

# Pacific Journal of Mathematics

**OSCILLATION THEOREMS FOR THE SOLUTIONS OF  
LINEAR, NONHOMOGENEOUS, SECOND-ORDER  
DIFFERENTIAL SYSTEMS**

LEONARD P. BURTON

# OSCILLATION THEOREMS FOR THE SOLUTIONS OF LINEAR, NONHOMOGENEOUS, SECOND-ORDER DIFFERENTIAL SYSTEMS

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1. **Introduction.** Oscillation theorems for the solutions of the equation

$$\frac{d}{dx} \left[ K(x) \frac{dy}{dx} \right] - G(x)y = 0.$$

are classical. It is the purpose of this paper to develop theorems of a similar nature for a class of equations of the type

$$\frac{d}{dx} \left[ K(x) \frac{dy}{dx} \right] - G(x)y = A(x).$$

It will be assumed that over an interval  $X: a \leq x \leq b$  ( $b > a$ ), the functions  $K(x)$ ,  $G(x)$ , and  $A(x)$  are continuous. All quantities used are assumed to be real. Primes will be used to indicate derivatives with respect to  $x$ .

Use will be made of the following lemma which gives a modified form of properties of the second-order linear homogeneous equation developed by W. M. Whyburn [3, pp. 633-634].

LEMMA 1. *Let  $y(x)$ , a solution of  $(Ky')' - Gy = 0$  over  $X$ , have the  $m$  zeros  $r_1, \dots, r_m$  ( $m > 2$ ) on  $X$ . Let the inequalities  $K > 0$ ,  $G < 0$  hold, and let  $GK$  be a nonincreasing function of  $x$  on  $X$ . If  $A$  is nonvanishing except possibly at  $a$ , and for  $x > a$  either one of the following is true over  $X$ :*

- (a)  $A > 0$  and  $A/G$  is a strictly decreasing function of  $x$ ,
- (b)  $A < 0$  and  $A/G$  is a strictly increasing function of  $x$ ,

then

$$\left| \int_{r_i}^{r_{i+1}} A(t)y(t) dt \right| < \left| \int_{r_{i+1}}^{r_{i+2}} A(t)y(t) dt \right| \quad (i = 1, \dots, m-2).$$

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In order to prove this lemma one needs only to make straightforward modifications in the arguments given by Whyburn.

LEMMA 2. *Under the hypotheses of Lemma 1, the zeros of*

$$F(x) = \int_a^x A(t)y(t)dt$$

*and  $y(x)$  separate each other on  $a < x \leq b$ .*

This result, which also was given by Whyburn, is an immediate consequence of Lemma 1.

LEMMA 3. *Let  $u(x)$  be any solution of the system  $(Ky')' - Gy = 0$ ,  $y(b) = 0$ . Under the hypotheses of Lemma 1,  $\int_x^b A(t)u(t)dt$  does not vanish in  $a \leq x < b$ .*

*Proof.* If  $u(x)$  has no zero except  $b$  on  $X$ , the conclusion is obvious. Otherwise, by Lemma 1, if  $q$  is the last zero of  $u(x)$  on  $X$  preceding  $b$ , then the integral  $\int_x^b A(t)u(t)dt$  has the sign of  $\int_q^b A(t)u(t)dt$ .

For the sake of brevity we shall henceforth let  $(H)$  represent the following set of conditions on  $X$ .

- $$(H) \left\{ \begin{array}{l} (1) \quad K(x) > 0, \quad G(x) < 0. \\ (2) \quad K(x)G(x) \text{ is a nonincreasing function of } x. \\ (3) \quad \text{Either one of the following is true:} \\ \quad (i) \quad \beta \leq 0, \quad A(x) > 0 \text{ for } x > a \text{ and } A(x)/G(x) \text{ is a strictly decreasing function of } x. \\ \quad (ii) \quad \beta \geq 0, \quad A(x) < 0 \text{ for } x > a \text{ and } A(x)/G(x) \text{ is a strictly increasing function of } x. \end{array} \right.$$

Let  $u_1(x)$  be any solution of  $(Ky')' - Gy = 0$  such that  $u_1(b) = 0$ . Choose another solution  $u_2(x)$  such that  $K(u_2u_1' - u_2'u_1) \equiv 1$  on  $X$ . As a final preliminary result we have the following:

LEMMA 4. *Under the hypotheses (H) if  $\beta \neq 0$ , then*

$$\frac{\beta}{u_2(b)} \int_x^b A(t)u_1(t)dt > 0$$

*over  $a \leq x < b$ .*

*Proof.* By Lemma 3,  $\int_x^b A(t)u_1(t)dt$  has the same sign over  $a \leq x < b$  as

$\int_{q_f}^b A(t)u_1(t)dt$ , where  $q_f$  is the last zero of  $u_1(x)$  on  $a \leq x < b$  (or where  $q_f = a$  if  $u_1(x)$  has no such zero). From  $K(u_2u_1' - u_2'u_1) \equiv 1$  we obtain  $1/u_2(b) = K(b)u_1'(b)$ . Hence

$$\frac{\beta}{u_2(b)} \int_{q_f}^b A(t)u_1(t)dt = K(b) \int_{q_f}^b [\beta A(t)][u_1'(b)u_1(t)]dt$$

and this latter expression is positive since the integrand is the product of two negative quantities.

Hereafter free use will be made of the facts that any solution of  $(Ky')' - Gy = 0$  can have only a finite number of zeros on  $X$  and that, under the hypothesis  $GK < 0$ , the zeros of any two linearly independent solutions separate each other.

**2. Oscillation theorems.** Let  $y_1(x)$  be any solution of  $(Ky')' - Gy = A$  over  $X$  which satisfies the condition  $y(b) = \beta$ . Then  $y_1(x)$  can be expressed in the form

$$y_1(x) = cu_1(x) + \frac{\beta}{u_2(b)} u_2(x) + u_1(x) \int_a^x A(t)u_2(t)dt + u_2(x) \int_x^b A(t)u_1(t)dt,$$

where  $u_1(x)$  and  $u_2(x)$  are as in Lemma 4, and  $c$  is a constant. We shall prove the following result.

**THEOREM 1.** *Under the hypotheses (H) the zeros of  $y_1(x)$  and  $u_1(x)$  separate each other on  $a \leq x < b$ .*

[If  $\beta \neq 0$  the restriction that  $A/G$  be strictly increasing or decreasing may be modified to the extent of allowing  $A/G$  to be a monotone increasing or decreasing function. Under the modified hypotheses it can be shown that

$$\frac{\beta}{u_2(b)} \int_x^b A(t)u_1(t)dt \geq 0,$$

and since  $\beta/u_2(b)$  is not zero the proof of the theorem is still valid.]

*Proof.* The functions  $y_1(x)$  and  $u_1(x)$  cannot vanish simultaneously on  $X$  except at  $b$ ; for, letting  $q$  be a zero of  $y_1(x)$  and  $u_1(x)$  one obtains

$$y_1(q) = u_2(q) \left[ \int_q^b A(t)u_1(t)dt + \beta/u_2(b) \right] = 0.$$

This is impossible since  $u_2(q) \neq 0$  and, by Lemmas 3 (if  $\beta = 0$ ) and 4 (if  $\beta \neq 0$ ), the expression in brackets never vanishes.

Suppose now that  $q$  and  $q' \neq b$ , ( $q < q'$ ), are consecutive zeros of  $u_1(x)$ , and that  $y_1(x)$  does not vanish at any point of  $q < x < q'$ . Then, by Rolle's Theorem,  $[u_1(x)/y_1(x)]'$  must vanish at least once in this interval. But

$$\left[ \frac{u_1(x)}{y_1(x)} \right]' = \frac{\int_x^b A(t)u_1(t)dt + \beta/[u_2(b)]}{K(x)y_1^2(x)}$$

and, as above, this expression never vanishes.

In a similar manner it can be shown that between two consecutive zeros of  $y_1(x)$ ,  $u_1(x)$  must vanish at least once.

**COROLLARY 1.** *If  $\beta \neq 0$ , the zeros of  $y_1(x)$  and  $u_1(x)$  separate each other on  $a \leq x \leq b$ .*

*Proof.* If  $\beta \neq 0$ , the above argument is valid with  $q' = b$ .

**COROLLARY 2.** *If  $u_1(x)$  has  $m$  zeros on  $X$ , then  $y_1(x)$  has either  $m - 1$ ,  $m$ , or  $m + 1$  zeros on  $X$ .*

*Proof.* Let  $q_0$  be the first zero of  $u_1(x)$  on  $X$  and  $q_f$  be the last zero of  $u_1(x)$  preceding  $b$ ;  $y_1(x)$  may or may not have a zero in  $a \leq x < q_0$ . In the interval  $q_0 < x < q_f$ ,  $y_1(x)$  has exactly  $m - 2$  zeros. If  $\beta \neq 0$ , then  $y_1(x)$  has exactly one zero in  $q_f < x < b$  by Corollary 1. If  $\beta = 0$ ,  $y_1(x)$  may or may not vanish in  $q_f < x < b$ . (See Theorem 4.)

The next theorem is applicable only if the system  $(Ky')' - Gy = 0$ ,  $y(a) = y(b) = 0$  is incompatible. In this case (i) one can select linearly independent solutions  $u_1(x)$  and  $u_2(x)$  of  $(Ky')' - Gy = 0$  such that  $u_1(b) = u_2(a) = 0$  and  $K(u_2u_1' - u_2'u_1) \equiv 1$  on  $X$  and (ii) the nonhomogeneous system  $(Ky')' - Gy = A$ ,  $y(a) = 0$ ,  $y(b) = \beta$  has a solution, say  $y_2(x)$ . We then have the following result.

**THEOREM 2.** *Let the hypotheses (H) be satisfied. Assume that  $u_2(x)$  oscillates on  $X$ , and let  $a = p_1, p_2, \dots, p_m$  ( $m > 3$ ) be its consecutive zeros. Then, for  $i \neq 1$ ,  $y_2(p_i) \neq 0$  and either  $y_2(x)$  has two zeros in  $(p_i, p_{i+1})$  and none in  $(p_{i+1}, p_{i+2})$  ( $2 \leq i \leq m - 2$ ), or vice versa. In the interval  $a < x < p_2$ ,  $y_2(x)$  has either no zero or one zero. In the former case it has two zeros in  $(p_2, p_3)$ , in the latter case it has no zero in  $(p_2, p_3)$ . If  $y_2(x)$  has two zeros in  $(p_{m-1}, p_m)$ , it has no zero in  $p_m < x < b$ .*

*Proof.* The function  $y_2(x)$  can be expressed in the form

$$y_2(x) = \frac{\beta}{u_2(b)} u_2(x) + u_1(x) \int_a^x A(t)u_2(t)dt + u_2(x) \int_x^b A(t)u_1(t) dt.$$

If  $y_2(x)$  has three or more zeros in  $(p_i, p_{i+1})$  ( $2 \leq i \leq m - 1$ ), Theorem 1 requires that  $u_1(x)$  have more than one zero in that interval; this is impossible since the zeros of  $u_1(x)$  and  $u_2(x)$  separate each other. Also,  $y_2(x)$  cannot have a single zero in  $(p_i, p_{i+1})$  ( $2 \leq i \leq m - 1$ ), for then  $y_2(p_i)y_2(p_{i+1}) < 0$  and such a product is always positive. To see this, notice that

$$y_2(p_i) = u_1(p_i) \int_a^{p_i} A(t)u_2(t)dt = u_1(p_i)F(p_i),$$

where  $F(x) = \int_a^x A(t)u_2(t)dt$  as in Lemma 2. Since the zeros of both  $u_1(x)$  and  $F(x)$  separate those of  $u_2(x)$ , the product  $u_1(p_i)F(p_i)$  ( $2 \leq i \leq m$ ) is consistently positive or negative. Thus  $y_2(p_i)y_2(p_{i+1}) > 0$  ( $2 \leq i \leq m - 1$ ).

The function  $u_1(x)$  has a zero in each of  $(p_i, p_{i+1})$  and  $(p_{i+1}, p_{i+2})$  ( $1 \leq i \leq m - 2$ ). By Theorem 1,  $y_2(x)$  must have a zero in  $(p_i, p_{i+2})$ . If  $y_2(x)$  has no zero in  $(p_i, p_{i+1})$ , it must have one, and therefore two, in  $(p_{i+1}, p_{i+2})$ . Now assume that  $y_2(x)$  has two zeros in  $(p_i, p_{i+1})$ . If  $y_2(x)$  also has two zeros in  $(p_{i+1}, p_{i+2})$ , then  $u_1(x)$  must have three zeros in  $(p_i, p_{i+2})$ ; but this is impossible. Hence  $y_2(x)$  has no zero in  $(p_{i+1}, p_{i+2})$ .

This same type of argument can be used to prove the part of the theorem pertaining to the interval  $a < x < p_2$  and the interval  $p_m < x < b$ .

REMARK. Theorems 1 and 2 are not true in case  $\beta \neq 0$  without the restriction  $\beta A(x) < 0, x > a$ . This is shown by the example

$$\left(\frac{1}{x} y'\right)' + xy = -x^3, \quad y(0) = 0, \quad y(\sqrt{9\pi}) = -9\pi.$$

Here  $\beta A(x) = 9\pi x^3 > 0$  on  $0 < x < \sqrt{9\pi}$ . The solution of the given system is  $y(x) = -x^2$ , which does not oscillate. However, each of  $u_1(x) = -\cos(x^2/2)$ ,  $u_2(x) = \sin(x^2/2)$  has five zeros on  $0 \leq x \leq \sqrt{9\pi}$ .

**3. Application to a system involving a parameter.** It will now be supposed that  $K, G,$  and  $A$  are continuous functions of  $(x, \lambda)$  when  $a \leq x \leq b, \Lambda_1 < \lambda < \Lambda_2$ . The system

$$[K(x, \lambda)y']' - G(x, \lambda)y = 0, \quad y(a, \lambda) = 0, \quad y(b, \lambda) = 0,$$

is a system of Sturmian type. Let  $K$  and  $G$  satisfy conditions sufficient to assure the validity of known oscillation theorems for this system [1, p.66] to the

extent that there exists an infinite set of characteristic numbers  $\lambda_i$ ,  $\Lambda_1 < \lambda_0 < \dots < \lambda_m < \dots < \Lambda_2$ , having no limit point except  $\Lambda_2$ , and such that if  $u_m$  is the characteristic function corresponding to  $\lambda_m$  then  $u_m$  has  $m$  zeros in  $a < x < b$ .

Let  $v_2(x, \lambda)$  be the solution of

$$[K(x, \lambda)y']' - G(x, \lambda)y = 0$$

satisfying the initial conditions  $v_2(a, \lambda) \equiv 0$ ,  $v_2'(a, \lambda) \equiv \sigma$ , where  $\sigma$  is a positive constant. By the fundamental existence theorem [1, p. 7],  $v_2(x, \lambda)$  is a continuous function of  $x$  and  $\lambda$ . It is well known [2, pp. 229, 232] that as  $\lambda$  increases from  $\Lambda_1$  a new zero of  $v_2(x, \lambda)$  appears at  $b$  for  $\lambda = \lambda_i$  ( $i = 0, 1, \dots$ ), and that each such zero moves continuously towards  $a$  as  $\lambda$  increases continuously.

For each  $\lambda$ , let  $v_1(x, \lambda)$  be a solution of

$$[K(x, \lambda)y']' - G(x, \lambda)y = 0$$

satisfying the condition  $v_1(b, \lambda) = 0$ . If  $\lambda = \lambda_i$  ( $i = 0, 1, \dots$ ),  $v_1(x, \lambda)$  is simply a constant multiple of  $v_2(x, \lambda)$ . For  $\lambda \neq \lambda_i$ ,  $v_1(x, \lambda)$  and  $v_2(x, \lambda)$  are linearly independent. It follows that on  $X$ , for  $\lambda < \lambda_0$ ,  $v_1(x, \lambda)$  has a zero only at  $b$ ; for  $\lambda_m \leq \lambda < \lambda_{m+1}$  ( $m = 0, 1, \dots$ ),  $v_1(x, \lambda)$  has  $m + 2$  zeros. Theorem 1 and its corollaries apply to give the following result.

**THEOREM 3.** *Let the system*

$$[K(x, \lambda)y']' - G(x, \lambda)y = A(x, \lambda), \quad y(b, \lambda) = \beta(\lambda),$$

for each fixed  $\lambda$  in  $(\Lambda_1, \Lambda_2)$ , satisfy the hypotheses (H). Let  $y_1(x, \lambda)$  be a solution. Over  $X$ :  $a \leq x \leq b$ , if

$\beta(\lambda) \neq 0$  and  $\lambda < \lambda_0$ , then  $y_1(x, \lambda)$  has either no zero or one zero,

$\lambda = \lambda_m$  ( $m \geq 0$ ), then  $y_1(x, \lambda)$  has  $m + 1$  zeros,

$\lambda_m < \lambda < \lambda_{m+1}$  ( $m \geq 0$ ), then  $y_1(x, \lambda)$  has  $m + 1$

or  $m + 2$  zeros;

$\beta(\lambda) = 0$  and  $\lambda < \lambda_0$ , then  $y_1(x, \lambda)$  has either one zero or two zeros,

$\lambda = \lambda_m$  ( $m \geq 0$ ), then  $y_1(x, \lambda)$  has  $m + 1$  or  $m + 2$  zeros,

$\lambda_m < \lambda < \lambda_{m+1}$  ( $m \geq 0$ ) then  $y_1(x, \lambda)$  has  $m + 1$ ,  $m + 2$ ,

or  $m + 3$  zeros.

COROLLARY. As  $\lambda$  increases in  $(\Lambda_1, \Lambda_2)$  the number of zeros of the solutions  $y_1(x, \lambda)$  increases indefinitely.

Interesting and more precise results can be obtained in connection with the two-point system

$$(S_\lambda) \quad \begin{aligned} [K(x, \lambda)y']' - G(x, \lambda)y &= A(x, \lambda), \\ y(a, \lambda) = 0, \quad y(b, \lambda) &= 0, \end{aligned}$$

where  $K, G, A$  conform to the hypotheses  $(H)$  for each fixed  $\lambda$  in  $(\Lambda_1, \Lambda_2)$ . If  $\lambda = \lambda_i$  ( $i = 0, 1, \dots$ ), then  $(S_\lambda)$  is of course incompatible. Otherwise, for each  $\lambda$  one can choose  $v_1(x, \lambda)$  such that

$$v_1(b, \lambda) = 0, \quad v_1'(b, \lambda) = \frac{1}{K(b, \lambda)v_2(b, \lambda)},$$

so that

$$\begin{aligned} K(x, \lambda) [v_2(x, \lambda)v_1'(x, \lambda) - v_2'(x, \lambda)v_1(x, \lambda)] \\ \equiv K(b, \lambda) [v_2(b, \lambda)v_1'(b, \lambda)] \equiv 1 \end{aligned}$$

on  $X$ . It follows that  $v_1(a, \lambda) = -1/[\sigma K(a, \lambda)]$  is negative for all  $\lambda$ .

The solution of  $(S_\lambda)$  can be expressed as

$$\begin{aligned} y_2(x, \lambda) &= v_1(x, \lambda) \int_a^x A(t, \lambda)v_2(t, \lambda) dt \\ &+ v_2(x, \lambda) \int_x^b A(t, \lambda)v_1(t, \lambda) dt. \end{aligned}$$

We now consider an interval  $L_m: \lambda_m < \lambda < \lambda_{m+1}$ , and let  $X_0$  represent the interval  $a < x < b$ . For a fixed  $\lambda$  in  $L_m$ , each of  $v_1(x, \lambda)$  and  $v_2(x, \lambda)$  has  $m + 1$  zeros on  $X_0$ . For the sake of definiteness let  $A(x, \lambda)$  be positive over  $X_0L_m$ ; and let  $m$  be odd so that, for  $\lambda$  in  $L_m$ ,  $v_1(x, \lambda)$  and  $v_2(x, \lambda)$  each has an even number,  $m + 1$ , of zeros in  $X_0$ . Then  $v_1'(b, \lambda) > 0$  and, by virtue of Lemma 3,

$$y_2'(a, \lambda) = \sigma \int_a^b A(t, \lambda)v_1(t, \lambda) dt$$

is negative. Let  $q_f(\lambda)$  represent the last zero of  $v_1(x, \lambda)$  preceding  $b$ . By Theorem 1 it follows that  $y_2[q_f(\lambda), \lambda]$  is positive over  $L_m$ . However,  $y_2'(b, \lambda)$  is



negative for  $\lambda$  sufficiently close to  $\lambda_m$ , positive for  $\lambda$  sufficiently close to  $\lambda_{m+1}$ , because

$$\gamma'_2(b, \lambda) = v'_1(b, \lambda) \int_a^b A(t, \lambda) v_2(t, \lambda) dt,$$

and by Lemma 3

$$\int_a^b A(t, \lambda_m) v_2(t, \lambda_m) dt < 0 \quad \text{and} \quad \int_a^b A(t, \lambda_{m+1}) v_2(t, \lambda_{m+1}) dt > 0.$$

Since  $\gamma_2(x, \lambda)$  is a continuous function of  $(x, \lambda)$  over  $XL_m$  [1, p. 114], it follows that there exist  $\epsilon_m > 0$  and  $\epsilon_{m+1} > 0$  such that for  $\lambda' - \lambda_m < \epsilon_m$  and  $\lambda_{m+1} - \lambda'' < \epsilon_{m+1}$ ,  $\gamma_2(x, \lambda')$  has no zero in  $q_f(\lambda') < x < b$ , and  $\gamma_2(x, \lambda'')$  has one zero in  $q_f(\lambda'') < x < b$ .

A similar argument can be made in case  $m$  is even or in case  $A(x, \lambda)$  is negative over  $X_0$ . This proves the following result.

**THEOREM 4.** *Let  $(S_\lambda)$  satisfy the hypotheses (H) for each  $\lambda$  in  $(\Lambda_1, \Lambda_2)$ . On  $X_0$ ,  $\gamma_2(x, \lambda)$  has  $m$  zeros for  $\lambda$  sufficiently close to  $\lambda_m$ ,  $m + 1$  zeros for  $\lambda$  sufficiently close to  $\lambda_{m+1}$  ( $m = 0, 1, 2, \dots$ ).*

Letting  $p_0(\lambda)$  be the first zero of  $v_2(x, \lambda)$  to the right of  $a$ , one readily sees that

$$\gamma_2[p_0(\lambda), \lambda] = v_1[p_0(\lambda), \lambda] \int_a^{p_0(\lambda)} A(t, \lambda) v_2(t, \lambda) dt$$

is positive or negative according as  $A$  is positive or negative. If  $A > 0$ , then  $\gamma'_2(a, \lambda) = \sigma \int_a^b A(t, \lambda) v_1(t, \lambda) dt$  is positive or negative over  $L_m$  according as  $m$  is even or odd. If  $A < 0$ ,  $\gamma'_2(a, \lambda)$  is negative or positive over  $L_m$  according as  $m$  is even or odd. If one uses these relations as well as Theorem 1 and Theorem 2 to sketch graphically several typical cases, he obtains a striking illustration of the effect of the discontinuities of the function  $\gamma_2(x, \lambda)$  at the characteristic values of  $\lambda$ . Finally, one may observe that, regardless of the sign of  $A$ , for an even value of  $m$  the first zero of  $v_2(x, \lambda)$  on  $a < x < b$  precedes the first zero of  $\gamma_2(x, \lambda)$ , and for an odd value of  $m$  the opposite is the case.

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Lars V. Ahlfors, <i>Remarks on the Neumann-Poincaré integral equation</i> . . . . .	271
Leonard P. Burton, <i>Oscillation theorems for the solutions of linear, nonhomogeneous, second-order differential systems</i> . . . . .	281
Paul Civin, <i>Multiplicative closure and the Walsh functions</i> . . . . .	291
James Michael Gardner Fell and Alfred Tarski, <i>On algebras whose factor algebras are Boolean</i> . . . . .	297
Paul Joseph Kelly and Lowell J. Paige, <i>Symmetric perpendicularity in Hilbert geometries</i> . . . . .	319
G. Kurepa, <i>On a characteristic property of finite sets</i> . . . . .	323
Joseph Lehner, <i>A diophantine property of the Fuchsian groups</i> . . . . .	327
Donald Alan Norton, <i>Groups of orthogonal row-latin squares</i> . . . . .	335
R. S. Phillips, <i>On the generation of semigroups of linear operators</i> . . . . .	343
G. Piranian, <i>Uniformly accessible Jordan curves through large sets of relative harmonic measure zero</i> . . . . .	371
C. T. Rajagopal, <i>Note on some Tauberian theorems of O. Szűsz</i> . . . . .	377
Halsey Lawrence Royden, Jr., <i>A modification of the Neumann-Poincaré method for multiply connected regions</i> . . . . .	385
George H. Seifert, <i>A third order irregular boundary value problem and the associated series</i> . . . . .	395
Herbert E. Vaughan, <i>Well-ordered subsets and maximal members of ordered sets</i> . . . . .	407
Hans F. Weinberger, <i>An optimum problem in the Weinstein method for eigenvalues</i> . . . . .	413
Shigeki Yano, <i>Note on Fourier analysis. XXXI. Cesàro summability of Fourier series</i> . . . . .	419