/2} \int_{r=a}^v r^{3/2} dB(t - 1/r) \\ &= v^{-1/2} \int_a^v r^{3/2} d[B(t - 1/r) - B(t - 0)] \\ &= r^{3/2} v^{-1/2} [B(t - 1/r) - B(t - 0)] \Big|_a^v \\ &\quad + \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)] r^{1/2} dr \\ &= \frac{B(t - 1/v) - B(t - 0)}{1/v} - \frac{B(t - 1/a) - B(t - 0)}{v^{1/2}} a^{3/2} \\ &\quad + \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)] r^{1/2} dr. \end{aligned}$$

As $v \rightarrow \infty$ the first expression on the right tends to $-B'(t-0)$, if this derivative exists, and the second vanishes. Now consider the integral term: given $\epsilon > 0$, choose T so large that

$$\left| B'(t-0) - \frac{B(t-0) - B(t-1/r)}{1/r} \right| < \epsilon \text{ if } r > T.$$

Then

$$\begin{aligned} & \frac{3}{2} v^{-1/2} \int_a^v [B(t-0) - B(t-1/r)] r^{1/2} dr \\ &= \frac{3}{2} v^{-1/2} \int_a^T [B(t-0) - B(t-1/r)] r^{1/2} dr \\ & \quad + \frac{3}{2} v^{-1/2} \int_T^v \frac{B(t-0) - B(t-1/r)}{1/r} r^{-1/2} dr. \end{aligned}$$

The first integral on the right $\rightarrow 0$ as $v \rightarrow \infty$, and

$$\begin{aligned} & \frac{3}{2} v^{-1/2} \int_T^v \frac{B(t-0) - B(t-1/r)}{1/r} r^{-1/2} dr \\ &= 3[B'(t-0) + \eta(T, v)] (v^{1/2} - T^{1/2}) v^{-1/2}, \end{aligned}$$

where $|\eta| < \epsilon$ for all values of $v > T$. Let $v \rightarrow \infty$, then let $\epsilon \rightarrow 0$; the right side of the above equation approaches $3B'(t-0)$. Consequently we now have

$$\lim_{v \rightarrow \infty} v^{-1/2} \alpha(v) = 2B'(t-0).$$

By applying the above-mentioned theorem with $\gamma = 1/2$, $A = \pi^{1/2} B'(t-0)$, we now obtain

$$\begin{aligned} & \overline{\lim}_{z \rightarrow 0} \left| z^{1/2} \int_0^\infty e^{-zv} d\alpha(v) - \pi^{1/2} B'(t-0) \right| \\ & \leq \overline{\lim}_{v \rightarrow \infty} \left| \frac{1}{2} \pi^{1/2} v^{-1/2} B(v) - \pi^{1/2} B'(t-0) \right| = 0. \end{aligned}$$

Hence

$$\lim_{x \rightarrow 0+} u(x, t) = \lim_{z \rightarrow 0} I = B'(t-0).$$

REFERENCES

1. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, New York, 1943.
2. W. Fulks, *On Integral Representations and Uniqueness of Solutions of the Heat Equation*, University of Minnesota Thesis, 1949.
3. P. Hartman and A. Wintner, *On the solutions of the equation of heat conduction*, Amer. J. Math. 72 (1950), 367-395.
4. D. V. Widder, *The Laplace Transform*, Princeton, 1941.

UNIVERSITY OF MINNESOTA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

R. M. ROBINSON

University of California
Berkeley 4, California

*R. P. DILWORTH

California Institute of Technology
Pasadena 4, California

E. F. BECKENBACH, Managing Editor

University of California
Los Angeles 24, California

*During the absence of Herbert Busemann in 1952.

ASSOCIATE EDITORS

R. P. DILWORTH

HERBERT FEDERER

MARSHALL HALL

P. R. HALMOS

HEINZ HOPF

R. D. JAMES

BØRGE JESSEN

PAUL LÉVY

GEORGE PÓLYA

J. J. STOKER

E. G. STRAUS

KÔSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA, BERKELEY

UNIVERSITY OF CALIFORNIA, DAVIS

UNIVERSITY OF CALIFORNIA, LOS ANGELES

UNIVERSITY OF CALIFORNIA, SANTA BARBARA

OREGON STATE COLLEGE

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

WASHINGTON STATE COLLEGE

UNIVERSITY OF WASHINGTON

• • •
AMERICAN MATHEMATICAL SOCIETY

NATIONAL BUREAU OF STANDARDS,

INSTITUTE FOR NUMERICAL ANALYSIS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December, by the University of California, Berkeley 4, California. The price per volume (4 numbers) is \$8.00; single issues, \$2.50. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

COPYRIGHT 1952 BY PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics

Vol. 2, No. 2

February, 1952

L. Carlitz, <i>Some theorems on Bernoulli numbers of higher order</i>	127
Watson Bryan Fulks, <i>On the boundary values of solutions of the heat equation</i>	141
John W. Green, <i>On the level surfaces of potentials of masses with fixed center of gravity</i>	147
Isidore Heller, <i>Contributions to the theory of divergent series</i>	153
Melvin Henriksen, <i>On the ideal structure of the ring of entire functions</i>	179
James Richard Jackson, <i>Some theorems concerning absolute neighborhood retracts</i>	185
Everett H. Languier, <i>Homology bases with applications to local connectedness</i>	191
Janet McDonald, <i>Davis's canonical pencils of lines</i>	209
J. D. Niblett, <i>Some hypergeometric identities</i>	219
Elmer Edwin Osborne, <i>On matrices having the same characteristic equation</i>	227
Robert Steinberg and Raymond Moos Redheffer, <i>Analytic proof of the Lindemann theorem</i>	231
Edward Silverman, <i>Set functions associated with Lebesgue area</i>	243
James G. Wendel, <i>Left centralizers and isomorphisms of group algebras</i> . . .	251
Kosaku Yosida, <i>On Brownian motion in a homogeneous Riemannian space</i>	263