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J. D. Niblett

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1. Introduction. T. W. Chaundy [3] has given some hypergeometric identities of which the most general is

$$(1)\qquad F(a, b; c; x) = h \sum_{n=0}^{\infty} \frac{(h-\alpha n+1)_{n-1}\,(e)_n}{n!\,(c)_n}$$

$$\times\ {}_4F_3\left[\begin{matrix} a,\ b,\ 1+h(1-\alpha)^{-1},\ -n \\ e,\ h(1-\alpha)^{-1},\ h-\alpha n+1 \end{matrix}\right](-x)^n\ F(e+n,\ h+(1-\alpha)n;\ c+n;\ x).$$

In this paper we give a generalisation of (1), namely,

$$(2)\qquad {}_{p+s}F_{q+t}\left[\begin{matrix} a_p,\ b_s; \\ c_q,\ d_t; \end{matrix}\ x\right] = h\sum_{n=0}^{\infty} \frac{(h-\alpha n+1)_{n-1}}{n!}\ \frac{(b_s)_n\,(e_q)_n}{(d_t)_n\,(c_q)_n}$$

$$\times\ {}_{p+2}F_{q+2}\left[\begin{matrix} a_p,\ 1+h(1-\alpha)^{-1},\ -n \\ e_q,\ h(1-\alpha)^{-1},\ h-\alpha n+1 \end{matrix}\right](-x)^n$$

$$\times\ {}_{s+q+1}F_{t+q}\left[\begin{matrix} b_s+n,\ e_q+n,\ h+(1-\alpha)n; \\ d_t+n,\ c_q+n; \end{matrix}\ x\right],$$

where $(h-\alpha n+1)_{-1}$ means $(h-\alpha n)^{-1}$ and a_λ, $(a_\lambda)_n$, $a_\lambda+n$ denote $a_1 \cdots, a_\lambda$; $(a_1)_n \cdots (a_\lambda)_n$; and $a_1+n, \cdots, a_\lambda+n$, respectively; and from (2), we deduce some other identities.

2. Proof of (2). The following is a simple extension of Dr. Chaundy's proof. Comparing the coefficients in (2) of $(a_p)_N/N!$, we have to prove that

$$\frac{(b_s)_N\,x^N}{(c_q)_N\,(d_t)_N} = \{h+(1-\alpha)N\}\sum_{n=N}^{\infty} \frac{(h-\alpha n+1)_{n-1}\,(b_s)_n\,(e_q)_n\,(-n)_N}{n!\,(d_t)_n\,(c_q)_N\,(e_q)_N\,(h-\alpha n+1)_N}\,(-x)^n$$

$$\times\ {}_{s+q+1}F_{t+q}\,.$$

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Writing $n = N + r$, we find that this reduces to

$$1 = \{h + (1-\alpha)N\} \sum_{r=0}^{\infty} \frac{[h+(1-\alpha)N+1-\alpha r]_{r-1}\,(b_s+N)_r\,(e_q+N)_r}{(d_t+N)_r\,(c_q+N)_r\,r!}(-x)^r$$

$$\times \; {}_{s+q+1}F_{t+q}\left[\begin{matrix} b_s+N+r,\; e_q+N+r,\; h+(1-\alpha)(N+r); \\ d_t+N+r,\; c_q+N+r; \end{matrix}\; x\right]$$

The term independent of x on the right is unity. It remains to be proved that the coefficient of any positive power of x vanishes on the right, that is, when $M > 0$,

$$\frac{(b_s+N)_M\,(e_q+N)_M}{(d_t+N)_M\,(c_q+N)_M} \sum_{r=0}^{M} (-1)^r \frac{[h+(1-\alpha)N+1-\alpha r]_{M-1}}{r!\,(M-r)!} = 0.$$

But this is the coefficient of x^{M-1} in

$$\frac{(b_s+N)_M\,(e_q+N)_M}{M\,(d_t+N)_M\,(c_q+N)_M}\,(1-x)^{-h-(1-\alpha)N-1}\,[1-(1-x)^{\alpha}]^M,$$

in which the lowest term is x^M.

This completes the formal proof of (2). The rearrangement of the infinite series requires absolute convergence, which is secured when x is "sufficiently small", at least for the case $p = q+1$, $s = t$, in which we are particularly interested.

3. A special case. If in (2) we write $s = t$, $b_k = d_k$ for $k = 1, 2, \cdots, s$, and $e_k = c_k$ for $k = 1, \cdots, q$, then we obtain

$$(3)\quad (1-x)^h\; {}_pF_q\left[\begin{matrix} a_p; \\ c_q; \end{matrix}\; x\right]$$

$$= h \sum_{n=0}^{\infty} \frac{(h-\alpha n+1)_{n-1}}{n!}\; {}_{p+2}F_{q+2}\left[\begin{matrix} a_p,\; 1+h(1-\alpha)^{-1},\; -n \\ c_q,\; h(1-\alpha)^{-1},\; h-\alpha n+1 \end{matrix}\right]\left(\frac{-x}{(1-x)^{1-\alpha}}\right)^n.$$

4. Other cases. If

$$(4)\qquad {}_{p+2}F_{q+2}\left[\begin{matrix} a_p,\; 1+h(1-\alpha)^{-1},\; -n \\ e_q,\; h(1-\alpha)^{-1},\; h-\alpha n+1 \end{matrix}\right] = \frac{(\sigma_\mu)_n}{(\rho_\nu)_n},$$

then (2) and (3) reduce to simpler expressions.

4.1. In the case $p = q + 1$, (2) becomes

$$(5)\quad {}_{q+s+1}F_{q+t}\left[\begin{matrix} a_{q+1}, & b_s; \\ c_q, & d_t; \end{matrix}\ x\right] = h\sum_{n=0}^{\infty}\frac{(h-\alpha n+1)_{n-1}\,(b_s)_n\,(e_q)_n\,(\sigma_\mu)_n}{n!\,(d_t)_n\,(c_q)_n\,(\rho_\nu)_n}(-x)^n$$

$$\times\ {}_{q+s+1}F_{q+t}\left[\begin{matrix} b_s+n, & e_q+n, & h+(1-\alpha)\,n; \\ d_t+n, & c_q+n; & \end{matrix}\ x\right];$$

and (3) becomes

$$(6)\qquad (1-x)^h\ {}_{q+1}F_q\left[\begin{matrix} a_{q+1}; \\ c_q; \end{matrix}\ x\right] = h\sum_{n=0}^{\infty}\frac{(h-\alpha n+1)_{n-1}}{n!}\,\frac{(\sigma_\mu)_n}{(\rho_\nu)_n}\left(\frac{-x}{(1-x)^{1-\alpha}}\right)^n,$$

which, for appropriate values of α, gives a relation between hypergeometric functions of argument x and $-x(1-x)^{\alpha-1}$.

4.2. In the case $q = 1$, $\alpha = 1/2$, $a_1 = a$, $a_2 = 2h$, $c = 2a$, (4) is summed by Watson's Theorem [1, p. 16], and vanishes for odd powers of n. Then (6) becomes (see [2, formula (4.22), with $\alpha + \beta = a$, $\alpha = h$])

$$(7)\qquad (1-x)^h\ {}_2F_1\left[\begin{matrix} a, & 2h; \\ 2a; & \end{matrix}\ x\right] = {}_2F_1\left[\begin{matrix} h, & a-h; \\ a+1/2; & \end{matrix}\ \frac{-x^2}{4(1-x)}\right]$$

and the corresponding formula (5) is

$$(8)\qquad {}_{s+2}F_{s+1}\left[\begin{matrix} a, & 2h, & b_s; \\ 2a; & d_s; & \end{matrix}\ x\right] = \sum_{m=0}^{\infty}\frac{(b_s)_{2m}\,(h)_m\,(a-h)_m}{(d_s)_{2m}\,m!\,(a+1/2)_m}\left(\frac{-x^2}{4}\right)^m$$

$$\times\ {}_{s+2}F_{s+1}\left[\begin{matrix} b_s+2m, & 2a+2m, & h+m; \\ d_s+2m, & 2a+m; & \end{matrix}\ x\right].$$

If $\alpha = -1$, $q = 2$, $a_1 = \beta$, $a_2 = \gamma$, $a_3 = \delta$, $e_1 = 1+\beta-\gamma$, $e_2 = 1+\beta-\delta$, $h = \beta$, (4) can be summed by Dougall's formula [1, p. 25],

$$(9)\qquad {}_5F_4\left[\begin{matrix} \beta, 1+\beta/2, \gamma, \delta, -n \\ \beta/2, 1+\beta-\gamma, 1+\beta-\delta, 1+\beta+n \end{matrix}\right] = \frac{(1+\beta)_n\,(1+\beta-\gamma-\delta)_n}{(1+\beta-\gamma)_n\,(1+\beta-\delta)_n};$$

equation (5) becomes

$$(10)\quad {}_{s+3}F_{s+2}\left[\begin{matrix}\beta, \gamma, \delta, \ b_s\,; \\ c_1,\ c_2,\ d_s\,;\end{matrix}\ x\right]$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1}\ (b_s)_n\ (1+\beta)_n\ (1+\beta-\gamma-\delta)_n}{n!\qquad (d_s)_n\ (c_1)_n\ (c_2)_n}\ (-x)^n$$

$$\times\ {}_{s+3}F_{s+2}\left[\begin{matrix}b_s+n,\ 1+\beta-\gamma+n,\ 1+\beta-\delta+n,\ \beta+2n; \\ d_s+n,\ c_1+n,\ c_2+n\,;\end{matrix}\ x\right];$$

and (6) becomes Whipple's formula [2, p. 250, where references are given]:

$$(11)\quad (1-x)^{\beta}\ {}_3F_2\left[\begin{matrix}\beta, & \gamma, & \delta; \\ 1+\beta-\gamma, & 1+\beta-\delta; &\end{matrix}\ x\right]$$

$$= {}_3F_2\left[\begin{matrix}\beta/2, & (1+\beta)/2, & 1+\beta-\gamma-\delta; \\ 1+\beta-\gamma, & & 1+\beta-\delta;\end{matrix}\ \frac{-4x}{(1-x)^2}\right].$$

4.3. If $\alpha = -1$, $q = 4$, $a_1 = \beta$, $a_2 = \gamma$, $a_3 = \delta$, $a_4 = \epsilon$, $a_5 = \theta$,

$e_1 = 1+\beta-\gamma$, $e_2 = 1+\beta-\delta$, $e_3 = 1+\beta-\epsilon$, $e_4 = 1+\beta-\theta$, $h = \beta$,

then using Whipple's transformation [1, p. 25],

$$(12)\quad {}_7F_6\left[\begin{matrix}\beta, & 1+\beta/2, & \gamma, & \delta, & \epsilon, & \theta, & -n \\ & \beta/2, & 1+\beta-\gamma, & 1+\beta-\delta, & 1+\beta-\epsilon, & 1+\beta-\theta, & 1+\beta+n\end{matrix}\right]$$

$$= \frac{(1+\beta)_n\ (1+\beta-\epsilon-\theta)_n}{(1+\beta-\epsilon)_n\ (1+\beta-\theta)_n}\ {}_4F_3\left[\begin{matrix}1+\beta-\gamma-\delta,\ \epsilon,\ \theta,\ -n \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ \epsilon+\theta-\beta-n\end{matrix}\right],$$

in place of (4), we obtain

$$(13)\quad {}_{s+5}F_{s+4}\left[\begin{matrix}\beta, \gamma, \delta, \epsilon, \theta, b_s\,; \\ c_1, c_2, c_3, c_4, d_s\,;\end{matrix}\ x\right]$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1}\ (b_s)_n\ (1-\beta-\gamma)_n\ (1-\beta-\delta)_n\ (1+\beta)_n\ (1+\beta-\epsilon-\theta)_n}{n!\qquad (d_s)_n\qquad (c_1)_n\ (c_2)_n\ (c_3)_n\ (c_4)_n}$$

$$\times\ {}_4F_3\left[\begin{matrix}1+\beta-\gamma-\delta,\ \epsilon,\ \theta,\ -n \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ \epsilon+\theta-\beta-n\end{matrix}\right](-x)^n\ \times$$

$$\times {}_{s+5}F_{s+4}\left[\begin{matrix} b_s+n,\ 1+\beta-\gamma+n,\ 1+\beta-\delta+n,\ 1+\beta-\epsilon+n,\ 1+\beta-\theta+n,\ \beta+2n; \\ d_s+n,\ c_1+n,\ c_2+n,\ c_3+n,\ c_4+n; \end{matrix}\ x\right].$$

If $b_k = d_k$ for $k = 1, \cdots, s$, $c_1 = 1+\beta-\gamma$, $c_2 = 1+\beta-\delta$, $c_3 = 1+\beta-\epsilon$, $c_4 = 1+\beta-\theta$, this reduces to

$$(14)\quad (1-x)^{\beta}\ {}_5F_4\left[\begin{matrix} \beta,\ \gamma,\ \delta,\ \epsilon,\ \theta; \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ 1+\beta-\epsilon,\ 1+\beta-\theta; \end{matrix}\ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1}\ (1+\beta)_n\ (1+\beta-\epsilon-\theta)_n}{n!\ (1+\beta-\epsilon)_n\ (1+\beta-\theta)_n}$$

$$\times {}_4F_3\left[\begin{matrix} 1+\beta-\gamma-\delta,\ \epsilon,\ \theta,\ -n \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ \epsilon+\theta-\beta-n \end{matrix}\right]\left(\frac{-x}{(1-x)^2}\right)^n.$$

If

$$\beta = \frac{1}{2}a - b,\ \gamma = 1-b,\ \delta = -\frac{1}{2}a,\ \epsilon = 1+\frac{1}{2}a,\ \theta = b,$$

by Bailey's result [1, p.30, formula (1.3)],

$$(15)\qquad {}_4F_3\left[\begin{matrix} a,\ 1+a/2,\ b,\ -n \\ a/2,\ 1+a-b,\ 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n\ (-b)_n}{(1+a-b)_n\ (-2b)_n},$$

this becomes

$$(16)\qquad (1-x)^{-b+a/2}\ {}_5F_4\left[\begin{matrix} -b+a/2,\ 1-b,\ -a/2,\ 1+a/2,\ b; \\ a/2,\ 1+a-b,\ -b,\ 1-2b+a/2; \end{matrix}\ x\right]$$

$$= {}_3F_2\left[\begin{matrix} (a-2b)/4,(a-2b+2)/4,\ a-2b; \\ 1-2b+a/2,\ 1+a-b; \end{matrix}\ \frac{-4x}{(1-x)^2}\right].$$

4.4. If we take $\alpha = 0$, $q = 0$ and use Vandermonde's theorem in place of (4), we obtain

$$(17)\qquad {}_{s+1}F_s\left[\begin{matrix} a,\ b_s; \\ d_s; \end{matrix}\ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(b_s)_n\ (h-a)_n}{n!\ (d_s)_n} (-x)^n\ {}_{s+1}F_s\left[\begin{matrix} b_s+n,\ h+n; \\ d_s+n; \end{matrix}\ x\right]$$

and if $b_k = d_k$ for $k = 1, \cdots s-1$, $b_s = b$, $d_s = h$ this reduces to Euler's identity,

$$(18)\qquad (1-x)^b\ {}_2F_1\left[\begin{matrix} a,\ b; \\ h; \end{matrix}\ x\right] = {}_2F_1\left[\begin{matrix} h-a,\ b; \\ h; \end{matrix}\ \frac{x}{x-1}\right].$$

4.5. Multiplying (7) by $(1-x)^{-h}$ and equating coefficients of x, we obtain

$$(19)\qquad {}_3F_2\left[\begin{matrix} a-h,\ -n/2,\ (1-n)/2 \\ a+1/2,\ 1-h-n \end{matrix}\right] = \frac{(a)_n\ (2h)_n}{(2a)_n\ (h)_n},$$

which is a particular case of Saalschutz' theorem.
Similarly from (16) we get

$$(20)\qquad {}_3F_2\left[\begin{matrix} a-2b,\ a/2-b+n,\ -n \\ 1+a/2-2b,\ 1+a-b \end{matrix}\right] = \frac{(1-b)_n\ (-a/2)_n\ (1+a/2)_n\ (b)_n}{(a/2)_n\ (1+a-b)_n\ (1+a/2-2b)_n}.$$

This is a special case of

$$(21)\qquad {}_3F_2\left[\begin{matrix} a,\ b,\ -n \\ e,\ 2+a+b-e-n \end{matrix}\right] = \frac{(e-b-1)_n\ (e-a-1)_n\ (\omega+1)_n}{(e)_n\ (e-a-b-1)_n\ (\omega)_n},$$

where

$$\omega = \frac{(e-a-1)\ (e-b-1)}{e-a-b-1},$$

which is, in Whipple's notation, a particular case of the relation between the quantities $F_p(0; 4, 5)$ and $F_p(2; 4, 5)$. [1, p. 85; 4]. This gives a generalisation of (16),

$$(22)\quad (1-x)^{2a}\ {}_5F_4\left[\begin{matrix} 2a,\ e-c-1,\ 2a-e+1,\ 1+\phi,\ 1+\theta; \\ 2a+c+2-e,\quad e,\quad \theta,\quad \phi; \end{matrix}\ x\right]$$

$$= {}_3F_2\left[\begin{matrix} a,\ a+1/2,\ c; \\ e,\ 2+c+2a-e; \end{matrix}\ \frac{-4x}{(1-x)^2}\right],$$

where θ, ϕ are the roots of $m^2 - 2am + (e-c-1)(2a+1-e) = 0$. Comparing with (14), we have

$$(23) \quad {}_4F_3\left[\begin{matrix} e-\theta-1, \ 1+\phi, \ e-c-1, \ -n \\ 2a-\theta, \ e, \ \phi+e-c-2a-n \end{matrix}\right] = \frac{(c)_n \quad (2a-\phi)_n}{(e)_n \ (1+2a-\phi-e+c)_n} \ .$$

This is a generalisation of (15); we obtain (15), (16) from (22), (23) by taking $a=(a-2b)/4, \ \ c=a-2b, \ \ e=1+a-b, \ \ \theta=-b, \ \ \phi=a/2$.

I should like to take this opportunity of thanking Dr. Chaundy for many kindnesses and especially for allowing me to see his most recent paper before it was published.

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