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A THIRD ORDER IRREGULAR BOUNDARY VALUE PROBLEM AND THE ASSOCIATED SERIES

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1. Introduction. Certain problems in aeroelastic wing theory [1] give rise to a third order irregular boundary value problem of the form given in equation (1) below. Questions have been raised [1] as to conditions under which functions have an expansion in terms of the associated characteristic functions. It is shown in this paper that the general approach by L. E. Ward [2] in dealing with a somewhat more specialized problem can be suitably modified to provide an answer to these questions.

We are concerned with the differential boundary value problem

$$(1) \quad L(u(x), \lambda) = u'''(x) + p(x)u'(x) + (q(x) + \lambda)u(x) = 0,$$

$$u(0) = u'(0) = u''(1) = 0,$$

where $p(x) = x\psi_1(x^3)$, $q(x) = \psi_2(x^3)$, and $\psi_1(z)$ and $\psi_2(z)$ are real for real z and analytic on $|z| \leq 1$. We seek conditions on $f(x)$ such that it be expandible in terms of the characteristic functions of (1) and its adjoint.

We shall first need a number of definitions and lemmas. Define:

$$i) \quad \delta_3(t) \equiv e^{\omega_1 t} - \omega_2 e^{\omega_2 t} - \omega_3 e^{\omega_3 t},$$

$$\delta_2(t) \equiv -\delta_3'(t),$$

$$\delta_1(t) \equiv -\delta_2'(t),$$

where $\omega_1 = -1$, $\omega_2 = e^{\pi i/3}$, $\omega_3 = e^{-\pi i/3}$;

$$ii) \quad \Delta(x, t, \rho) \equiv \rho^{-1} \delta_3[\rho(x-t)] r(t) - \delta_2[\rho(x-t)] p(t)$$

where $r(t) = q(t) - p'(t)$, and the complex number ρ satisfies

$$\rho^3 = \lambda, \quad |\arg \rho| \leq \pi/3;$$

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iii) the regions S_1 and S_2 of the ρ -plane by $0 \leq \arg \rho \leq \pi/3$ and $-\pi/3 \leq \arg \rho \leq 0$, respectively.

We shall be concerned with the integral equation

$$(2) \quad u(x, \xi, \rho) = \delta_3[\rho(x - \xi)] - \frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \rho) u(t, \xi, \rho) dt.$$

2. Lemmas. We shall use the following results.

LEMMA 1. Equation (2) has for fixed ρ a unique solution analytic in x and in ξ on $|x| \leq 1$ and $|\xi| \leq 1$, respectively, where x and ξ are complex variables.¹

Proof. For fixed ρ , define

$$f_1(x, \xi) \equiv \delta_3[\rho(x - \xi)],$$

$$f_j(x, \xi) \equiv -\frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \rho) f_{j-1}(t, \xi) dt.$$

Then

$$|f_1(x, \xi)| \leq M,$$

$$|f_2(x, \xi)| = \left| -\frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \rho) f_1(t, \xi) dt \right| < MN \int_{\xi}^x |dt| = MN|x - \xi|.$$

Hence, by induction,

$$|f_j(x, \xi)| < \frac{MN^{j-1}|x - \xi|^{j-1}}{(j-1)!} \quad (j = 2, 3, 4, \dots);$$

consequently,

$$\sum_{j=1}^{\infty} f_j(x, \xi) = w(x, \xi),$$

where $w(x, \xi)$ is analytic in x and in ξ in $|x| \leq 1$ and $|\xi| \leq 1$, respectively.

By direct substitution into (2), we see that $w(x, \xi)$ is a solution.

To show uniqueness, consider

¹The variables x and ξ will always be considered real, unless otherwise indicated, as here; in this case, as in subsequent cases, integration between complex limits, as in equation (2), may be taken along a straight line in the complex plane.

$$z(x, \xi) = u_1(x, \xi) - u_2(x, \xi),$$

where $u_1(x, \xi)$ and $u_2(x, \xi)$ are solutions of (2). Clearly $z(x, \xi)$ must satisfy the equation

$$z(x, \xi) = -\frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \xi) z(t, \xi) dt;$$

and for real x and ξ , $z(x, \xi)$ is easily seen to satisfy the system²

$$L(z(x, \xi), \lambda) = 0, \quad z(\xi, \xi) = z'(\xi, \xi) = z''(\xi, \xi) = 0.$$

Hence $z(x, \xi) = 0$ identically in x for any fixed ξ , for real x and ξ ; this implies $z(x, \xi) = 0$ identically for complex x and ξ and completes the proof.

LEMMA 2. For real x and ξ , (2) is equivalent to the system

$$(2a) \quad L(u(x, \xi), \lambda) = 0, \quad u(\xi, \xi) = u'(\xi, \xi) = 0, \quad u''(\xi, \xi) = 3\rho^2.$$

Proof. Substitution in (2a) of $u(x, \xi, \rho)$ as given by (2) shows that the unique solution of (2) is a solution of (2a). However, for fixed ξ and ρ , (2a) also has a unique solution. Clearly, these unique solutions must coincide, and our proof is complete.

LEMMA 3. Let $u(x, \xi, \rho)$ be a solution of (2). Then³

$$a) \quad u(x, \xi, \rho) = e^{\omega_3 \rho(x-\xi)} E(x, \xi, \rho)$$

provided $|\rho|$ is large enough $\rho \in S_1, x \geq \xi$;

$$b) \quad u(-\omega_2 x, -\omega_2 \xi, \rho) = -\omega_3 u(x, \xi, \rho);$$

$$c) \quad u''(1, 0, \rho) = \rho^2 e^{\omega_3 \rho} M(\rho),$$

where $|M(\rho)| \geq m > 0$, provided

$$\rho = \frac{2n + 2}{\sqrt{3}} \pi e^{i\theta} \quad (0 \leq \theta \leq \pi/3),$$

²Unless otherwise indicated, the prime will always denote differentiation with respect to the first indicated variable.

³Functions of ρ and other variables which are bounded for $|\rho|$ sufficiently large will be denoted by $E(\)$.

for sufficiently large n .

Proof of a). As in Lemma 2 of [3], p. 211, it follows that for $\rho \in S_1$, we have

$$u(x, \xi, \rho) = e^{\omega_3 \rho(x-\xi)} [-\omega_3 - \omega_2 e^{(\omega_2 - \omega_3) \rho(x-\xi)} + z(x, \xi, \rho)],$$

where $|z(x, \xi, \rho)| < M$ for $|\rho|$ sufficiently large and $x \geq \xi$. Hence

$$u(x, \xi, \rho) = e^{\omega_3 \rho(x-\xi)} E(x, \xi, \rho).$$

Proof of b). Using (2), we have

$$\begin{aligned} u(-\omega_2 x, -\omega_2 \xi, \rho) &= \delta_3 [-\omega_2 \rho(x-\xi)] \\ &\quad - \frac{1}{3\rho} \int_{-\omega_2 \xi}^{-\omega_2 x} \Delta(-\omega_2 x, s, \rho) u(s, -\omega_2 \xi, \rho) ds \\ &= -\omega_3 \delta_3 [\rho(x-\xi)] \\ &\quad + \frac{\omega_2}{3\rho} \int_{\xi}^x \Delta(-\omega_2 x, -\omega_2 t, \rho) u(-\omega_2 t, -\omega_2 \xi, \rho) dt. \end{aligned}$$

But

$$\begin{aligned} \Delta(-\omega_2 x, -\omega_2 t, \rho) &= -\frac{\omega_3}{\rho} \delta_3 [\rho(x-t)] r(t) \\ &\quad + \omega_2 \delta_2 [\rho(x-t)] (-\omega_2 p(t)) = -\omega_3 \Delta(x, t, \rho). \end{aligned}$$

Hence

$$\begin{aligned} u(-\omega_2 x, -\omega_2 \xi, \rho) &= -\omega_3 \delta_3 [\rho(x-\xi)] \\ &\quad - \frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \rho) u(-\omega_2 t, -\omega_2 \xi, \rho) dt. \end{aligned}$$

Multiplying this last equation by $-\omega_2$, we have

$$z(x, \xi, \rho) = \delta_3 [\rho(x-\xi)] - \frac{1}{3\rho} \int_{\xi}^x \Delta(x, t, \rho) z(t, \xi, \rho) dt,$$

where

$$z(x, \xi, \rho) = -\omega_2 u(-\omega_2 x, -\omega_2 \xi, \rho).$$

But by the uniqueness of the solutions of (2), we have

$$-\omega_2 u(-\omega_2 x, -\omega_2 \xi, \rho) = u(x, \xi, \rho);$$

upon multiplication by $-\omega_3$, this gives b).

Proof of c). We have, from (2),

$$\begin{aligned} u''(1, 0, \rho) &= \rho^2 [\delta_1(\rho) + \frac{e^{\omega_3 \rho}}{\rho} E_1(\rho)] \\ &= \rho^2 e^{\omega_3 \rho} \left[1 + e^{(\omega_2 - \omega_3)\rho} + \frac{E_2(\rho)}{\rho} \right] \end{aligned}$$

for $\rho \in S_1$. Let $\rho = x + iy$, and define $\Phi(\rho)$ and r_n by

$$\Phi(\rho) = 1 + e^{(\omega_2 - \omega_3)\rho} \text{ and } r_n = \frac{2(n+1)}{\sqrt{3}} \pi,$$

respectively. With $p = [3(r_n^2 - x^2)]^{1/2}$, we have

$$|\Phi(\rho)| \geq |1 + e^{-p} \cos(\sqrt{3}x)|,$$

provided $\rho = r_n e^{i\theta}$ where $0 \leq \theta \leq \pi/3$, and will show that

$$e^{-p} \cos(\sqrt{3}x) > -\frac{1}{2} \text{ for } \frac{r_n}{2} \leq x \leq r_n.$$

Since

$$\cos(\sqrt{3}x) \geq 0 \text{ for } r_n - \frac{\pi}{2\sqrt{3}} \leq x \leq r_n,$$

it is clearly sufficient to show that

$$e^{-p} \cos(\sqrt{3}x) > -\frac{1}{2} \text{ for } \frac{r_n}{2} \leq x \leq r_n - \frac{\pi}{2\sqrt{3}}.$$

Accordingly, we note that for x in this interval, we have

$$e^{-p} |\cos (\sqrt{3} x)| < \frac{1}{p} \leq \frac{1}{\sqrt{3} \left[r_n^2 - \left(r_n - \frac{\pi}{2\sqrt{3}} \right)^2 \right]^{1/2}} < \frac{1}{2}$$

for all $n > N$, provided N is sufficiently large. Taking N large enough we also have

$$\left| \frac{E_2(\rho)}{\rho} \right| < \frac{1}{4} \text{ for } \rho = r_n e^{i\theta} \quad (0 \leq \theta \leq \pi/3).$$

Hence

$$\left| \Phi(\rho) + \frac{E_2(\rho)}{\rho} \right| \geq |\Phi(\rho)| - \left| \frac{E_2(\rho)}{\rho} \right| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

This completes the proof of the lemma.

By the formal series for $f(x)$, we shall mean the series

$$\sum_{k=1}^{\infty} a_k u_k(x) \text{ where } a_k = \int_0^1 f(x) v_k(x) dx / \int_0^1 u_k(x) v_k(x) dx,$$

in which $u_k(x)$ and $v_k(x)$ are respectively the characteristic functions of the system (1) and its adjoint corresponding to the characteristic value λ_k .

LEMMA 4. *The sum of the first n terms of the formal series for $f(x)$ is given by*

$$\begin{aligned} I_n(x) &= \frac{1}{2\pi i} \int_{\gamma_n} \left[\int_0^x f(\xi) u(x, \xi, \rho) d\xi \right. \\ &\quad \left. - \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \int_0^1 f(\xi) u''(1, \xi, \rho) d\xi \right] d\rho \\ &= \frac{1}{2\pi i} \int_{\gamma_n} \left[\sigma(x) - \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \sigma''(1) \right] d\rho, \end{aligned}$$

where $\sigma(x) = \int_0^x f(\xi) u(x, \xi, \rho) d\xi$, and γ_n is the arc of the ρ -plane given by

$$\rho = \frac{2n+2}{\sqrt{3}} \pi e^{i\theta}, \quad -\pi/3 \leq \theta \leq \pi/3,$$

the ρ integration proceeding in a counter-clockwise direction.

We omit the proof of this lemma, as its details almost duplicate the discussion in [2], pp.424-426. We point out, however, that Lemma 2 is required in this proof.

LEMMA 5. *The function $\sigma(x)$ defined in the previous lemma satisfies the equation*

$$(3) \quad \sigma(x) = \int_0^x f(\xi) \delta_3[\rho(x-\xi)]d\xi - \frac{1}{3\rho} \int_0^x \Delta(x, t, \rho) \sigma(t)dt;$$

furthermore, $\sigma(x)$ is its unique solution, is analytic on $0 \leq x \leq 1$, and can be put into the form

$$\sigma(x) = u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho),$$

where

$$\Psi_2(x, \rho) = \frac{3f(x)}{\rho} + \frac{E_1(x, \rho)}{\rho^2}, \quad E_1''(x, \rho) = \rho^2 E_2(x, \rho),$$

provided $f(x) = x^2 \phi(x^3)$, where $\phi(z)$ is analytic on $|z| \leq 1$.

Proof. Using (2) in the expression for $\sigma(x)$, we obtain

$$\begin{aligned} \sigma(x) &= \int_0^x f(\xi) \delta_3[\rho(x-\xi)]d\xi \\ &\quad - \frac{1}{3\rho} \int_0^x f(\xi) \int_{\xi}^x \Delta(x, t, \rho) u(t, \xi, \rho) dt d\xi \\ &= \int_0^x f(\xi) \delta_3[\rho(x-\xi)]d\xi - \frac{1}{3\rho} \int_0^x \Delta(x, t, \rho) \sigma(t) dt \end{aligned}$$

on changing the order of integration in the second integral. Uniqueness of the solution $\sigma(x)$ can be shown in the usual manner. (See the proof of Lemma 1.)

We next substitute $u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho)$ into (3) for $\sigma(x)$, and obtain

$$\begin{aligned} u(x, 0, \rho) \Psi_1(\rho) + \Psi_2(x, \rho) &= \int_0^x f(\xi) \delta_3[\rho(x-\xi)]d\xi \\ &\quad - \frac{\Psi_1(\rho)}{3\rho} \int_0^x \Delta(x, t, \rho) u(t, 0, \rho) dt - \frac{1}{3\rho} \int_0^x \Delta(x, t, \rho) \Psi_2(t, \rho) dt, \end{aligned}$$

Using (2) with $\xi = 0$, and subtracting the term $u(x, 0, \rho) \Psi_1(\rho)$ from both sides, we obtain

$$(4) \quad \Psi_2(x, \rho) = \int_0^x f(\xi) \delta_3[\rho(x - \xi)] d\xi - \Psi_1(\rho) \delta_3(\rho x) \\ - \frac{1}{3\rho} \int_0^x \Delta(x, t, \rho) \Psi_2(t, \rho) dt.$$

On integrating by parts twice, we obtain

$$\int_0^x f(\xi) \delta_3[\rho(x - \xi)] d\xi = \frac{3f(x)}{\rho} + \rho^{-2} \int_0^x f''(\xi) \delta_2[\rho(x - \xi)] d\xi \\ = \frac{3f(x)}{\rho} + \rho^{-2} \delta_3(\rho x) \int_0^y f''(\xi) e^{\rho\xi} d\xi + \mathfrak{L}_3 f''(\xi) e^{\rho\xi} d\xi,$$

where y is a complex number to be determined later, and

$$\mathfrak{L}_3 F(t) dt = e^{\omega_1 \rho x} \int_y^x F(t) dt - \omega_2 e^{\omega_2 \rho x} \int_y^{-\omega_2 x} F(t) dt \\ - \omega_3 e^{\omega_3 \rho x} \int_y^{-\omega_3 x} F(t) dt.$$

It is in this step that we use the form of $f(x)$ as stated in the hypothesis of this lemma; for the details, see [2], pp. 428-429.

We also have

$$\int_0^x \Delta(x, t, \rho) \Psi_2(t, \rho) dt = \frac{1}{\rho} \int_0^x \delta_3[\rho(x - t)] r(t) \Psi_2(t, \rho) dt \\ + \int_0^x \delta_2[\rho(x - t)] p(t) \Psi_2(t, \rho) dt \\ = \frac{\delta_3(\rho x)}{\rho} \int_0^y r(t) e^{\rho t} \Psi_2(t, \rho) dt + \mathfrak{L}_3 r(t) e^{\rho t} \Psi_2(t, \rho) dt \\ + \delta_3(\rho x) \int_0^x p(t) e^{\rho t} \Psi_2(t, \rho) dt + \mathfrak{L}_3 p(t) e^{\rho t} \Psi_2(t, \rho) dt$$

$$= \delta_3(\rho x) \int_0^y R(t) e^{\rho t} \Psi_2(t, \rho) dt + \mathcal{L}_3 R(t) e^{\rho t} \Psi_2(t, \rho) dt,$$

where $R(t) = r(t)/\rho + p(t)$, and where we have made use of the properties of $p(t)$ and $r(t)$, and the fact that, from the form of $\Psi_2(t, \rho)$ in terms of $u(x, 0, \rho)$ and Lemma 3, part b, we have

$$\Psi_2(-\omega_2 t, \rho) = -\omega_3 \Psi_2(t, \rho).$$

Putting these results into equation (4), we obtain

$$\begin{aligned} \Psi_2(x, \rho) = & \frac{3f(x)}{\rho} + \delta_3(\rho x) \left[\Psi_1(\rho) - \frac{1}{\rho^2} \int_0^y f''(\xi) e^{\rho \xi} d\xi \right. \\ & \left. + \frac{1}{3\rho} \int_0^y R(t) e^{\rho t} \Psi_2(t, \rho) dt \right] \\ & + \frac{1}{\rho^2} \mathcal{L}_3 f''(t) e^{\rho t} dt - \frac{1}{3\rho} \mathcal{L}_3 R(t) e^{\rho t} \Psi_2(t, \rho) dt. \end{aligned}$$

This equation will certainly be satisfied if

$$(5) \quad \Psi_2(x, \rho) = \frac{3f(x)}{\rho} + \frac{1}{\rho^2} \mathcal{L}_3 f''(t) e^{\rho t} dt + \frac{1}{3\rho} \mathcal{L}_3 R(t) e^{\rho t} \Psi_2(t, \rho) dt$$

and

$$\Psi_1(\rho) = \frac{1}{\rho^2} \int_0^y f''(\xi) e^{\rho \xi} d\xi - \frac{1}{3\rho} \int_0^y R(t) e^{\rho t} \Psi_2(t, \rho) dt.$$

The proof of the existence of a unique solution $\Psi_2(x, \rho)$ of (5) will follow along the lines of the corresponding proof in [2], provided we can show that an expression of the form $|\mathcal{L}_3 F(t) e^{\rho t} dt|$ is bounded for complex ρ and $0 \leq x \leq 1$ whenever $|F(z)|$ is on $|z| \leq 1$ and we take $y = -e^{-i \arg \rho}$. For we have

$$\begin{aligned} |\mathcal{L}_3 F(t) e^{\rho t} dt| \leq & |e^{\omega_1 \rho x}| \int_y^x |F(t)| |e^{\rho t}| |dt| \\ & + |e^{\omega_2 \rho x}| \int_y^{-\omega_2 x} |F(t)| |e^{\rho t}| |dt| + |e^{\omega_3 \rho x}| \int_y^{-\omega_3 x} |F(t)| |e^{\rho t}| |dt| \end{aligned}$$

$$\leq \mu \left[|e^{\omega_1 \rho x} \int_y^x |e^{\rho t}| |dt| + |e^{\omega_2 \rho x} \int_y^{-\omega_2 x} |e^{\rho t}| |dt| + |e^{\omega_3 \rho x} \int_y^{-\omega_3 x} |e^{\rho t}| |dt| \right],$$

where $|F(z)| \leq \mu$ on $|z| \leq 1$; and since each integrand in this last expression assumes its maximum at its upper limit, we have

$$|\mathcal{L}_3 F(t) e^{\rho t} dt| \leq 6\mu.$$

We omit the rest of this existence proof. (See [2], pp. 429-430.)

For the asymptotic form of $\Psi_2(x, \rho)$, we substitute

$$\Psi_2(x, \rho) = \frac{3f(x)}{\rho} + v(x, \rho)$$

into (5). We obtain

$$(6) \quad v(x, \rho) = \frac{1}{\rho^2} \mathcal{L}_3 f''(t) e^{\rho t} dt - \frac{1}{3\rho} \mathcal{L}_3 R(t) e^{\rho t} \left[\frac{3f(t)}{\rho} + v(t, \rho) \right] dt.$$

For fixed ρ let $m = \max_{0 \leq x \leq 1} |v(x, \rho)|$; then

$$m \leq \frac{1}{|\rho|^2} |\mathcal{L}_3 [f''(t) + R(t)f(t)] e^{\rho t} dt| + \frac{1}{3|\rho|} |\mathcal{L}_3 R(t) e^{\rho t} v(t, \rho) dt|$$

$$\leq \frac{\mu_1}{|\rho|^2} + \frac{m\mu_1}{|\rho|} \leq \frac{\mu_1}{|\rho|^2} + \frac{m}{2},$$

provided $|\rho| \geq 2\mu_2$, where $|\mathcal{L}_3 R(t) e^{\rho t} dt| \leq \mu_2$. Hence for such ρ we have $m \leq 2\mu_1/|\rho|^2$, and it follows that $v(x, \rho) = \rho^{-2} E_1(x, \rho)$.

It remains to show that $v''(x, \rho) = E_2(x, \rho)$. Differentiating (6), we have

$$(7) \quad v'(x, \rho) = -\frac{1}{\rho} \left\{ \mathcal{L}_2 \left[f''(t) + R(t)f(t) + \frac{1}{3\rho} E_1(t, \rho) \right] e^{\rho t} dt \right\}$$

$$+ \frac{E_3(x, \rho)}{\rho^2},$$

where

$$\begin{aligned} \mathcal{L}_2 F(t) dt &= e^{\omega_1 \rho x} \int_y^x F(t) dt - \omega_3 e^{\omega_2 \rho x} \int_y^{-\omega_2 x} F(t) dt \\ &\quad - \omega_2 e^{\omega_3 \rho x} \int_y^{-\omega_3 x} F(t) dt, \end{aligned}$$

and we have used the fact that

$$\begin{aligned} |E_1(-\omega_2 x, \rho)| &= \left| \rho^2 \left(\Psi_2(-\omega_2 x, \rho) - \frac{3f(-\omega_2 x)}{\rho} \right) \right| \\ &= \left| -\rho^2 \omega_3 \left(\Psi_2(x, \rho) - \frac{3f(x)}{\rho} \right) \right| = |E_1(x, \rho)|. \end{aligned}$$

We can also show, as before in the case of the \mathcal{L}_3 operator, that if $|F(z)| \leq \mu$ on $|z| \leq 1$, then $|\mathcal{L}_2 F(t) e^{\rho t} dt| \leq m_2$.

Differentiating (7), we obtain

$$v''(x, \rho) = \mathcal{L}_1 \left[f''(t) + R(t)f(t) + \frac{1}{3\rho} E_1(t, \rho) \right] e^{\rho t} dt + \frac{E_4(x, \rho)}{\rho},$$

where

$$\begin{aligned} \mathcal{L}_1 F(t) dt &= e^{\omega_1 \rho x} \int_y^x F(t) dt + e^{\omega_2 \rho x} \int_y^{-\omega_2 x} F(t) dt \\ &\quad + e^{\omega_3 \rho x} \int_y^{-\omega_3 x} F(t) dt, \end{aligned}$$

and we have used the fact that $|E'_1(-\omega_2 x, \rho)| = |E'_1(x, \rho)|$ and that

$$|E'_1(x, \rho)| = |\rho^2 v'(x, \rho)| \leq |\rho| M$$

for $|\rho|$ sufficiently large.

Hence $v''(x, \rho) = E_2(x, \rho)$ since again $|F(z)| \leq \mu$ for $|z| \leq 1$ implies $|\mathcal{L}_1 F(t) e^{\rho t} dt| \leq m_1$, and the proof of the lemma is complete.

3. Theorem. We proceed now to the proof of the following theorem.

THEOREM. *If $f(x) = x^2 \phi(x^3)$, where $\phi(z)$ is analytic on $|z| \leq 1$, the formal series for $f(x)$ converges uniformly to $f(x)$ on $0 \leq x \leq 1$.*

Proof. Since, for real x and ξ , $u(x, \xi, \rho)$ is real for real ρ , by the principle of reflection we have $u(x, \xi, \rho^*) = [u(x, \xi, \rho)]^*$. This implies that the integrand in the expression for $I_n(x)$ given in Lemma 4 takes on values for ρ on $\gamma'_n = \gamma_n \cap S_1$ which are the complex conjugates of those it takes on for ρ on $\gamma''_n = \gamma_n \cap S_2$. It suffices, then, to consider only the ρ integration over γ'_n . Denoting the result by $I'_n(x)$, we have, by Lemmas 4 and 5,

$$I'_n(x) = \frac{1}{2\pi i} \int_{\gamma'_n} \left\{ \left[u(x, 0, \rho) \Psi_1(\rho) + \frac{3f(x)}{\rho} + \frac{E_1(x, \rho)}{\rho^2} \right] \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \left[u''(1, 0, \rho) \Psi_1(\rho) + \frac{3f''(x)}{\rho} + E_2(x, \rho) \right] \right\} d\rho;$$

and since, by Lemma 3, parts a) and c), we have

$$\left| \frac{u(x, 0, \rho)}{u''(1, 0, \rho)} \right| \leq \frac{M}{|\rho|^2}$$

for ρ on γ'_n and n sufficiently large, we obtain

$$I'_n(x) = \frac{1}{2\pi i} \int_{\gamma'_n} \left[\frac{3f(x)}{\rho} + \frac{E(x, \rho)}{\rho^2} \right] d\rho = \frac{f(x)}{2} + \epsilon'_n(x),$$

where

$$\lim_{n \rightarrow \infty} \epsilon'_n(x) = 0$$

uniformly in x . This proves the theorem.

At the expense of brevity, this theorem clearly could be generalized to problems involving somewhat more complicated boundary conditions and somewhat weaker analyticity conditions on $f(x)$, $p(x)$, and $q(x)$; in connection with the latter contention, see [2].

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