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LEFT CENTRALIZERS AND ISOMORPHISMS OF GROUP ALGEBRAS

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1. Introduction. Let G be a locally compact group with right invariant Haar measure m and real or complex group algebra $L(G)$[1]. A bounded linear transformation A of $L(G)$ to itself which commutes with all of the operations of left multiplication is called a *left centralizer.* Examples of transformations A which satisfy the defining equation $A(xy) = x\,Ay$, all x, $y \in L(G)$, include

(i) right multiplications: $Ax = xz$ for some fixed $z \in L(G)$;

(ii) right translations: $A = R_{g_0}$ = the operation of translation on the right by some fixed $g_0 \in G$; and

(iii) convolutions with measures: $Ax = x * \mu$, where μ is a countably additive set function of bounded variation defined on the Borel sets of G, and $x * \mu$ is defined by

$$(x * \mu)\ (g) = \int x(gh^{-1})\ \mu(dh), \qquad g \in G.$$

It is clear that (i) and (ii) are special cases of (iii); thus, given $z \in L(G)$ we may define the appropriate μ by $\mu(E) = \int_E z(g)\ m(dg)$, while for an assigned $g_0 \in G$ the corresponding measure is defined by $\mu(E) = 0$ or 1 according as E does not or does contain g_0^{-1}.

The principal result (Theorem 1) of Part I states that, conversely, *every left centralizer is a convolution with a regular measure.* Important auxiliary theorems (3 and 4) furnish a characterization of the right translations (up to scalar factors of unit modulus), and show that in the strong operator topology any left centralizer may be approximated by a finite linear combination of right translations.

In Part II these results are applied to obtain a generalization of a theorem proved in [5]. We showed there that if T is an isometric isomorphism of (real,

[1] We are using the notation and terminology of [5]. In particular, for elements x and y of $L(G)$ the symbol xy denotes the usual convolution-product.

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complex) $L(G)$ onto the (real, complex) algebra $L(G')$ belonging to a second locally compact group G' with right invariant Haar measure m', then there exists a topological isomorphism τ of G upon G' and a continuous character χ such that T is given by

$$(*) \qquad x'(g') \equiv (Tx)(\tau g) = c\chi(g)\, x(g),$$

for each $x \in L(G)$ and almost all $g \in G$, c being a measure-adjusting constant.[2] In the present paper we obtain the existence of τ and the validity of (*) by supposing only that *T is an isomorphism which does not increase norm.* (Of course, once (*) is established it follows that T must be an isometry after all.) This generalization was suggested by, and extends, some results of Helson [2], which the author had the privilege of reading in manuscript. Helson obtained the theorem to be proved here for the special case of abelian G, G' and complex algebras; his methods, unfortunately, seem to be essentially "abelian" in nature, in that he makes strong use of duality theory and the Bochner representation theorem for positive definite functions.

PART I. LEFT CENTRALIZERS

2. The principal theorem. We shall establish the following result:

THEOREM 1. *Let A be a left centralizer acting on (real, complex) $L(G)$. There exists a unique regular (real, complex) measure μ of bounded variation such that A is given by $Ax = x * \mu$; furthermore, $\|A\| = \operatorname{var}\mu$.*[3]

Proof. Let $\{V_\alpha\}$ be a basis for the neighborhoods of the identity element of G, with V_α compact; we write $\alpha \leq \beta$ in case $V_\alpha \supseteq V_\beta$, so that $\{\alpha\}$ is a directed set. Let e_α denote the characteristic function of V_α, normalized through division by $m(V_\alpha)$ so that $\|e_\alpha\| = 1$. It is well known that $\{e_\alpha\}$ constitutes an approximate identity, in the sense that $\lim_\alpha xe_\alpha = x$ for each $x \in L(G)$. Applying the transformation A, we obtain

$$\tilde{x} \equiv Ax = A \lim_\alpha xe_\alpha = \lim_\alpha A(xe_\alpha) = \lim_\alpha x\, Ae_\alpha = \lim_\alpha xf_\alpha,$$

where $f_\alpha \equiv Ae_\alpha$ and $\|f_\alpha\| \leq \|A\|\,\|e_\alpha\| = \|A\|$.

The elements f_α may be thought of as linear functionals on the space $C_0(G)$ of continuous functions vanishing at ∞; as such their norms are equal to their L norms, and thus are bounded. Since the unit sphere in $C_0(G)^*$ is weak*compact

[2] The notations G', m' used here replace Γ, μ of [5].

[3] Suggestions supplied by the referee have made possible considerable simplification in the proof of this result, as well as in that of Theorem 4.

the functionals f_α have a weak*limit point which, by the Kakutani-Riesz theorem is uniquely representable as a regular measure μ, of bounded variation; in fact, var $\mu \le \|A\|$.

Let K denote the subspace of $C_0(G)$ consisting of functions having compact support. If x and y belong to K, then so does the function z defined by

$$z(g) = \int y(h)\ x(hg^{-1})\ m(dh).$$

Let $\epsilon > 0$ and α_0 be arbitrary; since μ is a weak*limit point of the f_α, there is an $\alpha_1 > \alpha_0$ such that

$$\left| \int z(g)\ \mu(dg) - \int z(g)\ f_{\alpha_1}(g)\ m(dg)\right| < \epsilon.$$

Replacing $z(g)$ by its definition and rearranging the iterated integrals, we get

$$\left| \int y(h)\ m(dh) \int x(hg^{-1})\ \mu(dg) - \int y(h)\ m(dh) \int x(hg^{-1})\ f_{\alpha_1}(g)\ m(dg)\right| < \epsilon.$$

Since $xf_\alpha \longrightarrow \tilde{x}$ in $L(G)$ and therefore as functionals, we have, on passing to the limit through a suitable cofinal subset $\{\alpha_1\}$ of $\{\alpha\}$,

$$\left| \int y(h)\ m(dh) \int x(hg^{-1})\ \mu(dg) - \int y(h)\ \tilde{x}(h)\ m(dh)\right| \le \epsilon.$$

Therefore

$$\int y(h)\ m(dh) \int x(hg^{-1})\ \mu(dg) = \int y(h)\ \tilde{x}(h)\ m(dh);$$

this shows that $x * \mu$ and $\tilde{x}$ are equal as functionals on K, hence on $C_0(G)$, and so finally as elements of $L(G)$.

Thus we have shown that $Ax = x * \mu$, at least for $x \in K$; a density argument shows that the equation actually holds for all $x \in L(G)$. Since $\|A\| \le$ var μ it follows from the reverse inequality above that $\|A\| =$ var μ. If $x * \mu_1 = x * \mu_2$ for all $x \in K$, then it is easy to see that $\mu_1 = \mu_2$, from which it follows that μ is unique, and the proof is completed.

A result equivalent to Theorem 1 is the following:

THEOREM 2. *In the strong operator topology, the set of convolution operators on $L(G)$ is a closed subset of the ring of all bounded operators.*

Proof that Theorem 1 *implies Theorem* 2. Let $\{\mu_\alpha\}$ be a directed sequence of convolution operators which converges to an operator A in the strong topology. Clearly A is a left centralizer, and the result follows.

Proof that Theorem 2 *implies Theorem* 1. As in the proof of Theorem 1, we have $Ax = \lim_\alpha xf_\alpha$; but right multiplication by f_α may be regarded as a convolution operator μ_α, and μ_α tends strongly to A. Hence A is a convolution operator.

3. **Auxiliary theorems.** The right translations are characterized as follows:

THEOREM 3. *Let A be a bounded linear mapping of $L(G)$ to itself. A has the form $A = \lambda R_{g_0}$ for some scalar λ of unit modulus and some $g_0 \in G$ if and only if*

3A) *A is a left centralizer; and*

3B) *A preserves norm.*[4]

Proof. Since the mappings λR_{g_0} clearly have the indicated properties we have only to prove the reverse implication.

Let μ be the measure determined by A in accordance with Theorem 1. Then the assumption 3B) means that

$$\int m(dg)\ \left|\int x(gh^{-1})\ \mu(dh)\right| = ||x||, \qquad x \in L(G). \tag{1}$$

Let $|\mu|\ (E)$ = the total variation of μ on E; $|\mu|$ is a regular measure and

$$\left|\int x(gh^{-1})\ \mu(dh)\right| \leq \int |x(gh^{-1})|\ |\mu|\ (dh). \tag{2}$$

We assert that equality holds in (2) for almost all g. Supposing the contrary, let strict inequality hold on a set of positive measure, and integrate (2) over g with respect to m. From (1) we obtain, by means of Fubini's theorem,

$$\begin{aligned} ||x|| &= \int m(dg)\ \left|\int x(gh^{-1})\ \mu(dh)\right| < \int m(dg) \int |x(gh^{-1})|\ |\mu|\ (dh) \\ &= \int |\mu|\ (dh) \int |x(gh^{-1})|\ m(dg) = \int |\mu|\ (dh) \int |x(g)|\ m(dg) \\ &= (\mathrm{var}\ \mu)\ ||x|| = ||A||\ ||x|| = ||x||. \end{aligned}$$

This contradiction proves the assertion.

Let $x \in K$. Then both members of (2) are continuous functions of g, so that equality holds everywhere; set $g = i$ = identity of G and replace the function $x(h)$ by $x(h^{-1})$, which is again an element of K. We obtain

$$\left|\int x(h)\ \mu(dh)\right| = \int |x(h)|\ |\mu|\ (dh) \equiv J(x). \tag{3}$$

[4] A similar result has been given by Kawada [4], for real $L(G)$. In Kawada's theorem the condition $||Ax|| = ||x||$ is replaced by $Ax \geq 0$ if and only if $x \geq 0$, where of course the inequalities are meant in the almost everywhere sense. Theorem 3 can be deduced from this, and conversely.

For $J(x) \neq 0$ and x a nonnegative function, there is a unique constant λ_x of unit modulus such that

$$\int x(h)\ \mu(dh) = \lambda_x \int x(h)\ |\mu|\ (dh).$$

If y is a nonnegative function in K for which $J(y) \neq 0$, then similarly we may write

$$\int y(h)\ \mu(dh) = \lambda_y \int y(h)\ |\mu|\ (dh),$$

and

$$\int \{x(h) + y(h)\}\ \mu(dh) = \lambda_{x+y} \int \{x(h) + y(h)\}\ |\mu|\ (dh).$$

But the left member of the last equation is also equal to

$$\lambda_x \int x(h)\ |\mu|\ (dh) + \lambda_y \int y(h)\ |\mu|\ (dh).$$

Therefore $\lambda_x = \lambda_y = \lambda_{x+y}$. In other words, there is a unique constant λ of unit modulus such that for all nonnegative $x \in K$ for which $J(x) \neq 0$ we have

$$\int x(h)\ \mu(dh) = \lambda \int x(h)\ |\mu|\ (dh).$$

Hence for all Borel sets E we have $\mu(E) = \lambda\ |\mu|\ (E)$, and we may as well suppose that μ is nonnegative. Equation (3) then becomes

$$(4) \qquad \left|\int x(h)\ \mu(dh)\right| = \int |x(h)|\ \mu(dh), \qquad x \in K.$$

By the regularity of μ, this equation actually holds for all real continuous x having limits at ∞; that is, for all x in the space $C(G)$, where G is made compact, if necessary, by adjoining an ideal point.

We now appeal to a theorem due to Kakutani [3], stating that, if f is a functional of norm 1 on $C(\Omega)$ such that $x \geq 0$ implies $f(x) \geq 0$ and such that min $(x, y) = 0$ implies min $(f(x), f(y)) = 0$, then f is a point functional: $f(x) = x(\omega_0)$ for some fixed $\omega_0 \in \Omega$. We apply the theorem to the functional $f(x) = \int x(h)\ \mu(dh)$. Clearly $\|f\| = 1$, since var $\mu = 1$. The functional is certainly order-preserving. Finally, if min $(x, y) = 0$ then $x(g) + y(g) = |x(g) - y(g)|$, and therefore

$$\int \{x(g) + y(g)\}\ \mu(dg) = \int |x(g) - y(g)|\ \mu(dg) = \left|\int \{x(g) - y(g)\}\ \mu(dg)\right|$$

by (4). Consequently $f(x) + f(y) = |f(x) - f(y)|$, so that min $(f(x), f(y)) = 0$.

The functional $\int x(g)\ \mu(dg)$ is thus seen to satisfy the hypotheses of Kakutani's theorem, and therefore is a point functional, for some point other than the point at infinity, since the functional does not vanish identically on K. Therefore μ is concentrated at one point, and the operator A has the desired form.

Our next result states that the right translations span the space of all left centralizers. Precisely, we prove:

THEOREM 4. *Every left centralizer A is a strong limit point of the set of finite linear combinations of right translations.*

Proof. By the Hahn-Banach theorem, if suffices to show that, it F is any strongly continuous linear functional on the operators on $L(G)$, which vanishes on the right translations, then $F(A) = 0$. A strongly continuous linear functional F on the operators $\{T\}$ on a Banach space X is given by an expression of the form

$$F(T) = \sum_{i=1}^{n} x_i^*(Tx_i),$$

where $x_i \in X$, $x_i^* \in X^*$, $i = 1, 2, \cdots, n$ (see, for example, the proof of Theorem 2 in [1]). When $X = L(G)$, we have

$$F(T) = \sum_{i=1}^{n} \int v_i(g)\,(Tx_i)(g)\,m(dg),$$

where the v_i are bounded measurable functions.

Suppose now that F vanishes on right translations. This means that

$$\sum_{i=1}^{n} \int v_i(g)\,x_i(gh^{-1})\,m(dg) = 0 \qquad (h \in G).$$

Computing $F(A)$, we have

$$F(A) = \sum_{i=1}^{n} \int v_i(g)\,(Ax_i)(g)\,m(dg) = \sum_{i=1}^{n} \int v_i(g)\,m(dg) \int x_i(gh^{-1})\,\mu(dh)$$

$$= \int \mu(dh) \sum_{i=1}^{n} \int v_i(g)\,x_i(gh^{-1})\,m(dg) = 0,$$

as we wished to show.

PART II. ISOMORPHISM OF GROUP ALGEBRAS

4. The isomorphism theorem. In this and succeeding §§, G' is a second locally compact group having right invariant Haar measure m', group algebra $L(G')$, and right translation operators $R'_{g'}$. The chief result to be established is the following:

THEOREM 5. *Let T be an algebra isomorphism of $L(G)$ onto $L(G')$, both algebras real or complex, which does not increase norms. There exists a bicontinuous isomorphism τ of G upon G' and a continuous character χ (real or complex with $L(G)$) on G such that*

5A) $TR_gT^{-1} = \chi(g^{-1})\, R'_{\tau g}$, $\qquad g \in G,\ \tau g \in G'$;

5B) $(Tx)(\tau g) = c\chi(g)\, x(g)$;

hence T is actually an isometry. The number c is the constant value of the ratio $m(E)/m'(\tau E)$.

The proof of Theorem 5, which will be given in §6 after some necessary lemmas have been developed, is based on the following idea, due in part to Kawada [4]. First of all, it is clear that T induces a $1-1$ mapping of the left centralizers of $L(G)$ onto those of $L(G')$ by means of the formula

$$A' = TAT^{-1},$$

the boundedness of T^{-1} being guaranteed by that of T together with the $1-1$-ness. In particular, then, translations on $L(G)$ are carried into left centralizers. It turns out, moreover, that the image of an arbitrary translation R_g is an isometric left centralizer; this is proved in Lemma 1. Therefore, by Theorem 3, it follows that the operator TR_gT^{-1} is a scalar multiple of a translation on $L(G')$; we write

$$TR_gT^{-1} = \lambda_g\, R'_{\tau g},$$

where $|\lambda_g| = 1$. We then show (Lemmas 2, 3) that τ is $1-1$, onto, and bicontinuous, and that λ_g (or rather, its inverse) is a continuous character, thereby establishing 5A). The formula 5B) follows quickly with the aid of Theorem 4.

5. **The mapping of translations.** We shall first prove:

LEMMA 1. *Let R_g be a right translation on $L(G)$ and set $Z'_g = TR_gT^{-1}$. Then for some λ_g of modulus unity, and for some $g' \equiv \tau g \in G'$, we have*

$$Z'_g = TR_gT^{-1} = \lambda_g\, R'_{g'}.$$

Proof. In view of Theorem 3, we have only to show that Z'_g is isometric, since it certainly commutes with all left multiplications. As used in the proof of Theorem 1, let $\{e_\alpha\}$ be an approximate identity of $L(G)$, and put $Te_\alpha = e'_\alpha$. Choose $x' \in L(G')$; clearly $x'e'_\alpha \longrightarrow x'$, since $xe_\alpha \longrightarrow x$ for all $x \in L(G)$. Then

$$Z'_g x' = \lim_\alpha x' Z'_g e'_\alpha = \lim_\alpha x' \, TR_g T^{-1} T e_\alpha = \lim_\alpha x' \, TR_g e_\alpha,$$

and therefore

$$\|Z'_g x'\| \le \overline{\lim}_\alpha \|x' \, TR_g e_\alpha\| \le \|x'\| \, \overline{\lim}_\alpha \|TR_g e_\alpha\| \le \|x'\| \, \overline{\lim}_\alpha \, \|R_g e_\alpha\|$$
$$= \|x'\| \, \overline{\lim}_\alpha \, \|e_\alpha\| = \|x'\|,$$

which shows that Z'_g is a contraction. But $Z_g'^{-1} = Z'_{g^{-1}}$ is also a contraction, by the same argument. Therefore Z'_g is an isometry, as we had to show.

LEMMA 2. *The mappings $g \longrightarrow \lambda g$ and $g \longrightarrow \tau g$ defined above are continuous homomorphisms of G to, respectively, the multiplicative group of scalars of unit modulus and the group G'; τ is $1-1$.*

Proof. The fact that the mappings are homomorphisms follows from the equations

$$\lambda_{gh} \, R'_{\tau(gh)} = TR_{gh} T^{-1} = TR_g T^{-1} TR_h T^{-1} = \lambda_g R'_{\tau g} \, \lambda_h R'_{\tau h} = \lambda_g \lambda_h \, R'_{\tau g \, \tau h}.$$

The function τ is $1-1$; for if $\tau g = i'$, the identity of G', then

$$\lambda_g I' = TR_g T^{-1}, \text{ and } R_g = \lambda_g I, \text{ so that } g = i, \ \lambda_g = 1.$$

In order to prove that τ is continuous, we observe that it is the product of the mappings M_1, M_2, M_3 defined by:

$$M_1 : g \longrightarrow R_g, \qquad g \in G;$$

$$M_2 : R_g \longrightarrow TR_g T^{-1} = \lambda_g \, R'_{\tau g}, \qquad g \in G;$$

$$M_3 : \lambda R'_{g'} \longrightarrow g', \qquad \lambda \text{ a scalar of unit modulus, } g' \in G'.$$

It is well known that, in the strong operator topology, R_g is a continuous function of g, so that M_1 is continuous. M_2 is continuous, since T and its inverse are bounded. The mapping M_3 is a homomorphism of the group of operators $\{\lambda R'_{g'}\}$ onto G'; hence in order to prove its continuity everywhere it is sufficient to consider merely neighborhoods of the identity I'.

Let V' be an arbitrary neighborhood of $i' \in G'$; we shall construct a strong neighborhood of I' whose image under M_3 is contained in V'. Let W' be a neighborhood of i' having finite measure w and satisfying $W'W'^{-1} \subseteq V'$. Let $x' \in L(G')$ be the characteristic function of W'. We shall show that if $\|\lambda R'_{g'} x' - x'\| < 2w$ then $g' \in V'$. In fact, suppose that $g' \notin V'$; then W' and $W'g'$ are disjoint, so that

$$\begin{aligned} \|\lambda R'_{g'}x' - x'\| &= \int |\lambda x'(h'g') - x'(h')|\ m'(dh') \\ &= \int \{ |\lambda x'(h'g')| + |x'(h')| \}\ m'(dh') \\ &= \|\lambda R'_{g'} x'\| + \|x'\| = 2\|x'\| = 2w. \end{aligned}$$

Hence M_3 and *a fortiori* $\tau = M_3 M_2 M_1$ are continuous in g. Finally, λ_g is continuous since $\lambda_g I$ is the product of the uniformly bounded and continuous functions TR_gT^{-1} and $R'_{\tau g^{-1}}$.

LEMMA 3. *The mapping τ exhausts G' and is a homeomorphism of G onto G'.*

Proof. We first show that τG is closed in G'. Suppose that $\{\tau g_\alpha\}$ is a directed sequence of elements which converges to an element h' of G'. Then the corresponding translations $R'_{\tau g_\alpha}$ tend to $R'_{h'}$. Mapping back to the algebra of operators on $L(G)$, we see that $T^{-1}R'_{\tau g_\alpha}T$ tends to $T^{-1}R'_{h'}T$. But $T^{-1}R'_{\tau g}T = \lambda_g^{-1}R_g$ by definition of τ and λ. Therefore the operators $\lambda_{g_\alpha}^{-1}R_{g_\alpha}$ converge in the strong topology to an operator which is clearly on isometric left centralizer, and which therefore has the form λR_h for some scalar λ and $h \in G$. Returning to $L(G')$, we readily see that $\lambda TR_hT^{-1} = R'_{h'}$, so that $h' = \tau h$, $\lambda = \lambda_h^{-1}$.

Next we note that the continuity of τ^{-1} on τG follows from the equation

$$T^{-1}R'_{g'}T = \lambda^{-1}_{\tau^{-1}g'}R_{\tau^{-1}g'} \quad (g' = \tau g)$$

just as that of τ was obtained from its defining equation.

Finally we establish the fact that $\tau G = G'$. Suppose if possible that $h' \in G'$ has no counterimage in G; $T^{-1}R'_{h'}T \equiv A$ is nevertheless a left centralizer on $L(G)$, and therefore, by Theorem 4, may be expressed as the strong limit of a directed sequence $\{A_\alpha\}$ of finite linear combinations of translations. Then $TAT^{-1} = R'_{h'}$ is the strong limit of operators A_α each of which is a finite linear combination of translations $R'_{g'}$, with $g' \in \tau G$. Let W' be a neighborhood of i' so small that $m'(W') < \infty$ and $h'W'^{-1}W' \cap \tau G$ is empty; the existence of such a W' is assured by the fact that τG is closed. Let x' be the characteristic function of W', and set $x'_\alpha = A_\alpha x'$. Let k' be any element of $W'h'^{-1}$. Then $k'g' \notin W'$, for each $g' \in \tau G$. Hence $x'(k'h') = 1$ implies that $x'_\alpha(k') = 0$. Therefore

$$\begin{aligned} \|x'_\alpha - R'_{h'}x'\| &\geq \int_{W'h'^{-1}} |x'_\alpha(k') - x'(k'h')|\ m'(dk') \\ &= \int_{W'h'^{-1}} 1\ m'(dk') = m'(W'). \end{aligned}$$

But this contradicts the assertion that A'_α tends strongly to $R'_{h'}$. Hence no such h' can exist; this completes the proof of the lemma.

6. **Proof of the isomorphism theorem.** If we define $\chi(g) = \lambda_g^{-1}$, it is clear that the lemmas of the preceeding § establish the existence and properties of τ and χ and the validity of 5A). In order to prove 5B), let U be the isomorphism of $L(G)$ onto $L(G')$ which 5B) defines; that is,

$$(Ux)(\tau g) = c\chi(g)\, x(g), \quad x \in L(G).$$

Then

$$(U^{-1}x')(g) = c^{-1}\chi(g^{-1})\, x'(\tau g) \text{ for } x' \in L(G').$$

Hence

$$\begin{aligned}(UR_hU^{-1}x')(\tau g) &= c\chi(g)(R_hU^{-1}x')(g) = c\chi(g)(U^{-1}x')(gh)\\ &= c\chi(g)\, c^{-1}\chi(h^{-1}g^{-1})\, x'(\tau g\tau h) = \chi(h^{-1})\, x'(\tau g\,\tau h)\\ &= \chi(h^{-1})\,(R_{\tau h}x')(\tau g),\end{aligned}$$

showing that $UR_hU^{-1} = \chi(h^{-1})\, R_{\tau h}$, $h \in G$. Therefore $UR_gU^{-1} = TR_gT^{-1}$, and consequently $T^{-1}UR_g = R_gT^{-1}U$. Let $S = T^{-1}U$; we see that S is a bicontinuous automorphism of $L(G)$ which commutes with all right translations. We shall show that S is the identity mapping, which will prove that $U = T$, as desired.

Let $z \in L(G)$, and let A be the left centralizer defined by right multiplication by z: $Ax = xz$, all $x \in L(G)$. Let $\{A_\alpha\}$ be a directed sequence of combinations of translations which converges to A in the strong topology. We have

$$Ax = \lim_\alpha A_\alpha x.$$

Therefore

$$SAx = S \lim_\alpha A_\alpha x = \lim_\alpha SA_\alpha x = \lim_\alpha A_\alpha Sx = ASx.$$

In other words, using the fact that S is an automorphism, we have

$$Sx\, Sz = S(xz) = SAx = ASx = (Sx)z,$$

so that $Sz = z$. Since $z \in L(G)$ is arbitrary, $S = I$, and the proof is completed.

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