

Pacific Journal of Mathematics

**ON MATRICES HAVING THE SAME CHARACTERISTIC
EQUATION**

ELMER EDWIN OSBORNE

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1. **Introduction.** Let A and B be $n \times n$ matrices whose elements lie in an infinite perfect¹ field F . Alfred Brauer [1] and W. V. Parker [2] have considered the question: "When do A and B have the same characteristic equation?" Their results have been sufficiency conditions with special forms of A and B . W. T. Reid [3] has considered a related problem.

The present paper is concerned with the following theorem that contains the results of Brauer and Parker as special cases.

THEOREM. *A necessary and sufficient condition for matrices A and B to have the same characteristic equation is that there exist a nonsingular matrix P (with elements in F) such that for $N = A - P^{-1}BP$:*

Every polynomial g in A and N , each term of which contains N at least once, is nilpotent.

We introduce a special canonical form in §2 and give the proof in §3.

2. **Canonical forms.** For any matrix A , there exists a nonsingular matrix P_1 , with elements in F , such that

$$(2.1) \quad P_1^{-1}AP_1 = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_k,$$

where the characteristic equation of A_i is $[p_i(x)]^{\alpha_i} = 0$, and $p_i(x)$ is an irreducible polynomial over F . Moreover, for each A_i we have the decomposition by the nonsingular matrix P_{2i} with elements in F :

$$(2.2) \quad P_{2i}^{-1}A_iP_{2i} = A_{i1} \dot{+} A_{i2} \dot{+} \cdots \dot{+} A_{ik_i},$$

in which each $A_{i\mu}$ is nonderogatory with characteristic equation $[p_i(x)]^{\alpha_{i\mu}} = 0$ and is of the form [4, p. 750]

¹Every irreducible equation over F is separable.

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$$(2.3) \quad A_{i\mu} \equiv \begin{pmatrix} C_i & I_i & 0 & \dots & 0 \\ 0 & C_i & I_i & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I_i \\ 0 & \cdot & \cdot & \cdot & C_i \end{pmatrix};$$

where C_i is the companion matrix of $p_i(x)$, and occurs $\alpha_{i\mu}$ times down the main diagonal; I_i is the identity matrix of order the degree of $p_i(x)$. Clearly

$$\sum_{\mu=1}^{k_i} \alpha_{i\mu} = \alpha_i.$$

Letting $P_2 = P_{21} \dot{+} P_{22} \dots \dot{+} P_{2k}$ and $P = P_1 P_2$, we have a direct sum decomposition of A into matrices $A_{i\mu}$ of form (2.3). We shall indicate this by

$$(2.4) \quad P^{-1} A P = \dot{+} \sum_{i=1}^k \sum_{\mu=1}^{k_i} A_{i\mu}.$$

It should be pointed out that the existence of the canonical form (2.3) depends only on the perfectness of the field F .

3. Proof of the theorem. Necessity. Suppose A and B have the same characteristic equation

$$m(x) = \prod [p_i(x)]^{\alpha_i} = 0.$$

We may then find matrices P_a and P_b (see §2) such that

$$(3.1) \quad P_a^{-1} A P_a = \dot{+} \sum_{i=1}^k \sum_{\mu=1}^{k_i} A_{i\mu},$$

$$P_b^{-1} B P_b = \dot{+} \sum_{i=1}^k \sum_{\mu=1}^{h_i} A_{i\mu}^*;$$

where $A_{i\mu}$ and $A_{i\mu}^*$ (for the same subscript i) are of the form (2.3) and thus have the same blocks C_i on the main diagonal. Moreover $\dot{+} \sum_{\mu=1}^{k_i} A_{i\mu}$ and $\dot{+} \sum_{\mu=1}^{h_i} A_{i\mu}^*$ have the same order since A and B have the same characteristic equation.

Clearly $\dot{+} \sum_{\mu=1}^{k_i} A_{i\mu}$ is contained in the algebra of all $\alpha_i \times \alpha_i$ matrices, with elements in the field $F(C_i)$, whose elements below the main diagonal are

zero. Moreover,

$$N_i = \dagger \sum_{\mu=1}^{k_i} A_{i\mu} - \dagger \sum_{\mu=1}^{h_i} A_{i\mu}^*$$

is in the radical of this algebra since all elements on or below the main diagonal are zero. Thus $g(\dagger \sum_{\mu=1}^{k_i} A_{i\mu}, N_i)$, for g satisfying the conditions of the theorem, is a radical element and thus nilpotent. Hence, letting

$$N^1 = N_1 \dagger N_2 \dagger \dots \dagger N_k = P_a^{-1} A P_a - P_b^{-1} B P_b,$$

we see that $g(P_a^{-1} A P_a, N^1)$ is nilpotent. Finally, letting

$$P = P_b P_a^{-1} \quad \text{and} \quad N = P_a N^1 P_a^{-1} = A - P^{-1} B P,$$

we have the result that

$$(3.2) \quad P_a g(P_a^{-1} A P_a, N^1) P_a^{-1} = g(A, N)$$

is nilpotent. This completes the proof of the necessity.

Sufficiency. Assume that a P exists such that every polynomial g , satisfying the conditions of the theorem, is nilpotent. Define

$$A_\theta = A - \theta N \qquad (N = A - P^{-1} B P),$$

$m_\theta(\lambda) = |\lambda I - A_\theta| \equiv \lambda^n + a_1(\theta) \lambda^{n-1} + \dots + a_{n-1}(\theta) \lambda + a_n(\theta)$; where θ is an indeterminate and $a_i(\theta)$ ($i = 1, 2, \dots, n$) are polynomials in θ with coefficients in F .

Clearly, $m_0(\lambda) = 0$ and $m_1(\lambda) = 0$ are the characteristic equations of $A_0 \equiv A$ and $A_1 \equiv P^{-1} B P$, respectively.

If we now let θ assume values from F we have

$$m_0(A_\theta) = m_0(A) + h_\theta(A, N) = h_\theta(A, N);$$

moreover $h_\theta(A, N)$ contains N in each term and is nilpotent by hypothesis.

The characteristic roots of $m_0(A_\theta)$ are $m_0(\alpha_\theta^i)$ ($i = 1, \dots, n$), where the α_θ^i are the characteristic roots of A_θ . Since $m_0(A_\theta)$ is nilpotent we must have

$$(3.3) \quad m_0(\alpha_\theta^i) = 0 \qquad (i = 1, \dots, n).$$

From (3.3) it is clearly seen that there can be only a finite number of different

characteristic equations $m_\theta(\lambda) = 0$, since all the characteristic roots of A_θ are roots of $m_0(\lambda) = 0$. Since F is assumed to be infinite, this implies that $a_i(\theta)$ is a constant independent of θ . Thus $m_0(\lambda) \equiv m_1(\lambda)$, and the proof of the sufficiency is complete.

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Vol. 2, No. 2

February, 1952

L. Carlitz, <i>Some theorems on Bernoulli numbers of higher order</i>	127
Watson Bryan Fulks, <i>On the boundary values of solutions of the heat equation</i>	141
John W. Green, <i>On the level surfaces of potentials of masses with fixed center of gravity</i>	147
Isidore Heller, <i>Contributions to the theory of divergent series</i>	153
Melvin Henriksen, <i>On the ideal structure of the ring of entire functions</i>	179
James Richard Jackson, <i>Some theorems concerning absolute neighborhood retracts</i>	185
Everett H. Languier, <i>Homology bases with applications to local connectedness</i>	191
Janet McDonald, <i>Davis's canonical pencils of lines</i>	209
J. D. Niblett, <i>Some hypergeometric identities</i>	219
Elmer Edwin Osborne, <i>On matrices having the same characteristic equation</i>	227
Robert Steinberg and Raymond Moos Redheffer, <i>Analytic proof of the Lindemann theorem</i>	231
Edward Silverman, <i>Set functions associated with Lebesgue area</i>	243
James G. Wendel, <i>Left centralizers and isomorphisms of group algebras</i> . . .	251
Kosaku Yosida, <i>On Brownian motion in a homogeneous Riemannian space</i>	263