AN OPTIMUM PROBLEM IN THE WEINSTEIN METHOD FOR EIGENVALUES

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1. Introduction. The method of Weinstein [1] gives upper bounds for the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \) of the projection \( L' \) into a space \( \mathfrak{S} \) of a completely continuous positive symmetric operator \( L \) in a Hilbert space \( \mathfrak{H} \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \). These upper bounds are the eigenvalues \( \lambda_n^{(m)} \) of the projection of \( L \) into a space of finite index \( m \),

\[
\mathfrak{S} \ominus \{ p_1, \cdots, p_m \},
\]

where \( p_1, \cdots, p_m \) are any vectors in the space

\[
\mathfrak{N} = \mathfrak{S} \ominus \mathfrak{G}.
\]

The chief part of the Weinstein method is the explicit determination of the eigenvalues \( \lambda_n^{(m)} \) in the space (1) in terms of the eigenvalues and eigenvectors of \( L \) in \( \mathfrak{S} \). These satisfy

\[
\lambda_n^{(m)} \geq \lambda_n'.
\]

The values \( \lambda_n^{(m)} \) will, of course, depend on the choice of the vectors \( (p_1, \cdots, p_m) \). It is naturally desirable that the upper bound for a particular eigenvalue \( \lambda_n' \) should be as small as possible. This paper investigates how small it can be made for given \( n \) and \( m \) by a proper choice of the constraint vectors \( (p_1, \cdots, p_m) \).

Because of the minimax principle, \( \lambda_n^{(m)} \) must satisfy

\[
\lambda_n^{(m)} \geq \lambda_{n+m}.
\]

Our result is that the inequalities (3) and (4) are the only restrictions on the smallness of \( \lambda_n^{(m)} \). In other words, for given \( n \) and \( m \), there exist vectors \( (p_1, \cdots, p_m) \) such that the weaker of the inequalities (3) and (4) becomes an equality.

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2. The case of a single constraint. We first prove our result for the case of the first intermediate problem, that is, for \( m = 1 \).

Theorem 1. For any given \( n \), there is a vector \( p \) in the space
\[
\mathcal{P} = \mathcal{Q} \ominus \mathfrak{C}
\]
such that, if the projection of \( L \) into \( \mathcal{Q} \ominus \{ p \} \) has eigenvalues
\[
\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots,
\]
either
\[
\lambda_n^{(1)} = \lambda_n'
\]
or
\[
\lambda_n^{(1)} = \lambda_{n+1}',
\]
according as \( \lambda_n' \) or \( \lambda_{n+1}' \) is larger.

Proof. If \( \lambda_n' = \lambda_n \), then (6) is satisfied for any \( p \) and there is nothing to prove. Our theorem thus naturally splits into the two cases \( \lambda_n' > \lambda_n' \) and \( \lambda_n' < \lambda_{n+1}' \), which we shall prove separately.

3. The case \( \lambda_n' > \lambda_{n+1}' \). Let the eigenvector of \( L' \) corresponding to \( \lambda_n' \) be \( u_n' \). Its eigenvalue equation can be written in terms of the operator \( L \) as
\[
Lu_n' - \lambda_n' u_n' = p,
\]
where \( p \) is some vector in \( \mathcal{P} \). Let us assume for the moment that \( p \) is not a null vector. Then (8) is an eigenvalue equation for the projection of \( L \) into \( \mathfrak{C} \) but not for \( L \). Any eigenvector of \( L \) corresponding to the eigenvalue \( \lambda_n' \) must, because of (8), be orthogonal to \( p \) and hence must belong to \( \mathcal{Q} \ominus \{ p \} \). Thus, the multiplicity of \( \lambda_n' \) as an eigenvalue of the projection of \( L \) into \( \mathcal{Q} \ominus \{ p \} \) is one greater than its multiplicity as an eigenvalue of \( L \). Let the latter be \( r \geq 0 \). If \( r = 0 \), then \( \lambda_n' > \lambda_{n+1}' \) and \( \lambda_n' \) must be \( \lambda_n^{(1)} \) by the minimax principle. If \( r \geq 1 \), then \( \lambda_n > \lambda_{n+1} = \cdots = \lambda_{n+r} > \lambda_{n+r+1} \), and the minimax principle gives
\[
\lambda_{n-1}^{(1)} \geq \lambda_n > \lambda_n' > \lambda_{n+1} > \lambda_{n+r+1} \geq \lambda_{n+r+1}^{(1)}.
\]
Thus, since the multiplicity of \( \lambda_n' \) in \( \mathcal{Q} \ominus \{ p \} \) is \( r+1 \), we must have
(10) \[ \lambda_n^{(1)} = \lambda_n', \]

so that the vector \( p \) in (8) has the property stated in our theorem.

If \( \lambda_n' = \lambda_{n+1}' \), it is possible that the vector \( p \) in (8) is a null vector. This means that the eigenvector of \( L' \) corresponding to \( \lambda_n' \) is also an eigenvector of \( L \). Suppose that the same is also true of the eigenvalues \( \lambda_{n+1}', \ldots, \lambda_{n+s-1}' \) but not of \( \lambda_{n+s}' \). We then consider the projections \( \overline{L} \) and \( \overline{L}' \) into

\[ \mathcal{Q} \ominus \{ u_n', \ldots, u_{n+s-1}' \} \quad \text{and} \quad \mathcal{Q} \ominus \{ u_n', \ldots, u_{n+s-1}' \} \]

respectively, and call their eigenvalues \( \overline{\lambda}_i \) and \( \overline{\lambda}_i' \). Then \( \overline{L}' \) has the same eigenvalues as \( L' \), except that the eigenvalues \( \lambda_{n}', \ldots, \lambda_{n+s-1}' \) are removed. The same is true of \( \overline{L} \) and \( L \). Then

(11) \[ \overline{\lambda}_n' = \lambda_{n+s}' \leq \lambda_n'. \]

If there is a vector \( p \) in \( \mathcal{P} \) so that the \( n \)-th eigenvalue of \( L \) in

\[ \mathcal{Q} \ominus \{ u_n', \ldots, u_{n+s-1}', p \} \]

is at most \( \overline{\lambda}_n' \), then, because of (11), the \( n \)-th eigenvalue of \( L \) in \( \mathcal{Q} \ominus \{ p \} \) is \( \lambda_n' \) and equation (6) in our theorem will be proved. Now if

(12) \[ \overline{\lambda}_n' = \lambda_{n+s}' \geq \overline{\lambda}_n', \]

then, since by definition of \( s \) the eigenvector of \( L' \) corresponding to \( \lambda_{n+s}' \) is not an eigenvector of \( L \), the existence of such a vector \( p \) follows from the first part of this paragraph. If, on the other hand, we have

(13) \[ \lambda_n' < \overline{\lambda}_{n+1}', \]

the existence of this vector \( p \) will be assured by the results of the next paragraph.

A final possibility* is that there is no integer \( s \) such that the eigenvector \( u_{n+s}' \) of \( L' \) is not also an eigenvector of \( L \). In other words, all but the first \( n - 1 \) eigenvectors of \( L' \) are also eigenvectors of \( L \). Then, since \( \lambda_n' = \lambda_{n+1}' \), the vectors \( u_1', \ldots, u_{n-1}' \) are the only eigenvectors of \( L' \) which are not orthogonal to \( u_1, u_2, \ldots, u_n \). Therefore there is one linear combination \( p \) of \( u_2, \ldots, u_n \) which is orthogonal to all eigenvectors of \( L' \) and hence belongs to \( \mathcal{P} \). There can

*This possibility was pointed out by C. Arf in the course of an alternative proof of the results here presented.
be at most \( n - 1 \) eigenvectors of the projection of \( L \) into \( \mathbb{S} \oplus \{ p \} \) which are not orthogonal to \( u_1, \ldots, u_n \). Therefore, the \( n \)-th eigenvalue of this projection is \( \lambda_{n+1} = \lambda_n' \), and both equalities (6) and (7) hold.

4. The case \( \lambda_n' < \lambda_{n+1} \). We now show that if \( \lambda_n' < \lambda_{n+1} \) then the equation (7) can be made to hold. This will be done by induction. We first replace the space \( \mathbb{S} \) by a finite space. Since \( L \) is completely continuous it follows that \( \lambda_m \to 0 \) as \( m \to \infty \); therefore there is an integer \( m \) such that

\[
\lambda_{n+1} \geq \lambda_n' + \lambda_{m+1}. \tag{14}
\]

It has been shown by the author [1, 2] that if we let \( p_i \) be the projection in \( \mathbb{S} \) of \( u_i \), then the eigenvalues \( \lambda_{n}^{(m)} \) of \( L \) in \( \mathbb{S} \oplus \{ p_1, \ldots, p_m \} \) satisfy

\[
\lambda_{n}^{(m)} \leq \lambda_n' + \lambda_{m+1}. \tag{15}
\]

Combining this with (14), we obtain

\[
\lambda_{n}^{(m)} \leq \lambda_{n+1}. \tag{16}
\]

Thus, it will suffice to show that if the inequality (16) holds where \( \lambda_{n}^{(m)} \) is the \( n \)-th eigenvalue of \( L \) in a space \( \mathbb{S} \oplus \{ p_1, \ldots, p_m \} \), then there is a linear combination \( p \) of the vectors \( p_1, \ldots, p_m \) such that the \( n \)-th eigenvalue of \( L \) in \( \mathbb{S} \oplus \{ p \} \) is \( \lambda_{n+1} \). Our induction proof consists of showing that if (16) holds for \( m > 1 \) then there is a linear combination \( p' \) of \( p_{m-1} \) and \( p_m \) such that the \( n \)-th eigenvalue of \( L \) in \( \mathbb{S} \oplus \{ p_1, \ldots, p_{m-2}, p' \} \) is at most \( \lambda_{n+1} \). If \( \lambda_{n}^{(m-1)} \leq \lambda_{n+1} \), this is obviously true, for we must only take \( p' = p_{m-1} \). Thus, we need to examine only the case

\[
\lambda_{n}^{(m-2)} \geq \lambda_{n}^{(m-1)} > \lambda_{n+1} = \lambda_{n}^{(m)}. \tag{17}
\]

Since, by the minimax theorem,

\[
\lambda_{n+1} \geq \lambda_{n+1}^{(m-2)}, \tag{18}
\]

our induction step will be proved if we find \( p \) so that the \( n \)-th eigenvalue of \( L \) in \( \mathbb{S} \oplus \{ p_1, \ldots, p_{m-2}, p' \} \) is equal to either \( \lambda_{n}^{(m)} \) or \( \lambda_{n+1}^{(m-2)} \). In other words, the induction step is just Theorem 1 in the special case in which \( \mathbb{S} \) is a 2-space.

Thus if \( \lambda_{n}^{(m)} \geq \lambda_{n+1}^{(m-2)} \) the induction is proved by the results of \( \S 3 \). Note that in the case of a common eigenvector where one had to reduce the proof
in §3 to the proof of this section, the reduction is to the case $\lambda_n^{(m)} < \lambda_n^{(m-2)}$, which will now be treated.

If $\lambda_n^{(m)} < \lambda_n^{(m-2)}$, we must construct a linear combination $p'$ of $p_{m-1}$ and $p_m$ so that $\lambda_n^{(m-2)}$ is the $n$-th eigenvalue of $L$ in $\mathcal{S} \ominus \{ p_1, \ldots, p_{m-2}, p' \}$. To do this, we take for $p'$ the linear combination of $p_{m-1}$ and $p_m$ which is orthogonal to the eigenvector corresponding to $\lambda_n^{(m-2)}$. Then $\lambda_n^{(m-2)}$ is an eigenvalue of $L$ in $\mathcal{S} \ominus \{ p_1, \ldots, p_{m-2}, p' \}$. By the minimax principle, the $(n+1)$-st eigenvalue in this space is at most $\lambda_n^{(m)} < \lambda_n^{(m-2)}$. Therefore $\lambda_n^{(m-2)}$ must be the $n$-th eigenvalue in this space, and $p'$ has the desired property.

Thus, our induction step is proved and Theorem 1 has been shown to hold in all possible cases.

5. The general intermediate problem. We are now in a position to prove the more general result announced in the introduction.

**Theorem 2.** For any fixed integers $m$ and $n$, there are vectors $p_1, \ldots, p_m$ in $\mathcal{S}$ which, if used as constraints in the $n$-th intermediate problem, yield either

(19) $\lambda_n^{(m)} = \lambda_n'$

or

(20) $\lambda_n^{(m)} = \lambda_{n+m}$.

**Proof.** We first prove the possibility of the equality (20) when

(21) $\lambda_{n+m} > \lambda_n'$

According to Theorem 1 with $n + m - 1$ substituted for $n$, there is a vector $p_1$ such that

(22) $\lambda_{n+m-1}^{(1)} = \lambda_{n+m}$.

We then apply Theorem 1 to the projection of $L$ into $\mathcal{S} \ominus \{ p_1 \}$ to assert the existence of a vector $p_2$ such that

(23) $\lambda_{n+m-2}^{(2)} = \lambda_{n+m}$.

This process is repeated until the equality (20) is obtained. Inequality (21) assures us that the equality (7) of Theorem 1 will always be attainable.

If $\lambda_{n+m} \leq \lambda_n'$, then there is an integer $l \leq m$ such that
We shall show that there are \( I \) vectors \( p_1, \ldots, p_I \) for which
\[
\lambda_n^{(I)} = \lambda_n^*.
\]

The equality (19) will then hold for any \( m - 1 \) vectors \( p_{l+1}, \ldots, p_m \) appended to the first \( l \).

Since \( \lambda_n^* \geq \cdots \geq \lambda_{n+l-1}^* \), we can proceed as in the proof of (20) to show that there are \( l - 1 \) vectors \( p_1, \ldots, p_{l-1} \) for which
\[
\lambda_{n+1}^{(l+1)} = \lambda_{n+l}.
\]

We now apply Theorem 1 to \( L \) in \( \mathbb{B} \cap \{ p_1, \ldots, p_{l-1} \} \). According to (24) and (26) it is the equality (6) which can be made to hold by a constraint \( p_l \). We thus obtain (25), and Theorem 2 is proved.

**References**


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Lars V. Ahlfors, *Remarks on the Neumann-Poincaré integral equation* 271
Leonard P. Burton, *Oscillation theorems for the solutions of linear, nonhomogeneous, second-order differential systems* 281
Paul Civin, *Multiplicative closure and the Walsh functions* 291
James Michael Gardner Fell and Alfred Tarski, *On algebras whose factor algebras are Boolean* 297
Paul Joseph Kelly and Lowell J. Paige, *Symmetric perpendicularity in Hilbert geometries* 319
G. Kurepa, *On a characteristic property of finite sets* 323
Joseph Lehner, *A diophantine property of the Fuchsian groups* 327
Donald Alan Norton, *Groups of orthogonal row-latin squares* 335
R. S. Phillips, *On the generation of semigroups of linear operators* 343
G. Piranian, *Uniformly accessible Jordan curves through large sets of relative harmonic measure zero* 371
C. T. Rajagopal, *Note on some Tauberian theorems of O. Szász* 377
Halsey Lawrence Royden, Jr., *A modification of the Neumann-Poincaré method for multiply connected regions* 385
George H. Seifert, *A third order irregular boundary value problem and the associated series* 395
Herbert E. Vaughan, *Well-ordered subsets and maximal members of ordered sets* 407
Hans F. Weinberger, *An optimum problem in the Weinstein method for eigenvalues* 413
Shigeki Yano, *Note on Fourier analysis. XXXI. Cesàro summability of Fourier series* 419