A THIRD ORDER IRREGULAR BOUNDARY VALUE PROBLEM
AND THE ASSOCIATED SERIES

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1. Introduction. Certain problems in aeroelastic wing theory [1] give rise to a third order irregular boundary value problem of the form given in equation (1) below. Questions have been raised [1] as to conditions under which functions have an expansion in terms of the associated characteristic functions. It is shown in this paper that the general approach by L. E. Ward [2] in dealing with a somewhat more specialized problem can be suitably modified to provide an answer to these questions.

We are concerned with the differential boundary value problem

\[ L(u(x), \lambda) = u'''(x) + p(x)u'(x) + (q(x) + \lambda)u(x) = 0, \]
\[ u(0) = u'(0) = u''(1) = 0, \]

where \( p(x) = x \psi_1(x^3), \) \( q(x) = \psi_2(x^3), \) and \( \psi_1(z) \) and \( \psi_2(z) \) are real for real \( z \) and analytic on \(|z| < 1.\) We seek conditions on \( f(x) \) such that it be expansible in terms of the characteristic functions of (1) and its adjoint.

We shall first need a number of definitions and lemmas. Define:

i) \( \delta_3(t) = e^{\omega_1 t} - e^{\omega_2 t} - e^{\omega_3 t}, \)
\( \delta_2(t) = -\delta_3'(t), \)
\( \delta_1(t) = -\delta_3'(t), \)

where \( \omega_1 = -1, \) \( \omega_2 = e^{\pi i/3}, \) \( \omega_3 = e^{-\pi i/3}; \)

ii) \( \Delta(x, t, \rho) = \rho^{-1} \delta_3[\rho(x-t)] r(t) - \delta_2[\rho(x-t)] p(t) \)

where \( r(t) = q(t) - p'(t), \) and the complex number \( \rho \) satisfies

\( \rho^3 = \lambda, \ |\arg \rho| \leq \pi/3; \)

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iii) the regions $S_1$ and $S_2$ of the $\rho$-plane by $0 \leq \arg \rho \leq \pi/3$ and $-\pi/3 \leq \arg \rho \leq 0$, respectively.

We shall be concerned with the integral equation

$$u(x, \xi, \rho) = \delta_3 [\rho(x - \xi)] - \frac{1}{3\rho} \int_\xi^x \Delta(x, t, \rho)u(t, \xi, \rho)\,dt.$$  

2. Lemmas. We shall use the following results.

Lemma 1. Equation (2) has for fixed $\rho$ a unique solution analytic in $x$ and in $\xi$ on $|x| \leq 1$ and $|\xi| \leq 1$, respectively, where $x$ and $\xi$ are complex variables.\(^1\)

Proof. For fixed $\rho$, define

$$f_1(x, \xi) = \delta_3 [\rho(x - \xi)],$$

$$f_j(x, \xi) = -\frac{1}{3\rho} \int_\xi^x \Delta(x, t, \rho)f_{j-1}(t, \xi)\,dt.$$  

Then

$$|f_1(x, \xi)| \leq M,$$

$$|f_2(x, \xi)| = \left|-\frac{1}{3\rho} \int_\xi^x \Delta(x, t, \rho)f_1(t, \xi)\,dt\right| < MN \int_\xi^x |dt| = MN|x - \xi|.$$  

Hence, by induction,

$$|f_j(x, \xi)| < \frac{MN^{j-1}|x - \xi|^{j-1}}{(j - 1)!} \quad (j = 2, 3, 4, \cdots);$$

consequently,

$$\sum_{j=1}^\infty f_j(x, \xi) = w(x, \xi),$$

where $w(x, \xi)$ is analytic in $x$ and in $\xi$ in $|x| \leq 1$ and $|\xi| \leq 1$, respectively. By direct substitution into (2), we see that $w(x, \xi)$ is a solution.

To show uniqueness, consider

\(^1\)The variables $x$ and $\xi$ will always be considered real, unless otherwise indicated, as here; in this case, as in subsequent cases, integration between complex limits, as in equation (2), may be taken along a straight line in the complex plane.
\[ z(x, \xi) = u_1(x, \xi) - u_2(x, \xi), \]

where \( u_1(x, \xi) \) and \( u_2(x, \xi) \) are solutions of (2). Clearly \( z(x, \xi) \) must satisfy the equation

\[ z(x, \xi) = -\frac{1}{3\rho} \int_{\xi}^{x} \Delta(x, t, \xi) z(t, \xi) \, dt; \]

and for real \( x \) and \( \xi \), \( z(x, \xi) \) is easily seen to satisfy the system\(^2\)

\[ L(z(x, \xi), \lambda) = 0, \quad z(\xi, \xi) = z'(\xi, \xi) = z''(\xi, \xi) = 0. \]

Hence \( z(x, \xi) = 0 \) identically in \( x \) for any fixed \( \xi \), for real \( x \) and \( \xi \); this implies \( z(x, \xi) = 0 \) identically for complex \( x \) and \( \xi \) and completes the proof.

**Lemma 2.** For real \( x \) and \( \xi \), (2) is equivalent to the system

(2a) \[ L(u(x, \xi), \lambda) = 0, \quad u(\xi, \xi) = u'(\xi, \xi) = 0, \quad u''(\xi, \xi) = 3\rho^2. \]

**Proof.** Substitution in (2a) of \( u(x, \xi, \rho) \) as given by (2) shows that the unique solution of (2) is a solution of (2a). However, for fixed \( \xi \) and \( \rho \), (2a) also has a unique solution. Clearly, these unique solutions must coincide, and our proof is complete.

**Lemma 3.** Let \( u(x, \xi, \rho) \) be a solution of (2). Then\(^3\)

a) \[ u(x, \xi, \rho) = e^{\omega_3 \rho (x-\xi)} E(x, \xi, \rho) \]

provided \( |\rho| \) is large enough \( \rho \in S_1, x \geq \xi; \)

b) \[ u(-\omega_2 x, -\omega_2 \xi, \rho) = -\omega_3 u(x, \xi, \rho); \]

c) \[ u''(1, 0, \rho) = \rho^2 e^{\omega_3 \rho} M(\rho), \]

where \( |M(\rho)| \geq m > 0 \), provided

\[ \rho = \frac{2n + 2}{\sqrt{3}} n e^{i\theta} \quad (0 \leq \theta \leq \pi/3), \]

\(^2\)Unless otherwise indicated, the prime will always denote differentiation with respect to the first indicated variable.

\(^3\)Functions of \( \rho \) and other variables which are bounded for \( |\rho| \) sufficiently large will be denoted by \( E(\rho) \).
for sufficiently large \( n \).

**Proof of a.** As in Lemma 2 of [3], p. 211, it follows that for \( \rho \in S_1 \), we have

\[
 u(x, \xi, \rho) = e^{\omega_3 \rho(x-\xi)} \left[ -\omega_3 - \omega_2 e^{(\omega_3 - \omega_2) \rho(x-\xi)} + z(x, \xi, \rho) \right],
\]

where \( |z(x, \xi, \rho)| < M \) for \( |\rho| \) sufficiently large and \( x \geq \xi \). Hence

\[
 u(x, \xi, \rho) = e^{\omega_3 \rho(x-\xi)} E(x, \xi, \rho).
\]

**Proof of b.** Using (2), we have

\[
 u(-\omega_2 x, -\omega_2 \xi, \rho) = \delta_3 \left[ -\omega_2 \rho(x-\xi) \right] 
- \frac{1}{3} \int_{-\omega_2 \xi}^{\omega_2 x} \Delta(-\omega_2 x, s, \rho) u(s, -\omega_2 \xi, \rho) ds 
= -\omega_3 \delta_3 [\rho(x-\xi)] 
+ \frac{\omega_2}{3} \int_{-\omega_2 \xi}^{\omega_2 x} \Delta(-\omega_2 x, -\omega_2 t, \rho) u(-\omega_2 t, -\omega_2 \xi, \rho) dt.
\]

But

\[
 \Delta(-\omega_2 x, -\omega_2 t, \rho) = -\frac{\omega_3}{\rho} \delta_3 [\rho(x-t)] r(t) 
+ \omega_2 \delta_2 [\rho(x-t)] (-\omega_2 p(t)) = -\omega_3 \Delta(x, t, \rho).
\]

Hence

\[
 u(-\omega_2 x, -\omega_2 \xi, \rho) = -\omega_3 \delta_3 [\rho(x-\xi)] 
- \frac{1}{3} \int_{-\omega_2 \xi}^{\omega_2 x} \Delta(x, t, \rho) u(-\omega_2 t, -\omega_2 \xi, \rho) dt.
\]

Multiplying this last equation by \(-\omega_2\), we have

\[
 z(x, \xi, \rho) = \delta_3 [\rho(x-\xi)] - \frac{1}{3} \int_{-\omega_2 \xi}^{\omega_2 x} \Delta(x, t, \rho) z(t, \xi, \rho) dt,
\]
where

\[ z(x, \xi, \rho) = -\omega^2 u(-\omega^2 x, -\omega^2 \xi, \rho). \]

But by the uniqueness of the solutions of (2), we have

\[ -\omega^2 u(-\omega^2 x, -\omega^2 \xi, \rho) = u(x, \xi, \rho); \]

upon multiplication by \(-\omega^3\), this gives b).

Proof of c). We have, from (2),

\[
u''(1, 0, \rho) = \rho^2 \left[ \delta_1(\rho) + \frac{e^{\omega_3 \rho}}{\rho} E_1(\rho) \right] = \rho^2 e^{\omega_3 \rho} \left[ 1 + e^{(\omega_2 - \omega_3) \rho} + \frac{E_2(\rho)}{\rho} \right]
\]

for \(\rho \in S_1\). Let \(\rho = x + iy\), and define \(\Phi(\rho)\) and \(r_n\) by

\[
\Phi(\rho) = 1 + e^{(\omega_2 - \omega_3) \rho} \quad \text{and} \quad r_n = \frac{2(n + 1)}{\sqrt{3}} \pi,
\]

respectively. With \(p = [3(r_n^2 - x^2)]^{1/2}\), we have

\[
|\Phi(\rho)| \geq |1 + e^{-\rho} \cos(\sqrt{3}x)|,
\]

provided \(\rho = r_n e^{i\theta}\) where \(0 \leq \theta \leq \pi/3\), and will show that

\[
e^{-\rho} \cos(\sqrt{3}x) > -\frac{1}{2} \quad \text{for} \quad \frac{r_n}{2} \leq x \leq r_n.
\]

Since

\[
\cos(\sqrt{3}x) \geq 0 \quad \text{for} \quad r_n - \frac{\pi}{2\sqrt{3}} \leq x \leq r_n,
\]

it is clearly sufficient to show that

\[
e^{-\rho} \cos(\sqrt{3}x) > -\frac{1}{2} \quad \text{for} \quad \frac{r_n}{2} \leq x \leq r_n - \frac{\pi}{2\sqrt{3}}.
\]

Accordingly, we note that for \(x\) in this interval, we have
for all $n > N$, provided $N$ is sufficiently large. Taking $N$ large enough we also have

$$\left| \frac{E_2(\rho)}{\rho} \right| < \frac{1}{4} \text{ for } \rho = r_n e^{i\theta} \quad (0 \leq \theta \leq \pi/3).$$

Hence

$$\left| \Phi(\rho) + \frac{E_2(\rho)}{\rho} \right| \geq |\Phi(\rho)| - \left| \frac{E_2(\rho)}{\rho} \right| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

This completes the proof of the lemma.

By the formal series for $f(x)$, we shall mean the series

$$\sum_{k=1}^{\infty} a_k u_k(x) \text{ where } a_k = \int_0^1 f(x) v_k(x) \, dx / \int_0^1 u_k(x) v_k(x) \, dx,$$

in which $u_k(x)$ and $v_k(x)$ are respectively the characteristic functions of the system (1) and its adjoint corresponding to the characteristic value $\lambda_k$.

**Lemma 4.** The sum of the first $n$ terms of the formal series for $f(x)$ is given by

$$I_n(x) = \frac{1}{2\pi i} \oint_{\gamma_n} \left[ \int_0^x f(\xi) u(x, \xi, \rho) \, d\xi \right] d\rho$$

$$- \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \int_0^1 f(\xi) u''(1, \xi, \rho) \, d\xi \, d\rho$$

$$= \frac{1}{2\pi i} \oint_{\gamma_n} \left[ \sigma(x) - \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \sigma''(1, 0, \rho) \right] d\rho,$$

where $\sigma(x) = \int_0^x f(\xi) u(x, \xi, \rho) \, d\xi$, and $\gamma_n$ is the arc of the $\rho$-plane given by

$$\rho = \frac{2n + 2}{\sqrt{3}} \pi e^{i\theta}, -\pi/3 \leq \theta \leq \pi/3.$$
the $\rho$ integration proceeding in a counter-clockwise direction.

We omit the proof of this lemma, as its details almost duplicate the discussion in [2], pp. 424-426. We point out, however, that Lemma 2 is required in this proof.

**Lemma 5.** The function $\sigma(x)$ defined in the previous lemma satisfies the equation

$$
\sigma(x) = \int_{0}^{x} f(\xi) \delta_3 [\rho(x-\xi)] d\xi - \frac{1}{3\rho} \int_{0}^{x} \Delta(x, t, \rho) \sigma(t) dt;
$$

furthermore, $\sigma(x)$ is its unique solution, is analytic on $0 \leq x \leq 1$, and can be put into the form

$$
\sigma(x) = u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho),
$$

where

$$
\Psi_2(x, \rho) = \frac{3f(x)}{\rho} + \frac{E_1(x, \rho)}{\rho^2}, \quad E_1''(x, \rho) = \rho^2 E_2(x, \rho),
$$

provided $f(x) = x^2 \phi(x^3)$, where $\phi(z)$ is analytic on $|z| \leq 1$.

**Proof.** Using (2) in the expression for $\sigma(x)$, we obtain

$$
\sigma(x) = \int_{0}^{x} f(\xi) \delta_3 [\rho(x-\xi)] d\xi
$$

$$
- \frac{1}{3\rho} \int_{0}^{x} f(\xi) \int_{\xi}^{x} \Delta(x, t, \rho) u(t, \xi, \rho) dtd\xi
$$

$$
= \int_{0}^{x} f(\xi) \delta_3 [\rho(x-\xi)] d\xi - \frac{1}{3\rho} \int_{0}^{x} \Delta(x, t, \rho) \sigma(t) dt
$$

on changing the order of integration in the second integral. Uniqueness of the solution $\sigma(x)$ can be shown in the usual manner. (See the proof of Lemma 1.)

We next substitute $u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho)$ into (3) for $\sigma(x)$, and obtain

$$
u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho) = \int_{0}^{x} f(\xi) \delta_3 [\rho(x-\xi)] d\xi
$$

$$
- \frac{\Psi_1(\rho)}{3\rho} \int_{0}^{x} \Delta(x, t, \rho) u(t, 0, \rho) dt - \frac{1}{3\rho} \int_{0}^{x} \Delta(x, t, \rho) \Psi_2(t, \rho) dt,
$$
Using (2) with $\xi = 0$, and subtracting the term $u(x, 0, p) \Psi_1(p)$ from both sides, we obtain

$$(4) \quad \Psi_2(x, \rho) = \int_0^x f(\xi) \delta_3[\rho(x-\xi)] d\xi - \Psi_1(\rho) \delta_3(\rho x) - \frac{1}{3\rho} \int_0^x \Delta(x, t, \rho) \Psi_2(t, \rho) dt.$$ 

On integrating by parts twice, we obtain

$$\int_0^x f(\xi) \delta_3[\rho(x-\xi)] d\xi = \frac{3f(x)}{\rho} + \rho^-2 \int_0^x f''(\xi) \delta_2[\rho(x-\xi)] d\xi$$

$$= \frac{3f(x)}{\rho} + \rho^-2 \delta_3(\rho x) \int_0^x f''(\xi)e^{\rho \xi} d\xi + \int_0^x f''(\xi)e^{\rho \xi} d\xi,$$

where $y$ is a complex number to be determined later, and

$$L_3 F(t) dt = e^{\omega_1 \rho x} \int_0^x F(t) dt - \omega_2 e^{\omega_2 \rho x} \int_0^x e^{\omega_2 \xi} F(t) dt$$

$$- \omega_3 e^{\omega_3 \rho x} \int_0^x e^{\omega_3 \xi} F(t) dt.$$

It is in this step that we use the form of $f(x)$ as stated in the hypothesis of this lemma; for the details, see [2], pp. 428-429.

We also have

$$\int_0^x \Delta(x, t, \rho) \Psi_2(t, \rho) dt = \frac{1}{\rho} \int_0^x \delta_3[\rho(x-t)]r(t) \Psi_2(t, \rho) dt$$

$$+ \int_0^x \delta_2[\rho(x-t)]p(t) \Psi_2(t, \rho) dt$$

$$= \frac{\delta_3(\rho x)}{\rho} \int_0^x r(t)e^{\rho t} \Psi_2(t, \rho) dt + \int_0^x r(t)e^{\rho t} \Psi_2(t, \rho) dt$$

$$+ \delta_3(\rho x) \int_0^x p(t)e^{\rho t} \Psi_2(t, \rho) dt + \int_0^x p(t)e^{\rho t} \Psi_2(t, \rho) dt.$$
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\[ \psi_2(t, \rho) = -\omega_2 \psi_2(t, \rho). \]

Putting these results into equation (4), we obtain

\[
\psi_2(x, \rho) = \frac{3f(x)}{\rho} + \frac{\delta_3(\rho x)}{\rho} \left[ \psi_1(\rho) - \frac{1}{\rho^2} \int_0^\gamma f''(\xi)e^{\rho\xi} \, d\xi \right. \\
\left. + \frac{1}{3\rho} \int_0^\gamma \rho(\xi)e^{\rho\xi} \psi_2(t, \rho) \, dt \right] \\
+ \frac{1}{\rho^2} \int_0^\gamma f''(t)e^{\rho\xi} \, dt - \frac{1}{3\rho} \int_0^\gamma R(t)e^{\rho\xi} \psi_2(t, \rho) \, dt.
\]

This equation will certainly be satisfied if

\[ (5) \quad \psi_2(x, \rho) = \frac{3f(x)}{\rho} + \frac{\delta_3(\rho x)}{\rho^2} \int_0^\gamma f''(\xi)e^{\rho\xi} \, d\xi + \frac{1}{3\rho} \int_0^\gamma R(t)e^{\rho\xi} \psi_2(t, \rho) \, dt \\
+ \frac{1}{\rho^2} \int_0^\gamma f''(t)e^{\rho\xi} \, dt - \frac{1}{3\rho} \int_0^\gamma R(t)e^{\rho\xi} \psi_2(t, \rho) \, dt.
\]

The proof of the existence of a unique solution \( \psi_2(x, \rho) \) of (5) will follow along the lines of the corresponding proof in [2], provided we can show that an expression of the form \( |\int_0^\gamma F(t)e^{\rho t} \, dt| \) is bounded for complex \( \rho \) and \( 0 \leq x \leq 1 \) whenever \( |F(z)| \) is on \( |z| \leq 1 \) and we take \( y = -e^{-i} \arg \rho \). For we have

\[
|\int_0^\gamma F(t)e^{\rho t} \, dt| \leq |e^{\omega_1\rho x}| \int_0^x |F(t)| \, e^{\rho t} \, dt \\
+ |e^{\omega_2\rho x}| \int_y^{\omega_2\rho x} |F(t)| \, e^{\rho t} \, dt + |e^{\omega_3\rho x}| \int_y^{\omega_3\rho x} |F(t)| \, e^{\rho t} \, dt.
\]
\[
\begin{align*}
\leq \mu \left[ |e^{\omega_1 \rho x} \int_y^x |e^{\rho t}| \, dt| + |e^{\omega_2 \rho x} \int_y^{-\omega_2 x} |e^{\rho t}| \, dt| + |e^{\omega_3 \rho x} \int_y^{-\omega_3 x} |e^{\rho t}| \, dt| \right],
\end{align*}
\]

where \(|F(z)| \leq \mu\) on \(|z| \leq 1\); and since each integrand in this last expression assumes its maximum at its upper limit, we have

\[
|\mathcal{L}_3 F(t) e^{\rho t} dt| \leq 6\mu.
\]

We omit the rest of this existence proof. (See [2], pp. 429-430.)

For the asymptotic form of \(\Psi_2(x, \rho)\), we substitute

\[
\Psi_2(x, \rho) = \frac{3f(x)}{\rho} + v(x, \rho)
\]

into (5). We obtain

\[
(6) \quad v(x, \rho) = \frac{1}{\rho^2} \mathcal{L}_3 f''(t) e^{\rho t} dt
- \frac{1}{3\rho} \mathcal{L}_3 R(t) e^{\rho t} \left[ \frac{3f(t)}{\rho} + v(t, \rho) \right] dt.
\]

For fixed \(\rho\) let \(m = \max_{0 \leq x \leq 1} |v(x, \rho)|\); then

\[
m \leq \frac{1}{|\rho|^2} \left| \mathcal{L}_3 [f''(t) + R(t)f(t)] e^{\rho t} dt \right| + \frac{1}{3|\rho|} \left| \mathcal{L}_3 R(t) e^{\rho t} v(t, \rho) dt \right|
\]

\[
\leq \frac{\mu_1}{|\rho|^2} + \frac{m\mu_1}{|\rho|} \leq \frac{\mu_1}{|\rho|^2} + \frac{m}{2},
\]

provided \(|\rho| \geq 2\mu_2\), where \(|\mathcal{L}_3 R(t) e^{\rho t} dt| \leq \mu_2\). Hence for such \(\rho\) we have \(m \leq 2\mu_1/|\rho|^2\), and it follows that \(v(x, \rho) = \rho^{-2} E_1(x, \rho)\).

It remains to show that \(v''(x, \rho) = E_2(x, \rho)\). Differentiating (6), we have

\[
(7) \quad v'(x, \rho) = -\frac{1}{\rho} \left[ \mathcal{L}_2 \left[ f''(t) + R(t)f(t) + \frac{1}{3\rho} E_1(t, \rho) \right] e^{\rho t} dt \right]
+ \frac{E_3(x, \rho)}{\rho^2},
\]
where
\[ \mathcal{L}_2 F(t) dt = e^{-\omega_1 \rho x} \int_{-x}^{x} F(t) dt - \omega_3 e^{-\omega_2 \rho x} \int_{-x}^{x} F(t) dt \]
\[ - \omega_2 e^{-\omega_3 \rho x} \int_{-x}^{x} F(t) dt, \]
and we have used the fact that
\[ |E_i(-\omega_2 x, \rho)| = \left| \rho^2 \left( \Psi_2 (-\omega_2 x, \rho) - \frac{3f(-\omega_2 x)}{\rho} \right) \right| \]
\[ = \left| -\rho^2 \omega_3 \left( \Psi_2 (x, \rho) - \frac{3f(x)}{\rho} \right) \right| = |E_1(x, \rho)|. \]
We can also show, as before in the case of the \( \mathcal{L}_3 \) operator, that if \( | F(z) | \leq \mu \) on \( |z| \leq 1 \), then \( |\mathcal{L}_2 F(t) e^{\rho t} dt| \leq m_2 \).
Differentiating (7), we obtain
\[ v''(x, \rho) = \mathcal{L}_1 \left[ f''(t) + R(t)f(t) + \frac{1}{3\rho} E_1(t, \rho) \right] e^{\rho t} dt + \frac{E_4(x, \rho)}{\rho}, \]
where
\[ \mathcal{L}_1 F(t) dt = e^{-\omega_1 \rho x} \int_{-x}^{x} F(t) dt + e^{-\omega_2 \rho x} \int_{-x}^{x} F(t) dt \]
\[ + e^{-\omega_3 \rho x} \int_{-x}^{x} F(t) dt, \]
and we have used the fact that \( |E_1'(-\omega_2 x, \rho)| = |E_1'(x, \rho)| \) and that
\[ E_1'(x, \rho) = |\rho^2 v'(x, \rho)| \leq |\rho| M \]
for \( |\rho| \) sufficiently large.
Hence \( v''(x, \rho) = E_2(x, \rho) \) since again \( | F(z) | \leq \mu \) for \( |z| \leq 1 \) implies \( |\mathcal{L}_1 F(t) e^{\rho t} dt| \leq m_1 \), and the proof of the lemma is complete.

3. Theorem. We proceed now to the proof of the following theorem.

Theorem. If \( f(x) = x^2 \phi(x^3) \), where \( \phi(z) \) is analytic on \( |z| \leq 1 \), the formal series for \( f(x) \) converges uniformly to \( f(x) \) on \( 0 \leq x \leq 1 \).
Proof. Since, for real $x$ and $\xi$, $u(x, \xi, \rho)$ is real for real $\rho$, by the principle of reflection we have $u(x, \xi, \rho^*) = [u(x, \xi, \rho)]^*$. This implies that the integrand in the expression for $I_n(x)$ given in Lemma 4 takes on values for $\rho$ on $\gamma_n' = \gamma_n \cap S_1$ which are the complex conjugates of those it takes on for $\rho$ on $\gamma_n'' = \gamma_n \cap S_2$. It suffices, then, to consider only the $\rho$ integration over $\gamma_n'$. Denoting the result by $I_n'(x)$, we have, by Lemmas 4 and 5,

$$
I_n'(x) = \frac{1}{2\pi i} \int_{\gamma_n'} \left\{ \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \left[ \frac{3f(x)}{\rho} + \frac{E_1(x, \rho)}{\rho^2} \right] \right\} d\rho;
$$

and since, by Lemma 3, parts a) and c), we have

$$
\left| \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \right| \leq \frac{M}{|\rho|^2}
$$

for $\rho$ on $\gamma_n'$ and $n$ sufficiently large, we obtain

$$
I_n'(x) = \frac{1}{2\pi i} \int_{\gamma_n'} \left[ \frac{3f(x)}{\rho} + \frac{E(x, \rho)}{\rho^2} \right] d\rho = \frac{f(x)}{2} + \epsilon_n'(x),
$$

where

$$
\lim_{n \to \infty} \epsilon_n'(x) = 0
$$

uniformly in $x$. This proves the theorem.

At the expense of brevity, this theorem clearly could be generalized to problems involving somewhat more complicated boundary conditions and somewhat weaker analyticity conditions on $f(x)$, $p(x)$, and $q(x)$; in connection with the latter contention, see [2].

References


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