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SOME HYPERGEOMETRIC IDENTITIES

J. D. NIBLETT

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1. **Introduction.** T. W. Chaundy [3] has given some hypergeometric identities of which the most general is

$$(1) \quad F(a, b; c; x) = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (e)_n}{n! (c)_n} \\ \times {}_4F_3 \left[\begin{matrix} a, b, 1 + h(1 - \alpha)^{-1}, -n \\ e, h(1 - \alpha)^{-1}, h - \alpha n + 1 \end{matrix} \right] (-x)^n F(e + n, h + (1 - \alpha)n; c + n; x).$$

In this paper we give a generalisation of (1), namely,

$$(2) \quad {}_{p+s}F_{q+t} \left[\begin{matrix} a_p, b_s; \\ c_q, d_t; \end{matrix} x \right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}}{n!} \frac{(b_s)_n (e_q)_n}{(d_t)_n (c_q)_n} \\ \times {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, 1 + h(1 - \alpha)^{-1}, -n \\ e_q, h(1 - \alpha)^{-1}, h - \alpha n + 1 \end{matrix} \right] (-x)^n \\ \times {}_{s+q+1}F_{t+q} \left[\begin{matrix} b_s + n, e_q + n, h + (1 - \alpha)n; \\ d_t + n, c_q + n; \end{matrix} x \right],$$

where $(h - \alpha n + 1)_{-1}$ means $(h - \alpha n)^{-1}$ and $a_{\lambda}, (a_{\lambda})_n, a_{\lambda} + n$ denote $a_1 \dots, a_{\lambda}; (a_1)_n \dots (a_{\lambda})_n$; and $a_1 + n, \dots, a_{\lambda} + n$, respectively; and from (2), we deduce some other identities.

2. **Proof of (2).** The following is a simple extension of Dr. Chaundy's proof. Comparing the coefficients in (2) of $(a_p)_N/N!$, we have to prove that

$$\frac{(b_s)_N x^N}{(c_q)_N (d_t)_N} = \{h + (1 - \alpha)N\} \sum_{n=N}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (b_s)_n (e_q)_n (-n)_N}{n! (d_t)_n (c_q)_N (e_q)_N (h - \alpha n + 1)_N} (-x)^n \\ \times {}_{s+q+1}F_{t+q} \cdot$$

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Writing $n = N + r$, we find that this reduces to

$$1 = \{h + (1 - \alpha)N\} \sum_{r=0}^{\infty} \frac{[h + (1 - \alpha)N + 1 - \alpha r]_{r-1} (b_s + N)_r (e_q + N)_r}{(d_t + N)_r (c_q + N)_r r!} (-x)^r$$

$$\times {}_{s+q+1}F_{t+q} \left[\begin{matrix} b_s + N + r, e_q + N + r, h + (1 - \alpha)(N + r); \\ d_t + N + r, c_q + N + r; \end{matrix} \middle| x \right]$$

The term independent of x on the right is unity. It remains to be proved that the coefficient of any positive power of x vanishes on the right, that is, when $M > 0$,

$$\frac{(b_s + N)_M (e_q + N)_M}{(d_t + N)_M (c_q + N)_M} \sum_{r=0}^M (-1)^r \frac{[h + (1 - \alpha)N + 1 - \alpha r]_{M-1}}{r! (M - r)!} = 0.$$

But this is the coefficient of x^{M-1} in

$$\frac{(b_s + N)_M (e_q + N)_M}{M (d_t + N)_M (c_q + N)_M} (1 - x)^{-h-(1-\alpha)N-1} [1 - (1 - x)^\alpha]^M,$$

in which the lowest term is x^M .

This completes the formal proof of (2). The rearrangement of the infinite series requires absolute convergence, which is secured when x is “sufficiently small”, at least for the case $p = q + 1$, $s = t$, in which we are particularly interested.

3. A special case. If in (2) we write $s = t$, $b_k = d_k$ for $k = 1, 2, \dots, s$, and $e_k = c_k$ for $k = 1, \dots, q$, then we obtain

$$(3) (1 - x)^h {}_pF_q \left[\begin{matrix} a_p; \\ c_q; \end{matrix} \middle| x \right]$$

$$= h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}}{n!} {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, 1 + h(1 - \alpha)^{-1}, -n \\ c_q, h(1 - \alpha)^{-1}, h - \alpha n + 1 \end{matrix} \middle| \left(\frac{-x}{(1 - x)^{1-\alpha}} \right)^n \right].$$

4. Other cases. If

$$(4) {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, 1 + h(1 - \alpha)^{-1}, -n \\ e_q, h(1 - \alpha)^{-1}, h - \alpha n + 1 \end{matrix} \right] = \frac{(\sigma_\mu)_n}{(\rho_\nu)_n},$$

then (2) and (3) reduce to simpler expressions.

4.1. In the case $p = q + 1$, (2) becomes

$$(5) \quad {}_{q+s+1}F_{q+t} \left[\begin{matrix} a_{q+1}, b_s; \\ c_q, d_t; \end{matrix} x \right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (b_s)_n (e_q)_n (\sigma_\mu)_n}{n! (d_t)_n (c_q)_n (\rho_\nu)_n} (-x)^n \\ \times {}_{q+s+1}F_{q+t} \left[\begin{matrix} b_s + n, e_q + n, h + (1 - \alpha)n; \\ d_t + n, c_q + n; \end{matrix} x \right];$$

and (3) becomes

$$(6) \quad (1-x)^h {}_{q+1}F_q \left[\begin{matrix} a_{q+1}; \\ c_q; \end{matrix} x \right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (\sigma_\mu)_n}{n! (\rho_\nu)_n} \left(\frac{-x}{(1-x)^{1-\alpha}} \right)^n,$$

which, for appropriate values of α , gives a relation between hypergeometric functions of argument x and $-x(1-x)^{\alpha-1}$.

4.2. In the case $q = 1$, $\alpha = 1/2$, $a_1 = a$, $a_2 = 2h$, $c = 2a$, (4) is summed by Watson's Theorem [1, p.16], and vanishes for odd powers of n . Then (6) becomes (see [2, formula (4.22), with $\alpha + \beta = a$, $\alpha = h$])

$$(7) \quad (1-x)^h {}_2F_1 \left[\begin{matrix} a, 2h; \\ 2a; \end{matrix} x \right] = {}_2F_1 \left[\begin{matrix} h, a-h; \\ a+1/2; \end{matrix} \frac{-x^2}{4(1-x)} \right]$$

and the corresponding formula (5) is

$$(8) \quad {}_{s+2}F_{s+1} \left[\begin{matrix} a, 2h, b_s; \\ 2a, d_s; \end{matrix} x \right] = \sum_{m=0}^{\infty} \frac{(b_s)_{2m} (h)_m (a-h)_m}{(d_s)_{2m} m! (a+1/2)_m} \left(\frac{-x^2}{4} \right)^m \\ \times {}_{s+2}F_{s+1} \left[\begin{matrix} b_s + 2m, 2a + 2m, h + m; \\ d_s + 2m, 2a + m; \end{matrix} x \right].$$

If $\alpha = -1$, $q = 2$, $a_1 = \beta$, $a_2 = \gamma$, $a_3 = \delta$, $e_1 = 1 + \beta - \gamma$, $e_2 = 1 + \beta - \delta$, $h = \beta$,

(4) can be summed by Dougall's formula [1, p.25],

$$(9) \quad {}_5F_4 \left[\begin{matrix} \beta, 1 + \beta/2, \gamma, \delta, -n \\ \beta/2, 1 + \beta - \gamma, 1 + \beta - \delta, 1 + \beta + n \end{matrix} \right] = \frac{(1 + \beta)_n (1 + \beta - \gamma - \delta)_n}{(1 + \beta - \gamma)_n (1 + \beta - \delta)_n};$$

equation (5) becomes

$$\begin{aligned}
 (10) \quad {}_{s+3}F_{s+2} & \left[\begin{matrix} \beta, \gamma, \delta, b_s; \\ c_1, c_2, d_s; \end{matrix} x \right] \\
 &= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1} (b_s)_n (1+\beta)_n (1+\beta-\gamma-\delta)_n}{n! (d_s)_n (c_1)_n (c_2)_n} (-x)^n \\
 & \quad \times {}_{s+3}F_{s+2} \left[\begin{matrix} b_s+n, 1+\beta-\gamma+n, 1+\beta-\delta+n, \beta+2n; \\ d_s+n, c_1+n, c_2+n; \end{matrix} x \right];
 \end{aligned}$$

and (6) becomes Whipple’s formula [2, p. 250, where references are given]:

$$\begin{aligned}
 (11) \quad (1-x)^\beta {}_3F_2 & \left[\begin{matrix} \beta, \gamma, \delta; \\ 1+\beta-\gamma, 1+\beta-\delta; \end{matrix} x \right] \\
 &= {}_3F_2 \left[\begin{matrix} \beta/2, (1+\beta)/2, 1+\beta-\gamma-\delta; \\ 1+\beta-\gamma, 1+\beta-\delta; \end{matrix} \frac{-4x}{(1-x)^2} \right].
 \end{aligned}$$

4.3. If $\alpha = -1, q = 4, a_1 = \beta, a_2 = \gamma, a_3 = \delta, a_4 = \epsilon, a_5 = \theta,$
 $e_1 = 1 + \beta - \gamma, e_2 = 1 + \beta - \delta, e_3 = 1 + \beta - \epsilon, e_4 = 1 + \beta - \theta, h = \beta,$

then using Whipple’s transformation [1, p. 25],

$$\begin{aligned}
 (12) \quad {}_7F_6 & \left[\begin{matrix} \beta, 1+\beta/2, \gamma, \delta, \epsilon, \theta, -n \\ \beta/2, 1+\beta-\gamma, 1+\beta-\delta, 1+\beta-\epsilon, 1+\beta-\theta, 1+\beta+n \end{matrix} \right] \\
 &= \frac{(1+\beta)_n (1+\beta-\epsilon-\theta)_n}{(1+\beta-\epsilon)_n (1+\beta-\theta)_n} {}_4F_3 \left[\begin{matrix} 1+\beta-\gamma-\delta, \epsilon, \theta, -n \\ 1+\beta-\gamma, 1+\beta-\delta, \epsilon+\theta-\beta-n \end{matrix} \right],
 \end{aligned}$$

in place of (4), we obtain

$$\begin{aligned}
 (13) \quad {}_{s+5}F_{s+4} & \left[\begin{matrix} \beta, \gamma, \delta, \epsilon, \theta, b_s; \\ c_1, c_2, c_3, c_4, d_s; \end{matrix} x \right] \\
 &= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1} (b_s)_n (1-\beta-\gamma)_n (1-\beta-\delta)_n (1+\beta)_n (1+\beta-\epsilon-\theta)_n}{n! (d_s)_n (c_1)_n (c_2)_n (c_3)_n (c_4)_n} \\
 & \quad \times {}_4F_3 \left[\begin{matrix} 1+\beta-\gamma-\delta, \epsilon, \theta, -n \\ 1+\beta-\gamma, 1+\beta-\delta, \epsilon+\theta-\beta-n \end{matrix} \right] (-x)^n \times
 \end{aligned}$$

$$\times {}_{s+5}F_{s+4} \left[\begin{matrix} b_s + n, 1 + \beta - \gamma + n, 1 + \beta - \delta + n, \\ d_s + n, c_1 + n, c_2 + n, c_3 + n, \\ 1 + \beta - \epsilon + n, 1 + \beta - \theta + n, \beta + 2n; \\ c_4 + n; \end{matrix} x \right].$$

If $b_k = d_k$ for $k = 1, \dots, s$, $c_1 = 1 + \beta - \gamma$, $c_2 = 1 + \beta - \delta$, $c_3 = 1 + \beta - \epsilon$, $c_4 = 1 + \beta - \theta$, this reduces to

$$\begin{aligned} (14) \quad (1-x)^\beta {}_5F_4 & \left[\begin{matrix} \beta, \gamma, \delta, \epsilon, \theta; \\ 1 + \beta - \gamma, 1 + \beta - \delta, 1 + \beta - \epsilon, 1 + \beta - \theta; \end{matrix} x \right] \\ &= \sum_{n=0}^{\infty} \frac{(\beta + n + 1)_{n-1} (1 + \beta)_n (1 + \beta - \epsilon - \theta)_n}{n! (1 + \beta - \epsilon)_n (1 + \beta - \theta)_n} \\ & \quad \times {}_4F_3 \left[\begin{matrix} 1 + \beta - \gamma - \delta, \epsilon, \theta, -n \\ 1 + \beta - \gamma, 1 + \beta - \delta, \epsilon + \theta - \beta - n \end{matrix} \right] \left(\frac{-x}{(1-x)^2} \right)^n. \end{aligned}$$

If

$$\beta = \frac{1}{2} a - b, \gamma = 1 - b, \delta = -\frac{1}{2} a, \epsilon = 1 + \frac{1}{2} a, \theta = b,$$

by Bailey's result [1, p.30, formula (1.3)],

$$(15) \quad {}_4F_3 \left[\begin{matrix} a, 1 + a/2, b, -n \\ a/2, 1 + a - b, 1 + 2b - n \end{matrix} \right] = \frac{(a - 2b)_n (-b)_n}{(1 + a - b)_n (-2b)_n},$$

this becomes

$$\begin{aligned} (16) \quad (1-x)^{-b+a/2} {}_5F_4 & \left[\begin{matrix} -b + a/2, 1 - b, -a/2, 1 + a/2, b; \\ a/2, 1 + a - b, -b, 1 - 2b + a/2; \end{matrix} x \right] \\ &= {}_3F_2 \left[\begin{matrix} (a - 2b)/4, (a - 2b + 2)/4, a - 2b; \\ 1 - 2b + a/2, 1 + a - b; \end{matrix} \frac{-4x}{(1-x)^2} \right]. \end{aligned}$$

4.4. If we take $\alpha = 0$, $q = 0$ and use Vandermonde's theorem in place of (4), we obtain

$$(17) \quad {}_{s+1}F_s \left[\begin{matrix} a, b_s; \\ d_s; \end{matrix} x \right]$$

$$= \sum_{n=0}^{\infty} \frac{(b_s)_n (h-a)_n}{n! (d_s)_n} (-x)^n {}_{s+1}F_s \left[\begin{matrix} b_s + n, h + n; \\ d_s + n; \end{matrix} x \right]$$

and if $b_k = d_k$ for $k = 1, \dots, s - 1$, $b_s = b, d_s = h$ this reduces to Euler's identity,

$$(18) \quad (1-x)^b {}_2F_1 \left[\begin{matrix} a, b; \\ h; \end{matrix} x \right] = {}_2F_1 \left[\begin{matrix} h-a, b; \\ h; \end{matrix} \frac{x}{x-1} \right].$$

4.5. Multiplying (7) by $(1-x)^{-h}$ and equating coefficients of x , we obtain

$$(19) \quad {}_3F_2 \left[\begin{matrix} a-h, -n/2, (1-n)/2 \\ a+1/2, 1-h-n \end{matrix} \right] = \frac{(a)_n (2h)_n}{(2a)_n (h)_n},$$

which is a particular case of Saalschutz' theorem.

Similarly from (16) we get

$$(20) \quad {}_3F_2 \left[\begin{matrix} a-2b, a/2-b+n, -n \\ 1+a/2-2b, 1+a-b \end{matrix} \right] = \frac{(1-b)_n (-a/2)_n (1+a/2)_n (b)_n}{(a/2)_n (1+a-b)_n (1+a/2-2b)_n}.$$

This is a special case of

$$(21) \quad {}_3F_2 \left[\begin{matrix} a, b, -n \\ e, 2+a+b-e-n \end{matrix} \right] = \frac{(e-b-1)_n (e-a-1)_n (\omega+1)_n}{(e)_n (e-a-b-1)_n (\omega)_n},$$

where

$$\omega = \frac{(e-a-1)(e-b-1)}{e-a-b-1},$$

which is, in Whipple's notation, a particular case of the relation between the quantities $F_p(0; 4, 5)$ and $F_p(2; 4, 5)$. [1, p. 85; 4]. This gives a generalisation of (16),

$$(22) \quad (1-x)^{2a} {}_5F_4 \left[\begin{matrix} 2a, e-c-1, 2a-e+1, 1+\phi, 1+\theta; \\ 2a+c+2-e, e, \theta, \phi; \end{matrix} x \right] \\ = {}_3F_2 \left[\begin{matrix} a, a+1/2, c; \\ e, 2+c+2a-e; \end{matrix} \frac{-4x}{(1-x)^2} \right],$$

where θ, ϕ are the roots of $m^2 - 2am + (e-c-1)(2a+1-e) = 0$. Comparing with (14), we have

$$(23) \quad {}_4F_3 \left[\begin{matrix} e - \theta - 1, 1 + \phi, e - c - 1, -n \\ 2a - \theta, e, \phi + e - c - 2a - n \end{matrix} \right] = \frac{(c)_n (2a - \phi)_n}{(e)_n (1 + 2a - \phi - e + c)_n} .$$

This is a generalisation of (15); we obtain (15), (16) from (22), (23) by taking $a = (a - 2b)/4$, $c = a - 2b$, $e = 1 + a - b$, $\theta = -b$, $\phi = a/2$.

I should like to take this opportunity of thanking Dr. Chaundy for many kindnesses and especially for allowing me to see his most recent paper before it was published.

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