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**AN OPTIMUM PROBLEM IN THE WEINSTEIN METHOD FOR
EIGENVALUES**

HANS F. WEINBERGER

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1. Introduction. The method of Weinstein [1] gives upper bounds for the eigenvalues $\lambda'_1 \geq \lambda'_2 \geq \dots$ of the projection L' into a space \mathfrak{G} of a completely continuous positive symmetric operator L in a Hilbert space \mathfrak{H} with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$. These upper bounds are the eigenvalues $\lambda_n^{(m)}$ of the projection of L into a space of finite index m ,

$$(1) \quad \mathfrak{H} \ominus \{p_1, \dots, p_m\},$$

where p_1, \dots, p_m are any vectors in the space

$$(2) \quad \mathfrak{B} = \mathfrak{H} \ominus \mathfrak{G}.$$

The chief part of the Weinstein method is the explicit determination of the eigenvalues $\lambda_n^{(m)}$ in the space (1) in terms of the eigenvalues and eigenvectors of L in \mathfrak{H} . These satisfy

$$(3) \quad \lambda_n^{(m)} \geq \lambda'_n.$$

The values $\lambda_n^{(m)}$ will, of course, depend on the choice of the vectors (p_1, \dots, p_m) . It is naturally desirable that the upper bound for a particular eigenvalue λ'_n should be as small as possible. This paper investigates how small it can be made for given n and m by a proper choice of the constraint vectors (p_1, \dots, p_m) .

Because of the minimax principle, $\lambda_n^{(m)}$ must satisfy

$$(4) \quad \lambda_n^{(m)} \geq \lambda_{n+m}.$$

Our result is that the inequalities (3) and (4) are the only restrictions on the smallness of $\lambda_n^{(m)}$. In other words, for given n and m , there exist vectors (p_1, \dots, p_m) such that the weaker of the inequalities (3) and (4) becomes an equality.

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2. The case of a single constraint. We first prove our result for the case of the first intermediate problem, that is, for $m = 1$.

THEOREM 1. *For any given n , there is a vector p in the space*

$$(5) \quad \mathfrak{P} = \mathfrak{S} \ominus \mathfrak{G}$$

such that, if the projection of L into $\mathfrak{S} \ominus \{p\}$ has eigenvalues

$$\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \dots,$$

either

$$(6) \quad \lambda_n^{(1)} = \lambda'_n$$

or

$$(7) \quad \lambda_n^{(1)} = \lambda_{n+1},$$

according as λ'_n or λ_{n+1} is larger.

Proof. If $\lambda'_n = \lambda_n$, then (6) is satisfied for any p and there is nothing to prove. Our theorem thus naturally splits into the two cases $\lambda_n > \lambda'_n \geq \lambda_{n+1}$ and $\lambda'_n < \lambda_{n+1}$, which we shall prove separately.

3. The case $\lambda_n > \lambda'_n \geq \lambda_{n+1}$. Let the eigenvector of L' corresponding to λ'_n be u'_n . Its eigenvalue equation can be written in terms of the operator L as

$$(8) \quad Lu'_n - \lambda'_n u'_n = p,$$

where p is some vector in \mathfrak{P} . Let us assume for the moment that p is not a null vector. Then (8) is an eigenvalue equation for the projection of L into \mathfrak{G} but not for L . Any eigenvector of L corresponding to the eigenvalue λ'_n must, because of (8), be orthogonal to p and hence must belong to $\mathfrak{S} \ominus \{p\}$. Thus, the multiplicity of λ'_n as an eigenvalue of the projection of L into $\mathfrak{S} \ominus \{p\}$ is one greater than its multiplicity as an eigenvalue of L . Let the latter be $r \geq 0$. If $r = 0$, then $\lambda'_n > \lambda_{n+1}$, and λ'_n must be $\lambda_n^{(1)}$ by the minimax principle. If $r \geq 1$, then $\lambda_n > \lambda_{n+1} = \dots = \lambda_{n+r} > \lambda_{n+r+1}$, and the minimax principle gives

$$(9) \quad \lambda_{n-1}^{(1)} \geq \lambda_n > \lambda'_n \geq \lambda_{n+1} > \lambda_{n+r+1} \geq \lambda_{n+r+1}^{(1)}.$$

Thus, since the multiplicity of λ'_n in $\mathfrak{S} \ominus \{p\}$ is $r + 1$, we must have

$$(10) \quad \lambda_n^{(1)} = \lambda'_n,$$

so that the vector p in (8) has the property stated in our theorem.

If $\lambda'_n = \lambda_{n+1}$, it is possible that the vector p in (8) is a null vector. This means that the eigenvector of L' corresponding to λ'_n is also an eigenvector of L . Suppose that the same is also true of the eigenvalues $\lambda'_{n+1}, \dots, \lambda'_{n+s-1}$ but not of λ'_{n+s} . We then consider the projections \bar{L} and \bar{L}' into

$$\mathfrak{H} \ominus \{u'_n, \dots, u'_{n+s-1}\} \quad \text{and} \quad \mathfrak{G} \ominus \{u'_n, \dots, u'_{n+s-1}\}$$

respectively, and call their eigenvalues $\bar{\lambda}_i$ and $\bar{\lambda}'_i$. Then \bar{L}' has the same eigenvalues as L' , except that the eigenvalues $\lambda'_n, \dots, \lambda'_{n+s-1}$ are removed. The same is true of \bar{L} and L . Then

$$(11) \quad \bar{\lambda}'_n = \lambda'_{n+s} \leq \lambda'_n.$$

If there is a vector p in \mathfrak{H} so that the n -th eigenvalue of L in

$$\mathfrak{H} \ominus \{u'_n, \dots, u'_{n+s-1}, p\}$$

is at most $\bar{\lambda}'_n$, then, because of (11), the n -th eigenvalue of L in $\mathfrak{H} \ominus \{p\}$ is λ'_n and equation (6) in our theorem will be proved. Now if

$$(12) \quad \bar{\lambda}'_n = \lambda'_{n+s} \geq \bar{\lambda}_n,$$

then, since by definition of s the eigenvector of L' corresponding to λ'_{n+s} is not an eigenvector of L , the existence of such a vector p follows from the first part of this paragraph. If, on the other hand, we have

$$(13) \quad \lambda'_n < \bar{\lambda}_{n+1},$$

the existence of this vector p will be assured by the results of the next paragraph.

A final possibility* is that there is no integer s such that the eigenvector u_{n+s} of L' is not also an eigenvector of L . In other words, all but the first $n - 1$ eigenvectors of L' are also eigenvectors of L . Then, since $\lambda'_n = \lambda_{n+1}$, the vectors u'_1, \dots, u'_{n-1} are the only eigenvectors of L' which are not orthogonal to u_1, u_2, \dots, u_n . Therefore there is one linear combination p of u_2, \dots, u_n which is orthogonal to all eigenvectors of L' and hence belongs to \mathfrak{H} . There can

* This possibility was pointed out by C. Arf in the course of an alternative proof of the results here presented.

be at most $n - 1$ eigenvectors of the projection of L into $\mathfrak{E} \ominus \{p\}$ which are not orthogonal to u_1, \dots, u_n . Therefore, the n -th eigenvalue of this projection is $\lambda_{n+1} = \lambda'_n$, and both equalities (6) and (7) hold.

4. **The case $\lambda'_n < \lambda_{n+1}$.** We now show that if $\lambda'_n < \lambda_{n+1}$ then the equation (7) can be made to hold. This will be done by induction. We first replace the space \mathfrak{F} by a finite space. Since L is completely continuous it follows that $\lambda_m \rightarrow 0$ as $m \rightarrow \infty$; therefore there is an integer m such that

$$(14) \quad \lambda_{n+1} \geq \lambda'_n + \lambda_{m+1}.$$

It has been shown by the author [1, 2] that if we let p_i be the projection in \mathfrak{F} of u_i , then the eigenvalues $\lambda_n^{(m)}$ of L in $\mathfrak{E} \ominus \{p_1, \dots, p_m\}$ satisfy

$$(15) \quad \lambda_n^{(m)} \leq \lambda'_n + \lambda_{m+1}.$$

Combining this with (14), we obtain

$$(16) \quad \lambda_n^{(m)} \leq \lambda_{n+1}.$$

Thus, it will suffice to show that if the inequality (16) holds where $\lambda_n^{(m)}$ is the n -th eigenvalue of L in a space $\mathfrak{E} \ominus \{p_1, \dots, p_m\}$, then there is a linear combination p of the vectors p_1, \dots, p_m such that the n -th eigenvalue of L in $\mathfrak{E} \ominus \{p\}$ is λ_{n+1} . Our induction proof consists of showing that if (16) holds for $m > 1$ then there is a linear combination p' of p_{m-1} and p_m such that the n -th eigenvalue of L in $\mathfrak{E} \ominus \{p_1, \dots, p_{m-2}, p'\}$ is at most λ_{n+1} . If $\lambda_n^{(m-1)} \leq \lambda_{n+1}$, this is obviously true, for we must only take $p' = p_{m-1}$. Thus, we need to examine only the case

$$(17) \quad \lambda_n^{(m-2)} \geq \lambda_n^{(m-1)} > \lambda_{n+1} \geq \lambda_n^{(m)}.$$

Since, by the minimax theorem,

$$(18) \quad \lambda_{n+1} \geq \lambda_{n+1}^{(m-2)},$$

our induction step will be proved if we find p so that the n -th eigenvalue of L in $\mathfrak{E} \ominus \{p_1, \dots, p_{m-2}, p'\}$ is equal to either $\lambda_n^{(m)}$ or $\lambda_{n+1}^{(m-2)}$. In other words, the induction step is just Theorem 1 in the special case in which \mathfrak{F} is a 2-space.

Thus if $\lambda_n^{(m)} \geq \lambda_{n+1}^{(m-2)}$ the induction is proved by the results of §3. Note that in the case of a common eigenvector where one had to reduce the proof

in §3 to the proof of this section, the reduction is to the case $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$, which will now be treated.

If $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$, we must construct a linear combination p' of p_{m-1} and p_m so that $\lambda_{n+1}^{(m-2)}$ is the n -th eigenvalue of L in $\mathfrak{S} \ominus \{p_1, \dots, p_{m-2}, p'\}$. To do this, we take for p' the linear combination of p_{m-1} and p_m which is orthogonal to the eigenvector corresponding to $\lambda_n^{(m-2)}$. Then $\lambda_{n+1}^{(m-2)}$ is an eigenvalue of L in $\mathfrak{S} \ominus \{p_1, \dots, p_{m-2}, p'\}$. By the minimax principle, the $(n+1)$ -st eigenvalue in this space is at most $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$. Therefore $\lambda_{n+1}^{(m-2)}$ must be the n -th eigenvalue in this space, and p' has the desired property.

Thus, our induction step is proved and Theorem 1 has been shown to hold in all possible cases.

5. The general intermediate problem. We are now in a position to prove the more general result announced in the introduction.

THEOREM 2. *For any fixed integers m and n , there are vectors p_1, \dots, p_m in \mathfrak{P} which, if used as constraints in the n -th intermediate problem, yield either*

$$(19) \quad \lambda_n^{(m)} = \lambda'_n$$

or

$$(20) \quad \lambda_n^{(m)} = \lambda_{n+m}.$$

Proof. We first prove the possibility of the equality (20) when

$$(21) \quad \lambda_{n+m} > \lambda'_n$$

According to Theorem 1 with $n + m - 1$ substituted for n , there is a vector p_1 such that

$$(22) \quad \lambda_{n+m-1}^{(1)} = \lambda_{n+m}.$$

We then apply Theorem 1 to the projection of L into $\mathfrak{S} \ominus \{p_1\}$ to assert the existence of a vector p_2 such that

$$(23) \quad \lambda_{n+m-2}^{(2)} = \lambda_{n+m}.$$

This process is repeated until the equality (20) is obtained. Inequality (21) assures us that the equality (7) of Theorem 1 will always be attainable.

If $\lambda_{n+m} \leq \lambda'_n$, then there is an integer $l \leq m$ such that

$$(24) \quad \lambda_{n+l} \leq \lambda'_n < \lambda_{n+l-1}.$$

We shall show that there are l vectors p_1, \dots, p_l for which

$$(25) \quad \lambda_n^{(l)} = \lambda'_n$$

The equality (19) will then hold for any $m-1$ vectors p_{l+1}, \dots, p_m appended to the first l .

Since $\lambda'_n \geq \dots \geq \lambda'_{n+l-1}$, we can proceed as in the proof of (20) to show that there are $l-1$ vectors p_1, \dots, p_{l-1} for which

$$(26) \quad \lambda_{n+1}^{(l+1)} = \lambda_{n+l}.$$

We now apply Theorem 1 to L in $\mathfrak{S} \ominus \{p_1, \dots, p_{l-1}\}$. According to (24) and (26) it is the equality (6) which can be made to hold by a constraint p_l . We thus obtain (25), and Theorem 2 is proved.

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