SOME HYPERGEOMETRIC IDENTITIES

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1. **Introduction.** T. W. Chaundy [3] has given some hypergeometric identities of which the most general is

$$F(a, b; c; x) = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (e)_{n}}{n! (c)_{n}}$$

$$\times \binom{a, b, 1 + h (1 - \alpha)^{-1}, -n}{e, h (1 - \alpha)^{-1}, h - \alpha n + 1} (-x)^n F(e + n, h + (1 - \alpha) n; c + n; x).$$

In this paper we give a generalisation of (1), namely,

$$F_{p+q} \left[ \begin{array}{c} a_p, b_s; c_q, d_t; x \\ \end{array} \right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (b_s)_{n} (e_q)_{n}}{(d_t)_{n} (c_q)_{n}}$$

$$\times \binom{a_p, 1 + h (1 - \alpha)^{-1}, -n}{e_q, h (1 - \alpha)^{-1}, h - \alpha n + 1} (-x)^n$$

$$\times \binom{b_s + n, e_q + n, h + (1 - \alpha) n; d_t + n, c_q + n}{x},$$

where \((h - \alpha n + 1)_{-1}\) means \((h - \alpha n)^{-1}\) and \(a_\lambda, (a_\lambda)_n, a_\lambda + n\) denote \(a_1 \ldots, a_\lambda; (a_\lambda)_n \ldots (a_\lambda)_n;\) and \(a_1 + n, \ldots, a_\lambda + n,\) respectively; and from (2), we deduce some other identities.

2. **Proof of (2).** The following is a simple extension of Dr. Chaundy's proof. Comparing the coefficients in (2) of \((a_p)_N/N!,\) we have to prove that

$$\frac{(b_s)_N x^N}{(c_q)_N (d_t)_N} = \frac{1}{h + (1 - \alpha) N} \sum_{n=N}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (b_s)_n (e_q)_n (-n)_N}{n! (d_t)_n (c_q)_N (e_q)_N (h - \alpha n + 1)_N} (-x)^n$$

$$\times s^{q+1} F_{t+q}.$$
Writing \( n = N + r \), we find that this reduces to

\[
1 = \left\{ h + (1 - \alpha) N \right\} \sum_{r=0}^{\infty} \frac{[h + (1 - \alpha) N + 1 - \alpha r]_{r-1} \left( b_s + N \right)_r (e_q + N)_r}{(d_t + N)_r (c_q + N)_r} (-x)^r
\]

\[
\times \binom{s+q+1}{q} \left[ b_s + N + r, \ e_q + N + r, \ h + (1 - \alpha)(N + r); \ x \right] \]

The term independent of \( x \) on the right is unity. It remains to be proved that the coefficient of any positive power of \( x \) vanishes on the right, that is, when \( M > 0 \),

\[
\frac{(b_s + N)_M (e_q + N)_M}{(d_t + N)_M (c_q + N)_M} \sum_{r=0}^{M} (-1)^r \frac{[h + (1 - \alpha) N + 1 - \alpha r]_{M-1}}{r! (M - r)!} = 0.
\]

But this is the coefficient of \( x^{M-1} \) in

\[
\frac{(b_s + N)_M (e_q + N)_M}{(d_t + N)_M (c_q + N)_M} (1 - x)^{-h-(1-\alpha)N-1} \left[ 1 - (1 - x)^{\alpha} \right]^M,
\]

in which the lowest term is \( x^M \).

This completes the formal proof of (2). The rearrangement of the infinite series requires absolute convergence, which is secured when \( x \) is "sufficiently small", at least for the case \( p = q + 1, \ s = t \), in which we are particularly interested.

3. A special case. If in (2) we write \( s = t, \ b_k = d_k \) for \( k = 1, 2, \ldots, s, \) and \( e_k = c_k \) for \( k = 1, \ldots, q, \) then we obtain

\[
(3) \ (1 - x)^h \ \binom{a_p;}{c_q; \ x} \]

\[
= h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_n}{n!} \binom{a_p, 1 + h (1 - \alpha)^{-1}, - n}{c_q, h (1 - \alpha)^{-1}, h - \alpha n + 1} \left( \frac{-x}{(1 - x)^{1-\alpha}} \right)^n.
\]

4. Other cases. If

\[
(4) \ \binom{a_p, 1 + h (1 - \alpha)^{-1}, - n}{e_q, h (1 - \alpha)^{-1}, h - \alpha n + 1} = \frac{(\sigma_\mu)_n}{(\rho_\nu)_n},
\]

then (2) and (3) reduce to simpler expressions.
4.1. In the case \( p = q + 1 \), (2) becomes

\[
\sum_{n=0}^{\infty} \frac{(h - \alpha)n + 1}{n!} (b_s)_n (e_q)_n (\sigma_\mu)_n (-x)^n
\]

and (3) becomes

\[
(1-x)^h \binom{a_q+1}{c_q; x} = h \sum_{n=0}^{\infty} \frac{(h - \alpha)n + 1}{n!} (\sigma_\mu)_n \frac{(-x)}{(1-x)^{1-\alpha}}
\]

which, for appropriate values of \( \alpha \), gives a relation between hypergeometric functions of argument \( x \) and \( -x(1-x)^{\alpha-1} \).

4.2. In the case \( q = 1, \alpha = 1/2, a_1 = a, a_2 = 2h, c = 2a \), (4) is summed by Watson's Theorem [1, p. 16], and vanishes for odd powers of \( n \). Then (6) becomes (see [2, formula (4.22), with \( \alpha + \beta = a, \alpha = h \)])

\[
(1-x)^h \binom{a, 2h; x}{a_1} = \binom{h, a-h; -x^2}{a + 1/2; 4(1-x)}
\]

and the corresponding formula (5) is

\[
\sum_{m=0}^{\infty} \frac{(b_s)_{2m} (h)_m (a-h)_m}{(d_s)_{2m} m! (a + 1/2)_m} \frac{(-x^2)^m}{4} \binom{b_s + 2m, 2a + 2m, h + m; x}{d_s + 2m, 2a + m;}
\]

If \( \alpha = -1, q = 2, a_1 = \beta, a_2 = \gamma, a_3 = \delta, e_1 = 1 + \beta - \gamma, e_2 = 1 + \beta - \delta, h = \beta \), (4) can be summed by Dougall's formula [1, p. 25],

\[
\binom{\beta, 1+\beta/2, \gamma, \delta, -n}{\beta/2, 1+\beta-\gamma, 1+\beta-\delta, 1+\beta+n} = \frac{(1+\beta)_n (1+\beta-\gamma-\delta)_n}{(1+\beta-\gamma)_n (1+\beta-\delta)_n};
\]

equation (5) becomes
(10)  \[
_3F_2 \left[ \begin{array}{c}
\beta, \gamma, \delta, b_s; \\
c_1, c_2, d_s;
\end{array} \right] x
\]

\[
= \beta \sum_{n=0}^{\infty} \frac{(\beta + n + 1)_{n-1} (b_s)_n (1 + \beta)_n (1 + \beta - \gamma - \delta)_n}{n! (d_s)_n (c_1)_n (c_2)_n} (-x)^n
\]

\[
\times _3F_2 \left[ \begin{array}{c}
b_s + n, 1 + \beta - \gamma + n, 1 + \beta - \delta + n, \beta + 2n; \\
d_s + n, c_1 + n, c_2 + n;
\end{array} \right] x
\]

and (6) becomes Whipple's formula [2, p. 250, where references are given]:

(11)  \[
(1 - x)^{\beta} _3F_2 \left[ \begin{array}{c}
\beta, \gamma, \delta; \\
1 + \beta - \gamma, 1 + \beta - \delta;
\end{array} \right] x
\]

\[
= _3F_2 \left[ \begin{array}{c}
\beta/2, (1 + \beta)/2, 1 + \beta - \gamma - \delta; \\
1 + \beta - \gamma, 1 + \beta - \delta; \\
\end{array} \right] \frac{4x}{(1 - x)^2}.
\]

4.3. If \( \alpha = -1, q = 4, a_1 = \beta, a_2 = \gamma, a_3 = \delta, a_4 = \epsilon, a_5 = \theta, \)

\( e_1 = 1 + \beta - \gamma, e_2 = 1 + \beta - \delta, e_3 = 1 + \beta - \epsilon, e_4 = 1 + \beta - \theta, h = \beta, \)

then using Whipple's transformation [1, p. 25],

(12)  \[
_7F_6 \left[ \begin{array}{c}
\beta, 1 + \beta/2, \gamma, \delta, \epsilon, \theta, -n \\
\beta/2, 1 + \beta - \gamma, 1 + \beta - \delta, 1 + \beta - \epsilon, 1 + \beta - \theta, 1 + \beta + n
\end{array} \right]
\]

\[
= \frac{(1 + \beta)_n (1 + \beta - \epsilon - \theta)_n}{(1 + \beta - \epsilon)_n (1 + \beta - \theta)_n} _4F_3 \left[ \begin{array}{c}
1 + \beta - \gamma - \delta, \epsilon, \theta, -n \\
1 + \beta - \gamma, 1 + \beta - \delta, \epsilon + \theta - \beta - n
\end{array} \right],
\]

in place of (4), we obtain

(13)  \[
_5F_4 \left[ \begin{array}{c}
\beta, \gamma, \delta, \epsilon, \theta, b_s; \\
c_1, c_2, c_3, c_4, d_s;
\end{array} \right] x
\]

\[
= \beta \sum_{n=0}^{\infty} \frac{(\beta + n + 1)_{n-1} (b_s)_n (1 - \beta - \gamma)_n (1 - \beta - \delta)_n (1 + \beta)_n (1 + \beta - \epsilon - \theta)_n}{n! (d_s)_n (c_1)_n (c_2)_n (c_3)_n (c_4)_n}
\]

\[
\times _4F_3 \left[ \begin{array}{c}
1 + \beta - \gamma - \delta, \epsilon, \theta, -n \\
1 + \beta - \gamma, 1 + \beta - \delta, \epsilon + \theta - \beta - n
\end{array} \right] (-x)^n \
\]
\[ \times s + 5 \mathcal{F}_{5}^{4} \left[ \begin{array}{l}
 b_{s} + n, 1 + \beta - \gamma + n, 1 + \beta - \delta + n, \\
 d_{s} + n, c_{1} + n, c_{2} + n, c_{3} + n,
\end{array} \right] \]
\[ 1 + \beta - \epsilon + n, 1 + \beta - \theta + n, \beta + 2n; \]
\[ c_{4} + n; \]

If \( b_{k} = d_{k} \) for \( k = 1, \ldots, s \), \( c_{1} = 1 + \beta - \gamma, c_{2} = 1 + \beta - \delta, c_{3} = 1 + \beta - \epsilon, c_{4} = 1 + \beta - \theta \), this reduces to

\[(14) \quad (1 - x)^{\beta} \mathcal{F}_{4}^{5} \left[ \begin{array}{l}
 \beta, \gamma, \delta, \epsilon, \theta; \\
 1 + \beta - \gamma, 1 + \beta - \delta, 1 + \beta - \epsilon, 1 + \beta - \theta; \end{array} \right] \]
\[ = \sum_{n=0}^{\infty} \frac{(\beta + n + 1)_{n-1}}{n!} \frac{(1 + \beta + \epsilon - \theta)_{n}}{(1 + \beta - \epsilon)_{n}} \]
\[ \times \mathcal{F}_{3}^{4} \left[ \begin{array}{l}
 1 + \beta - \gamma - \delta, \epsilon, \theta, -n \\
 1 + \beta - \gamma, 1 + \beta - \delta, \epsilon + \theta - \beta - n
\end{array} \right] \left( \frac{-x}{(1 - x)^{2}} \right)^{n}. \]

If
\[ \beta = \frac{1}{2} a - b, \gamma = 1 - b, \delta = -\frac{1}{2} a, \epsilon = 1 + \frac{1}{2} a, \theta = b, \]
by Bailey’s result [1, p. 30, formula (1.3)],

\[(15) \quad \mathcal{F}_{3}^{4} \left[ \begin{array}{l}
 a, 1 + a/2, b, -n \\
 a/2, 1 + a - b, 1 + 2b - n
\end{array} \right] = \frac{(a - 2b)_{n} (-b)_{n}}{(1 + a - b)_{n} (-2b)_{n}}, \]

this becomes

\[(16) \quad (1 - x)^{-b + a/2} \mathcal{F}_{4}^{5} \left[ \begin{array}{l}
 -b + a/2, 1 - b, -a/2, 1 + a/2, b; \\
 a/2, 1 + a - b, -b, 1 - 2b + a/2; \end{array} \right] \]
\[ = \mathcal{F}_{2}^{3} \left[ \begin{array}{l}
 (a - 2b)/4, (a - 2b + 2)/4, a - 2b; \\
 -4x \end{array} \right] \left( 1 - 2b + a/2, 1 + a - b; \right) \left( \frac{1}{1 - x} \right)^{2}. \]

4.4. If we take \( \alpha = 0, q = 0 \) and use Vandermonde’s theorem in place of (4), we obtain

\[(17) \quad \mathcal{F}_{5}^{4} \left[ \begin{array}{l}
 a, b_{s}; \\
 d_{s}; \end{array} \right] \]

\[
\sum_{n=0}^{\infty} \frac{(b_s)_n (h-a)_n}{n! (d_s)_n} (-x)^n_{s+1} F_s \left[ \begin{array}{c}
  b_s + n, h + n; \\
  d_s + n;
\end{array} \right] x
\]

and if \( b_k = d_k \) for \( k = 1, \ldots, s - 1 \), \( b_s = b, d_s = h \) this reduces to Euler's identity,

\[
(1-x)^b \ 2F_1 \left[ \begin{array}{c}
  a, b; \\
  h; 
\end{array} \right] = 2F_1 \left[ \begin{array}{c}
  h-a, b; \\
  h; 
\end{array} \right].
\]

4.5. Multiplying (7) by \((1-x)^{-h}\) and equating coefficients of \( x \), we obtain

\[
3F_2 \left[ \begin{array}{c}
  a-h, -n/2, (1-n)/2; \\
  a + 1/2, 1-h-n;
\end{array} \right] = \frac{(a)_n (2h)_n}{(2a)_n (h)_n},
\]

which is a particular case of Saalschutz' theorem.

Similarly from (16) we get

\[
3F_2 \left[ \begin{array}{c}
  a - 2b, a/2 - b + n, -n; \\
  1 + a/2 - 2b, 1 + a - b;
\end{array} \right] = \frac{(1-b)_n (-a/2)_n (1+a/2)_n (b)_n}{(a/2)_n (1+a-b)_n (1+a/2-2b)_n}.
\]

This is a special case of

\[
3F_2 \left[ \begin{array}{c}
  a, b, -n; \\
  e, 2 + a + b - e - n;
\end{array} \right] = \frac{(e-b-1)_n (e-a-1)_n (\omega+1)_n}{(e)_n (e-a-b-1)_n (\omega)_n},
\]

where

\[
\omega = \frac{(e-a-1) (e-b-1)}{e-a-b-1},
\]

which is, in Whipple's notation, a particular case of the relation between the quantities \( F_p (0; 4, 5) \) and \( F_p (2; 4, 5) \). [1, p. 85; 4]. This gives a generalisation of (16),

\[
(1-x)^{2a} \ 5F_4 \left[ \begin{array}{c}
  2a, e-c-1, 2a-e+1, 1+\phi, 1+\theta; \\
  2a+c+2-e, e, \theta, \phi; \\
\end{array} \right] x
\]

\[
= 3F_2 \left[ \begin{array}{c}
  a, a+1/2, c; \\
  e, 2+c+2a-e; \\
\end{array} \right] \frac{-4x}{(1-x)^2},
\]

where \( \theta, \phi \) are the roots of \( m^2 - 2am + (e-c-1) (2a+1-e) = 0 \). Comparing with (14), we have
This is a generalisation of (15); we obtain (15), (16) from (22), (23) by taking
\(a = (a - 2b)/4, \ c = a - 2b, \ e = 1 + a - b, \ \theta = - b, \ \phi = a/2.\)

I should like to take this opportunity of thanking Dr. Chaundy for many
kindnesses and especially for allowing me to see his most recent paper before
it was published.

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Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

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