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ON THE SINGULARITIES OF TAYLOR SERIES WITH RECIPROCAL COEFFICIENTS

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1. Introduction.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be a Taylor series with a nonvanishing radius of convergence and such that $a_n \neq 0$. Put

$$f_{-1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{a_n}, \tag{1.1'}$$

and suppose that the latter series also has a positive radius of convergence. Now $f_{-1}(z)$ can be considered as the inverse of $f(z)$ under the Hadamard "multiplication" of series, and it is natural to inquire into the existence of a simple relation between the singularities of the two functions. This problem was treated by Soula [3], who discovered such a relation for the singularities of the two series on their circles of convergence. His result is as follows:

THEOREM S. *Let $f(z)$ and $f_{-1}(z)$ be defined by* (1.1) *and* (1.1′), *where a_n is real, $a_n \neq 0$, and where, furthermore,*

$$\lim_{n=\infty} |a_n|^{1/n} = 1. \tag{1.2}$$

(Thus the unit circle is the circle of convergence for both series.) *If $z = 1$ is the only singularity of $f(z)$ on the unit circle then either the unit circle is a cut for $f_{-1}(z)$, or $f_{-1}(z)$ also has $z = 1$ as its only singularity on the unit circle. Moreover, in the latter case we have*:

(i) $\lim (a_n/a_{n+1}) = 1$;

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(ii) $z = 1$ *is a singularity "without contact"* (with the unit circle) *for both functions* $f(z)$ *and* $f_{-1}(z)$. *That is, there exist* $\delta > 0$ *and* ϕ, *with* $0 < \phi < \pi/2$, *such that* $f(z)$ *and* $f_{-1}(z)$ *are analytic in the sector*

$$0 < |z-1| < \delta, \quad \phi < |\arg(z-1)| \leq \pi.$$

We remark that if the radius of convergence of $f(z)$ is 1 (which is no loss of generality) then it is not difficult to see that one cannot expect to have a dependence even between the singularities of the two series on their circles of convergence unless the radius of convergence of $f_{-1}(z)$ is also 1. Thus (1.2) is a necessary restriction. Also it seems that the condition that $f(z)$ should have only one singular point, say $z = 1$, on its circle of convergence is essential for the simple character of the result. However, the condition on the reality of the coefficients in Soula's theorem is superfluous. All that is needed in Soula's proof when the coefficients are complex is the use of Lemma 5 of this paper. In what follows we shall refer to this more general result as Theorem S.

We propose in this paper to obtain a relation between the singularities of $f(z)$ and $f_{-1}(z)$ outside the unit circle. To this end it is necessary to have some information on the location of the singularities of $f(z)$ outside the unit circle. We shall impose on $f(z)$ the somewhat restrictive condition that it be holomorphic in the whole plane cut along the line $1 \leq x < \infty$. With this condition, however, we shall derive a surprisingly simple result concerning the location of the singularities of $f_{-1}(z)$ in the whole plane.

2. **Preliminary considerations.** We collect in this section the definitions and lemmas which we shall need in the proof of our main result. Some of these lemmas are well-known theorems.

DEFINITION. Let $f(z)$ be given in the neighborhood of the origin by the Taylor series (1.1). The *star of holomorphy* of $f(z)$ (Mittag-Leffler star) is defined as the domain composed of all segments $te^{i\theta}$, $0 \leq t < \rho(\theta)$, where $\rho(\theta)e^{i\theta}$ is the first singularity of $f(z)$ on the ray $te^{i\theta}$ $(0 \leq t < \infty)$ when $f(z)$ is continued analytically along this ray. The function $\rho(\theta)$ shall be called the *star-function* and shall be defined by periodicity for all values of θ.

It follows readily from the definition that $\rho(\theta)$ is a lower semicontinuous function, and as such it attains its lower bound in any finite interval.

LEMMA 1. (Hadamard's multiplication theorem for stars [1, p.300]). *Let*

$$(2.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

have the radii of convergence R_f *and* R_g, *respectively, and star-functions* $\rho_f(\theta)$ *and* $\rho_g(\theta)$. *Put*

$$(2.2) \qquad h(z) = [f, g] = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Then $h(z)$ *converges for* $|z| < R_f R_g$ *and can be continued analytically along the segment* $te^{i\theta}$, $0 \le t < r(\theta)$, *for any* $0 \le \theta < 2\pi$, *where*

$$r(\theta) = \min_{0 \le u < 2\pi} \rho_f(u)\, \rho_g(\theta - u).$$

The following is a simple lemma on the separation of singularities of an analytic function.

LEMMA 2. *Let* $f(z)$ *be an analytic function in the neighborhood of the origin where it has the Taylor development* (1.1), *and let* $\rho(\theta)$ *be its star-function. Then, given* θ_1 *and* θ_2, $0 < \theta_1 < \theta_2 < 2\pi$, *and* $\epsilon > 0$, *there exist two analytic functions* $g_1(z)$ *and* $g_2(z)$ *with developments*

$$g_1(z) = \sum_{n=0}^{\infty} a_n' z^n, \quad g_2(z) = \sum_{n=0}^{\infty} a_n'' z^n,$$

such that

(i) $f(z) = g_1(z) + g_2(z)$;

(ii) *the star-function* $\rho_i^g(\theta)$ *of* $g_i(z)$ *satisfy*

$$(2.3) \qquad \begin{cases} \rho_1^g(\theta) = \rho(\theta) \text{ and } \rho_2^g(\theta) = \infty \text{ for } \theta_1 < \theta < \theta_2, \\ \rho_1^g(\theta) = \infty \text{ and } \rho_2^g(\theta) = \rho(\theta) \text{ for } \theta_2 < \theta < \theta_1 + 2\pi, \\ \rho_i^g(\theta_j) > \rho(\theta_j) - \epsilon \text{ for } i, j = 1, 2. \end{cases}$$

We shall indicate an easy proof of Lemma 2. Let C be a star-shaped rectifiable Jordan curve enclosing the origin, contained in the star of holomorphy of

$f(z)$, and whose defining function $R = R(\theta)$ $(0 \leq \theta < 2\pi)$ satisfies

$$R(\theta_i) > \rho(\theta_i) - \epsilon \qquad (i = 1, 2).$$

Let C_1 be the part of C in the sector $\theta_1 \leq \theta \leq \theta_2$, and C_2 the complementary part of C. Then by Cauchy's theorem we have

$$(2.4) \qquad f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= g_1(z) + g_2(z).$$

It follows now readily that the functions $g_1(z)$ and $g_2(z)$ satisfy the conditions of Lemma 2.

The following two lemmas are known. (For example, see [2, p. 103, Th. III], where the lemmas are generalized to Dirichlet series). We remark, however, that we make a trivial addition (without proof) to the lemmas by not assuming the angle in question to be symmetric with respect to the positive axis.

LEMMA 3. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a Taylor series having the unit circle for its circle of convergence. Suppose that $f(z)$ *can be continued analytically in a domain* $D_{\alpha\beta}$ *whose boundary is composed of the two spirals*:

$$\rho = \exp[(\tan \alpha)\theta] \quad (0 \leq \theta \leq \theta_0),$$

and

$$\rho = \exp[(\tan \beta)(2\pi - \theta)] \quad (\theta_0 \leq \theta \leq 2\pi),$$

where $0 < \alpha$, $\beta < \pi/2$ *and* $\theta_0 = (2\pi \tan \beta)/(\tan \alpha + \tan \beta)$, *and where* $z = \rho e^{i\theta}$. *Then, given* $0 < \alpha' < \alpha$ *and* $0 < \beta' < \beta$, *there exists an "interpolation function"* $G(u)$, *analytic in the angle* $-\beta' \leq \arg u \leq \alpha'$, *such that*

$$(2.5) \qquad G(n) = a_n \qquad (n = 0, 1, \cdots),$$

and

$$\limsup_{|u|=\infty}\left(\frac{\log|G(u)|}{|u|}\right)\le 0, \tag{2.5'}$$

uniformly in $-\beta'\le \arg u \le \alpha'$.

The next lemma is a kind of converse of Lemma 3.

LEMMA 3′. *Let* $G(u)$ *be an analytic function of exponential type in the infinite sector* $-\beta\le \arg u\le \alpha$, $|u|\ge R_0$ $(0<\alpha,\beta<\pi/2)$ *satisfying* (2.5′). *Let*

$$f(z)=\sum_{n>R_0} G(n)z^n.$$

Then $f(z)$ *can be continued analytically in the domain* $D_{\alpha\beta}$.

We shall prove now the following lemma.

LEMMA 4. *Let* $G(u)$ *be an analytic function of exponential type in the angle* $-\beta\le \arg u\le\alpha$ $(0<\alpha,\ \beta<\pi/2)$. *Put*

$$\tau_G(\theta)=\limsup_{r=\infty}\left(\frac{\log|G(re^{i\theta})|}{r}\right). \tag{2.6}$$

Suppose that

$$\text{(i)}\qquad \begin{cases}\tau_G(\theta)\le \Omega_1\sin\theta \text{ for } 0\le\theta\le\alpha,\\ \tau_G(\theta)\le -\Omega_2\sin\theta \text{ for } -\beta\le\theta\le 0,\end{cases}$$

where $\Omega_1\ge 0$, $\Omega_2\ge 0$ *and* $\Omega_1+\Omega_2<2\pi$, *and that*

$$\text{(ii)}\qquad G(n)=0 \text{ for } n=0,1,\cdots.$$

Let α^* *and* β^* $(0<\alpha^*,\ \beta^*<\pi/2)$ *be defined by*

$$\begin{cases}\tan\alpha^*=\dfrac{2\pi-(\Omega_1+\Omega_2)}{\Omega_1\cot\beta+(2\pi-\Omega_2)\cot\alpha},\\[2ex] \tan\beta^*=\dfrac{2\pi-(\Omega_1+\Omega_2)}{(2\pi-\Omega_1)\cot\beta+\Omega_2\cot\alpha}\end{cases} \tag{2.7}$$

Then we have $\tau_G(\theta)<0$ *for* $-\beta^*<\theta<\alpha^*$, *and, in particular,*

$$G(u) = O(e^{-\delta|u|}), \tag{2.8}$$

uniformly in any angle $-\beta^* < -\bar{\beta} \leq \arg u \leq \bar{\alpha} < \alpha^*$ *for some* $\delta = \delta(\bar{\alpha}, \bar{\beta}) > 0$.

Proof. Put

$$F(u) = \frac{G(u)}{\sin \pi u}. \tag{2.9}$$

It is easily seen that $F(u)$ is analytic and of exponential type in the angle $-\beta \leq \arg u \leq \alpha$. Let $\tau_F(\theta)$ be the "type function" (2.6) of $F(u)$. It follows from (i) and (2.9) that

$$\tau_F(\alpha) \leq (\Omega_1 - \pi) \sin \alpha \quad \text{and} \quad \tau_F(-\beta) \leq (\Omega_2 - \pi) \sin \beta.$$

Applying a well-known result of Phragmén and Lindelöf [5, p. 183], we deduce from the last two inequalities that

$$\tau_F(\theta) \leq A \cos \theta + B \sin \theta, \tag{2.10}$$

where A and B are the solutions of

$$A \cos \alpha + B \sin \alpha = (\Omega_1 - \pi) \sin \alpha,$$

$$A \cos \beta - B \sin \beta = (\Omega_2 - \pi) \sin \beta.$$

That is:

$$A = \frac{\Omega_1 + \Omega_2 - 2\pi}{\cot \alpha + \cot \beta}, \quad B = \frac{(\Omega_1 - \pi) \cot \beta - (\Omega_2 - \pi) \cot \alpha}{\cot \alpha + \cot \beta}. \tag{2.10'}$$

Hence, from (2.9), (2.10), and (2.10′), we get

$$\tau_G(\theta) \leq A \cos \theta + B \sin \theta + \pi |\sin \theta| \qquad (-\beta \leq \theta \leq \alpha). \tag{2.11}$$

The assertion of the lemma now follows from (2.11) if we note that the right side of (2.11) is a continuous negative function for $-\beta^* < \theta < \alpha^*$. (We also make use of the well-known fact that an analytic function of exponential type satisfying (2.11) also satisfies

$$G(u) = O(\exp[|u| (A \cos \theta + (B + \pi) \sin \theta + \epsilon)])$$

uniformly in the angle.)

Finally, we shall need the following lemma which is a generalization of a lemma due to Soula [3, p. 38]. It is stated in a somewhat more general form than is needed for our purpose here; however, the lemma in this form is required for the completion of Soula's theorem (Theorem S) mentioned in § 1.

LEMMA 5. *Let* $h(z)$ *be an analytic function in the infinite sector* $|z| \geq R_0$, $|\arg z| \leq \alpha < \pi/2$, *where it satisfies*

(i) $$\lim_{n=\infty} \frac{\Re\{h(n)\}}{n} = 0 \qquad (n \text{ being an integer});$$

(ii) *there exists a nonnegative, continuous, and increasing function* $\delta(\theta)$, $0 \leq \theta \leq \alpha$, *with* $\delta(0) = 0$, *such that*

$$\Re\{h(re^{i\theta})\} \leq [\delta(|\theta|)\,|\sin\theta| + \epsilon]r,$$

for any $\epsilon > 0$, $r \geq r_0(\epsilon)$ *large enough, and* $|\theta| \leq \alpha$. *Then*

$$\lim_{|z|\to\infty} \left(\frac{h(z)}{z}\right) = 0, \tag{2.12}$$

uniformly in any sector $|\arg z| \leq \alpha' < \alpha$, $|z| \geq R_0$.

Proof. We shall make use of the well-known inequality

$$|f(z) - f(0)| \leq \frac{2|z|(A(R) - A(0))}{R - |z|}, \tag{2.13}$$

where $f(z)$ is analytic for $|z| \leq R$, and

$$A(R) = \max_{|z|=R} \Re\{f(z)\}.$$

We shall first show that

$$\lim_{n=\infty} [h(n+\zeta) - h(n)] = 0, \tag{2.14}$$

uniformly for ζ in any bounded set. Indeed, choose η such that $0 < \eta < \alpha$, and let $|\zeta| \leq C$. Let $n + z = re^{i\theta}$ be a point inside the circle of radius $R = n \sin\eta$ and center $z = n$ inscribed inside the angle $|\arg z| \leq \eta$. On account of (ii), taking $\epsilon = \delta(\eta)\sin\eta$ and n large enough, we have

$$(2.15)\qquad \Re\{h(re^{i\theta})\} \le 2\delta(\eta)\sin\eta\, r \le 2\delta(\eta)\sin\eta(1+\sin\eta)\,n$$

$$\le (4\delta(\eta)\sin\eta)n.$$

Applying (2.13) to $f(z)=h(n+z)$ for $z=\zeta$, $R=n\sin\eta$, and using (2.15), we get

$$|h(n+\zeta)-h(n)| \le \frac{2|\zeta|[(4\delta(\eta)\sin\eta)n+|\Re\{h(n)\}|]}{n\sin\eta-|\zeta|}$$

$$\le \frac{8C\sin\eta\,\delta(\eta)}{\sin\eta-C/n}+\frac{2C}{\sin\eta-C/n}\left|\Re\left\{\frac{h(n)}{n}\right\}\right|.$$

Sending n to infinity and using (i), we find that

$$\limsup |h(n+\zeta)-h(n)| \le 8C\,\delta(\eta),$$

uniformly for $|\zeta|\le C$. Letting η tend to zero, and recalling that $\delta(0)=0$, we arrive at (2.14). In what follows we shall need only the weaker result

$$(2.16)\qquad \lim_{x=\infty}\left(\frac{h(x)}{x}\right)=0 \qquad (x \text{ real}),$$

which follows in an obvious way from (2.14).

We next show that $h(z)/z$ is bounded in every sector

$$|\arg z| \le \alpha' < \alpha,\ |z| \ge R_0.$$

This will be established if we prove that $h(z)/z$ is bounded uniformly in the circles C_ξ: $|z-\xi|\le \xi\sin\alpha'$ (ξ large enough). Now, as before, for ξ positive and large enough, and for z such that $|z|\le\xi\sin\alpha$, we have

$$\Re\{h(\xi+z)\} \le (4\delta(\alpha)\sin\alpha)\xi.$$

Applying (2.13), for $|z|\le\xi\sin\alpha'$ we find

$$|h(\xi+z)-h(\xi)| \le \frac{8\delta(\alpha)\sin\alpha'\sin\alpha}{\sin\alpha-\sin\alpha'}\,\xi+\frac{2\sin\alpha'}{\sin\alpha-\sin\alpha'}\,|\Re\{h(\xi)\}|,$$

from which we get that

$$(2.17)\qquad |h(\xi+z)| \le C_1\xi + C_2\,|h(\xi)|,$$

where C_1 and C_2 are constants depending only on α and α'. We have only to divide (2.17) by $\xi + z$ and use (2.16) in order to obtain the uniform boundedness of $h(\xi + z)/(\xi + z)$ for $|z| \leq \xi \sin \alpha'$ $(\xi \longrightarrow \infty)$, and therefore the boundedness of $h(z)/z$ in any interior sector. To complete the lemma we apply a well-known result of Phragmén and Lindelöf by which the boundedness and (2.16) imply (2.12). We also note that by successive applications of the last lemma it follows that the result still holds if the angle containing the positive axis is not supposed to be symmetric.

3. **The main theorem.** We pass now to our main result:

THEOREM 1. *Let $f(z)$ be an analytic function in the whole plane cut along the line $1 \leq x < \infty$. Suppose that the coefficients of its Taylor expansion* (1.1) *are different from zero and satisfy* (1.2). *Let $f_{-1}(z)$ be the "inverse" series defined by* (1.1′), *and denote by $\rho_{-1}(\theta)$ its star-function* (Def. §2). *Then there exist two numbers α, β, with $0 \leq \alpha \leq \pi/2$ and $0 \leq \beta \leq \pi/2$, such that for any $0 < \theta < 2\pi$ we have*

$$\rho_{-1}(\theta) = \min\left\{e^{(\tan \alpha)\theta},\ e^{(\tan \beta)(2\pi-\theta)}\right\}, \tag{3.1}$$

and (*trivially*) $\rho_{-1}(0) = 1$.

The theorem states, in other words, that the star of holomorphy of $f_{-1}(z)$ consists in general of the two logarithmic spirals:

$$\rho_{-1}(\theta) = \exp[(\tan \alpha)\theta] \text{ for } 0 \leq \theta \leq \psi,$$

and

$$\rho_{-1}(\theta) = \exp[(\tan \beta)(2\pi - \theta)] \text{ for } \psi \leq \theta \leq 2\pi,$$

where ψ is given by

$$\psi = \frac{2\pi \tan \beta}{\tan \alpha + \tan \beta}. \tag{3.1′}$$

(We shall treat $\theta = 0$ separately in order to account for the case where either α or β is equal to $\pi/2$, in which case we agree that the product of a positive number and infinity is infinity.)

It is possible to distinguish the following cases which correspond to limiting values of α and β: (i) $\alpha = 0$ or $\beta = 0$. In this case we have $\rho_{-1}(\theta) \equiv 1$, and the unit circle is a cut for $f_{-1}(z)$; this is a particular case of Theorem S.

(ii) $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$. In this case the star of holomorphy of $f_{-1}(z)$ is the domain $D_{\alpha\beta}$ (introduced in Lemma 3), whose boundary consists of the two spirals (3.1′); this domain is also the region of existence of $f_{-1}(z)$.

(iii) $0 < \alpha < \pi/2$ while $\beta = \pi/2$. In this case all the points of the spiral $\rho = \exp[(\tan\,\alpha)\theta]$ $(0 \le \theta < 2\pi)$ are singularities. (However, this does not exclude the possibility of analytic continuation through the segment $1 < x < \exp[(\tan\,\alpha)2\pi]$.)

(iv) $\alpha = \pi/2$ while $0 < \beta < \pi/2$. This case is similar to the preceding one; the only difference is that the roles of α and β are interchanged.

(v) $\alpha = \beta = \pi/2$. In this case $f_{-1}(z)$ has the properties of $f(z)$, and is analytic in the whole plane cut along $1 \le x < \infty$.

Proof of Theorem 1. Since $f(z)$ satisfies the conditions of Theorem S, it follows that our results will go beyond those of Soula only when $z = 1$ is the only singularity of $f_{-1}(z)$ on the unit circle, and when it is, furthermore, "without contact". We shall assume in what follows that this is indeed the case. Now, the proof of the theorem is somewhat long and will be divided into two parts.

Part I. Let us define α, β, with $0 < \alpha \le \pi/2$ and $0 < \beta \le \pi/2$, by the relations

$$(3.2)\qquad \begin{cases} \tan\alpha = \limsup\limits_{h \to +0} \dfrac{\log \rho_{-1}(h)}{h}, \\ \tan\beta = \limsup\limits_{h \to +0} \dfrac{\log \rho_{-1}(-h)}{h}. \end{cases}$$

We shall establish in this part of the proof that:

(a) The interval $0 \le \theta \le 2\pi$ can be divided into two disjoint intervals I_1 and I_2, where I_1 is the interval $0 \le \theta < \omega$ or $0 \le \theta \le \omega$, and I_2 is the interval $\omega \le \theta \le 2\pi$ or $\omega < \theta \le 2\pi$ (one of the intervals can consist of a single point), such that $\rho_{-1}(\theta)$ is increasing in I_1 and decreasing in I_2, and such that furthermore, the equality $\rho_{-1}(\theta_1) = \rho_{-1}(\theta_2)$ for $\theta_1 < \theta_2$, θ_1 and θ_2 both in to the same interval, can hold only if $\rho_{-1}(\theta) \equiv \infty$ for $\theta_1 \le \theta \le \theta_2$.

(b) The following inequalities hold:

$$(3.3)\qquad \begin{cases} \log \rho_{-1}(\theta) \geq (\tan \alpha)\theta \ \text{ in } I_1, \\ \\ \log \rho_{-1}(\theta) \geq (\tan \beta)(2\pi - \theta) \ \text{ in } I_2. \end{cases}$$

Now, both (a) and (b) are consequences of the following inequality which we shall establish later:

$$(3.4)\qquad \rho_{-1}(\theta) \geq \min \left\{ \rho_{-1}(\theta_1)\, \rho_{-1}(\theta - \theta_1),\ \rho_{-1}(\theta_2)\, \rho_{-1}(\theta - \theta_2) \right\},$$

where $0 < \theta_1 < \theta < \theta_2 < 2\pi$. Indeed, it is easily seen that (a) will be proved if we can show that the minimum of $\rho_{-1}(\theta)$ in an interval $0 < \theta_1 \leq \theta \leq \theta_2 < 2\pi$ is attained only at one of the end-points. To establish this let us assume, by way of contradiction, that the minimum is attained at a certain inner point $\overline{\theta}$. Using now (3.4) for $\theta = \overline{\theta}$, and using the fact that both $\rho_{-1}(\overline{\theta} - \theta_1)$ and $\rho_{-1}(\theta_2 - \overline{\theta})$ are greater than one, we get the absurdity

$$\rho_{-1}(\overline{\theta}) > \min \left\{ \rho_{-1}(\theta_1),\ \rho_{-1}(\theta_2) \right\} \geq \min_{\theta_1 \leq \theta \leq \theta_2} \rho_{-1}(\theta) = \rho_{-1}(\overline{\theta}).$$

This proves (a). We note also that from (a) follows that the points where $\rho_{-1}(\theta) = \infty$ constitute one interval.

Let us now establish (b). We shall limit ourselves to proving the first inequality; the second will follow in a similar fashion. It is clear from the above that it is enough to prove the first inequality (3.3) for θ which is interior to I_1 and such that $\rho_{-1}(x)$ is finite for $0 \leq x \leq \theta + \epsilon$, ϵ being a small positive number depending on θ (if no such θ exists, then $\rho_{-1}(\theta) \equiv \infty$ for $\theta \in I_1$, $\theta \neq 0$, and the inequality is satisfied with $\alpha = \pi/2$). Let now $h > 0$ be such that $\theta + 2h \in I_1$, and let us apply (3.4) to $\theta_1 (=) \theta$, $\theta (=) \theta + h$ and $\theta_2 (=) \theta + 2h$. We get

$$\rho_{-1}(\theta + h) \geq \min \left\{ \rho_{-1}(\theta)\, \rho_{-1}(h),\ \rho_{-1}(\theta + 2h)\, \rho_{-1}(-h) \right\}.$$

Since $\rho_{-1}(-h) > 1$ and $\rho_{-1}(\theta + 2h) \geq \rho_{-1}(\theta + h)$, we obviously must have

$$\rho_{-1}(\theta + h) \geq \rho_{-1}(\theta)\, \rho_{-1}(h).$$

Passing to logarithms, from the last inequality we get

$$(3.5)\qquad \limsup_{h \to +0} \frac{\log \rho_{-1}(\theta + h) - \log \rho_{-1}(\theta)}{h} \geq \tan \alpha,$$

where α was defined by (3.2). But, since $\log \rho_{-1}(\theta)$ is an increasing function in I_1, its derivative exists almost everywhere in I_1^*, the largest sub-interval of I_1 where $\rho_{-1}(\theta)$ is finite. Making use of (3.5), we see that almost everywhere in I_1^*, we have

$$[\log \rho_{-1}(\theta)]' \geq \tan \alpha. \tag{3.6}$$

We now employ a simple inequality applicable to any nondecreasing function $g(x)$ [5, pp. 361, 373]:

$$g(b) - g(a) \geq \int_a^b g'(x)\,dx,$$

where the integral is taken in the Lebesgue sense. When applying this to the function $\log \rho_{-1}(\theta) - (\tan \alpha)\theta$ in $[\theta_1, \theta_2]$, using (3.6), we conclude that $\log \rho_{-1}(\theta) - (\tan \alpha)\theta$ is nondecreasing in I_1^*, and *a fortiori* in the interval I_1. The desired first inequality (3.3) now follows if we note that $\log \rho_{-1}(\theta) - (\tan \alpha)\theta$ vanishes for $\theta = 0$.

We still have to establish (3.4). Let θ_1 and θ_2 satisfy $0 < \theta_1 < \theta_2 < 2\pi$, and let ϵ be an arbitrary positive number. By Lemma 2 there exist two analytic functions $g_1(z)$ and $g_2(z)$, with star-functions $\rho_1^g(\theta)$ and $\rho_2^g(\theta)$, such that

$$f_{-1}(z) = g_1(z) + g_2(z),$$

and such that the corresponding star-functions satisfy the relations (2.3). Now the Hadamard multiplication of the two "inverse" functions $f(z)$ and $f_{-1}(z)$ is the "unit" function $1/(1-z)$, so that if we define

$$h_1(z) = [f(z), g_1(z)] \text{ and } h_2(z) = [f(z), g_2(z)], \tag{3.7}$$

we get

$$\frac{1}{1-z} = [f(z), f_{-1}(z)] = [f(z), g_1(z) + g_2(z)]$$

$$= [f(z), g_1(z)] + [f(z), g_2(z)] = h_1(z) + h_2(z).$$

Let us now denote by $\rho_1^h(\theta)$ and $\rho_2^h(\theta)$ the star-functions of $h_1(z)$ and $h_2(z)$. It follows again from (3.7) and Hadamard's multiplication theorem, since $f(z)$

is analytic in the whole plane cut along $1 \leq x < \infty$, that

$$\rho_1^h(\theta) \geq \rho_1^g(\theta) \text{ and } \rho_2^h(\theta) \geq \rho_2^g(\theta). \tag{3.8}$$

Using properties (2.3) of $\rho_i^g(\theta)$, we see that

$$\begin{cases} \rho_1^h(\theta) \equiv \infty \text{ for } \theta_2 < \theta < \theta_1 + 2\pi, \\ \rho_2^h(\theta) \equiv \infty \text{ for } \theta_1 < \theta < \theta_2. \end{cases} \tag{3.9}$$

(In other words, $h_1(z)$ and $h_2(z)$ are analytic in the angles $\theta_2 < \arg z < \theta_1 + 2\pi$ and $\theta_1 < \arg z < \theta_2$ respectively.) We also find that

$$\rho_i^h(\theta_j) \geq \rho_{-1}(\theta_j) - \epsilon \qquad (i, j = 1, 2). \tag{3.10}$$

Moreover, since $h_1(z)$ and $h_2(z)$ add to $1/(1-z)$, a function having its only singularity at $z = 1$, we conclude that

$$\rho_1^h(\theta) \equiv \infty \text{ for } \theta_1 < \theta < \theta_2. \tag{3.11}$$

Let us consider now the Hadamard multiplication of $f_{-1}(z)$ and $h_1(z)$. Clearly, we have

$$\begin{aligned} (3.12) \quad [f_{-1}(z), h_1(z)] &= [f_{-1}(z), [f(z), g_1(z)]] \\ &= [[f_{-1}(z), f(z)], g_1(z)] = \left[\frac{1}{1-z}, g_1(z)\right] = g_1(z). \end{aligned}$$

Using once more Hadamard's theorem, taking into account (3.9)–(3.12) for $\rho_1^h(\theta)$, and also remembering that $\rho_1^g(\theta) = \rho_{-1}(\theta)$ for $\theta_1 < \theta < \theta_2$, we obtain

$$\begin{aligned} \rho_{-1}(\theta) = \rho_1^g(\theta) &\geq \min\left\{\rho_1^h(\theta_1)\rho_{-1}(\theta - \theta_1), \rho_1^h(\theta_2)\rho_{-1}(\theta - \theta_2)\right\} \\ &\geq \min\left\{(\rho_{-1}(\theta_1) - \epsilon)\rho_{-1}(\theta - \theta_1), (\rho_{-1}(\theta_2) - \epsilon)\rho_{-1}(\theta - \theta_2)\right\}. \end{aligned}$$

We now have only to send ϵ in the last inequality to zero in order to arrive at the desired result (3.4).

Part II. It follows from Part I that if α and β are defined by (3.2), then

$$\log \rho_{-1}(\theta) \geq \min\{(\tan \alpha)\theta, (\tan \beta)(2\pi - \theta)\}. \tag{3.13}$$

Theorem 1 will, therefore, be established if we show that (3.13) is actually an equality. Now, we may assume that α and β are not both equal to $\pi/2$. For, if this were the case, then we would have $\rho_{-1}(\theta) \equiv \infty$ for $0 < \theta < 2\pi$, and the theorem would be proved. In what follows we shall assume that the theorem is false and that (3.13) is a strict inequality for a certain $\theta = \theta_0$. This we shall show will lead to a contradiction. Now, from the lower semicontinuity of $\rho_{-1}(\theta)$ it follows that if we do not have everywhere equality in (3.13), then for infinitely many points we have a strict inequality. This allows us to assume that θ_0 differs from ψ, where ψ is defined by (3.1′). There is also no loss of generality in assuming $0 < \theta_0 < \psi$, since otherwise we have only to replace $f(z)$ and $f_{-1}(z)$ by $\overline{f(\overline{z})}$ and $\overline{f_{-1}(\overline{z})}$, respectively. We first note that in the interval $[0, \psi)$

$$(\tan \alpha)\theta < (\tan \beta)(2\pi - \theta),$$

and hence there exists $\delta > 0$ small enough that

$$\log \rho_{-1}(\theta_0) \geq (\tan \alpha)\theta_0 + \delta.$$

We shall now define the domain $D = D(\alpha, \beta, \theta_0, \delta)$ as the set of points $z = re^{i\theta}$ satisfying

$$(3.14) \qquad \begin{cases} \log r < (\tan \alpha)\theta \ \text{ for } \ 0 \leq \theta \leq \theta_0, \\ \log r < \min\{(\tan \alpha)\theta + \delta, (\tan \beta)(2\pi - \theta)\} \ \text{ for } \ \theta_0 < \theta < 2\pi. \end{cases}$$

Let us denote by $R(\theta)$ the star-function corresponding to the boundary of D. We claim that

$$(3.15) \qquad R(\theta) \leq \rho_{-1}(\theta).$$

Indeed, this is obvious from the definition of D and from (3.13) so long as $0 \leq \theta \leq \theta_0$. If $\theta_0 < \theta < 2\pi$, then we have to distinguish between two cases:

(i) θ belongs to the interval I_1 introduced in Part I; then both θ_0 and $\theta(\theta_0 < \theta)$ belong to I_1, and since we have established before that $\log \rho_{-1}(\theta) - (\tan \alpha)\theta$ is an increasing function in I_1, we have

$$\log \rho_{-1}(\theta) - (\tan \alpha)\theta \geq \log \rho_{-1}(\theta_0) - (\tan \alpha)\theta_0 \geq \delta,$$

so that

$$\log \rho_{-1}(\theta) \geq (\tan \alpha)\,\theta + \delta \geq \log R(\theta).$$

(ii) $\theta \in I_2$. In this case we see from (3.14) and (3.3) that

$$R(\theta) \leq \exp[(\tan \beta)(2\pi - \theta)] \leq \rho_{-1}(\theta).$$

This establishes (3.15).

Let now β', θ', δ' be such that $0 < \beta' < \beta$, $\theta_0 < \theta' < \psi$ and $0 < \delta' < \delta$, where β', θ', and δ' are chosen so near to β, θ_0, and zero, respectively, that

$$(3.16) \qquad (\tan \beta')(2\pi - \theta') > (\tan \alpha)\theta' + \delta'.$$

Let ϵ and ϵ_1 be two small positive numbers, and let

$$D^* = D^*(\alpha - \epsilon, \beta', \theta', \delta', \epsilon_1)$$

be the domain defined by:

$$(3.17) \begin{cases} \log r < [\tan(\alpha - \epsilon)]\theta \text{ for } 0 \leq \theta \leq \theta', \\ \log r < \min\{[\tan(\alpha - \epsilon)]\theta + \delta', (\tan \beta')(2\pi - \theta)\} \text{ for } \theta' \leq \theta < 2\pi, \\ \log|z - 1| > \epsilon_1 . \end{cases}$$

Because of (3.15) it is clear that $f_{-1}(z)$ is analytic in the closure of D^*. Let C^* be the boundary of D^*, and set

$$(3.18) \qquad G(u) = \frac{1}{2\pi i} \int_{C^*} f_{-1}(z)\, e^{-(\log z)u} \frac{dz}{z},$$

where the determination of $\log z$ is chosen in the following manner: let Φ be the argument corresponding to the vertex V of C^* where the two spirals

$$\log r = [\tan(\alpha - \epsilon)]\theta + \delta' \text{ and } \log r = (\tan \beta)(2\pi - \theta)$$

intersect, that is

$$(3.19) \qquad \Phi = \frac{2\pi \tan \beta' - \delta'}{\tan(\alpha - \epsilon) + \tan \beta'}$$

Then we choose $-(2\pi - \Phi) \leq \arg z \leq \Phi$. It is readily seen that $G(u)$ is an entire function of exponential type. Furthermore, for ϵ_1 small enough, $G(u)$ is independent of ϵ_1. That is, if we change only ϵ_1 and leave the other parameters in the definition of C^* fixed, then the value of the integral (3.18) will remain unchanged. This follows easily from Cauchy's theorem if we note that the curves so obtained all have the same vertex V. Finally, if n is a nonnegative integer, then from (3.18) and (1.1′) we obtain

$$G(n) = \frac{1}{a_n}.$$

We shall now study the growth of $G(u)$ more closely. For this purpose let us put

$$u = |u| e^{i\phi}$$

and

$$\Lambda_\phi(z) = (\arg z) \sin \phi - (\log |z|) \cos \phi . \tag{3.20}$$

From (3.18) we get

$$\log |G(u)| \leq |u| \max_{z \in C^*} \Lambda_\phi(z) + K, \tag{3.21}$$

where K is a constant.

Now, the curve C^* is composed of five analytic arcs,

$$C^* = \sum_{i=1}^{5} C_i,$$

on each of which we shall evaluate the maximum of $\Lambda_\phi(z)$.

(i) On C_1, the arc of circle

$$|z-1| = \epsilon_1, \ \eta_1 \leq \arg z \leq \eta_2,$$

since η_1 and η_2 tend to zero with ϵ_1, we have

$$\begin{aligned} |\Lambda_\phi(z)| &\leq |\arg z| \, |\sin \phi| + |\log |z|| \, |\cos \phi| \\ &\leq 2\pi \max(\eta_1, \eta_2) + \log(1+\epsilon_1) = \eta_3, \end{aligned} \tag{3.22}$$

where η_3 tends to zero with ϵ_1.

(ii) On C_2, the spiral arc

$$\log |z| = (\arg z) \tan(\alpha - \epsilon) \text{ for } \eta_2 \leq \arg z \leq \theta',$$

we have

$$\Lambda_\phi(z) = (\arg z) \sin \phi\, (1 - \cot \phi \tan(\alpha - \epsilon)). \tag{3.23}$$

Hence

$$\max_{z \in C_2} \Lambda_\phi(z) \leq 0 \text{ for } -\frac{\pi}{2} \leq \phi \leq 0. \tag{3.24}$$

On the other hand, if $0 < \phi^* < \pi/2$ is defined by

$$\tan \phi^* = \tan(\alpha - \epsilon) + \frac{\delta'}{\Phi} = \tan(\alpha - \epsilon) + \mu^*, \tag{3.25}$$

we get from (3.23) and (3.25), for $0 \leq \phi \leq \phi^*$,

$$\max_{z \in C_2} \Lambda_\phi(z) \leq \theta' \sin \phi\,(1 - \cot \phi^* \tan(\alpha - \epsilon)$$
$$= \theta' \sin \phi \left(1 - \frac{\tan(\alpha - \epsilon)}{\tan(\alpha - \epsilon) + \mu^*}\right) = \Omega^* \sin \phi, \tag{3.26}$$

where we put

$$\Omega^* = \frac{\theta' \mu^*}{\tan(\alpha - \epsilon) + \mu^*}. \tag{3.27}$$

(iii) On C_3, the segment

$$\exp[i\theta' t], \text{ where } \theta' \tan(\alpha - \epsilon) \leq t \leq \theta' \tan(\alpha - \epsilon) + \delta'$$

we obtain the same inequalities as in the preceding case.

(iv) On C_4, the spiral

$$\log |z| = \delta' + (\arg z) \tan(\alpha - \epsilon),\ \theta' \leq \arg z \leq \Phi,$$

we have

$$\Lambda_\phi(z) = (\arg z)\sin\phi\left(1 - \cot\phi\left[\tan(\alpha-\epsilon) + \frac{\delta'}{\arg z}\right]\right),$$

from which, using (3.25), we get

$$\max_{z\in C_4} \Lambda_\phi(z) \le 0 \quad \text{for } -\frac{\pi}{2} \le \phi \le \phi^*. \tag{3.28}$$

(v) Finally, on C_5, the spiral

$$\log|z| = -(\arg z)\tan\beta', \; -(2\pi - \Phi) \le \arg z \le -\eta_1,$$

we have

$$\Lambda_\phi(z) = (\arg z)\sin\phi\,(1 + \cot\phi\tan\beta'),$$

from which it follows that

$$\max_{z\in C_5} \Lambda_\phi(z) \le 0 \quad \text{for } -\beta' \le \phi \le \pi/2. \tag{3.29}$$

By combining the inequalities (3.22)–(3.29), we conclude that

$$\begin{cases} \max\limits_{z\in C^*} \Lambda_\phi(z) \le \max(\eta_3, \Omega^*\sin\phi) & \text{for } 0 \le \phi \le \phi^*, \\ \max\limits_{z\in C^*} \Lambda_\phi(z) \le \eta_3 & \text{for } -\beta' \le \phi \le 0. \end{cases} \tag{3.30}$$

But η_3 tends to zero with ϵ_1, while Ω^* and $G(z)$ are independent of ϵ_1. This, with (3.30) and (3.21), implies that the type-function (2.6) of $G(u)$ satisfies

$$\begin{cases} \tau_G(\phi) \le \Omega^*\sin\phi & \text{for } 0 \le \phi \le \phi^*, \\ \tau_G(\phi) \le 0 & \text{for } -\beta' \le \phi \le 0. \end{cases} \tag{3.31}$$

Now, let γ be a number such that $\max\{\alpha-\epsilon, \beta'\} < \gamma < \pi/2$. Since $f(z)$ is analytic in the whole plane cut along the half-line $1 \le x < \infty$, there exists by Lemma 3 an analytic function $F(u)$ of exponential type in the angle $|\arg u| \le \gamma$, such that

$$F(n) = a_n \qquad (n = 0, 1, \cdots), \tag{3.32}$$

and

$$\tau_F(\phi) \leq 0 \qquad \text{for } |\phi| \leq \gamma. \tag{3.33}$$

Let us put

$$H(u) = F(u)\,G(u) - 1. \tag{3.34}$$

Then $H(u)$ is analytic and of exponential type in the angle $-\beta' \leq \arg u \leq \phi^*$. Furthermore, because of (3.31)–(3.34), the following relations hold:

(i) $\quad H(n) = 0 \qquad (n = 0, 1, \cdots),$

(ii) $\quad \tau_H(\phi) \leq \Omega^* \sin\phi \;\text{ for }\; 0 \leq \phi \leq \phi^*,$

and

$$\tau_H(\phi) \leq 0 \;\text{ for }\; -\beta' \leq \phi \leq 0.$$

Thus $H(u)$ satisfies the conditions of Lemma 4 with $\Omega_1 = \Omega^*$, $\Omega_2 = 0$, $\alpha = \phi^*$ and $\beta = \beta'$. Applying this lemma, we conclude that $H(u)$ tends uniformly to zero in any angle interior to the angle $A_{\alpha^*\beta^*}$: $-\beta^* < \arg u < \alpha^*$, where α^* and β^* $(0 < \alpha^* < \pi/2,\ 0 < \beta^* < \pi/2)$ are defined by

$$\begin{cases} \tan\alpha^* = \dfrac{2\pi - \Omega^*}{\Omega^* \cot\beta' + 2\pi\cot\phi^*}, \\[2ex] \tan\beta^* = \dfrac{2\pi - \Omega^*}{(2\pi - \Omega^*)\cot\beta'} = \tan\beta'. \end{cases} \tag{3.35}$$

From the last property, and from (3.34), it follows in particular that $F(u)$ can have only a finite number of zeros in any angle interior to $A_{\alpha^*\beta^*}$. Let now $\bar\alpha$, $\bar\beta$ satisfy $0 < \bar\alpha < \alpha^*$ and $\theta < \bar\beta < \beta^*$, and let R_0 be large enough so that $F(u) \neq 0$ in the sector

$$\Sigma_{\bar\alpha,\bar\beta,R_0} : \; -\bar\beta \leq \arg u \leq \bar\alpha, \; |u| \geq R_0.$$

In this sector any determination of $\log F(u)$ is analytic and satisfies, because of (3.32), (3.33), and (1.2), the conditions of Lemma 5. Hence

(3.36) $$\lim \frac{\log |F(u)|}{|u|} = 0,$$

uniformly in any sector interior to $\Sigma_{\bar{\alpha},\bar{\beta},R_0}$. Let us put $F_{-1}(u) = 1/F(u)$. Then $F_{-1}(u)$ is analytic in $\Sigma_{\bar{\alpha},\bar{\beta},R_0}$ and satisfies

(i) $$F_{-1}(n) = \frac{1}{a_n} \text{ for } n > R_0,$$

(ii) $$\lim_{|u|\to\infty} \frac{\log |F_{-1}(u)|}{|u|} = 0,$$

uniformly in any sector interior to $\Sigma_{\bar{\alpha},\bar{\beta},R_0}$. We can now apply Lemma 3′, and conclude that

$$\sum_{n > R_0} F_{-1}(n) z^n = \sum_{n > R_0} \frac{z^n}{a_n}$$

is analytic in the domain $D_{\bar{\alpha},\bar{\beta}}$ bounded by the two spirals

$$r = \exp[(\tan \bar{\alpha})\theta] \text{ for } 0 \le \theta \le \bar{\psi},$$

and

$$r = \exp[(\tan \bar{\beta})(2\pi - \theta)] \text{ for } \bar{\psi} \le \theta \le 2\pi,$$

where $\bar{\psi}$ is the expression (3.1′) with bars. Obviously the function $f_{-1}(z)$ will be analytic in the same region. Moreover, since $\bar{\alpha}$ and $\bar{\beta}$ can be chosen as near as we please to α^* and β^*, respectively, it follows that $f_{-1}(z)$ is analytic in D_{α^*,β^*}. This gives us, in particular, the inequality

$$\log \rho_{-1}(\theta) \ge (\tan \alpha^*)\theta \text{ for } 0 \le \theta \le \psi^*,$$

where ψ^* is the expression (3.1′) with asterisks. The last inequality and (3.2) lead to the inequality

(3.37) $$\tan \alpha \ge \tan \alpha^*.$$

Now, from (3.27), (3.35), and (3.25) we find that

$$(3.38)\quad \tan \alpha^* = \frac{2\pi - \Omega^*}{\Omega^* \cot \beta' + 2\pi \cot \phi^*}$$

$$= \frac{2\pi - \dfrac{\theta' \mu^*}{\tan(\alpha - \epsilon) + \mu^*}}{\dfrac{\theta' \mu^*}{\tan(\alpha - \epsilon) + \mu^*} \cot \beta' + \dfrac{2\pi}{\tan(\alpha - \epsilon) + \mu^*}}$$

$$= \frac{2\pi \tan(\alpha - \epsilon) + \mu^* [2\pi - \theta']}{2\pi + \theta' \mu^* \cot \beta'}$$

$$= \tan(\alpha - \epsilon) + \frac{\mu^* [2\pi - \theta' - \theta' \tan(\alpha - \epsilon) \cot \beta']}{2\pi + \theta' \mu^* \cot \beta'}$$

Combining (3.37) with (3.38), and sending ϵ to zero, we get

$$(3.39)\qquad \tan \alpha \geq \tan \alpha + \mu_0 \frac{2\pi - \theta' \dfrac{\tan \alpha + \tan \beta'}{\tan \beta'}}{2\pi + \theta' \mu_0 \cot \beta'},$$

where, by (3.25), we have $\mu_0 = \delta'/\Phi_0 > 0$, and where Φ_0 is given by

$$\Phi_0 = \frac{2\pi \tan \beta' - \delta'}{\tan \alpha + \tan \beta'}.$$

Since from (3.16) we also have

$$\theta'(\tan \alpha + \tan \beta') < 2\pi \tan \beta',$$

we find that the last term in (3.39) is positive. This, however, leads us to $\tan \alpha > \tan \alpha$ $(0 < \alpha < \pi/2)$, an absurdity. Thus the assumption that (3.13) is not always an equality leads to a contradiction. This establishes the theorem.

4. **Some further results.** In Theorem 1 the existence only of the constants α and β was proved. More careful analysis leads to the following more explicit result concerning the constants. Let γ be a number such that $0 < \gamma < \pi/2$, and let $F_\gamma(u)$ be any interpolation function of $f(z)$ defined in the angle $|\arg u| \leq \gamma$. (Thus $F_\gamma(u)$ is of exponential type in the angle, verifying there (3.32) and (3.33)). Then we have:

(i) The unit circle is a cut for $f_{-1}(z)$ if, and only if, the positive axis is a direction of condensation of zeros for $F_\gamma(u)$. (That is, any angle $|\arg u| \leq \epsilon$ contains infinitely many zeros of $F_\gamma(u)$.)

(ii) If the positive axis is not a direction of condensation of zeros, then let the two numbers $\phi^+(\gamma)$ and $\phi^-(\gamma)$ be defined in the following way: $0 < \phi^+ \leq \gamma$ is such that $F_\gamma(u)$ has only a finite number of zeros in any angle $0 \leq \arg u \leq \phi^+ - \epsilon$, and infinitely many zeros in any angle $0 \leq \arg u \leq \phi^+ + \epsilon$. (We put $\phi^+ = \gamma$ if $F_\gamma(u)$ has a finite number of zeros in any angle $0 \leq \arg u \leq \gamma - \epsilon$.) Similarly we define ϕ^- $(0 < \phi^- \leq \gamma)$ by the property that $F_\gamma(u)$ has a finite number of zeros in any angle $-\phi^- + \epsilon \leq \arg u \leq 0$ and infinitely many zeros in $-\phi^- - \epsilon \leq \arg u \leq 0$. Then, if $\phi^+ < \gamma$, the constant α of Theorem 1 is the number ϕ^+ just defined. Similarly, if $\phi^- < \gamma$ we have $\beta = \phi^-$. Furthermore, if γ_n is an increasing sequence such that $\gamma_n \longrightarrow \pi/2$, then we always have

$$\alpha = \lim_{n=\infty} \phi^+(\gamma_n), \quad \beta = \lim_{n=\infty} \phi^-(\gamma_n).$$

We shall omit here the proof of this result.

In Theorem 1 it was assumed that $f(z)$ is analytic in the whole plane cut along the line $1 \leq x < \infty$. Suppose now that we know only that $f(z)$ has the point $z = 1$ as its only singularity on the unit circle, and that it is, furthermore, semi-isolated. That is, there exists $\rho > 1$ such that $f(z)$ is analytic in the region bounded by the circle $|z| = \rho$ and the segment $1 \leq x \leq \rho$. It was shown by Pólya [4, p. 738] that in this case the singularities of $f(z)$ can be "separated" in the following way:

$$f(z) = f^*(z) + f^{**}(z) = \sum a_n^* a^n + \sum a_n^{**} z^n,$$

where $f^*(z)$ is analytic in the whole plane cut along $1 \leq x < \infty$ while $f^{**}(z)$ is analytic in the circle $|z| < \rho$. Obviously, we have $a_n = a_n^* + a_n^{**}$ with

$$\limsup |a_n^{**}|^{1/n} = \frac{1}{\rho}.$$

Now, if the sequence $\{a_n\}$ satisfies (1.2), it is easily seen that $\{a_n^*\}$ also satisfies (1.2). There is also no loss of generality in assuming $a_n^* \neq 0$. From

$$\frac{1}{a_n} = \frac{1}{a_n^* + a_n^{**}} = \frac{1}{a_n^*} - \frac{a_n^{**}}{a_n a_n^*},$$

it follows that

$$\limsup \left| \frac{1}{a_n} - \frac{1}{a_n^*} \right|^{1/n} \leq \rho^{-1} .$$

Hence, if we put

$$f_{-1}^*(z) = \sum \frac{1}{a_n^*} z^n ,$$

we find that $f_{-1}(z) - f_{-1}^*(z)$ is analytic in $|z| < \rho$. Applying Theorem 1 to $f_{-1}^*(z)$, we arrive at the following conclusion:

THEOREM 2. *Let $f(z)$ be analytic in the domain bounded by the circle $|z| < \rho$ $(\rho > 1)$ and the segment $1 \leq x \leq \rho$. Let (1.1) be the Taylor expansion of $f(z)$ in $|z| < 1$, where $a_n \neq 0$ and where (1.2) is satisfied. Let $f_{-1}(z)$ be the "inverse" function defined by (1.1′). Then either the unit circle is a cut for $f_{-1}(z)$, or there exist two constants α and β $(0 < \alpha \leq \pi/2$ and $0 < \beta \leq \pi/2)$, such that $f_{-1}(z)$ can be continued analytically along any ray $te^{i\theta}$, $0 \leq t < r(\theta)$, where*

$$r(\theta) = \min \left\{ e^{(\tan \alpha)\theta},\ e^{(\tan \beta)(2\pi - \theta)},\ \rho \right\} \quad \text{for} \quad 0 < \theta < 2\pi,$$

and

$$r(0) = 1.$$

Furthermore, $r(\theta)e^{i\theta}$ is actually a singularity of $f_{-1}(z)$ if $r(\theta) < \rho$.

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THE RICE INSTITUTE

A GENERALIZATION OF NORMED RINGS

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1. Introduction. A normed ring is, as is well known, a linear algebra A over the complex numbers or the real numbers with a norm having, besides the usual properties of a norm, also the "ring" property

1.1. $$\|xy\| \leq \|x\| \, \|y\|.$$

The generalization studied here is that instead of merely one norm defined on A there is a family of them, each satisfying 1.1; but of course it is natural to permit $\|x\| = 0$ even though $x \neq 0$, to which attention is drawn by prefixing the word 'pseudo.'

The theory can be briefly summed up by saying that a pseudo-ring-normed algebra A is an "inverse limit" of normed algebras. The main tool, which is rather obvious, is the fact that for a given pseudo-norm V (we avoid the use of the double bars since an additional symbol would still be needed to distinguish the various pseudo-norms) those x for which $V(x) = 0$ form a two-sided ideal Z_V, and that V can be used to define a norm in A/Z_V. When A is complete some questions, such as whether x has an inverse, can be reduced to the corresponding question for the *completion* B_V of A/Z_V. It is of course profitable to be able to reduce questions to B_V because B_V is a Banach algebra, while A/Z_V need not be complete. However, it seems to be difficult to say in general what questions can be so reduced to the case of Banach algebras. (We have spent much time vainly trying to discover whether the question of an element's having a *right* inverse in A is reducible to the same question for the various B_V.)

When a pseudo-ring-normed algebra A has a unit, then the latter may not be an interior point of the set of regular elements, but inversion is nevertheless continuous on the set of regular elements. On the other hand, there are many dense proper ideals. We devote some time to the topologization of the space of maximal, nondense (and hence closed), left ideals. From this a "structure space" of the topologically significant primitive ideals can easily be obtained, although we do not pursue the latter topic.

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In the commutative case, for each $\mathfrak{l}$ in L and x in A, one can define $x(\mathfrak{l})$ to be a complex number, just as for the normed case. There is given a sufficient condition that the $x(\ )$ be continuous, and this leads directly to be a characterization, among pseudo-ring-normed linear algebras over the complex numbers, of the space of continuous functions on a locally compact paracompact Hausdorff space.

Except for the part having to do with the paracompactness, which depends on the existence of a "locally-finite partition of unity," this paper was presented to the American Mathematical Society in November, 1946 (Bull. Amer. Math. Soc. Abstract 53-1-93). A forthcoming memoir of the American Mathematical Society, being prepared independently by Dr. Ernest Michael [16], on the subject of generalizations of normed rings, will treat many of these topics in greater detail.

2. **Pseudo-valuations and pseudo-norms.** A *pseudo-valuation* in a ring A is a nonnegative real-valued function V satisfying

$$V(x+y) \leq V(x) + V(y),\ V(xy) \leq V(x)V(y),\ V(-x) = V(x),\ V(0) = 0.$$

If A is a linear algebra over the field K (the real or complex numbers), and we have

$$V(\lambda x) = |\lambda|\, V(x)$$

as well as the other properties, we call V a *pseudo-ring-norm*. In a topological ring we shall call a pseudo-valuation *continuous* if the set on which $V(x) < e$ is open for each real e.

We shall call a ring A *pseudo-valuated* if there is a family $\mathfrak{V}$ of pseudo-valuations such that $V(x) = 0$ for all V in $\mathfrak{V}$ only if $x = 0$. It is not hard to see that A becomes a topological ring if the various translations of the sets on which $V(x) < e$, where e is real and V ranges through $\mathfrak{V}$, are taken as a subbase [14] for open sets. A pseudo-valuated ring A is called *complete* when it is complete with respect to the uniform structure defined by the various relations $V(x-y) < e$.

2.1. THEOREM. *Let U be a neighborhood of* 0 *in a topological linear algebra A. Then there is a continuous pseudo-ring-norm, V, such that U contains the set on which $V(x) < 1$ if and only if U is convex and UU lies in U.*

We leave the proof to the reader, except that we give a formula for V when U is given:

$$V(x) = \sup_{|\lambda|=1} \inf\{r;\ r \geq 0,\ x \in \lambda r U\}$$

3. **Quasi-inversion.** In a ring A, y is a *right quasi-inverse* of an element x if

$$x + y + xy = 0$$

[cf. 12, 11], and x is a *left quasi-inverse of* y.

The methods of normed rings can be adapted to establish the following continuity property of quasi-inversion.

3.1. THEOREM. *Let R be the set of elements having right quasi-inverses in a pseudo-valuated ring A. Let y be a left quasi-inverse of a limit point z of R. For each x in R, select a right quasi-inverse x' of x. Then*

3.2.
$$y = \lim_{x \to z,\ x \in R} x'.$$

Proof. The expressions to be written down will seem to involve the assumption that A has a unity element 1. However, such equations as we shall write down can always be freed of this assumption by expansion and cancellation.

Now by hypothesis we have,

$$(1 + y)(1 + z) = 1 \text{ and } (1 + x)(1 + x') = 1$$

for x in R. Let $u = z - x$ and $v = x' - y$. Then

$$(1 + z + u)(1 + y + v) = 1.$$

Multiplying this on the left by $1 + y$ we obtain

3.3.
$$v = (1 + y)uv + (1 + y)u(1 + y).$$

Let V be a pseudo-valuation. Since

$$V((1 + y)u) = V(u + yu) \leq V(u) + V(y)V(u) = (1 + V(y))V(u),$$

we get

$$V(v) \leq (1 + V(y))V(u)V(v) + (1 + V(y))^2 V(u).$$

If $V(u) \longrightarrow 0$, then presently

$$V(u) < (1 + V(y))^{-1},$$

and then

$$V(v) \leq (1 + V(y))^2 V(u)(1 - (1 + V(y))V(u))^{-1};$$

this shows that $V(v) \longrightarrow 0$.

Since, in 3.1, $1 + x$ tends to $1 + z$ and $1 + x'$ tends to $1 + y$, the continuity of multiplication shows that $(1 + x)(1 + x')$ tends to $(1 + z)(1 + y)$, so that y is also a *right* quasi-inverse of z. For the sake of clarity we reformulate this result for the special case of a pseudo-valuated ring with unity element.

3.4. COROLLARY. *If z is a left regular limit of right regular elements, then z is also right regular, and right inversion is continuous at z.*

A topological ring in which inversion is not continuous at 1, and which is (consequently) not pseudo-valuated, is L^{ω} [2, p. 629].

4. **Expansions for quasi-inverses.** With the hypotheses of 3.1, not only is right quasi-inversion continuous at z, it is analytic, in a sense which we shall not further define.

4.1. THEOREM. *Let the hypothesis and notation of* 3.2 *be assumed. For each x in R and each n, let*

$$y_n(x) = \sum_{i=0}^{n} ((1 + y)(z - x))^i (1 + y) - 1.$$

Then for any pseudo-valuation V and any x in R such that

$$V(x - z) < (1 + V(y))^{-1},$$

we have

$$\lim_{n \to \infty} V(y_n(x) - x') = 0.$$

Proof. Using u and v as in the proof of 3.2, we rewrite 3.3 as

4.2. $$(1 - (1 + y)u)(1 + x') = 1 + y.$$

Let $v_n = y_n(x) - x'$. Substituting here the expansion for $y_n(x)$, multiplying by $(1 - (1 + y)u)$ on the left, and using 4.2, we obtain

$$v_n = (1 + y)u\, v_n - ((1 + y)u)^{n+1}(1 + y).$$

If

$$V(u) \leq r < (1 + V(y))^{-1},$$

it follows readily that

$$(1 - r)V(v_n) \leq (1 + V(y))^{n+2} \; V(u)^{n+1},$$

from which the conclusion follows.

REMARK. The infinite series obtained by setting $n = \infty$ in y_n is of no use in showing the *existence* of right-quasi-inverses even when A is complete, as is done in the theory of Banach algebras. The reason is that a formal power product series in A has something like a radius of convergence for each V, and if these are not bounded away from 0 then the series may not converge in A.

5. **Direct operators.** Let L be an abelian group, and suppose there are defined in L a number of real-valued functions P such that

$$P(x) \geq 0, \; P(0) = 0, \; P(x - y) \leq P(x) + P(y).$$

A special case of this are the "pseudo-norms" of convex topological linear spaces [cf. 15]. Let $\mathfrak{P}$ be any set of such P's defined in L. Then an endomorphism α of L into itself will be called $\mathfrak{p}$-*direct* if for every P in $\mathfrak{P}$ and each positive e there is a positive d such that $P(x) < d$ implies $P(\alpha x) < e$. The implication of this requirement evidently depends on the size of the family $\mathfrak{P}$. For example, if L is a convex topological linear space, and $\mathfrak{P}$ is the class of all continuous pseudo-norms in L, then a $\mathfrak{p}$-direct linear operator in L is necessarily a scalar multiple of the identity.

There is another application of the idea of direct operators which we mention in passing. Let L be a Banach space, and let $\mathcal{E}$ be a Boolean ring (with unit) of projections in L. For E in $\mathcal{E}$, we can define a pseudo-ring-norm P_E by

$$P_E(x) = ||Ex||.$$

The result we wish to state is the following.

5.1. THEOREM. *A bounded operator α in L is direct with respect to the pseudo-norms P_E if and only if $\alpha E = E\alpha$ for all E in $\mathcal{E}$.*

Proof. For each E in $\mathcal{E}$, we have

$$||E\alpha x|| \leq C_E \; ||Ex||.$$

Now let $x = (1 - E)y$. Then

$$||E\alpha(1-E)y|| \leq C_E \; ||E(1-E)y|| = 0$$

for all y, or $E\alpha = E\alpha E$. Similarly $(1-E)\alpha = (1-E)\alpha(1-E)$. Expanding this and comparing with the former yields $E\alpha = \alpha E$, as desired.

Continuing with the general discussion, let us suppose that L is a linear space, and that $\mathfrak{P}$ is a fixed family of pseudo-norms. Let $D_{\mathfrak{P}}(L)$, or more briefly $D(L)$, be the family of $\mathfrak{P}$-direct linear operators in L.

5.2. THEOREM. *The family $D(L)$ is a linear algebra with unit element, and the V_P, where*

$$V_P(\alpha) = \sup_{P(x) \leq 1} P(\alpha x),$$

form pseudo-ring-norms for $D(L)$.

We shall omit the proof, which is easy.

For our purposes, a linear space L with a family $\mathfrak{P}$ of pseudo-norms P shall be called *complete* if

a) $P(x) = 0$ for every P in $\mathfrak{P}$ implies $x = 0$ and

b) whenever $P(x_\mu - x_\nu)$ converges to 0 for some directed set x_μ in L, and every P in $\mathfrak{P}$, there is an x in L such that $P(x_\mu - x)$ converges to 0 for every P.

This definition obviously applies to ring-pseudo-normed linear algebras. Concerning $D(L)$ we may assert the following, again leaving the proof to the reader.

5.3. THEOREM. *If L is complete (with respect to the pseudo-norms P in $\mathfrak{P}$), then $D(L)$ is complete with respect to the V_P.*

The purpose of the preceding discussion is to make possible the following statement.

5.4. THEOREM. *Let A be a linear algebra with unit element and a family $\mathfrak{V}$ of pseudo-ring-norms such that $V(1) = 1$ for each V in $\mathfrak{V}$. Then A is isomorphic, with preservation of pseudo-norms, to a subalgebra of $D(B)$, where B is A regarded as a pseudo-normed linear space with family $\mathfrak{V}$ of distinguished pseudo-norms.*

For the proof of 5.4 we represent each x in A by the operator that sends y into xy.

6. Completeness of kernel quotients. Let A be a ring with a family $\mathfrak{V}$ of pseudo-valuations, and suppose $V_1, \cdots, V_n$ belong to $\mathfrak{V}$. Then

$$V(x) \equiv \max(V_1(x), \cdots, V_n(x))$$

defines a pseudo-valuation in A. Those x with $V(x) = 0$ form a two-sided ideal Z_V, a *kernel ideal* of A (with respect to $\mathfrak{V}$). We could have limited our attention to the case $n = 1$ by assuming that $V \in \mathfrak{V}$ whenever $V_1, \cdots, V_n \in \mathfrak{V}$, but it is convenient not to assume this. The quotient ring $A_V = A/Z_V$ is a *kernel quotient*, and V may be defined in it in an obvious way.

When V is a pseudo-ring-norm, A_V is a normed ring.

The canonical homomorphism of A onto A_V is continuous when the topology described in § 2 is used in A, and that defined by V is used in A_V. The completion in that topology of A_V will be denoted by $\overline{A_V}$ and called a *completed kernel quotient*. In the ring-pseudo-norm case, the completed kernel quotients are all Banach algebras.

We shall now give several examples to show that we have no right to suppose that A_V is complete even when A is. In these examples, which are algebras A of complex-valued continuous functions f on various spaces X, we presuppose pseudo-ring-norms of the following type. Let $\mathcal{K}$ be a class of compact sets whose interiors cover X. For each K in $\mathcal{K}$ let $V_K(f)$ be the maximum of $|f(t)|$ for t in K (the topology in A is then the k-topology, and X necessarily is locally compact).

6.1. THEOREM. *Let T be completely regular, and let $C(T)$ be the ring of continuous functions on T. Then $C(T)$ is complete, and each $C(T)_{V_K}$ is complete.*

6.2. THEOREM. *Let $H(D)$ be the holomorphic functions on an open set D in the plane. Then $H(D)$ is complete; but if K has at least one limit point, then $H(D)_{V_K}$ is not complete.*

6.3. THEOREM. *Let $BC(R_1)$ be the ring of bounded continuous functions on the real line. Then $BC(R_1)$ is not complete, but each $BC(R_1)_{V_K}$ is complete.*

In 6.1, $C(T)$ is well-known to be complete [1]; Z_{V_K} consists of those functions which vanish on K, and so $C(T)_{V_K}$ is naturally isomorphic to a subalgebra of $C(K)$. By consideration of the Stone-Čech compactification, or otherwise, one can extend any function continuous on K to all of T (as a matter of fact, without increase of bound). Hence the subalgebra in question is all of $C(K)$ which is complete in its norm.

Morera's theorem shows that $H(D)$ is closed in $C(D)$, and so it is complete. Again $H(D)_{V_K}$ is isomorphic to a subalgebra S of $C(K)$. (Of course Z_K contains only 0.) This subalgebra is not closed, however, for we can select an analytic function f holomorphic on K but with a singularity somewhere on D. It is the uniform limit of polynomials on K and hence a limit of S in $C(K)$, whence S is not closed. We emphasize that this example shows that Z_V may be 0 alone, while the topology of A_V is not the same as that of A.

Finally, $BC(R_1)$ is not complete since it is dense in $C(R_1)$; but each $BC(R_1)_{V_K}$ is complete for the very reasons given for 6.1.

7. **Right inverses.** In a later section we want to show that each maximal ideal of a commutative complete pseudo-valuated ring A is a "divisor" of some kernel ideal Z_V. The following theorem together with Gelfand's principle yields this result. It obviously implies that, if A is complete, an element x which has a (two-sided) quasi-inverse in each $\overline{A_V}$ (the completion of A_V) has a quasi-inverse in A, since a two-sided quasi-inverse is a unique right quasi-inverse. We have not been able to drop the requirement of "unique" in 7.1, since there seem to be difficulties in combining the various right inverses which are supposed to exist. If it should be in fact impossible to prove 7.1 without the word "unique," then this would be the first indication of a serious divergence between the theory of pseudo-valuated rings and that of normed rings, after which it is patterned.

After Theorem 7.1, we present a theorem like 7.1 in which the word "unique" is omitted, but there are other hypotheses which are by no means always fulfilled.

7.1. THEOREM. *Let x be an element of a complete pseudo-valuated ring A. Then x has a unique right quasi-inverse in A if and only if its image in each completed kernel quotient has a unique right quasi-inverse there.*

Proof: There is no loss in generality here in supposing the class $\mathfrak{V}$ of pseudo-valuations to contain

$$V(x) \equiv \max\,(V_1(x), \cdots, V_n(x))$$

when it contains $V_1, \cdots, V_n$. Let X_V be the image of x in the kernel quotient A_V, and let Y_V be the right quasi-inverse of X_V in $\overline{A_V}$. For each positive integer n, one can find an element $y_{V,n}$ in A such that its image $Y_{V,n}$ in $\overline{A_V}$ is close to Y_V :

$$V(Y_{V,n} - Y_V) < 1/n.$$

The index-pairs V, n on these $y_{V,n}$ may be partially ordered by setting $(U, m) \leq (V, n)$ whenever $m \leq n$, and $U(z) \leq V(z)$ for all z in A (this latter we abbreviate $U \leq V$). When $U \leq V$, we have $Z_V \subset Z_U$; moreover we have a natural mapping (of bound 1) of $\bar{A}_V$ into $\bar{A}_U$, and hence we can act as if an element originally introduced as belonging to $\bar{A}_V$ (such as Y_V) also belongs to $\bar{A}_U$. As a matter of fact, with this convention, we have $Y_V = Y_U$ when $U \leq V$ because Y_V is clearly a right quasi-inverse of X_U in $\bar{A}_U$, and this was supposed to be *unique*. Making use of this fact, we shall show that $\{y_{V,n}\}$ forms a Cauchy system. Let U belong to $\mathfrak{V}$, and suppose $V, W \geq U$. Then

$$U\left(y_{V,m} - y_{W,n}\right) = U\left(Y_{V,m} - Y_{W,n}\right)$$

$$\leq U\left(Y_{V,m} - Y_U\right) + U\left(Y_U - Y_{W,n}\right)$$

$$\leq \frac{1}{m} + \frac{1}{n}.$$

Because of the assumption made at the outset about max (V_1, V_2) belonging to $\mathfrak{V}$ with V_1 and V_2, the indices form a directed set; and the $y_{V,n}$ form a Cauchy directed system, which must converge to a y in A since A is complete. A calculation similar to that just performed shows that

$$V(xy + x + y) = 0$$

for all V, whence y is a right quasi-inverse for x, as desired. This proves 7.1.

Let A be a pseudo-valuated ring, and suppose that for each V in $\mathfrak{V}$ there is selected an element u_V of A such that

7.2. for each W in $\mathfrak{V}$ there is a finite set $\mathfrak{V}_W$ such that $W(u_V) \neq 0$ only for those V which belong to $\mathfrak{V}_W$, and $W(u_V) \leq 1$;

7.3. $W(y - \sum y u_V) = 0$, the sum being extended over all V in $\mathfrak{V}_W$;

7.4. for a fixed V, we have $W(u_V) \neq 0$ only when W belongs to $\mathfrak{V}_V$.

Then we shall call u_V a *locally finite partition of unity.*

The partial sums of the series $\sum u_V$ clearly form a Cauchy system, so that when A is complete the existence of a locally finite partition of unity ensures the existence of a unity element, and makes it possible to talk about inverses rather than quasi-inverses.

We can exhibit nontrivial examples of such partitions.

7.5. THEOREM. *Let $C(X)$ be the ring of continuous complex-valued functions on a locally compact, paracompact* [see 8] *Hausdorff space T. Then a* *family* $\mathfrak{P}$ *of pseudo-norms can be defined in* $C(T)$ *so that the ring* $D(C(T))$ *of direct operators in* $C(T)$ *pseudo-normed as in* 5.2, *possesses a locally finite partition of unity, and is complete.*

Proof. According to the hypothesis we can obtain a neighborhood-finite family $\{G\}$ of open sets which cover X and whose closures are compact.

Using Theorem 6 of [8] and the method of Bourbaki (*partition continue de l'unité*) we construct a family of continuous real-valued nonnegative functions f_G, where $f_G(t) \neq 0$ only for x in G and $\sum f_G(t) = 1$. As pseudo-norms in $C(T)$, take

$$P_G(f) = \sup_{t \in G} |f(t)|.$$

The topology thus obtained is the k-topology, in which $C(T)$ is complete, and hence $D(C(T))$ is complete. The operators u_V defined by $u_V(f) = f_G f$, where $V = V_{P_G}$ (see 5.2), are surely direct. Now let G, H belong to $\{G\}$, and let V, W be V_{P_G}, V_{P_H} respectively. Then $W(u_V) \neq 0$ only if H is one of the finitely many sets of $\{G\}$ which meet the compact closure of G, and only if G is one of the finitely many sets of G which meet the closure of H. Except for details, this proves 7.5.

7.6. THEOREM. *Let* x *be an element of a complete pseudo-valuated ring* A *possessing a locally finite partition of unity. Then* x *has a right inverse in* A *if and only if its image in each completed kernel quotient has a right inverse there.*

Proof. We adopt the notation of 7.2-7.4. We do not suppose that the class $\mathfrak{V}$ here is closed under the maximum formation mentioned in the proof of 7.1, because this would require a complicated reformulation of 7.4. For each V in $\mathfrak{V}$, define $V_1(z) = \max W(z)$ for all W in $\mathfrak{V}_V$, and suppose V itself to be adjoined to $\mathfrak{V}_V$. Let X_{V_1} be the image of x in $\bar{A}_{V_1}$, and let Y_V be a right inverse of X_{V_1} in $\bar{A}_{V_1}$. Select $y_{V,n}$ in A so that if $Y_{V,n}$ is its image in $\bar{A}_{V_1}$, then for $n = 1, 2, \cdots$, we have

$$7.7. \qquad (V_1(x) + 1)\, V_1(Y_{V,n} - Y_V) < 2^{-n-1}.$$

By the local finiteness, $y_n = \sum_V y_{V,n} u_V$ converges. Let W belong to $\mathfrak{V}_V$. Then

$$W(y_{n+1} - y_n) \leq \sum W(y_{V,n+1} - y_{V,n})\, W(u_V).$$

Since this sum needs to be extended only over the V in $\mathfrak{V}_W$, and since then W

lies in $\mathfrak{V}_V$ so that $W \leq V_1$, we have from 7.7 that the left member is less than 2^{-n} times the number of elements in $\mathfrak{V}_W$. Hence the sequence of y_n is Cauchy and converges to some y in A. Now

$$W(xy - 1) = \lim W(xy_n - 1),$$

and

$$xy_n - 1 = \sum (xy_{V,n} - 1)\, u_V ,$$

so that

7.8. $$W(xy_n - 1) \leq \sum W(xy_{V,n} - 1),$$

where this sum needs to be extended only over the V in $\mathfrak{V}_W$. But then in each case W belongs to $\mathfrak{V}_V$, and so $W \leq V_1$ and

$$W(xy_{V,n} - 1) \leq V_1(xy_{V,n} - 1) = V_1(x_{V_1}, Y_{V,n} - X_{V_1} Y_V)$$
$$\leq V_1(x) V_1(Y_{V,n} - Y_V) < 2^{-n-1}.$$

Since 7.8 involves only a fixed finite number of terms of this sort, we conclude that $W(xy_n - 1)$ tends to 0, whence $xy = 1$ as desired.

8. **Ideals.** In topological rings, naturally the closed ideals play a more important part than the others. Much of the success of Banach algebras is due to the fact that maximal (that is, maximally proper) ideals are closed. The same is true for pseudo-valuated rings with only a finite set of pseudo-valuations. However, it is not true in general. For example, if in the case of the ring in 6.1, when T is not compact, the ideal of functions each vanishing outside some compact set is swelled (by Zorn's lemma) to a maximal ideal M, then this ideal is certainly not closed. For every closed ideal in $C(T)$ can easily be shown to consist of all functions vanishing at a suitable point of T, and for each t of T there is an f in M which is not zero there. This ideal is consequently dense in $C(T)$.

Our idea is to reduce certain parts of the ideal theory (and at some future time, of the representation theory) of pseudo-valuated rings to that of the completed kernel quotients, in which some of the techniques of Banach algebras can be applied. For terms used below but not defined, see [11].

8.1. THEOREM. *In a pseudo-valuated ring, every nondense left and/or regular and/or two-sided ideal L is contained in a closed left and/or regular* *and/or two-sided ideal N which contains a kernel ideal.*

Proof. Let L be a nondense ideal. Then there is a u in A, a V in $\mathfrak{V}$, and a positive e, such that $V(u-x) < e$ implies $x \notin L$. Let M be the sum of L and the kernel ideal Z_V. Let $x \in M$. Then $x = y + z$, $y \in L$, and $z \in Z_V$. Now

$$e \leq V(u-y) = V(u-x+z) \leq V(u-x),$$

whence u does not belong to the closure N of M. Hence N is the desired ideal. However, if L is regular with relative right unit v, we must show that there is a relative right unit modulo N. But v itself can obviously be chosen for this purpose.

8.2. COROLLARY. *Every nondense maximal (two-sided) or maximal left ideal is closed and contains a kernel ideal.*

9. **An abstract approach to structure spaces.** We note that the ideas of Stone and Jacobson can be generalized to topologizing suitable subsets of partially ordered systems. In what follows one may think of $\mathfrak{J}$ as the class of ideals of a ring, $\mathfrak{I}$, the two-sided ideals, and $\mathfrak{L}$, the maximal left ideals.

Let $\mathfrak{J}$ be a partially ordered set, $\mathfrak{I}$ a subset of $\mathfrak{J}$ which forms a complete lattice with greatest element A, and $\mathfrak{L}$ an arbitrary subset of $\mathfrak{J}$. We want to define a closure operation in $\mathfrak{L}$. For $\mathfrak{M} \subset \mathfrak{L}$, let

$$\mathfrak{i}(\mathfrak{M}) = \sup \{\mathfrak{a};\ \mathfrak{a} \in \mathfrak{I},\ \mathfrak{a} \leq \mathfrak{l} \text{ for every } \mathfrak{l} \text{ in } \mathfrak{M}\}.$$

For $\mathfrak{a}$ in $\mathfrak{I}$, let

$$\mathfrak{u}(\mathfrak{a}) = \{\mathfrak{l};\ \mathfrak{l} \in \mathfrak{L},\ \mathfrak{l} \geq \mathfrak{a}\}.$$

One can easily verify that

9.1. $$\mathfrak{u}(\sup \mathfrak{a}_\alpha) = \bigcap \mathfrak{u}(\mathfrak{a}_\alpha).$$

For $\mathfrak{M} \subset \mathfrak{L}$, the closure $\overline{\mathfrak{M}}$ shall be $\mathfrak{u}\mathfrak{i}(\mathfrak{M})$.

The next three propositions, whose proofs are omitted, mention all the closure axioms needed for a topological (T-) space; Λ is the void set:

9.2. $\overline{\Lambda} = \Lambda$ *if and only if A has no upper bound in $\mathfrak{L}$.*

9.3. $\mathfrak{M} \subset \overline{\mathfrak{M}}$; $\mathfrak{M} \subset \mathfrak{N}$ *implies* $\overline{\mathfrak{M}} \subset \overline{\mathfrak{N}}$; *and* $\overline{\overline{\mathfrak{M}}} = \overline{\mathfrak{M}}$.

9.4. $\overline{(\mathfrak{M} \cup \mathfrak{N})} \subset \overline{\mathfrak{M}} \cup \overline{\mathfrak{N}}$ *holds generally in $\mathfrak{L}$ provided for each $\mathfrak{l}$ in $\mathfrak{L}$, the inequality* $\mathfrak{l} \geq \mathfrak{a} \wedge \mathfrak{b}$ *implies* $\mathfrak{l} \geq \mathfrak{a}$ *or* $\mathfrak{l} \geq \mathfrak{b}$.

9.5. THEOREM. *If the conditions of* 9.4 *and* 9.2 *are satisfied,* $\mathfrak{L}$ *becomes a T-space.*

As remarked by Jacobson, $\mathfrak{L}$ need not be a T_1-space, nor a T_2-space even when it is a T_1-space. As a matter of fact, it need not even be a T_0-space, but this defect is smaller than the other two, for it may be "removed" by a process of identification.

These spaces $\mathfrak{L}$ often have compactness properties.

9.6. THEOREM. *The set* $\mathfrak{u}(\mathfrak{a})$ *is compact if and only if given* $\mathfrak{a}_\alpha$ *in* $\mathfrak{J}$ *with* $\mathfrak{a}_\alpha \geq \mathfrak{a}$ *such that every finite collection has an upper bound in* $\mathfrak{L}$, *then the entire collection has an upper bound in* $\mathfrak{L}$.

10. **Application to rings.** Now let A be a ring and let $\mathfrak{J}$ be the class of two-sided ideals. Let $\mathfrak{L}$ be a class of ideals of A each of which is either

10.1. prime and proper,

10.2. maximal left, or

10.3. primitive.

Then the condition of 9.4 holds (case 10.3 was considered by Jacobson, and the others are obvious), and $\mathfrak{L}$ becomes a space of 9.5.

The special case in which $\mathfrak{L}$ consists of the maximal left ideals is interesting. It is fairly easy to see that $\mathfrak{i}(\{\mathfrak{l}\})$ is always primitive and that we thus obtain an open continuous mapping on Jacobson's structure space.

In topological rings, $\mathfrak{L}$ may be chosen to contain only closed ideals, while $\mathfrak{J}$ may be chosen as before. This does not adversely affect the fulfillment of the closure axioms, but naturally when the class of ideals in $\mathfrak{L}$ is restricted, limit points are lost and compactness is affected. Indeed, even when A is commutative and has a unit, $\mathfrak{L}$ may be noncompact (see 8.1). Hence we prove the following about pseudo-valuated rings.

10.4. THEOREM. *Let* $\mathfrak{L}$ *be the class of nondense maximal left ideals, and* $\mathfrak{J}$ *the two-sided ideals, of a pseudo-valuated ring* A. *Let* $\mathfrak{a}$ *be a regular member of* $\mathfrak{J}$, *and let* Z_V *be a kernel ideal. Then* $\mathfrak{u}(\mathfrak{a}) \cap \mathfrak{u}(Z_V)$ *is compact.*

Proof. Without loss of generality (see 9.1) we may suppose $\mathfrak{a} \supset Z_V$, and thus $\mathfrak{u}(\mathfrak{a}) \subset \mathfrak{u}(Z_V)$. Now let $\mathfrak{a}_\alpha \supset \mathfrak{a}$. Suppose that for $\alpha_1, \cdots, \alpha_n$ there is an $\mathfrak{l}$ in $\mathfrak{L}$ containing the $\mathfrak{a}_{\alpha_i}$ for $i = 1, \cdots, n$. Let $B = A/\mathfrak{a}$, pseudo-valuated by means of V. Since $\mathfrak{a}$ is regular, B has a unit, and $\mathfrak{l}$ does not map onto B. The multiplicative properties of V ensure that the closure of the image of $\mathfrak{l}$ is also an ideal in the completion $\bar{B}$. Thus the images of $\mathfrak{a}_{\alpha_1}, \cdots, \mathfrak{a}_{\alpha_n}$ generate a proper left ideal in $\bar{B}$. Hence, by the argument of Banach algebras (cf. [11]), the images of the $\mathfrak{a}_\alpha$ all fall into one (closed) maximal left ideal of $\bar{B}$. The inverse image of this in A provides a bound for the $\mathfrak{a}_\alpha$, and so 9.6 applies, finishing the proof.

As in an earlier section, we can go further with the assumption of a locally finite partition of unity.

10.5. Lemma. *Let $\mathfrak{L}$, $\mathfrak{J}$, and A be as in 10.4. Let A have a locally finite partition of unity, $\{u_V\}$. Let G_V be the (open) complement of $\mathfrak{u}(\mathfrak{a}_V)$, where $\mathfrak{a}_V$ is the two-sided ideal generated by u_V. Then G_V does not meet $\mathfrak{u}(Z_W)$ for W not in $\mathfrak{V}_V$.*

Proof. Clearly $u_V \in Z_W$ when $W \in \mathfrak{V}_V$. Hence $\mathfrak{a}_V \subset Z_W$, and so

$$\mathfrak{u}(\mathfrak{a}_V) \supset \mathfrak{u}(Z_W),$$

from which the conclusion follows.

10.6. Theorem. *Let $\mathfrak{L}$, $\mathfrak{J}$, A, and G_V be as in 10.5. Then each G_V has a compact closure in $\mathfrak{L}$, and the G_V form a star-finite open covering of $\mathfrak{L}$, which is consequently a paracompact* locally compact space.*

Proof. By 8.2, the $\mathfrak{u}(Z_V)$ cover $\mathfrak{L}$, and so G_V must be contained in the union of those finitely many $\mathfrak{u}(Z_W)$ for which W belongs to $\mathfrak{V}_V$, and this union is compact by 10.4. (Recall that A has a unit.) Now suppose that G_V and G_W intersect in a nonvoid set. They must intersect in a point of some $\mathfrak{u}(Z_U)$, whence $U \in \mathfrak{V}_V$, $U \subset \mathfrak{V}_W$. For V fixed, this rules out all but a finite set of possibilities for W. This shows that the G_V form a star-finite system. Now let $\mathfrak{l}$ belong to $\mathfrak{L}$. Then for some V, we have $u_V \not\subset \mathfrak{l}$; for otherwise we would have $1 \in \mathfrak{l}$ since the latter is closed. Then $\mathfrak{l} \in G_V$. Hence the G_V cover $\mathfrak{L}$, and $\mathfrak{L}$ is therefore locally compact.

Now let an arbitrary open covering C of $\mathfrak{L}$ be given. For each V select a finite number of these open sets to cover G_V, and cut these sets down so that they lie in the union of those G_W which meet G_V. The class of sets so obtained is easily seen to form a neighborhood finite refinement of C. This completes the proof of 10.6.

11. Characterization of the ring of continuous functions. We are now in a position to characterize the ring $A = C(T, K)$ (K is here the complex field),

*In the generalized sense obtained by removing the stipulation of Hausdorff separation from Dieudonné's definition.

where T is a locally compact, paracompact Hausdorff space, as a pseudo-ring-normed ring in which the topology is that of the k-topology. From 7.5 we say about $C(X, K)$ that

11.1. it has a locally finite partition of unity; and moreover, if x^* is defined by $x^*(t) = \overline{x(t)}$, then

11.2. it has a semilinear operation * such that $(\lambda x + yz)^* = \overline{\lambda} x^* + z^* y^*$, $x^{**} = x$, and

11.3. $V(xx^*) \geq k_V\, V(x)\, V(x^*)$, where k_V is some positive number, for each V in $\mathfrak{V}$.

In $C(X, K)$, all the $k_V = 1$.

The main theorem is a converse of these observations.

11.4. THEOREM. *A commutative complete pseudo-ring-normed linear algebra A over the complex numbers K satisfying* 11.1, 11.2, 11.3 *is equivalent to a $C(T, K)$, where T is a locally compact, paracompact Hausdorff space which is homeomorphic to the space $\mathfrak{L}$ of closed maximal ideals of A, topologized as in* 10.6.

Proof. Since each closed maximal ideal contains some Z_V, the corresponding residue class ring is a normed field, which must be the field of complex numbers K. For $\mathfrak{l}$ in $\mathfrak{L}$, define $x(\mathfrak{l}) = a$ if $x - a \cdot 1$ belongs to $\mathfrak{l}$. Now A_V is isomorphic to a subset of $C(X_V, K)$ by Theorem 1 of [4], which is essentially the Gelfand-Neumark lemma. It follows that

11.5. $$k_V^2\, V(x) \leq \sup_{\mathfrak{l} \in \mathfrak{u}(Z_V)} |x(\mathfrak{l})| \leq V(x),\ x^*(\mathfrak{l}) = \overline{x(\mathfrak{l})}.$$

Let $p_1, p_2, \cdots$ be a sequence of ordinary polynomials with real coefficients such that

$$|p_m(a) - |a|| < 2^{-m}$$

for a any real number with $|a| \leq m$. These can be constructed by Weierstrass' approximation theorem. It follows from 11.5 that if $x = x^*$ in A, then $p_m(x)$ is a Cauchy system, and for the limit y we surely have $|x(\mathfrak{l})| = y(\mathfrak{l})$ for each $\mathfrak{l}$ in $\mathfrak{L}$. Denote this y by $|x|$.

We must now establish that $x(\ )$ is continuous on $\mathfrak{L}$. Let $\mathfrak{l} \in \mathfrak{L}$. Since A has a unit we may suppose that $x(\mathfrak{l}) = 0$. We may also suppose that $x = x^*$.

From the possibility of taking absolute values, it follows that

$$y \equiv 1 - |x| + |1 - |x||$$

belongs to A, and it has the value 2 at $\mathfrak{l}$. Now let $\mathfrak{a}$ be the principal ideal generated by y, and suppose $\mathfrak{m} \in \mathfrak{u}(\mathfrak{a})$. Then $y(\mathfrak{m}) = 0$, so that $\mathfrak{m} \neq \mathfrak{l}$. Suppose that $|x(\mathfrak{m})| \geq 1$ for some $\mathfrak{m}$ in $\mathfrak{L}$. Then $y(\mathfrak{m}) = 0$, or $\mathfrak{m} \in \mathfrak{u}(\mathfrak{a})$. Hence the complement of $\mathfrak{u}(\mathfrak{a})$ is a neighborhood of $\mathfrak{l}$ on which the absolute value of value of $x(\ )$ is less than 1. In view of the possibility of scalar multiplication, this shows that $x(\ )$ is continuous.

We next show that the topology of A is the same as the k-topology for the corresponding functions. If $\mathfrak{K}$ is a compact subset of $\mathfrak{L}$, it is contained in finitely many of the G_V of 10.6, and by 10.5 it is contained in the union of some finite class of $\mathfrak{U}(Z_V)$'s. Hence convergence in all pseudo-norms implies uniform convergence on $\mathfrak{K}$ by 11.5, hence in the k-topology. The other way around is simpler, depending on 11.5 and the fact that each $\mathfrak{u}(Z_V)$ is compact. An application of a generalized form [5, p. 765] of Kakutani's method for the Stone-Weierstrass theorem completes the proof of 11.4.

One could go on to generalize the numerous variations of the Gelfand-Neumark lemma involving only real scalars, or quaternions, and so on, but the method of reducing these questions to the corresponding case of normed rings is now clear. The purpose of the partition of unity is of course to enable one to disclose the topology of A as the k-topology, and thus has no nontrivial counterpart in the "normed" theory. If that part of the previous proof which involves the G_V is ignored, we obtain the following:

11.6. THEOREM. *A commutative complete pseudo-ring-normed linear algebra A, with unit over the complex numbers satisfying* 11.2 *and* 11.3, *is isomorphic to a $C(T, K)$, where T is a completely regular space homeomorphic to the space of closed maximal ideals, and such that the topology of A corresponds to some topology in $C(T, K)$ which has at most the open sets of the k-topology.*

This result, while perhaps more easily applicable, is not "a characterization of those $C(T, K)$ with a topology $t \leq k$," since such $C(T, K)$ do not need to be complete. In [6, p. 234] there is exhibited a space T in which all compact sets are finite (although *not* always *open* as the next sentence in that paper should have said). Consequently, the completion of the space $C(T, K)$ in any topology t which is $\leq k$ includes discontinuous functions.

When 11.3 does not hold, we have no way of knowing that the functions $x(\ )$ are continuous on $\mathfrak{L}$; in fact, even in the norm case they are sometimes not continuous (cf. Gelfand and Šilov). Of course, one can force them to be continuous by abandoning the topology in $\mathfrak{L}$ and introducing a new one *ad hoc*, defining just enough sets to be open so that they are continuous. The result is a completely regular space, and in it we can make this statement:

11.7. THEOREM. *Let A be a commutative complete pseudo-normed linear algebra over the complex or real numbers. For each x in A, define a function $x(\ )$ on the space of closed maximal regular ideals. Then x has a quasi-inverse in A if and only if $x(\mathfrak{l}) \neq -1$ for each $\mathfrak{l}$ in $\mathfrak{L}$.*

The proof follows from the preceding remarks, Theorem 7.1, and 8.2.

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INTERSECTION THEORY FOR CYCLES OF AN ALGEBRAIC VARIETY

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Introduction. For a number of years intersection theory represented one of the most debated subjects in the field of algebraic geometry; also one of the main reasons for seeing in the whole structure of algebraic geometry an inherent flimsiness which even discouraged the study of this branch of mathematics. This situation came to an end when the methods of algebra began to be successfully applied to geometry, mainly by van der Waerden and Zariski; in the specific case of intersection theory, a completely general and rigorous treatment of the subject was given by Chevalley [3] in 1945. This rebuilding of algebraic geometry on firm foundations has often taken a form quite different from what the classical works would have led one to expect. Thus it is not surprising that Chevalley's solution of the problem has no evident link with the methods that, according to the suggestions of the classical geometers, should have been used in order to define the intersection multiplicity (for a sketch of these methods and suggestions see, for instance, [4]); rather, it is linked to the analytical approach, and it is therefore a strictly "local" theory, thus having the advantage of providing an intersection multiplicity also for algebroid varieties. The method by A. Weil [5] is another example of local theory.

The classical approach to the problem is illustrated in the introduction to [2] (see "first approach"), and carried out in the present paper. After an introduction dealing with algebraic correspondences (§1) we study in §2 a particular algebraic system related to any given cycle $\mathfrak{z}$ of a projective space, namely the system consisting of all the cycles obtained from $\mathfrak{z}$ by projective transformations of the ambient space, plus the "limit cycles" which must be added in order to complete the algebraic system (and which would correspond to the degenerate projective transformations). This system, called the homographic system of $\mathfrak{z}$, is used in §3 to obtain the principal results, namely Lemma 3.1 and Theorem 3.2. The wording of these results, as of the other results of §3, is complicated by the fact that we do not restrict ourselves to varieties over an algebraically

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closed field, or to varieties in the sense of [5]; the gist of them, however, is the following:

Given the irreducible *cycles* $\mathfrak{y}$, $\mathfrak{z}$ of a projective space, let $\mathfrak{Z}$ be the "generic" element of the homographic system of $\mathfrak{z}$, and let P be an isolated component (of the right dimension) of the intersection of the *varieties* $\mathfrak{y}$ and $\mathfrak{z}$. Then the number of those intersections of the varieties $\mathfrak{y}$ and $\mathfrak{Z}$ which approach P when $\mathfrak{Z}$ approaches $\mathfrak{z}$ is, by definition, the intersection multiplicity of $\mathfrak{y}$ and $\mathfrak{z}$ at P; this number does not change if $\mathfrak{z}$ is allowed to vary in any "admissible" algebraic system rather than in its homographic system; and finally, the number is the same when $\mathfrak{z}$ varies in any algebraic system, provided that then we already count each intersection of $\mathfrak{y}$ and $\mathfrak{Z}$ with a certain multiplicity, to be computed by means of an "admissible" system. Also, the same number is obtained if $\mathfrak{y}$, or both $\mathfrak{y}$ and $\mathfrak{z}$, are allowed to vary.

The fact that we allow our varieties to be defined over an arbitrary field is not just a refinement of debatable usefulness, but a plain necessity: in fact, the general element of an algebraic system is *never* defined over an algebraically closed field (unless the system consists of just one element).

This definition takes care of the intersection of cycles of a projective space; the next step (carried out in §5) is the extension of the definition and of the related results to the cycles of an arbitrary (irreducible) variety V. Should it be possible to find, for any given cycle $\mathfrak{z}$ of V, an algebraic system of cycles of V, containing $\mathfrak{z}$, and playing the same role as the homographic system, then the theory on V would not differ from the theory on a projective space; more generally, it would be enough to find another cycle $\mathfrak{x}$ which does not contain the intersection U in which we are interested, and such that $\mathfrak{z} + \mathfrak{x}$ is contained in such a general algebraic system. Now, it is well known that this is not the case in general, but that one very wide class of cycles $\mathfrak{z}$ through U which fulfill the condition is the set of the cycles of V which are locally (at U) intersections of V and of a cycle of the ambient space; and this, in turn, is always the case if U is simple on V and the ground field is algebraically closed. As a consequence, we define the intersection multiplicity of $\mathfrak{y}$ and $\mathfrak{z}$ at U on V only for the case in which $\mathfrak{y}$ and $\mathfrak{z}$ are intersections, at U, of V with cycles Y, Z of the ambient space S; for this case the algebraic system containing $\mathfrak{z} + \mathfrak{x}$ (with $\mathfrak{x}$ not passing through U) which can be used in order to define the intersection multiplicity is the system of the intersections of V with the elements of the homographic system of $\mathfrak{Z}$; it is not even necessary, however, to consider this system: since the intersection of $\mathfrak{y}$ and $\mathfrak{Z}$ in S is already defined, the multiplicity of U in this intersection can be assumed to be, by definition, the multiplicity of U in the intersection of $\mathfrak{y}$ and $\mathfrak{z}$ on V. This is an outline of the content of §5, but one more detail needs to be mentioned here: it may happen, a priori at least, that although $\mathfrak{z}$ is not an intersection at U, it becomes such by a suitable birational transformation of V which is regular at U; this is taken into account after Theorem 5.9. Finally, since we are using rational cycles, it must be remarked that such cases as the vertex of a quadric cone are naturally taken care of by the theory: a line $\mathfrak{z}$ through the vertex U of a quadric cone V is the intersection at U of V with the cycle $\mathfrak{Z}/2$ of the 3-space containing V, $\mathfrak{Z}$ being the tangent plane to V along $\mathfrak{z}$.

Bezout's theorem is proved in §4 by means of one of the usual geometric methods, namely by letting the two cycles degenerate completely into cycles consisting of linear varieties only; other proofs of a more algebraic nature would display the relations of Bezout's theorem to that property of the divisors which is called the "product formula" by number theorists; the present proof, however, offers the advantage of being extremely simple.

The main advantage of the present geometrical theory of intersections is the fact that it can readily be applied to problems "in the large"; although throughout this paper the local intersection number is stressed, the theory finds easy and immediate application to the construction of the algebraic system determined by two cycles over any connected component of their intersection which happens to have a dimension larger than expected; in particular, the characteristic system of an irreducible subvariety of a variety and its virtual degree could easily be established. These topics, however, would find their natural place in a paper dealing with algebraic equivalence.

1. **Preliminary results.** We shall use the same definitions and notations as in [1] and [2], paying attention to the fact that some of the definitions or notations of [1] have been modified in [2]. A few additional modifications or generalizations will be explained now. In [1] "cycle" meant "integral effective cycle" (that is, with positive integers as coefficients); in [2] it meant "rational effective cycle"; it shall now mean "rational (effective or virtual) cycle". More precisely, a cycle is an expression of the form

$$\mathfrak{z} = \sum_{i=1}^{n} a_i V_i ,$$

where $n \geq 1$, the a_i's are nonzero rational numbers, and the V_i's are mutually distinct irreducible pseudosubvarieties of a pseudovariety over a field; $\mathfrak{z}$ is *unmixed* if all the V_i's have the same dimension (called the dimension of the cycle). The set of s-dimensional cycles becomes an additive group by addition of the zero cycle $0 = 0V$ for any s-dimensional irreducible pseudosubvariety V. The above expression $\sum_{i=1}^{n} a_i V_i$ is called the *minimal representation of* $\mathfrak{z}$; any expression $0V$ is a minimal representation of 0. If V is an s-dimensional irreducible pseudo-subvariety, the *multiplicity of* V *in* $\mathfrak{z}$ is zero if $V \neq V_i$ for each i or if $\mathfrak{z} = 0$, and equals a_i if $V = V_i$. The cycle $\mathfrak{z}$ is *irreducible* if $n = a_1 = 1$. The identification, used in [1] and [2], of an irreducible cycle $\mathfrak{z} = 1V$ with the irreducible pseudovariety V is no longer valid. If $\sum_{i=1}^{n} a_i V_i$ is the minimal representation of the cycle $\mathfrak{z} \neq 0$, then each V_i is called a *component variety of* $\mathfrak{z}$, and each $1V_i$ is a *component of* $\mathfrak{z}$; the cycle $\mathfrak{y}$ whose minimal representation is $\sum_{j=1}^{m} b_j W_j$ is *part of* $\mathfrak{z}$ if $m \leq n$, and if it is possible to establish a $1-1$ correspondence $j \longrightarrow i(j)$ such that $a_{i(j)} = b_j$, $V_{i(j)} = W_j$ for $j = 1, \cdots, m$; the only part of 0 is 0.

If U is a subvariety of a projective space S over k, two cycles $\mathfrak{y}$, $\mathfrak{z}$ of S whose minimal representations are

$$\mathfrak{y} = \sum_{i=1}^{n} a_i V_i, \quad \mathfrak{z} = \sum_{j=1}^{m} b_j W_j$$

are said to *coincide locally at* U if either (1) no component of U is a subvariety of any V_i of of any W_j, or (2) if, say, $V_1, \cdots, V_r$ and $W_1, \cdots, W_s$ are the component varieties of $\mathfrak{y}$ and $\mathfrak{z}$ respectively which contain some component of U, then $r = s$, $V_i = W_i$ for $i = 1, \cdots, r$, and $a_i = b_i$ for $i = 1, \cdots, r$; the cycle

$$\sum_{i=1}^{r} a_i V_i = \sum_{j=1}^{s} b_j W_j$$

in case (2), or the cycle 0 in case (1), is called the *U-part of* $\mathfrak{z}$ (or *of* $\mathfrak{y}$); the *radical* rad $\mathfrak{z}$ *of* $\mathfrak{z}$ is the join of the component varieties of $\mathfrak{z}$ if $\mathfrak{z} \neq 0$, and is the empty variety if $\mathfrak{z} = 0$.

An algebraic correspondence is a cycle, not a pseudovariety. In the expressions $[D;\, V,\, G]$, $\{D;\, V,\, G]$, $(D;\, V,\, G)$, $D[G]$, $D(G)$, $\Delta[v]$, $\Delta(v)$, the symbols D and Δ are cycles, while the expressions themselves are pseudovarieties. In the expressions $\{D;\, V,\, G\}$, $\{D;\, V,\, G\}^*$, $D\{G\}$, $D\{G\}^*$, $\Delta\{v\}$, $\Delta\{v\}^*$, D and Δ are cycles, and so are the expressions themselves. In the expressions $e(D^*/D;\, V,\, G)$, $e(D^*/D;\, V,\, G)^*$, D is a cycle, D^* a pseudovariety. In the expressions ord $\mathfrak{z}$, deg $\mathfrak{z}$, red $\mathfrak{z}$, $\mathfrak{z}$ can be either a cycle or a variety; in the expressions ins $\mathfrak{z}$, exp $\mathfrak{z}$, $h(\mathfrak{z})$, $\mathfrak{z}$ can be either an irreducible cycle or an irreducible pseudovariety.

It is thus evident that if $\mathfrak{y}$, $\mathfrak{z}$ are cycles, then rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ is the variety which is the intersection of the varieties rad $\mathfrak{y}$ and rad $\mathfrak{z}$ (point-set theoretic), while $\mathfrak{y} \cap \mathfrak{z}$ has not been defined so far; and when it will be defined, it will be a cycle, not a variety.

Let V, F be varieties over k, F being irreducible, and let D be an unmixed algebraic correspondence between F and V, every component of which operates on the whole F; let G be an irreducible subvariety of F, D^* an irreducible component of $[D; V, G]$. The symbol $e(D^*/D; V, G)^*$ has been defined (when it exists) in [2] under the assumption that V and D be irreducible. We shall extend it now to a more general case. Let D be unmixed, and let $D = \sum_i a_i D_i$ be its minimal representation. Let v be a valuation of $k(F)$ over k, of the same dimension as G over k, and whose center on F is G; let $\{x^{(i)}\}$ be the h.g.p. (homogeneous general point) of D_i, and denote by $C_i(v)$ the complete set of extensions of v to $k(D_i)$ with respect to $\{x^{(i)}\}$ (see [2, § 3]). Assume $\dim D^* = \dim D - \dim F + \dim G$, and call $n_i(v)$ the number (≥ 0) of elements of $C_i(v)$ whose center on D_i is D^*. If

$$\sum_i a_i\, n_i\,(v) \text{ ins } D_i[F]\ (\text{ord } D^*[G])^{-1}$$

does not depend on v, this number will be denoted by

$$e(D^*/D;\ V,\ G)^* = e(D^*/D;\ G,\ V)^*.$$

Clearly, if D' is another unmixed algebraic correspondence between F and V, having the same dimension as D, and if $e(D^*/D; V, G)^*$ and $e(D^*/D'; V, G)^*$ both exist, then $e(D^*/aD + bD'; V, G)^*$ exists and equals

$$ae(D^*/D;\ V,\ G)^* + be(D^*/D';\ V,\ G)^*$$

for any pair of rational numbers a, b. As a consequence of statement 5 of Theorem 3.1 of [2], we have the result: if $v_{ij}\,(j = 1, 2, \cdots)$ are the distinct elements of $C_i(v)$ whose center on D_i is D^*, then

$$(1)\qquad e(D^*/D;\ V,\ G)^* = \sum_{ij} a_i[\Gamma_{v_{ij}}\colon \Gamma_v]\,[K_{v_{ij}}\colon k(D^*)]\,[K_v\colon k(G)]^{-1}.$$

If D^* has the dimension $\dim D - \dim F + \dim G$, but it is not a component of $[D; V, G]$, then we set, by definition, $e(D^*/D; V, G)^* = 0$. This is in accordance with (1), since in this case no element of any $C_i(v)$ has the center D^* on D.

According to [2], instead of saying that $e(D^*/D; V, G)^* = \alpha$, we shall also say that α is the *multiplicity of* D^* *in* $\{D; V, G\}^*$, even if $\{D; V, G\}^*$ does not exist; this will be extended to the other expressions, like "$\mathfrak{z}$ is part of $\{D; V, G\}^*$" and similar ones.

Let $\sum_i a_i V_i$ be the minimal representation of an unmixed cycle $\mathfrak{h}$ over k. If K is an extension of k, and $V_{ij}\,(j = 1, 2, \cdots)$ are the distinct components of $(V_i)_K$, the extension of $\mathfrak{h}$ over K has been defined in [2] to be

$$\mathfrak{y}_K = \sum_{ij} a_i \exp V_i (\exp V_{ij})^{-1} V_{ij},$$

the exponent of V_{ij} being independent of j. This had the advantage that $\Psi_{t,y}\ \mathfrak{y} = \Psi_{t,y}\ \mathfrak{y}_K$, and that $\deg \mathfrak{y} = \deg \mathfrak{y}_K$. We shall often need, however, to consider the cycle

$$\mathfrak{y}' = \sum_{ij} a_i \text{ ins } V_i (\text{ins } V_{ij})^{-1} V_{ij};$$

this, as remarked in [2, §1] is an alternate definition of the extension of a cycle. The cycle $\mathfrak{y}'$ shall be called the *modified extension of* $\mathfrak{y}$ *over* K, and no special symbol will be used to denote it. We have ord $\mathfrak{y}'$ = ord $\mathfrak{y}$. Let finally $\mathfrak{Y}$ be a cycle over K. We say that $\mathfrak{Y}$ is a *partial extension of* $\mathfrak{y}$ *over* K if $\mathfrak{Y} = \sum_i \mathfrak{Y}_i$, where each component variety of $\mathfrak{Y}_i$ is a component of $(V_i)_K$, and a_i ord V_i = ord $\mathfrak{Y}_i$.

LEMMA 1.1. *Let D, D^*, F, V, G, k have the same meanings as in formula (1). Let F' be birationally equivalent to F, and such that if G' is any irreducible subvariety of F' which corresponds to G, and which has the same dimension as G, then $Q(G/F) \subseteq Q(G'/F')$. Let D' be the algebraic correspondence between F' and V such that $D'\{F'\}^* = D\{F\}^*$; for each G' let D_1^*, D_2^*, $\cdots$ be the pseudo-varieties which correspond to D^* and such that $1D_i^*$ operates on G', and assume F' to be such that $e(D_i^*/D'; V, G')^*$ exists for each G' and each i. Then $e(D^*/D; V, G)^*$ exists if and only if*

$$\alpha = \sum_i e(D_i^*/D'; V, G')^* \text{ ord}(1D_i^*)[G']$$

does not depend on G'; that is, if and only if $\sum_i e(D_i^*/D'; V, G')^*\ D_i^*$ is a partial extension of a fixed multiple of $1D^*$ over $k(G')$ for any G'. In such case, we have

$$e(D^*/D; V, G)^* = \alpha(\text{ord}(1D^*)[G])^{-1}.$$

Proof. The proof of this lemma is an immediate application of (1), since the varieties G' are the centers on F' of the valuations v of formula (1).

COROLLARY. *Maintain the notations of Lemma 1.1, and let $\{\zeta\}$ be a set of parameters of $Q(G/F)$; then $\{\zeta\}$ is a set of parameters of each $Q(D^*/D_i)$. If $e(D^*/D; V, G)^*$ exists, it equals*

$$\sum_i a_i\ e(Q(D^*/D_i); \zeta)\ e(Q(G/F); \zeta)^{-1}.$$

Proof. In Lemma 1.1 choose for F' a normal associate to F, so that each $e(D_j^*/1D_i'; V, G')^*$ exists (by statement 1 of Theorem 5.3 of [2]) and equals $e(Q(D_j^*/D_i'); \zeta)\ e(Q(G'/F'); \zeta)^{-1}$. As a consequence of the lemma we then have

$$e(D^*/D\,;\ V,\ G)^* = \sum_{ij} a_i\ e(Q(D_j^*/D_i');\ \zeta)\ e(Q(G'/F');\ \zeta)^{-1} \times$$
$$\mathrm{ord}(1D_j^*)[G'](\mathrm{ord}(1D^*)[G])^{-1}$$

for any G'. There are finitely many varieties G' in this case, and we shall denote them by G_1', G_2', $\cdots$, while the D_j^*'s which operate on G_m' shall be denoted by D_{mj}^* $(j = 1, 2, \cdots)$. We have:

$$e(D^*/D;\ V,\ G)^* \sum_m e(Q(G_m'/F');\ \zeta)[k(G_m'):\ k(G)]\ \mathrm{ord}(1D^*)[G]$$
$$= \sum_i a_i \sum_{jm} e(Q(D_{mj}^*/D_i');\ \zeta)\ \mathrm{ord}(1D_{mj}^*)[G_m'][k(G_m'):\ k(G)],$$

or also

$$e(D^*/D\,;\ V,\ G)^* \sum_m e(Q(G_m'/F');\zeta)[k(G_m'):\ k(G)]$$
$$= \sum_i a_i \sum_{jm} e(Q(D_{mj}^*/D_i');\ \zeta)[k(D_{mj}^*):\ k(D^*)].$$

Now, by Lemma 2.2 of [2], we have

$$\sum_m e(Q(G_m'/F');\ \zeta)[k(G_m'):\ k(G)] = e(Q(G/F);\ \zeta)$$

and

$$\sum_{jm} e(Q(D_{mj}^*/D_i');\ \zeta)[k(D_{mj}^*):\ k(D^*)] = e(Q(D^*/D_i);\ \zeta),$$

Q.E.D.

We now maintain the same notations, and assume that V is irreducible and that each component of D, as well as $1D^*$, operates on the whole V. In this case $e(D^*/D;\ V,\ G)^*$ does not actually depend on D, but depends only on $D\{V\}^*$, by the above corollary, since $Q(D^*/D_i)$ contains $k(V)$. Accordingly, if Δ denotes $D\{V\}^*$ and Δ^* denotes $(1D^*)[V]$, we shall denote $e(D^*/D;\ V,\ G)^*$ also by $e(\Delta^*/\Delta)^*$. We remark that Δ^* can be described as a component of the intersection of rad Δ and $G_{k(V)}$ such that $1\Delta^*$ operates on the whole G. Let Δ_1^*, Δ_2^*, $\cdots$ be components of rad $\Delta \cap G_{k(V)}$ such that each $1\Delta_j^*$ operates on the whole G. If $\alpha_i = e(\Delta_i^*/\Delta)^*$ exists for each i, we shall say that $\sum_i \alpha_i \Delta_i^*$ is *part of the intersection* $G' \cap \Delta$ *of* G' *with* Δ, G' being the modified extension of $1G$ over $k(V)$. Notice that the symbol $\cap$ now links two cycles, so that no confusion may arise with rad Δ $\cap$ rad G'. This notation, as will appear later, is in agreement with the general intersection theory.

LEMMA 1.2. *Let K be an algebraic function field over k, Δ an algebraic correspondence between K and an irreducible variety F over k, every component of which operates on the whole F. Let K' be an algebraic function field over k*

containing K, and Δ' the modified extension of Δ over K'. Let G be an irreducible subvariety of F, Z and Z' the modified extensions of $1G$ over K and K' respectively. Let Δ^* be a component of rad $\Delta \cap$ rad Z, *such that* $1\Delta^*$ *operates on the whole G, and let Δ_i^* $(i = 1, 2, \cdots)$ be the distinct components of $\Delta^*_{K'}$; then each Δ_i^* is a component of* rad $\Delta' \cap$ rad Z', *and each $1\Delta_i^*$ operates on the whole G. The multiplicity $e(\Delta^*/\Delta)^*$ exists if and only if $e(\Delta_i^*/\Delta')^*$ exists for some i, in which case this exists and is the same for each i. If this is the case, then the modified extension $\Delta^{*\prime}$ of $e(\Delta^*/\Delta)^*\ \Delta^*$ over K' is part of $Z' \cap \Delta'$.*

Proof. Obviously each Δ_i^* is a component of rad $\Delta' \cap$ rad Z', and dim Δ^* = dim Δ_i^* for each i. Therefore, if Δ^* has the dimension dim Δ + dim G – dim F, so does Δ_i^*, and conversely. The contention which needs to be proved is the last one. Now, if K' is purely transcendental over K, also this contention becomes obvious, since in such a case there is exactly one Δ_i^*. We shall therefore assume K' to be an algebraic extension of K. Again, a well-known artifice makes it possible to prove the last contention if it is known that it holds true for each K' which is normal over K. Hence we restrict our attention further to the case in which K' is normal over K (the word "normal" does not imply separability).

Under these assumptions, let v be a valuation of $k(F)$ over k of dimension equal to dim G, and whose center on F is G. Clearly we may further assume Δ to be irreducible. Let then w be an extension of v to K(rad Δ), having the center Δ^* on rad Δ; let Δ_i' $(i = 1, 2, \cdots)$ be the component varieties of Δ', and let w' be an extension of w to $K'(\Delta_1')$, whose center on Δ_1' will therefore be, say, Δ_1^*. Each automorphism σ of the Galois group $\mathfrak{G}$ of K' over K can be interpreted, in a natural way, as an operator which transforms, isomorphically and transitively, the fields $K'(\Delta_i')$ into each other. Then $\sigma w'$ has a meaning, and when σ ranges in $\mathfrak{G}$, $\sigma w'$ ranges among all the extensions of w to $K'(\Delta_i')$, for each i, while the centers of these range among all the Δ_j^*. As a consequence, $[\Gamma_{\sigma w'}: \Gamma_w]$ and $[K_{\sigma w'}: K_w]$ are the same for each σ. The ramification theory gives then

$$[\Gamma_{\sigma w'}:\Gamma_w][K_{\sigma w'}:K_w] = [K'(\Delta_1'):K(\text{rad}\,\Delta)]\,m^{-1}\,n^{-1},$$

n being the number of distinct extensions of w to $K'(\Delta_1')$ whose center on Δ_1' is Δ_1^*, and m being the number of distinct $\sigma\Delta_1^*$ which are subvarieties of Δ_1'. Now, let $\alpha(w)$ be the sum of all the expressions $[\Gamma_{w''}: \Gamma_w][K_{w''}: K'(\Delta_1^*)]$ when w'' ranges over the distinct extensions of w to $K'(\Delta_i')$ whose center on Δ_i' is Δ_1^*, and $i = 1, 2, \cdots$. If m' denotes the number of distinct Δ_i' which contain Δ_1^*, from what precedes we obtain

$$\alpha(w) = nm'[\Gamma_{w'}:\Gamma_w][K_{w'}:K'(\Delta_1^*)]$$

$$= m'[K'(\Delta_1'):K(\text{rad }\Delta)]\, m^{-1}[K_w:K(\Delta^*)]\,[K'(\Delta_1^*):K(\Delta^*)]^{-1}.$$

Now, there is the relation

$$m \times \text{number of distinct } \Delta_i' = m' \times \text{number of distinct } \Delta_j^*;$$

that is,

$$m' m^{-1} = \text{red }\Delta \text{ red }\Delta_1^*(\text{red }\Delta_1' \text{ red }\Delta^*)^{-1};$$

on the other hand,

$$[K'(\Delta_1^*):K(\Delta^*)] = \text{ord }\Delta_1^*[K':K]\,(\text{ord }\Delta^*)^{-1},$$

and likewise for $[K'(\Delta_1'):K(\text{rad }\Delta)]$. Hence

$$\alpha(w) = \text{ins }\Delta_1' \text{ ins }\Delta^*(\text{ins }\Delta \text{ ins }\Delta_1^*)^{-1}[K_w:K(\Delta^*)].$$

If we denote by $\beta(v)$ the right side of formula (1), which would equal $e(\Delta^*/\Delta)^*$ if it were independent of v when $C(v/F) = G$, and by $\gamma(v)$ the similar expression for $e(\Delta_i^*/\Delta')^*$, then we have the relation:

$$\begin{aligned}\gamma(v) &= \textstyle\sum_w \text{ins }\Delta(\text{ins }\Delta_1')^{-1}\,\alpha(w)\,[\Gamma_w:\Gamma_v]\,[K_v:k(G)]^{-1}\\ &= \text{ins }\Delta^*(\text{ins }\Delta_1^*)^{-1}\textstyle\sum_w[\Gamma_w:\Gamma_v]\,[K_w:K(\Delta^*)]\,[K_v:k(G)]^{-1}\\ &= \text{ins }\Delta^*(\text{ins }\Delta_1^*)^{-1}\,\beta(v),\end{aligned}$$

where w ranges over all the extensions of v to $K(\text{rad }\Delta)$ whose center on rad Δ is Δ^*. This proves that $\gamma(v)$ is independent of v if and only if $\beta(v)$ has the same property, and, because of (1), also proves all the statements of Lemma 1.2. Q.E.D.

THEOREM 1.1. *Let D be an unmixed algebraic correspondence between the irreducible variety F over k and the variety V over k, every component of which operates on the whole F. Let P and G be irreducible subvarieties of F, P also being a subvariety of G, and let D' be a component of $[D;\, V,\, P]$ such that $e(D'/D;\, V,\, P)^*$ exists. Let $D_1^*, D_2^*, \cdots$ be the components of $[D;\, V,\, G]$ which contain D'; then*

$$\dim D_i^* = \dim D - \dim F + \dim G.$$

Assume $e(D_i^*/D;\, V,\, G)^*$ to exist for each i, and set

$$D^* = \textstyle\sum_i e(D_i^*/D;\, V,\, G)^*\, D_i^*.$$

Then $e(D'/D^*;\, V,\, P)^*$ exists and equals $e(D'/D;\, V,\, P)^*$.

Proof. If $r = \dim D_i^*$, then we have $\dim D' \geq r - \dim G + \dim P$. Since dim $D' =$ dim $D -$ dim $F +$ dim P, it follows that $r \leq$ dim $D -$ dim $F +$ dim G, and therefore the equal sign must hold. This proves the statement concerning the dimension. We shall give a proof of the main result under the assumption that D is irreducible; the proof in the general case would proceed exactly in the same way.

Let v be a valuation of $k(F)$ over k, of dimension equal to dim G, whose center of F is G, and let w_1' be a valuation of K_v over k, of dimension equal to dim P, which compounded with v gives a valuation of $k(F)$, of dimension equal to dim P, and whose center on F is P. Let u be the valuation of $k(G) \subseteq K_v$ induced by w_1', and let $w_1', w_2', \cdots$ be the distinct extensions of u to K_v. Denote by w_i the valuation of $k(F)$ which is compounded of v and w_i', so that $\mathrm{C}(w_i/F) = P$. For each i, let $v_{i1}, v_{i2}, \cdots$ be the distinct extensions of v to $k(\text{rad } D)$ having the center D_i^* on rad D, and let $u_{i1}, u_{i2}, \cdots$ be the distinct extensions of u to $k(D_i^*)$ having the center D' on D_i^*. For given i, j, r, l, let w'_{lijrs} $(s = 1, 2, \cdots)$ be the distinct extensions of u_{ir} to $K_{v_{ij}}$ which induce w_l' in K_v, and call w_{lijrs} the valuation of $k(\text{rad } D)$ compounded of v_{ij} and w'_{lijrs}. For a given l, the w_{lijrs} are all the distinct extensions of w_l to $k(\text{rad } D)$ which have the center D' on D; therefore formula (1) gives

$$e(D'/D; P, V)^* [K_{w_l'} : k(P)] = \sum_{ijrs} [\Gamma_{w_{lijrs}} : \Gamma_{w_l}] [K_{w'_{lijrs}} : k(D')];$$

now,

$$[\Gamma_{w_{lijrs}} : \Gamma_{w_l}] = [\Gamma_{v_{ij}} : \Gamma_v] [\Gamma_{w'_{lijrs}} : \Gamma_{w_l'}],$$

so that

$$e(D'/D; P, V)^* [K_{w_l'} : K_u] [K_u : k(P)] [\Gamma_{w_l'} : \Gamma_u]$$
$$= \sum_{ijrs} [\Gamma_{v_{ij}} : \Gamma_v] [\Gamma_{w'_{lijrs}} : \Gamma_{u_{ir}}] [\Gamma_{u_{ir}} : \Gamma_u] [K_{w'_{lijrs}} : K_{u_{ir}}] \times [K_{u_{ir}} : k(D')].$$

We now sum with respect to l, and use the formulas

$$\sum_l [K_{w_l'} : K_u] [\Gamma_{w_l'} : \Gamma_u] = [K_v : k(G)]$$

and

$$\sum_{ls} [K_{w'_{lijrs}} : K_{u_{ir}}] [\Gamma_{w'_{lijrs}} : \Gamma_{u_{ir}}] = [K_{v_{ij}} : k(D_i^*)],$$

obtaining

$$e(D'/D;\ P,\ V)^*\,[K_u\colon k(P)]\,[K_v\colon k(G)]$$
$$= \sum_i \Big(\sum_j [\Gamma_{v_{ij}}\colon \Gamma_v]\,[K_{v_{ij}}\colon k(D_i^*)]\Big)\Big(\sum_r [\Gamma_{u_{ir}}\colon \Gamma_u]\,[K_{u_{ir}}\colon k(D')]\Big).$$

This proves Theorem 1.1, since

$$e(D_i^*/D;\ G,\ V)^*\,[K_v\colon k(G)] = \sum_j [\Gamma_{v_{ij}}\colon \Gamma_v]\,[K_{v_{ij}}\colon k(D_i^*)], \qquad \text{Q.E.D.}$$

It is hardly worth mentioning that if w is a valuation of $k(F)$ compounded of a valuation v of $k(F)$ and a valuation u of K_v, then

$$\Delta\{w\}^* = (\Delta\{v\}^*)\{u\}^*;$$

the proof of this fact is an immediate consequence of the obvious relation $\Delta\{w\} = (\Delta\{v\})\{u\}$. Another result which will be used later is the following: If Δ is an algebraic correspondence between the algebraic function field K over k and the variety V over k, let k' be an extension of k, K' a composite of K and k' over k (that is, the quotient field of the homomorphic image of $K \times k'$ over k modulo one of its prime ideals), Δ' the modified extension of Δ over K', so that Δ' is an algebraic correspondence between K' and $V' = V_{k'}$. If v is a valuation of K over k, v' any extension of v to K' over k', then $\Delta'\{v'\}^*$ is the modified extension of $\Delta\{v\}^*$ over $K_{v'}$. This fact also is derived from the analogous result concerning $\Delta\{v\}$, namely: if $\Delta' = \Delta_{K'}$, then $\Delta'\{v'\}$ is the extension of $\Delta\{v\}$ over $K_{v'}$.

Finally, the extension of the meaning of $e(D^*/D;\ V,\ G)^*$ to the case in which D is reducible, and in particular the corollary to Lemma 1.1, affords a generalization of the reduction theorem (Theorem 5.4 of [2]) in the following sense:

THEOREM. 1.2. *In the statement of Theorem* 4.2 *of* [2], *let us replace the assumption of the existence of* $\{D;\ V_j,\ W_i\}^*$ *and* $\{D_h^{(i)};\ W_j,\ W_i\}^*$ *by the following assumption:*

$$e(D_h^{(i)}/D;\ V_j,\ W_i)^* \text{ exists for each } h,\ i,$$

and if

$$D^{(i)} = \sum_h e(D_h^{(i)}/D;\ V_j,\ W_i)^*\, D_h^{(i)},$$

then $e(U/D^{(i)};\ W_j,\ W_i)^*$ *exists for each* i. *Let us replace, moreover, the assumption that* D *is irreducible by the assumption that* D *is unmixed. Then* $e(U/D^{(i)};\ W_j,\ W_i)^*$ *does not depend on* i.

2. The homographic system. An irreducible algebraic system $\mathfrak{C}$ of integral effective cycles is in one-to-one correspondence with the irreducible variety $G = G(\mathfrak{C})$ (see [1]); therefore we shall apply to $\mathfrak{C}$ the language adapted to varieties. For instance, if G is a variety over k, we shall write $k(\mathfrak{C})$ in place of $k(G)$, $M(\mathfrak{C})$ in place of $M(G)$ (this denotes the set of the places of G; see [1]); the cycle $\Delta = \Delta(\mathfrak{C})$ shall be referred to as the *general element* or *general cycle of* $\mathfrak{C}$.

A *linear variety* is an irreducible variety L over a field k such that ord $L = 1$, or, equivalently, such that deg $L = 1$. From the definition of order or degree [1, § 2; 2, § 1], it appears that an r-dimensional irreducible subvariety V of the projective space $S = S_n(k)$ is linear if and only if $\wp(V/k[X])$ has a basis consisting of linear (i.e. of degree 1) forms in the X's $\{X\}$ being the h.g.p. of S; and a minimal basis will consist then of n-r linear forms. After an obvious identification, it also follows that a linear variety is a projective space. A *linear cycle* is an irreducible cycle whose radical is a linear variety.

Let S be an n-dimensional projective space over k, $\{x\}$ its h.g.p., and let X denote the one-column matrix $(x_0, \cdots, x_n)$, while $U = (u_{ij})$ is a square matrix of order $n+1$ with elements in k. Set $X' = UX$, and let $x_0', \cdots, x_n'$ be the elements of the one-column matrix X'; let $\mathfrak{U}$ be the homomorphic mapping of $k[x]$ such that $\mathfrak{U}a = a$ if $a \in k$, $\mathfrak{U}x_i = x_i'$ $(i = 0, \cdots, n)$; if det $U \neq 0$, $\mathfrak{U}$ is an automorphism and transforms in an obvious way an ideal of $k[x]$ into an ideal of $k[x]$, a subvariety of S into a subvariety of S, and a cycle of S into a cycle of S. U will be called the *matrix* of $\mathfrak{U}$; two $\mathfrak{U}$'s whose matrices have proportional elements have the same effect on homogeneous ideals, subvarieties, and cycles, and shall be identified; $\mathfrak{U}$ is called a *nondegenerate homography of* S. If $\mathfrak{z}$ is a cycle of S, then $\mathfrak{U}\mathfrak{z}$ is called a *homographic transform of* $\mathfrak{z}$.

Maintaining the same notations, assume the u_{ij}'s to be indeterminates; then $\mathfrak{U}$ is a nondegenerate homography of $S_n(k(u))$, and will be referred to as the *general homography of* S. Set $K' = k(u)$, so that K' is homogeneous for the set $\{u_{00}, \cdots, u_{nn}\}$; let K be the subfield of K' consisting of all the homogeneous elements of degree zero of K'. If $\mathfrak{z}$ is an unmixed cycle of S, set $\mathfrak{Z}' = \mathfrak{U}\mathfrak{z}_{K'}$; then $\mathfrak{Z}'$ is a cycle of $S_{K'}$, and it is the extension over K' of a cycle $\mathfrak{Z}$ of S_K. Clearly $\mathfrak{Z}$ is an unmixed algebraic correspondence between K and S, and is called the *general homographic transform of* $\mathfrak{z}$. Assume $\mathfrak{z}$ to be integral and effective; if $\bar{k}$ is the algebraic closure of k, and $\bar{\mathfrak{z}}$, $\bar{\mathfrak{Z}}$ are the extensions of $\mathfrak{z}$, $\mathfrak{Z}$ over $\bar{k}$, $K\bar{k}$ respectively, then $\bar{\mathfrak{Z}}$ is related to $\bar{\mathfrak{z}}$ as $\mathfrak{Z}$ is to $\mathfrak{z}$, and the set $\mathfrak{H}$ of the cycles $\bar{\mathfrak{Z}}\{v\}$, where v ranges over the places of $K\bar{k}$ over $\bar{k}$, is an algebraic system of cycles on[1] S, called the *homographic system of* $\mathfrak{z}$.

[1] Note that, according to [1] or [2], a cycle *on* S means a cycle *of* the extension of S over the algebraic closure of k.

LEMMA 2.1. *The homographic system of $\mathfrak{z}$ is the smallest algebraic system of cycles on S containing all the homographic transforms of $\overline{\mathfrak{z}}$.*

Proof. Set $\overline{\overline{K}} = K\overline{k}$, and let $v \in M(\overline{K})$. Let u_{00} be such that $v(u_{ij}\, u_{00}^{-1}) \geq 0$ for every i, j. Let σ be the homomorphic mapping of R_v whose kernel is $\mathfrak{P}_v$, and set $u_{ij}(v) = \sigma(u_{ij}\, u_{00}^{-1})$; since u_{00} is not necessarily the only u_{rs} such that $v(u_{ij}\, u_{rs}^{-1}) \geq 0$ for each i, j, the set $\{u_{ij}(v)\}$ is determined but for a nonzero factor in $\overline{k}$. Let $U(v)$ be the matrix obtained after replacing, in U, each u_{ij} by the corresponding $u_{ij}(v)$: if det $U(v) \neq 0$, then $U(v)$ is the matrix of a nondegenerate homography $\upsilon(v)$. These notations will be used throughout this section.

We contend that $\upsilon(v)\,\overline{\mathfrak{z}} = \overline{\mathfrak{Z}}\{v\}$, and this will completely prove the lemma. Let $\psi(t, y)$ be a determination of $\Psi t, y\overline{\mathfrak{z}} = \Psi t, y\mathfrak{z}$ (see [1, §2]); denote by Y the one-column matrix $(y_0, \cdots, y_{r+1})$, r being the dimension of $\mathfrak{z}$, and by T the matrix (t_{ij}), so that $Y = TX$; υ can be extended in a natural way to $\overline{k}(t, u, x)$, and we have

$$\upsilon Y = \upsilon(TX) = T(\upsilon X) = TUX = (\tau T)X = \tau(TX) = \tau Y,$$

where by τ we denote the automorphism of $\overline{k}(t, u, x)$ over $\overline{k}(u, x)$ such that $\tau T = TU$. If v has the previous meaning, $T(v)$ and $\tau(v)$ will be related to T, τ, v as $U(v)$, $\upsilon(v)$ are to U, υ, v. If $\overline{\mathfrak{z}}$ is irreducible, set

$$\mathfrak{p} = \wp(\text{rad}\ \overline{\mathfrak{z}}/\overline{k}[x])\,\overline{K}'(t)[x],$$

where $\overline{K}' = K'\overline{k}$; we have, by definition,

$$\psi(t, y)\,\overline{K}'(t)[y] = \mathfrak{p} \cap \overline{K}'(t)[y],$$

hence

$$\psi(t, \upsilon y)\,\overline{K}'(t)[\upsilon y] = \upsilon\mathfrak{p} \cap \overline{K}'(t)[\upsilon y].$$

Applying τ^{-1}, and using the fact that $\upsilon y = \tau y$, we obtain

$$\psi(\tau^{-1} t, y)\,\overline{K}'(t)[y] = \upsilon\mathfrak{p} \cap \overline{K}'(t)[y],$$

which proves that $\psi(\tau^{-1} t, y)$ is a determination of $\Psi_{t,y}\,\overline{\mathfrak{Z}}'$; hence $\psi(\tau^{-1}(v)\,t, y)$ is a determination of $\Psi_{t,y}\,\overline{\mathfrak{Z}}\{v\}$. But, since

$$\tau^{-1}(v) = (\tau(v))^{-1},$$

we see in like manner that $\psi(\tau^{-1}(v), y)$ is a determination of $\Psi_{t,y}\,\upsilon(v)\,\overline{\mathfrak{z}}$. It is thus proved that $\upsilon(v)\,\overline{\mathfrak{z}} = \overline{\mathfrak{Z}}\{v\}$ if $\overline{\mathfrak{z}}$ is irreducible. If $\overline{\mathfrak{z}}$ is not irreducible, the same relation is easily established as a consequence of its validity for irreducible cycles, Q.E.D.

LEMMA 2.2. *The homographic system of* $\mathfrak{z}$ *contains the homographic system of each of its cycles.*

Proof. Let $\mathfrak{H}$ be the homographic system of $\mathfrak{z}$, and let $\mathfrak{z}_1 \in \mathfrak{H}$, so that

$$\Psi_{t,y}\ \mathfrak{z}_1 = \psi(\tau^{-1}(v)\ t,\ y)$$

(but for a proportionality coefficient) for some $v \in M(\overline{K})$; here $\tau^{-1}(v)t_{ij}$ has to be interpreted as the ij-th element of the matrix $TU^{-1}(v)$, which has a meaning even if $\det U(v) = 0$. Let $\mathfrak{Z}_1'$, $\mathfrak{Z}_1$ be obtained from $\mathfrak{z}_1$ as $\overline{\mathfrak{Z}}'$, $\overline{\mathfrak{Z}}$ are from $\overline{\mathfrak{z}}$; we have

$$\Psi_{t,y}\ \mathfrak{Z}_1' = \psi(\tau^{-1}(v)\ \tau^{-1}t,\ y).$$

For any $v' \in M(K)$ we have therefore

$$\Psi_{t,y}\ \mathfrak{Z}_1\{v'\} = \psi(\tau^{-1}(v)\ \tau^{-1}(v')\ t,\ y).$$

Now, there exists a place $v'' \in M(\overline{K})$ such that

$$\tau^{-1}(v)\ \tau^{-1}(v')T = \tau^{-1}(v'')\ T,$$

so that

$$\Psi_{t,y}\ \mathfrak{Z}_1\{v'\} = \psi(\tau^{-1}(v'')\ t,\ y) = \Psi_{t,y}\ \overline{\mathfrak{Z}}\{v''\},$$

or $\mathfrak{Z}_1\{v'\} = \overline{\mathfrak{Z}}\{v''\}$, Q.E.D.

LEMMA 2.3. *Let* $\mathfrak{z}$ *be an unmixed integral effective cycle of* $S = S_n(k)$, *and let* $\mathfrak{Z}$ *be the general homographic transform of* $\mathfrak{z}$. *Set* $G = G_{\mathfrak{Z}}$ *(see* [1], *Lemma* 4.2); *let* Δ *be the algebraic correspondence between* G *and* S *induced by* $\mathfrak{Z}$ *according to Lemma* 4.2 *of* [1], *and set* $Z = D_{\Delta,G}$. *Let* $\overline{k}$ *be the algebraic closure of* k, *and let* P *be a point of* G *such that* $(Z\{P\})_{\overline{k}}$ *is a homographic transform of* $\mathfrak{z}_{\overline{k}}$. *Then* G *is analytically irreducible at* P.

Proof. Let $\mathfrak{H}$ be the homographic system of $\mathfrak{z}$, and set $\overline{G} = G(\mathfrak{H})$; then $\overline{G}$ is a component of the extension of G over the algebraic closure $\overline{k}$ of k. Assume the lemma to be true when k is algebraically closed. In this case, $\overline{G}$ is analytically irreducible at each $\overline{P} \in \overline{G}$ such that $\overline{P}$ is the image point of a homographic transform of $\overline{\mathfrak{z}} = \mathfrak{z}_{\overline{k}}$. Let P be the point mentioned in the lemma, $R = Q(P/G)$, $\overline{P}$ the image (on $\overline{G}$) of $(Z\{P\})_{\overline{k}}$, $\overline{R} = Q(\overline{P}/\overline{G})$. If $\mathfrak{m} = \mathfrak{P}(P/G)$, $\overline{\mathfrak{m}} = \mathfrak{P}(\overline{P}/\overline{G})$, we have that $\mathfrak{m}\overline{k}$ is a primary ideal of $R\overline{k}$ belonging to $\overline{\mathfrak{m}} \cap R\overline{k}$, and that $\overline{\mathfrak{m}}^h \cap R \subseteq \mathfrak{m}^l$, where $l \longrightarrow \infty$ when $h \longrightarrow \infty$. Therefore the topology induced in R by the $\bar{R}$-topology is the R-topology, so that the completion R' of R is a subring of the completion $\bar{R}'$ of $\bar{R}$. Since, by assumption, $\bar{R}'$ is an integral domain, so is R'; that is, G is analytically irreducible at P. This shows that it is enough to prove the statement under the further assumption that k be algebraically closed.

Under this assumption, let $\mathfrak{z}'$ be a homographic transform of $\mathfrak{z}$, and set $P = P(\mathfrak{z})$, $P' = P(\mathfrak{z}')$, so that $\mathfrak{z} = Z\{P\}$, $\mathfrak{z}' = Z\{P'\}$. Let K have the previous usual meaning. For each $v_0 \in M(K)$ whose center on G is P' we have $\det U(v_0) \neq 0$; let π be the automorphism of $k(u)$ over k such that $\pi U = U^{-1}(v_0)\, U$. We have, for $v \in M(K)$:

$$\Psi_{t,y}\, \mathfrak{Z}\{\pi v\} = \psi(\tau^{-1}(\pi v)\, t, y).$$

Now,

$$\tau T = TU, \quad \tau^{-1} T = TU^{-1},$$

$$\tau^{-1}(\pi v)\, T = TU^{-1}(\pi v) = T(\pi^{-1} U^{-1})(v) = TU^{-1}(v)\, U^{-1}(v_0)$$

$$= \tau^{-1}(v)\, \tau^{-1}(v_0)\, T,$$

so that

$$\Psi_{t,y}\, \mathfrak{Z}\{\pi v\} = \psi(\tau^{-1}(v)\, \tau^{-1}(v_0)\, t, y).$$

On the other hand, as we have already seen, $\Psi_{t,y}\, \upsilon(v_0)\, \mathfrak{Z}\{v\}$ is obtained from

$$\Psi_{t,y}\, \mathfrak{Z}\{v\} = \psi(\tau^{-1}(v)\, t, y)$$

by replacing $\{t\}$ with $\{\tau^{-1}(v_0)\, t\}$, so that

$$\Psi_{t,y}\, \upsilon(v_0)\, \mathfrak{Z}\{v\} = \psi(\tau^{-1}(v)\, \tau^{-1}(v_0)\, t, y) = \Psi_{t,y}\, \mathfrak{Z}\{\pi v\}.$$

It follows that

$$\mathfrak{Z}\{\pi v\} = \upsilon(v_0)\, \mathfrak{Z}\{v\},$$

and this proves that $\mathfrak{C}(\pi v/G)$ depends only on $\mathfrak{C}(v/G)$. Then the same is true for $\mathfrak{C}(v/\pi^{-1} G)$ and $\mathfrak{C}(v/G)$. Let H be the smallest subfield of K containing $k(G)$ and $\pi^{-1}(k(G)) = k(\pi^{-1} G)$; the embedding of $k(G)$ and $k(\pi^{-1} G)$ in H gives an irreducible algebraic correspondence C between G and $\pi^{-1} G$, and the above-proved property shows that C has the same dimension as G, and that $k(\operatorname{rad} C)$ is purely inseparable over $k(G)$. Besides, if $P = P\{\mathfrak{z}\} \in G$, then $C[P]$ is the point $\pi^{-1} P'$ of $\pi^{-1} G$, and $P = C[\pi^{-1} P']$. Now, by Lemma 2.1, P' can be chosen in such a way that G is analytically irreducible at P', and therefore $\pi^{-1} G$ is analytically irreducible at $\pi^{-1}P'$. Let G^* be a normal associate to G, C^* the irreducible algebraic correspondence between $\pi^{-1}G$ and G^* generated by the embedding of $k(\pi^{-1}G)$ and $k(G^*)$ in H. Should G be not analytically irreducible at P, $C^*[\pi^{-1}P']$ would contain two distinct points, which is impossible by Theorem 4.1 of [1]. Hence G is analytically irreducible at P. By Lemma 2.1, however, we can choose for P the image of any cycle $\mathfrak{z}''$ of $\mathfrak{H}$ whose homographic system is $\mathfrak{H}$, Q.E.D.

THEOREM 2.1. *Maintain the same notation as in Lemma 2.3. If V is an irreducible subvariety of S, then $Z\{V\}$ exists, and each component of the total transform $\{Z; V, G]$ operates on the whole V.*

Proof. Let D be a component of $\{Z; V, G]$, P a point of V on which $1D$ operates, and assume

$$\dim D > \dim Z - \dim S + \dim V.$$

If then D' is a component of $[D; P, G]$, we have

$$\dim D' \geq \dim D - \dim V > \dim Z - \dim S,$$

and D' belongs to $(Z; P, G)$. If, therefore, we show that each component of $[Z; P, G]$ has dimension equal to $\dim Z - \dim S$, it is also proved that each component of $\{Z; V, G]$ has dimension equal to $\dim Z - \dim S + \dim V$, and that as a consequence $Z\{V\}$ exists, because V is simple on S (see statement f of Lemma 4.2 of [1]). In order to show that $\{Z; P, G]$ has the pure dimension $\dim Z - \dim S$, we proceed as follows: let $\bar{k}$ be the algebraic closure of k, and let $\bar{\mathfrak{Z}}$ be the general homographic transform of $\mathfrak{z}$; let K have the usual meaning, and set $\bar{K} = K\bar{k}$, $\bar{\mathfrak{Z}} = \mathfrak{Z}_{\bar{K}}$, $\bar{\mathfrak{z}} = \mathfrak{z}_{\bar{k}}$; let $\bar{G}$, $\bar{Z}$ be related to $\bar{\mathfrak{Z}}$ as G, Z are to $\mathfrak{Z}$, so that $\bar{G}$ is a component of the extension of G over $\bar{k}$. Let $P_1, P_2, \cdots$ be the components of $P_{\bar{k}}$; we have $\bar{Q} \in [\bar{Z}; P_i, \bar{G}]$ for some i if and only if there exists a $Q \in [Z; P, G]$ such that $\bar{Q}$ is a component of $Q_{\bar{k}}$. Therefore $[Z; P, G]$ has the pure dimension $\dim Z - \dim S$ if and only if each $[\bar{Z}; P_i, \bar{G}]$ has the same property. As a consequence, it is sufficient to prove the statement under the further assumption that k is algebraically closed. Under this assumption, let $P' \in S$, and let π be a non-degenerate homography of S such that $\pi P = P'$. Let M be the matrix of π, so that $\pi X = MX$ (X being the one-column matrix $(x_0, \cdots, x_n)$). Let σ be the automorphism of $k(u)$ over k such that $\sigma U = MU$. Then it is possible to prove (by the same method used in the proof of Lemma 2.3) the following: if $v \in M(K)$ and $P \in \text{rad}(\mathfrak{Z}\{v\})$, then $P' \in \text{rad}(\mathfrak{Z}\{\sigma^{-1}v\})$; in other words, $Z[P']$ is the total transform of $\sigma^{-1}Z[P]$ in the algebraic correspondence C (between $\sigma^{-1}G$ and G) generated by the embedding of $k(G)$ and $k(\sigma^{-1}G)$ in K. Now, C is the same as the algebraic correspondence C used in the proof of Lemma 2.3, concerning which it was proved that it does not have fundamental points either on G or on $\sigma^{-1}G$. Therefore C has no fundamental variety either on G or on $\sigma^{-1}G$. Since P' can be chosen in such a way that $Z[P']$ has the pure dimension $\dim Z - \dim S$, it follows that $Z[P]$ also has the pure dimension $\dim Z - \dim S$, as asserted.

Suppose that a component D of $[Z; V, G]$ operates on $W \subset V$, so that it is also a component of $[Z; W, G]$. From the above proof it follows that

$$\dim D = \dim Z - \dim S + \dim W < \dim Z - \dim S + \dim V,$$

a contradiction, Q.E.D.

We say that a cycle or a subvariety $\mathfrak{z}$ of S is *degenerate* if each component of $\mathfrak{z}$ is a linear cycle or subvariety.

LEMMA 2.4. *The homographic system of an unmixed cycle of $S = S_n(k)$ contains some degenerate cycle.*

Proof. We may assume k to be algebraically closed, since we are dealing with an algebraic system. In view of Lemma 2.2, the statement is true if it is true when $\mathfrak{z}$ is irreducible. Therefore we assume $\mathfrak{z}$ to be irreducible. Set $r = \dim \mathfrak{z}$, and let F be a linear subvariety of S such that rad $\mathfrak{z} \cap F$ consists of finitely many points; we also require F to have dimension $n - r$. Such an F certainly exists, because by repeated application of the theorem according to which each minimal prime of a principal ideal is maximal dimensional, one can easily establish that the intersection of rad $\mathfrak{z}$ with a linear subvariety of S of dimension s has dimension $\geq r + s - n$, and that there exists some s-dimensional linear subvariety of S whose intersection with rad $\mathfrak{z}$ has the pure dimension $r + s - n$ if this number is not negative.

Let $\{x\}$ be the h.g.p. of S, and let $l_1, \cdots, l_r$ be the linear forms in the x's forming a basis of $\wp(F/k[x])$. The system of equations $l_i = 0 (i = 1, \cdots, r)$ can be solved for r among the x's, say $x_{n-r+1}, \cdots, x_n$, and the solution is written in the form

$$x_i = \sum_{j=0}^{n-r} a_{i-n+r,j}\, x_j \qquad (a_{p,q} \in k,\ i = n - r + 1, \cdots, n).$$

Let U' be the square matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdots & & \cdots & & \cdots & \cdot \\ 0 & \cdots & & 1 & 0 & \cdots & 0 \\ a_{1,0} & \cdots & & a_{1,n-r} & 0 & \cdots & 0 \\ \cdot & \cdots & & \cdots & & \cdots & \cdot \\ a_{r,0} & \cdots & & a_{r,n-r} & 0 & \cdots & 0 \end{pmatrix}$$

of order $n + 1$. Let $v \in M(K)$ be such that $U(v) = U'$, and set

$$\mathfrak{p} = \wp(\text{rad } \mathfrak{z}/k[x]).$$

Let T be the projective space over k whose h.g.p. is $\{u\}$, and set

$$u'_{ij} = u_{ij}\, u_{00}^{-1},$$

so that v is at finite distance for $\{u'\}$. If $\{p_1(x), p_2(x), \cdots\}$ is a basis of $\mathfrak{p}$, set

$$x'_i = \sum_j u'_{ij}\, x_j,$$

and let $\mathfrak{D}$ be the radical of the ideal of $k[x, u']$ whose basis is

$$\{p_1(x'), p_2(x'), \cdots\}.$$

If $D = \wp(\mathfrak{D})$, then $1D$ is an algebraic correspondence between T and S, and it differs from $Z = D_{\mathfrak{Z},T}$ at most for components which do not operate on the whole T. Set

$$P = \mathrm{C}(v/T), \quad \mathfrak{q} = \mathrm{C}(v/k[u']),$$

and let σ be the homomorphic mapping of $k[x, u']$ whose kernel is $\mathfrak{q}k[x, u']$. Then $\{\sigma x'\}$ is the h.g.p. of F, and $\{p(\sigma x')\}$ is the basis of an ideal of $k[x]$ whose radical is $\wp((1D)[P]/k[x])$. However, since $\{\sigma x'\}$ is the h.g.p. of F, $\{p(\sigma x')\}$ is also the basis of an ideal of $k[x_0, \cdots, x_{n-r}]$ whose radical $\mathfrak{R}$ is $\wp(\text{rad } \mathfrak{z} \cap F/k[x_0, \cdots, x_{n-r}])$; therefore $\mathfrak{R}$ is purely 0-dimensional. Also, $\mathfrak{R}$ can be extended to an ideal $\mathfrak{R}k[x]$ of $k[x]$, and

$$\mathfrak{R}k[x] = \wp((1D)[P]/k[x]).$$

Now, $\mathfrak{R}k[x]$ is purely r-dimensional; besides, each minimal prime of $\mathfrak{R}$, being a 0-dimensional ideal of $k[x_0, \cdots, x_{n-r}]$, has a basis consisting of linear forms in the x's with coefficients in k, and the same must be true of each minimal prime of $\mathfrak{R}k[x]$. This proves that $(1D)[P]$ is a degenerate r-dimensional variety. Since $Z[P] \subseteq (1D)[P]$, and since each component of $Z[P]$ has dimension $\geq r$, this also proves that (1) $Z[P]$ is purely r-dimensional, and (2) $Z[P]$ is degenerate. From (1), and from the fact that T is locally normal at P, follows that $Z[P] = \mathfrak{Z}[v]$, so that (2) implies that $\mathfrak{Z}[v]$, which is the radical of a cycle of the homographic system of $\mathfrak{z}$, is degenerate, Q.E.D.

LEMMA 2.5. *Maintain the notations of Theorem* 2.1, *and assume k to be algebraically closed, and $\mathfrak{z}$ to be irreducible. Then $Z[V]$ is irreducible.*

Proof. Since, by Theorem 2.1, $Z\{V\}$ exists, and so does $Z\{P\}$ if $P \in V$, by Theorem 1.1 we have that $(Z\{V\})\{P\}$ exists, and that it is enough to prove the lemma under the additional assumption that V is a point. Besides, the same argument used in the proof of Theorem 2.1 shows that $Z[P]$ is either irreducible for each $P \in G$, or reducible for each $P \in G$. Set $D = D_{\mathfrak{Z},T}$, T having the same meaning as in the proof of Lemma 2.4. In order to prove that $Z[P]$ is irreducible, it is enough to prove that $D[P]$ is irreducible for some (hence for each) $P \in S$. Let W be the subvariety of T consisting of the centers on T of those $v \in M(T)$ for which $\det U(v) = 0$. We shall show first that if it is true that $D[P]$ has only one component outside W for $P \in S$, then it is also true that $D[P]$ is irreducible. In fact, let $\mathfrak{A}$ be the prime algebraic system of cycles of T whose general element is $D\{S\}$ (after extending it over $k(S)$). If $D[P]$ is reducible for each $P \in S$, then $\mathfrak{A}$ is not simple; according to Theorem 5.4 of [1], $\mathfrak{A}$ is then composed with a simple algebraic system $\mathfrak{A}'$ and an involution $\mathfrak{I}$ on $G(\mathfrak{A}')$; $\mathfrak{A}'$ contains cycles which have no component variety on W (because not every element of $\mathfrak{A}$ has the radical in W), and $\mathfrak{I}$ contains cycles which have no component variety in any one given proper subvariety of $G(\mathfrak{A}')$. Therefore $\mathfrak{A}$ contains cycles which have no component variety in W, and this proves that for some (hence for each) $P \in S$, $D[P]$ is irreducible, as claimed.

For any point $Q \in T - W$ we shall write $\mathfrak{v}(Q)$ instead of $\mathfrak{v}(v)$, $v \in M(T)$, $\mathcal{C}(v/T) = Q$. Then $D[P] - (D[P] \cap W)$ consists of the $Q \in T - W$ such that $\mathfrak{v}^{-1}(Q) \in \operatorname{rad} \mathfrak{z}$. Let $\mathfrak{P}$ be the general homographic transform of P constructed with the general homography $\mathfrak{v}^{-1}$ (rather than $\mathfrak{v}$), and set $E = D_{\mathfrak{P},T}$; then $\mathfrak{v}^{-1}(Q)P = E[Q]$ if $Q \in T - W$, so that $D[P] - (D[P] \cap W) = L - (L \cap W)$, where L is the subvariety of T on which $E[\operatorname{rad} \mathfrak{z}]$ operates. If we prove that $E[\operatorname{rad} \mathfrak{z}]$ is irreducible, it will follow that L is irreducible, as desired. Now, the same argument used at the beginning of this proof shows that $E[\operatorname{rad} \mathfrak{z}]$ is irreducible if $E[P']$ is irreducible for some (hence for each) $P' \in S$, or also if $E[P']$ has only one component outside W for a $P' \in S$, say $P' = P$. But this is obviously true, since the set of the $Q \in T - W$ for which $\upsilon^{-1}(Q)P = P$, that is, for which $\upsilon(Q)P = P$, is a linear variety less its intersection with W, Q.E.D.

THEOREM 2.2. *Notations as in Theorem* 2.1. *Set* $n = \dim S$, $r = \dim \mathfrak{z}$, $s = \dim V$. *If* $r + s - n \geq 0$, *then each component of* $[Z; V, G]$ *operates on the whole* G, $V \cap \operatorname{rad} \mathfrak{z}$ *is not empty, and each of its components has dimension* $\geq r + s - n$.

Proof. The proof of this result, like that of Theorem 2.1, is readily reduced to the case in which k is algebraically closed and $\mathfrak{z}$ is irreducible. In this case, according to Lemma 2.5, $D = [Z; V, G]$ is irreducible, and, by Theorem 2.1, $D = \{Z; V, G\}$. If P is a point of G such that $Z\{P\} = \mathfrak{z}'$, then $V \cap \operatorname{rad} \mathfrak{z}' = (1D)[P]$ by Theorem 2.1. Set $d = \dim D$, and let F be the irreducible subvariety of G on which $1D$ operates. Then $d = r + m - n + s$, where $m = \dim G$. Therefore, $(1D)[P]$ is empty if $P \notin F$, while if $P \in F$ each component of $(1D)[P]$, hence of $V \cap \operatorname{rad} \mathfrak{z}$, has dimension $\geq r + s - n + m - \dim F$. By Lemma 2.4, the homographic system $\mathfrak{H}$ of $\mathfrak{z}$ contains some degenerate cycle $\mathfrak{z}''$, and therefore, by Lemma 2.2, it contains the homographic system $\mathfrak{H}'$ of $\mathfrak{z}''$. According to the first part of the proof of Lemma 2.4, $\mathfrak{H}'$ contains some cycle $\mathfrak{z}_0$ such that $V \cap \operatorname{rad} \mathfrak{z}_0$ is nonempty and has pure dimension $r + s - n$. If $P_0 \in G$ is such that $\mathfrak{z}_0 = Z\{P_0\}$, it follows that $P_0 \in F$ and that

$$r + s - n + m - \dim F \leq r + s - n,$$

that is, that $\dim F = m$, $F = G$. Hence $1D$ operates on the whole G, as claimed, and each component of $(1D)[P]$, for any P, has dimension $\geq r + s - n$, Q.E.D.

3. **Intersection of cycles of a projective space.** *In this section* S *denotes an* n*-dimensional projective space over* k.

If $\mathfrak{y}$, $\mathfrak{z}$ are unmixed cycles of S, of dimensions r, s respectively, such that $r + s - n \geq 0$, then a component V of $\operatorname{rad} \mathfrak{z} \cap \operatorname{rad} \mathfrak{y}$ is said to be a *component variety of* $\mathfrak{z} \cap \mathfrak{y}$ or *of* $\mathfrak{y} \cap \mathfrak{z}$ if $\dim V = r + s - n$.

Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed integral effective cycles of S, of dimensions r, s respectively; assume V to be a component variety of $\mathfrak{y} \cap \mathfrak{z}$, and let $\mathfrak{y} = \sum_i a_i \mathfrak{y}_i$ be the minimal representation of $\mathfrak{y}$. Let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$, $G = G_{\mathfrak{Z}}$, Z the algebraic correspondence between $k(G)$ and S induced by $\mathfrak{Z}$ according to Lemma 4.2 of [1]. Let P be the (unique) point of G such that $Z\{P\} = \mathfrak{z}$. Then P is a rational point, so that $V \times P$ is irreducible, and $V \times P$ is a pseudosubvariety of $\{Z; \mathfrak{y}_i, G]$ for some i; Theorems 2.1 and 2.2 imply then that $V \times P$ is a component of $[\{Z; \mathfrak{y}_i, G\}; \mathfrak{y}_i, P]$ for some i. Now assume $Z = \sum_j c_j Z_j$ to be a minimal representation of Z, and let $\mathfrak{z}_1, \mathfrak{z}_2, \cdots$ be the distinct component varieties of $\mathfrak{z}$. Then each $\mathfrak{z}_i \times P$ is component of exactly one $[1Z_j; S, P]$, say $[1Z_{j(i)}; S, P]$; and if some Z_j is such that $[1Z_j; S, P]$ has more than one component, say $\mathfrak{z}_1$ and $\mathfrak{z}_2$, then $h(\mathfrak{z}_1) = h(\mathfrak{z}_2)$. This being established, set $c_j^* = c_j h(\mathfrak{z}_i)(h((1Z_j)[G]))^{-1}$, i being such that $j(i) = j$; set also $Z^* = \sum_j c_j^* Z_j$. Then we have $Z^*\{P\}^* = Z\{P\} = \mathfrak{z}$. By Theorem 2.1, $\{Z^*; \mathfrak{y}_i, G\}^*$ exists for each i. Since G is analytically irreducible at P by Lemma 2.3,

$$\alpha_i = e(V \times P/\{Z^*; \mathfrak{y}_i, G\}^*; \mathfrak{y}_i, P\}^*$$

exists for each i by Theorem 5.3 of [2]. The number $\sum_j a_j \alpha_j$ is denoted by $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ and called the *intersection multiplicity of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *at* V *on* S. We set $i(V, \mathfrak{y} \cap \mathfrak{z}, S) = 0$ if $\dim V = r + s - n$ but V is not a subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$. If each component V_j of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ has the dimension $r + s - n$, we set

$$\mathfrak{y} \cap \mathfrak{z} = \sum_j i(V_j, \mathfrak{y} \cap \mathfrak{z}, S) V_j;$$

$\mathfrak{y} \cap \mathfrak{z}$ is called the *intersection of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *on* S (although S does not appear, at this stage, in the symbol $\mathfrak{y} \cap \mathfrak{z}$). Evidently, if $i(V, \mathfrak{y}_1 \cap \mathfrak{z}, S)$ and $i(V, \mathfrak{y}_2 \cap \mathfrak{z}, S)$ both exist and have the same dimension, then $i(V, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap \mathfrak{z}, S)$ also exists, and equals $\sum_j i(V, \mathfrak{y}_j \cap \mathfrak{z}, S)$.

A cycle $\mathfrak{W}$ of S whose minimal representation is $\mathfrak{W} = \sum_j e_j W_j$ is said to be a *part of* $\mathfrak{y} \cap \mathfrak{z}$ (whether $\mathfrak{y} \cap \mathfrak{z}$ exists or not) if (1) each W_j is a component variety of $\mathfrak{y} \cap \mathfrak{z}$, and (2) $e_j = i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)$. The same cycle $\mathfrak{W}$ is said *to coincide locally at* U *with* $\mathfrak{y} \cap \mathfrak{z}$ (U being a subvariety of S) if (1) each W_j which contains some component of U is a component variety of $\mathfrak{y} \cap \mathfrak{z}$, (2) each component of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ which contains some component of U coincides with some W_j, and (3) $e_j = i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)$ for each W_j which contains some component of U. Also, $\mathfrak{y} \cap \mathfrak{z}$ is said *to exist locally* at U if $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ exists for each component V of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ which contains some component of U; the *local part of* $\mathfrak{y} \cap \mathfrak{z}$ *at* U is $\sum_j i(X_j, \mathfrak{y} \cap \mathfrak{z}, S) X_j$, where X_j ranges among all the components of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ each of which contains some component of U.

LEMMA 3.1. *Let* $\mathfrak{y}$, $\mathfrak{z}$ *be unmixed integral effective cycles of* $S = S_n(k)$, *of dimensions* r, s *respectively. If* $r + s - n \geq 0$, *let* V *be a component variety of* $\mathfrak{y} \cap \mathfrak{z}$. *Let* θ *be an unmixed algebraic correspondence between an algebraic function field* K *over* k *and* S, *such that the set* $N(\theta)$ *of the* $v \in M(K)$ *for which* $\theta\{v\}^*$ *is the modified extension of* $\mathfrak{z}$ *over* K_v *is nonempty. If* $\mathfrak{y}'$ *is the modified extension of* $\mathfrak{y}$ *over* K, *let* Λ_j $(j = 1, 2, \cdots)$ *be the component varieties of* $\mathfrak{y}' \cap \theta$, *and set*

$$\Lambda_\theta = \sum_j i(\Lambda_j, \mathfrak{y}' \cap \theta, S_K) \Lambda_j.$$

If $v \in N(\theta)$, *then a partial extension of* $i(V, \mathfrak{h} \cap \mathfrak{z}, S)\ V$ *over* K_v *is part of* $\Lambda_\theta\{v\}^*$.

Proof. The statement is clearly true if it is true when $\mathfrak{h}$ is irreducible; accordingly, we shall assume $\mathfrak{h}$ to be irreducible, and put $Y = \text{rad}\ \mathfrak{h}$.

In order to avoid lengthy repetitions, we shall say that the set $\{K, \theta\}$ is "admissible" if (1) every component of θ operates on the whole S, (2) $N(\theta)$ is not empty, and (3) each component of rad $\theta \cap Y_K$ has dimension $r + s - n$ and operates on the whole Y. And we shall say that an admissible set $\{K, \theta\}$ is "satisfactory" if the following statement is true: Set

$$\Gamma_\theta = \sum_j e(\Lambda_j/\theta)^* \Lambda_j ;$$

then, for each $v \in N(\theta)$, a partial extension of $i(V, \mathfrak{h} \cap \mathfrak{z}, S)\ V$ over K_v is part of $\Gamma_\theta\{v\}^*$.

Step 1. Let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$, $G = G_{\mathfrak{Z}}$, $K = k(G)$, θ' the algebraic correspondence between K and S induced by $\mathfrak{Z}$ according to Lemma 4.2 of [1]. If $\theta' = \sum_j a'_j \theta_j$ is the minimal representation of θ', set

$$D_j = D_{1\theta_j, G},$$

and let $P \in G$ be such that $\sum_j a'_j D_j\{P\} = \mathfrak{z}$. Set

$$a_j = a'_j h(\mathfrak{z}_i)(h(\theta_j))^{-1},$$

$\mathfrak{z}_i$ being any component variety of $D_j\{P\}$; finally, put

$$\theta = \sum_j c_j \theta_j .$$

Then clearly $\{K, \theta\}$ is admissible by Theorems 2.1, 2.2, and $N(\theta)$ is the set of the $v \in M(K)$ whose center on G is P. If $\Gamma = \Gamma_\theta$ and $C = D_{\Gamma, G}$, then by definition we have

$$i(V, \mathfrak{h} \cap \mathfrak{z}, S) = e(V \times P/C;\ Y, P)^*;$$

if $v \in N(\theta)$, by formula (1) and by the corollary to Theorem 5.1 of [2] it follows that $\Gamma\{v\}^*$ is a partial extension of $C\{P\}^*$, and therefore $\{K, \theta\}$ is satisfactory. This is the contention of Step 1.

Step 2. Let K^* be an algebraic function field over K, θ^* the modified extension of θ over K^*. By means of Lemma 1.2 it is a simple matter to prove that $\{K^*, \theta^*\}$ is admissible if and only if $\{K, \theta\}$ is admissible; in this case, $N(\theta^*)$ consists of the extensions to K^* of the elements of $N(\theta)$; and clearly, if $\{K^*,$

$\theta^*\}$ and $\{K, \theta\}$ are admissible, then $\{K^*, \theta^*\}$ is satisfactory if and only if $\{K, \theta\}$ is such.

Step 3. We work again with two sets $\{K, \theta\}$ and $\{K^*, \theta^*\}$, on which we make the following assumptions: (1) if $G = G_\theta$, $G^* = G_{\theta *}$, then $K = k(G)$, $K^* = k(G^*)$; (2) $\{K, \theta\}$ is (admissible and) satisfactory; (3) if $Z = D_{\theta, G}$, $Z^* = D_{\theta *, G*}$, then $G \subseteq G^*$ and $Z = \{Z^*; S, G\}^*$. We wish to prove that $\{K^*, \theta^*\}$ is admissible and satisfactory.

Clearly each component of θ^* operates on the whole S. If $N = N(\theta)$, $N^* = N(\theta^*)$, let $v \in N$, and let w be a valuation of K^* whose dimension equals dim G, and such that $C(w/G^*) = G$. Then any valuation of K^* compounded of w and of an extension of v to K_w belongs to N^*, so that N^* is nonempty. Let C_i^* be a component of $\{Z^*; Y, G^*]$ such that $1C_i^*$ operates on the whole G^*, and let Y' be the subvariety of Y on which $1C_i^*$ operates. Since $(1C_i^*)[G^*]$ is a component of $Z^*[G^*] \cap Y_{K*}$, by Theorem 2.2 it has dimension $\geq r+s-n$, so that dim $C_i^* \geq r+s-n+\dim G^*$. Let C_j be a component of $[1C_i^*; Y, G]$, so that

$$\dim C_j \geq \dim C_i^* - \dim G^* + \dim G \geq r + s - n + \dim G.$$

Since C_j is also a pseudosubvariety of $\{Z; Y, G]$, and since $1C_j$ operates on the whole G, by assumption C_j is also a pseudosubvariety of $[Z; Y, G]$, and therefore

$$\dim C_j \leq r+s-n+\dim G.$$

This proves that

$$\dim C_j = r+s-n+\dim G;$$

hence C_j is a component of $[Z; Y, G]$ and $1C_j$ operates on the whole G; therefore $1C_j$ operates on the whole Y, and the same must be true of $1C_i^*$. As a consequence, $\{K^*, \theta^*\}$ is admissible. We remark that we have also proved that G is not fundamental for $1C_i^*$.

Let now G'^* be a normal associate to G^*, and call G' an irreducible subvariety of G'^* which corresponds to G in the birational correspondence between G^* and G'^*. Set

$$Z'^* = D_{\theta *, G'*}.$$

If $C_i'^*$ is any component of $\{Z'^*; Y, G'^*]$, and if $1C_i'^*$ operates on the whole G'^*, since $\{K^*, \theta^*\}$ is admissible and S is normal we have that $e(C_j'^*/Z'^*; Y, G'^*)^*$ exists. Set

$$C'^* = \sum_i e(C_i'^*/Z'^*; Y, G'^*)^* C_i'^*.$$

Then $\Gamma^* = \Gamma_{\theta*}$ equals $C'^*\{G'^*\}^*$. Set also $Z' = \{Z'^*;\ S,\ G'\}^*$, so that $Z'\{G'\}^*$ is the modified extension of $Z\{G\}^*$ over $k(G')$. Since G is not fundamental for $1\,C_i^*$ (as previously remarked), G' is not fundamental for C'^*, and G'^* is locally normal at G'. Hence, by Theorem 4.1 of [1], $C' = \{C'^*;\ Y,\ G'\}^*$ exists. The component varieties C_j' of C' are those components A of $\{Z';\ Y,\ G']$ such that $1A$ operates on the whole G'; but then C_j', Z'^*, Y, G' can replace respectively U, D, W_1, W_2 in Theorem 1.2, and the result is that C' equals the cycle obtained from $[Z';\ Y,\ G']$ in the same way as C'^* is obtained from $[Z'^*;\ Y,\ G'^*]$.

Let C be obtained in the same way from $[Z;\ Y,\ G]$; then

$$\Gamma = \Gamma_\theta = C\{G\}^*,$$

and, by Lemma 1.2, $C'\{G'\}^*$ is the modified extension of Γ over $k(G')$.

If $v^* \in N^*$ and $P = \mathfrak{C}(v^*/G^*)$, then $Z^*\{P\}^*$ is the modified extension of $\mathfrak{z}$ over $k(P)$; therefore $P \in G$; hence $P' = \mathfrak{C}(v^*/G'^*)$ belongs to one of the irreducible subvarieties of G'^* (say G') which correspond to G. Since $C'\{G'\}^*$ is the modified extension of Γ over $k(G')$, and because of formula (1), there are components V_j $(j = 1, 2, \cdots)$ of $V \times P$ such that $e(V_j/C';\ Y,\ P')^*$ exists, and such that

$$i(V,\ \mathfrak{h} \cap \mathfrak{z},\ S)\ \mathrm{ord}\ V = \sum_j e(V_j/C';\ Y,\ P')^*\ \mathrm{ord}\,(1\,V_j)\{P'\}^*.$$

Hence each V_j is a component of $[C'^*;\ Y,\ P']$, and $e(V_j/C'^*;\ T,\ P')^*$ exists since G'^* is normal (see Theorem 5.3 of [2]). But then Theorem 1.1 yields

$$e(V_j/C'^*;\ Y,\ P')^* = e(V_j/C';\ Y,\ P')^*.$$

As a consequence, a partial extension of

$$\sum_j e(V_j/C'^*;\ Y,\ P')^*\,(1\,V_j)\{P'\}^*$$

(which is also a partial extension of $i(V,\ \mathfrak{h} \cap \mathfrak{z},\ S)V$) over K_{v^*} is part of $\Gamma^*\{v^*\}^*$. This means that $\{K^*,\ \theta^*\}$ is satisfactory, as announced.

Step 4. If $\{K,\ \theta\}$ is the set given in the statement of Lemma 3.1, let θ' be the general homographic transform of $\mathfrak{z}$, and set

$$K' = k(\cdots,\ u_{ij}\,u_{00}^{-1},\ \cdots),$$

the u_{ij}'s playing the usual role in the definition of θ'. Let K^*, θ^* be obtained from K, θ as K', θ' are from k, $\mathfrak{z}$, and by means of the same u_{ij}'s. Set also

$$G = G_{\theta'},\quad G^* = G_{\theta*},\quad K_1 = k(G),\quad K_1^* = k(G^*),$$

and let θ_1, θ_1^* be the algebraic correspondences between respectively, K_1, K_1^*, and S of which θ', θ^* are modified extensions over respectively, K' and K^*.

If $v \in N = \mathrm{N}(\theta)$ and w is the unique extension of v over K^*, since $\theta\{v\}^*$ is the modified extension of $\mathfrak{z}$ over K_v it follows that $\theta^*\{w\}^*$ is the modified extension of θ' over K_w. This also means that $G \subseteq G^*$, and that $Z_1 \doteq \{Z_1^*; S, G\}^*$, where we have set

$$Z_1 = D_{\theta_1, G}, \quad Z_1^* = D_{\theta_1^*, G^*}.$$

By Step 1, $\{K_1, \theta_1\}$ is satisfactory; since $\{K_1^*, \theta_1^*\}$ has been shown to be related to $\{K_1, \theta_1\}$ as $\{K^*, \theta^*\}$ is related to $\{K, \theta\}$ in Step 3, it follows that $\{K_1^*, \theta_1^*\}$ is satisfactory. Step 2 implies then that $\{K^*, \theta^*\}$ is satisfactory.

Step 5. Let K, θ, K^*, θ^*, N have the same meanings as in Step 4, and set

$$N^* = N(\theta^*), \quad \Gamma^* = \Gamma_{\theta^*}.$$

Let w be a valuation of K^* over K such that $u_{ij}(w) = 0$ if $i \neq j$, $u_{ii}(w) = 1$, and $K_w = K$. If $v \in N$, let v^* be the place of K^* over k which is compounded of w and v, so that $v^* \in N^*$. Since $\{K^*, \theta^*\}$ is satisfactory by Step 4, a modified extension of $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ over $K_{v^*} = K_v$ is part of

$$\Gamma^*\{v^*\}^* = (\Gamma^*\{w\}^*)\{v\}^*,$$

Since w is a place of K^* over K, and $K_w = K$, and since θ^* is the general homographic transform of θ, the following statement is true by definition: If Λ_j is a component variety of $\mathfrak{y}' \cap \theta$, then $i(\Lambda_j, \mathfrak{y}' \cap \theta, S_K)\, \Lambda_j$ is part of $\theta^*\{w\}^*$. As a consequence, $\Lambda = \Lambda_\theta$ is part of $\Gamma^*\{w\}^*$, and its component varieties are all the components of rad $\mathfrak{y}' \cap$ rad θ' of dimension $r + s - n$. If V_l is a component of the extension of V over K_v, and if Λ' is a component variety of $\Gamma^*\{w\}^*$ such that $(1\Lambda')\{v\}^*$ has V_l as a component variety, certainly

$$\dim \Lambda' = \dim V_l = r + s - n;$$

that is, Λ' is a component variety of Λ. This proves that $(\Gamma^*\{w\}^*)\{v\}^*$ and $\Lambda\{v\}^*$ coincide locally at V_l; hence a partial extension of $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ over K_v is part of $\Lambda\{v\}^*$, Q.E.D.

REMARK 1. Maintaining the same notations, by comparing Steps 3 and 5 we see that if $\{K, \theta\}$ is admissible, then $\Gamma_\theta = \Lambda_\theta$, and $\{K, \theta\}$ is satisfactory.

REMARK 2. Remark 1 shows that the use of the word "intersection" and of the symbol $\cap$ in Lemma 1.2 agrees with the present definition of intersection of cycles.

REMARK 3. Remark 1 also shows that in defining the intersection $\mathfrak{y} \cap \mathfrak{z}$, any admissible set $\{K, \theta\}$ can be used in place of the set $\{K, \theta\}$ of Step 1 of the proof of Lemma 3.1. Step 1 itself shows that admissible sets do exist.

THEOREM 3.1. *If $\mathfrak{y}_1$, $\mathfrak{y}_2$ are r-dimensional cycles of S, and $\mathfrak{z}_1$, $\mathfrak{z}_2$ are s-dimensional cycles of S, and V is a component variety of*

$$(\mathfrak{y}_1 + \mathfrak{y}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2),$$

then

$$i(V, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \textstyle\sum_{lj} i(V, \mathfrak{y}_l \cap \mathfrak{z}_j, S).$$

Proof. (See Remark 4 at the end of this proof). Assume $\mathfrak{y}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ to be integral effective cycles. By definition,

$$i(V, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \textstyle\sum_l i(V, \mathfrak{y}_l \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S).$$

Hence it is enough to prove that if $\mathfrak{y}$ denotes either $\mathfrak{y}_l$, then

$$i(V, \mathfrak{y} \cap (\mathfrak{z}_1 \cap \mathfrak{z}_2), S) = \textstyle\sum_j i(V, \mathfrak{y} \cap \mathfrak{z}_j, S).$$

Now, let θ_j be the general homographic transform of $\mathfrak{z}_j$, and set

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots).$$

Then $\theta_1 + \theta_2$ is the general homographic transform of $\mathfrak{z}_1 + \mathfrak{z}_2$. In the notations of Lemma 3.1 and its proof,

$$N = N(\theta_1 + \theta_2) = N(\theta_1) \cap N(\theta_2),$$

and

$$\{K, \theta_j\} \ (j = 1, 2), \ \{K, \theta_1 + \theta_2\}$$

are satisfactory. We also have

$$\Gamma_{\theta_1 + \theta_2} = \Gamma_{\theta_1} + \Gamma_{\theta_2},$$

so that Lemma 3.1 itself and Remark 3 imply

$$i(V, \mathfrak{y} \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \textstyle\sum_j i(V, \mathfrak{y} \cap \mathfrak{z}_j, S), \text{ Q.E.D.}$$

REMARK 4. So far, Theorem 3.1 has a meaning only if $\mathfrak{y}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ are integral effective cycles. This particular case is sufficient, however, to give a meaning to $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ when $\mathfrak{y}$, $\mathfrak{z}$ are rational virtual cycles: in fact, for some integer m it is true that

$$m\mathfrak{y} = \mathfrak{y}_1 - \mathfrak{y}_2, \quad m\mathfrak{z} = \mathfrak{z}_1 - \mathfrak{z}_2,$$

where $\mathfrak{y}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ are integral effective cycles; Theorem 3.1 shows that the number

$$m^{-1}[i(V, \mathfrak{y}_1 \cap \mathfrak{z}_1, S) - i(V, \mathfrak{y}_1 \cap \mathfrak{z}_2, S) - i(V, \mathfrak{y}_2 \cap \mathfrak{z}_1, S) + i(V, \mathfrak{y}_2 \cap \mathfrak{z}_2, S)]$$

depends only on V, $\mathfrak{y}$, $\mathfrak{z}$. This number will be denoted by $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ and called the *intersection multiplicity of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *at* V *on* S; all the other notations and definitions concerning $\mathfrak{y} \cap \mathfrak{z}$ are extended likewise. With this definition it is easily proved that Theorem 3.1 remains true in general. As a matter of fact, Lemma 3.1 itself remains true after removing the assumption that $\mathfrak{y}$ and $\mathfrak{z}$ are integral effective cycles.

Let $\mathfrak{z}$ be an unmixed r-dimensional cycle of an irreducible n- dimensional variety U over k, and let V be an irreducible subvariety of U of dimension $\leq r$, $R = Q(V/U)$; let $\mathfrak{z} = \sum_i b_i \mathfrak{z}_i$ be the minimal representation of $\mathfrak{z}$, and set $\mathfrak{p}_i = \mathfrak{P}(\mathfrak{z}_i/U) \cap R$ for each i such that $R \subseteq Q(\mathfrak{z}_i/U)$ (that is, such that $V \subseteq \mathfrak{z}_i$). We say that $\mathfrak{z}$ is a *complete intersection at* V *on* U if there exists a subset $\{\zeta\}$ of a set of parameters of R such that (1) the $\mathfrak{p}_i$'s are all the distinct minimal primes of the ideal of R whose basis is $\{\zeta_1, \zeta_2, \cdots\}$, and (2) we have

$$b_i = e(R_{\mathfrak{p}_i}; \zeta) = e(Q(\mathfrak{z}_i/U); \zeta)$$

for each i for which $\mathfrak{p}_i$ exists. Any such set $\{\zeta\}$ is called a *set of representatives of* $\mathfrak{z}$ *at* V *on* U. Also, $\{\zeta\}$ is assumed to consist of units of R if $V \nsubseteq \operatorname{rad} \mathfrak{z}$. A complete intersection at V obviously coincides, locally at V, with an integral effective cycle.

LEMMA 3.2. *Maintain the notations of Lemma* 3.1; *assume* $\mathfrak{z}$ *to be irreducible, and* $\mathfrak{y}$ *to be a complete intersection at* V *on* S. *Let* $\{\zeta\}$ *be a set of representatives of* $\mathfrak{y}$ *at* V *on* S, *and set* $\mathfrak{p} = \mathfrak{P}(\operatorname{rad} \mathfrak{z}/S) \cap Q(V/S)$. *If* σ *is the homomorphic mapping of* $Q(V/S)$ *whose kernel is* $\mathfrak{p}$, *then* $\{\sigma\zeta\}$ *is a set of parameters of* $Q(V/\operatorname{rad} \mathfrak{z})$, *and* $i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(Q(V/\operatorname{rad} \mathfrak{z}); \sigma\zeta)$.

Proof. Let θ be the general homographic transform of $\mathfrak{z}$,

$$K = k(\cdots, u_{ij} u_{00}^{-1}, \cdots),$$

T the projective space whose h.g.p. is $\{u\}$, P the point of T at which $u_{ij} = 0$ for $i \neq j$, $u_{ii} = 1$. Let $\{\eta\}$ be a regular set of parameters of $Q(P/T)$. Set $Z = D_{\theta, T}$, and let $\mathfrak{Y} = \sum_j a_j \mathfrak{Y}_j$ be the minimal representation of $\mathfrak{Y}$. Set $C_j = \{Z; \mathfrak{Y}_j, T\}^*$. If C_{jl} ($l = 1, 2, \cdots$) are the component varieties of C_j, then from the corollary to Lemma 1.1 follows that

$$C_j = \sum_l e(Q(C_{jl}/\text{rad } Z); \zeta)\, e(Q(\mathfrak{Y}_j/S); \zeta)^{-1}\, C_{jl}.$$

According to Lemma 3.1 and its proof, we also have, by Theorem 3.1:

$$\begin{aligned} i(V, \mathfrak{Y} \cap \mathfrak{Z}, S) &= \sum_j a_j\, e(V \times P/C_j; \mathfrak{Y}_j, P)^* \\ &= \sum_{jl} a_j\, e(Q(V \times P/C_{jl}); \eta)\, e(Q(P/T); \eta)^{-1}\, e(Q(C_{jl}/\text{rad } Z); \zeta) \\ &\qquad \times e(Q(\mathfrak{Y}_j/S); \zeta)^{-1}. \end{aligned}$$

Since $a_j = e(Q(\mathfrak{Y}_j/S); \zeta)$, this also gives

$$i(V, \mathfrak{Y} \cap \mathfrak{Z}, S) = \sum_{jl} e(Q(V \times P/C_{jl}); \eta)\, e(Q(C_{jl}/\text{rad } Z); \zeta).$$

The ideals $\mathfrak{P}(C_{jl}/\text{rad } Z) \cap Q(V \times P/\text{rad } Z)$ are all the minimal primes of the ideal of $Q(V \times P/\text{rad } Z)$ whose basis is $\{\zeta\}$; therefore the associativity formula (Theorem 2.1 of [2]) gives

$$i(V, \mathfrak{Y} \cap \mathfrak{Z}, S) = e(Q(V \times P/\text{rad } Z), \zeta, \eta).$$

The only minimal prime of the ideal of $Q(V \times P/\text{rad } Z)$ whose basis is $\{\eta\}$ is the ideal $\mathfrak{P}(\text{rad } \mathfrak{Z} \times P/\text{rad } Z) \cap Q(V \times P/\text{rad } Z)$, and

$$e(Q(\text{rad } \mathfrak{Z} \times P/\text{rad } Z); \eta) = e(\text{rad } \mathfrak{Z} \times P/\text{rad } Z; S, P)^* = 1;$$

therefore, if τ denotes the homomorphic mapping of $Q(V \times P/\text{rad } Z)$ whose kernel is said prime, the associativity formula gives

$$i(V, \mathfrak{Y} \cap \mathfrak{Z}, S) = e(Q(V \times P/\text{rad } \mathfrak{Z} \times P; \tau\zeta) = e(Q(V/\text{rad } \mathfrak{Z}); \sigma\zeta), \text{ Q.E.D.}$$

Notice that the fact expressed in Lemma 3.2 is the basic reason for which $\Gamma_\theta = \Lambda_\theta$ when θ is admissible (see Remark 1 and the proof of Lemma 3.1).

LEMMA 3.3. *Let $\mathfrak{Y}$, $\mathfrak{Z}$ be unmixed cycles of S, and let V be a component variety of $\mathfrak{Y} \cap \mathfrak{Z}$. Let Δ be the general homographic transform of $\mathfrak{Y}$,*

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots),$$

v any place of K over k such that $K_v = k$, $u_{ij}(v) = 0$ if $i \neq j$, $u_{ii}(v) = 1$. If $\mathfrak{Z}'$ is the modified extension of $\mathfrak{Z}$ over K, and Λ_j ($j = 1, 2, \cdots$) are all the component varieties of $\Delta \cap \mathfrak{Z}'$, set

$$\Lambda = \sum_j i(\Lambda_j, \Delta \cap \mathfrak{z}', S_K)\Lambda_j.$$

Then $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ *is part of* $\Lambda\{v\}^*$.

Proof. Assume first $\mathfrak{y}$ and $\mathfrak{z}$ to be integral effective cycles. Let u'_{ij} be the reciprocal element of u_{ij} in the matrix $U = (u_{ij})$; if $\{X\}$ is the h.g.p. of S, let σ be the non-degenerate homography of S_K such that

$$\sigma X_i = \sum_j u'_{ij}\, u_{00}\, X_j.$$

Then $\sigma\Delta$ is the modified extension $\mathfrak{y}'$ of $\mathfrak{y}$ over K, and $\theta = \sigma\,\mathfrak{z}$ is the general homographic transform of $\mathfrak{z}$. If Λ_j is a component variety of $\Delta \cap \mathfrak{z}'$, then $\sigma\Lambda_j$ is a component variety of $\mathfrak{y}' \cap \theta$, and

$$i(\Lambda_j, \Delta \cap \mathfrak{z}', S_K) = i(\sigma\Lambda_j, \mathfrak{y}' \cap \theta, S_K).$$

If

$$\psi(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, t, y) = \Psi_{t,y}\,\Lambda,$$

and τ is constructed from $\{u'_{ij}\, u_{00}\}$ as τ is constructed from $\{u_{ij}\}$ in § 2, then by § 2 we have

$$\Psi_{t,y}(\sigma\Lambda) = \psi(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \tau^{-1}t, y).$$

If we replace here each u_{ij} by $u_{ij}(v)$, we obtain

$$\Psi_{t,y}((\sigma\Lambda)\{v\}^*) = \psi(u_{ij}(v), t, y) = \Psi_{t,y}(\Lambda\{v\}^*),$$

which goes to show that

$$(\sigma\Lambda)\{v\}^* = \Lambda\{v\}^*.$$

But $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ is part of $(\sigma\Lambda)\{v\}^*$ by Lemma 3.1, so that it is also part of $\Lambda\{v\}^*$, as asserted.

If $\mathfrak{y}$ or $\mathfrak{z}$ are not integral effective cycles, the proof of Lemma 3.3 is easily derived from the above special case, Q.E.D.

THEOREM 3.2. *Let K be an algebraic function field over k, Δ and θ two unmixed algebraic correspondences between K and S, of dimensions r, s respectively. Let $\mathfrak{y}$, $\mathfrak{z}$ be two cycles of S such that the set N of the $v \in M(K)$ for which $\Delta\{v\}^*$, $\theta\{v\}^*$ are the modified extensions over K_v of $\mathfrak{y}$, $\mathfrak{z}$ respectively is nonempty. Let V be a component variety of $\mathfrak{y} \cap \mathfrak{z}$, and let Λ_j $(j = 1, 2, \cdots)$ be all the component varieties of $\theta \cap \Delta$; set*

$$\alpha_j = i(\Lambda_j, \theta \cap \Delta, S_K), \; \Lambda = \sum_j \alpha_j \Lambda_j.$$

Then the set $\{\Lambda_j\}$ *is nonempty, and, for each* $v \in N$*, a partial extension of* $i(V, \mathfrak{z} \cap \mathfrak{y}, S)$ *over* K_v *is part of* $\Lambda\{v\}^*$.

THEOREM 3.3. *If* $\mathfrak{y}$, $\mathfrak{z}$ *are unmixed cycles of* S*, and* V *is a component variety of* $\mathfrak{y} \cap \mathfrak{z}$*, then*

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = i(V, \mathfrak{z} \cap \mathfrak{y}, S).$$

Proof. Theorems 3.2 and 3.3 will be proved together in a number of steps. We shall prove them under the additional assumption that $\mathfrak{y}$, $\mathfrak{z}$ are integral effective cycles. The transition to the general case is obvious.

Step 1. In the notation of Theorem 3.2, let k' be the algebraic closure of k in K, and let K' be a field isomorphic to K over k'; then the direct product $K \times K'$ over k' is an integral domain. Let E be the quotient field of $K \times K'$. Let Δ' be a "copy" of Δ over K'. Given a $v \in N$, select elements $x_1, \cdots, x_m \in K$ such that (1) $K = k'(x)$, (2) $k'[x] \subseteq R_v$, (3) if $\mathfrak{p} = \mathfrak{C}(v/k'[x])$, then $k'[x]_{\mathfrak{p}}$ contains all the coefficients of

$$\Psi_{t,y}\, \theta \in K[t, y],$$

after one of these has been made equal to 1, and (4) $k'[x]_{\mathfrak{p}}$ contains all the coefficients of

$$\Psi_{t,y}\, \Delta \in K[t, y],$$

after one of these has been made equal to 1.

Let $x_1', \cdots, x_m'$ be the elements of K' which correspond to $x_1, \cdots, x_m$ in the isomorphism between K and K'. Then $E = k'(x, x')$. The ideal $\mathfrak{P}$ of $k'[x, x']$ whose basis is $\{x_1' - x_1, \cdots, x_m' - x_m\}$ is prime; let u be any valuation of E over k whose center on $k'[x, x']$ is $\mathfrak{P}$, and whose dimension over k equals $\dim \mathfrak{P}/k = \text{transc } K/k$. Then u is a place of E over K. Let Δ^* be the modified extension of Δ' over K: then we see that $\Delta^*\{u\}^*$ is the modified extension of Δ over K_u. Let Λ^* be obtained from Δ^* and θ as Λ_θ (in Lemma 3.1) is obtained from θ and $\mathfrak{y}$. Then, by Lemma 3.1, a partial extension of Λ over K_u is part of $\Lambda^*\{u\}^*$.

Step 2. Let $k'[z]$ be the integral closure of $k'[x, x']$, and let v' be the place of K' which corresponds to v in the isomorphism between K and K'. Set $\mathfrak{q} = \mathfrak{C}(v'/k'[x'])$, and let $\mathfrak{Q}$ be the minimal prime of $\mathfrak{q}k'[x, x']$. Denote by v^* the place of E over k which is compounded of u and of an extension of v to K_u.

Then $\mathfrak{Q} \subset \mathfrak{C}(v^*/k'[x, x'])$, and therefore some minimal prime $\mathfrak{Q}'$ of $\mathfrak{Q}k'[z]$ is contained in $\mathfrak{C}(v^*/k'[z])$. We select a place w of E over K whose center on $k'[z]$ is $\mathfrak{Q}'$; then there exists a place w^* of E over k whose center on $k'[z]$ is $\mathfrak{C}(v^*/k'[z])$, and which is compounded of w and of some place v_1 of K_w over k. If v_0 is the place of K over k induced by v_1, we have

$$\mathfrak{C}(w^*/k'[x, x']) = \mathfrak{C}(v^*/k'[x, x']),$$

hence

$$\mathfrak{C}(v_0/k'[x]) = \mathfrak{C}(w^*/k'[x]) = \mathfrak{C}(v^*/k'[x]) = \mathfrak{C}(v/k'[x]).$$

As a consequence, because of the choice of $\{x\}$, v_0 and v have, on G_θ and G_Δ, the same centers; since $v \in N$, we deduce that $v_0 \in N$. We also have

$$\mathfrak{C}(w/k'[x']) = \mathfrak{Q}' \cap k'[x'] = \mathfrak{q} = \mathfrak{C}(v'/k'[x']);$$

therefore, since $v \in N$, it follows that $\Delta^*\{w\}^*$ is the modified extension of $\mathfrak{y}$ over K_w. Let Λ' be obtained from θ, $\mathfrak{y}'$ (= modified extension of $\mathfrak{y}$ over K) as Λ_θ (in Lemma 3.1) is obtained from $\mathfrak{y}$, θ respectively. Now we can replace, in Lemma 3.1 $\mathfrak{y}$ by θ, $\mathfrak{z}$ by $\mathfrak{y}'$, θ by Δ^*, Λ_θ by Λ^*, and the result is that a partial extension of Λ' over K_w is part of $\Lambda^*\{w\}^*$.

Step 3. We now make the assumption that a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)V$ over K_{v_0} is part of $\Lambda'\{v_0\}^*$. Since we also have that

$$\Lambda^*\{w^*\}^* = (\Lambda^*\{w\}^*)\{v_1\}^*$$

is the modified extension of $\Lambda'\{v_0\}^*$ over K_{w^*}, we deduce that a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\ V$ over K_{v_1} is part of $\Lambda^*\{w^*\}^*$.

Let F be the irreducible variety over k' whose n.h.g.p. is $\{z\}$; set

$$L = D_{\Lambda^*, F}, \ \mathfrak{P} = \mathfrak{C}(w^*/F) = \mathfrak{C}(v^*/F),$$

and let U be the subvariety of S on which L operates. Since F is normal, for any component C of $[L;\ U, P]$ of dimension $r + s - n$ the number $e(C/L;\ U, P)^*$ exists. The previous result shows that among the C's there are pseudosubvarieties V_j of $S_{k'} \times F$ such that $(1V_j)[P]$ is a component of $V_{k(P)}$; and it also shows that if

$$V' = \sum_j e(V_j/L;\ U, P)^*\,(1V_j)[P],$$

then a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\ V$ over $k(P)$ is part of V'.

The concluding statement of Step 1 shows that a partial extension of $\Lambda\{v\}^*$ over K_{v^*} is part of $\Lambda^*\{v^*\}^*$. If V'' is the part of $\Lambda\{v\}^*$ whose component varieties are components of the extension of V over K_v, this also means that a partial extension of V'' over K_{v*} coincides with a partial extension of V' over K_{v*}, and therefore also with a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$ over K_{v*}. But then V'' itself is a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$ over K_v, and this proves that *Theorem* 3.2 *is true if the assumption made at the beginning of Step* 3 *is true.*

Step 4. We now apply the content of Steps 1, 2, 3, to the following case: assume $\mathfrak{y}$, $\mathfrak{z}$ to be irreducible; let θ' be the general homographic transform of $\mathfrak{z}$,

$$H = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots);$$

let Δ' be the general homographic transform of $\mathfrak{y}$, constructed with an independent set $\{u'_{ij}\}$ of indeterminates, and set

$$H' = k(\cdots, u'_{ij}\, u'^{-1}_{00}, \cdots);$$

set

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, u'_{ij}\, u'^{-1}_{00}, \cdots),$$

and let Δ, θ be the modified extensions of Δ', θ' over K. We select places p, p' of H, H' over k such that

$$K_p = K_{p'} = k,\ u_{ij}(p) = u'_{ij}(p') = 0 \text{ if } i \neq j,\ u_{ii}(p) = u'_{ii}(p') = 1.$$

We further select for v the place of K over k which is compounded of the unique extension p^* of p over H', and of p'. In this case the set $\{x\}$ can be selected to coincide with the set

$$\{\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, u'_{ij}\, u'^{-1}_{00}, \cdots\}$$

(see Step 1), and $k' = k$. Besides, $k[x, x']$ is integrally closed, so that $\{z\} = \{x, x'\}$ (see Step 2). The fact that $k[z] = k[x, x']$ implies that we can select $v_0 = v$ in Step 2. Hence we can replace, in Lemma 3.3, S by $S_{H'}$, $\mathfrak{y}$ by the modified extension $\mathfrak{z}''$ of $\mathfrak{z}$ over H', $\mathfrak{z}$ by the modified extension $\mathfrak{y}''$ of $\mathfrak{y}$ over H', K by K, v by p^*, Δ by θ, Λ by Λ', V by the extension V'' of V over H', and Lemma 3.3 yields that $i(V'', \mathfrak{z}'' \cap \mathfrak{y}'', S_{H'})\, V''$ is part of $\Lambda'\{p^*\}^*$. Now, from the definition of intersection multiplicity follows that

$$i(V'', \mathfrak{z}'' \cap \mathfrak{y}'', S_{H'}) = i(V, \mathfrak{z} \cap \mathfrak{y}, S);$$

therefore $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$ is part of

$$(\Lambda'\{p^*\}^*)\{p'\}^* = \Lambda'\{v\}^* = \Lambda'\{v_0\}^*.$$

Hence, in this particular case, the assumption made at the beginning of Step 3 is true, and therefore, by Step 3, $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$ is part of $\Lambda\{v\}^*$. If Λ_j is any component variety of Λ, then $1\Lambda_j$, considered as an algebraic correspondence between K and S_H, operates on the whole rad θ' (see Remark 1), while, considered as an algebraic correspondence between K and $S_{H'}$, $1\Lambda_j$ operates on the whole rad Δ'. If $\{\zeta\}$, $\{\delta\}$ are sets of regular parameters of $Q(\text{rad } \theta'/S_H)$ and $Q(\text{rad } \Delta'/S_{H'})$ respectively, it follows that θ, Δ are complete intersections at Λ_j on S_K, and that $\{\zeta\}$, $\{\delta\}$ are sets of representatives of θ, Δ, respectively, on S_K. Set

$$R = Q(\Lambda_j/S_K).$$

If σ, τ are the homomorphic mappings of R whose kernels are $\mathfrak{P}(\text{rad } \Delta/S_K) \cap R$ and $\mathfrak{P}(\text{rad } \theta/S_K) \cap R$ respectively, Lemma 3.2 implies that

$$i(\Lambda_j, \theta \cap \Delta, S_K) = e(\sigma R; \sigma\zeta),$$

and this equals $e(R; \zeta, \delta)$ by the associativity formula (Theorem 2.1 of [2]); therefore, again by the associativity formula and Lemma 3.2, we have

$$i(\Lambda_j, \theta \cap \Delta, S_K) = e(R; \zeta, \delta) = e(\tau R; \tau\delta) = i(\Lambda_j, \Delta \cap \theta, S_K).$$

This shows that Λ is unaffected when Δ, θ are interchanged, that is, when $\mathfrak{y}$, $\mathfrak{z}$ are interchanged; hence $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ is also part of $\Lambda\{v\}^*$, and this amounts to saying that

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = i(V, \mathfrak{z} \cap \mathfrak{y}, S).$$

Theorem 3.3 is thus completely proved when $\mathfrak{y}$ and $\mathfrak{z}$ are irreducible, and therefore also when they are not irreducible, because of Theorem 3.1.

Step 5. We go back to the general case considered in Steps 1, 2, 3, and prove that the assumption made at the beginning of Step 3 is always true. According to Theorem 3.3, proved in Step 4, the equality

$$i(\Lambda_j', \theta \cap \mathfrak{y}', S_K) = i(\Lambda_j', \mathfrak{y}' \cap \theta, S_K)$$

is true for any component variety Λ_j' of Λ'. Therefore we can replace, in Lemma 3.1, $\mathfrak{y}$ by $\mathfrak{y}$, $\mathfrak{z}$ by $\mathfrak{z}$, θ by θ, Λ_θ by Λ', and the result is that a partial extension of

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V = i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$$

over K_{v_0} is part of $\Lambda'\{v_0\}^*$, since, as it was proved in Step 2, $v_0 \in N$. This completes the proof of Theorem 3.2, Q.E.D.

THEOREM 3.4. *Let $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ be three unmixed cycles of $S = S_n(k)$, of dimensions r, s, t respectively such that*

$$r + s + t - 2n \geq 0;$$

let V be a component of rad $\mathfrak{x} \cap$ rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ *of dimension $r+s+t-2n$. Let U_1, U_2, $\cdots$ be the components of* rad $\mathfrak{x} \cap$ rad $\mathfrak{y}$ *which contain V, and let W_1, W_2, $\cdots$ be the components of* rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ *which contain V. Then*

$$\dim U_j = r + s - n, \dim W_j = s + t - n,$$

so that

$$U = \sum_j i(U_j, \mathfrak{x} \cap \mathfrak{y}, S)\, U_j \quad \text{and} \quad W = \sum_j i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)\, W_j$$

exist. Moreover,

$$i(V, \mathfrak{x} \cap W, S) = i(V, U \cap \mathfrak{z}, S).$$

This number shall be denoted by $i(V, \mathfrak{x} \cap \mathfrak{y} \cap \mathfrak{z}, S)$, and a similar notation will be used when more that three cycles are involved.

Proof. We may assume, by Theorem 3.1, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ to be irreducible. Let $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ be the general homographic transforms of $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, respectively, constructed with three independent sets of indeterminates $\{u_{ij}\}$, $\{v_{ij}\}$, $\{w_{ij}\}$, and set

$$H = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots),\ J = k(\cdots, v_{ij}\, v_{00}^{-1}, \cdots),\ L = k(\cdots, w_{ij}\, w_{00}^{-1}, \cdots),$$
$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots, \cdots, w_{ij}\, w_{00}^{-1}, \cdots).$$

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be the modified extensions of $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ respectively over K. Then $(\mathfrak{X} \cap \mathfrak{Y}) \cap \mathfrak{Z}$ and $\mathfrak{X} \cap (\mathfrak{Y} \cap \mathfrak{Z})$ exist. Let $\{\xi\}$, $\{\eta\}$, $\{\zeta\}$ be sets of regular parameters of $Q(\text{rad } \mathfrak{X}'/S_H)$, $Q(\text{rad } \mathfrak{Y}'/S_J)$, $Q(\text{rad } \mathfrak{Z}'/S_L)$ respectively. If Λ_j is any component variety of $\Lambda = \mathfrak{X} \cap \mathfrak{Y}$, $1\Lambda_j$ operates on the whole rad $\mathfrak{X}'$ and the whole rad $\mathfrak{Y}'$, so that $\mathfrak{X}$ and $\mathfrak{Y}$ are complete intersections at Λ_j on S_K, and $\{\xi\}$, $\{\eta\}$ are their sets of representatives at Λ_j on S_K. Therefore, by Lemma 3.2, Theorem 3.3, and the associativity formula (Theorem 2.1 of [2]), we have

$$i(\Lambda_j, \mathfrak{X} \cap \mathfrak{Y}, S_K) = e(Q(\Lambda_j/S_K); \xi, \eta).$$

If Γ_l is a component variety of $\Gamma = \Lambda \cap \mathfrak{Z}$, this also shows that Λ is a complete intersection at Γ_l on S_K, and that $\{\xi, \eta\}$ is a set of representatives of Λ at Γ_l on S_K; since $\mathfrak{Z}$ is also a complete intersection at Γ_l on S_K, and $\{\zeta\}$ is a set of representatives of $\mathfrak{Z}$ at Γ_l on S_K, the same argument gives

$$i(\Gamma_l, \Lambda \cap \mathfrak{Z}, S_K) = e(Q(\Gamma_l/S_K); \xi, \eta, \zeta).$$

If now $\Delta = \mathfrak{Y} \cap \mathfrak{Z}$, we can prove by the same method that

$$i(\Gamma_l, \mathfrak{X} \cap \Delta, S_K) = e(Q(\Gamma_l/S_K); \xi, \eta, \zeta),$$

so that $\Lambda \cap \mathfrak{Z} = \mathfrak{X} \cap \Delta$. Let now v be a place of K over k such that,

$$u_{ij}(v) = v_{ij}(v) = w_{ij}(v) = 0 \text{ if } i \neq j, u_{ii}(v) = v_{ii}(v) = w_{ii}(v) = 1, K_v = k.$$

Theorem 3.2 implies that U is part of $\Lambda\{v\}^*$, and therefore also that $i(V, U \cap \mathfrak{z}, S)V$ is part of $\Gamma\{v\}^*$; for the same reason, $i(V, \mathfrak{x} \cap W, S)V$ is part of $\Gamma\{v\}^*$, Q.E.D.

4. Further properties of the intersection multiplicity in a projective space. *Throughout this section, S will be an n-dimensional projective space over the field k.*

THEOREM 4.1. *If $\mathfrak{x}$, $\mathfrak{y}$ are unmixed integral effective cycles of S, and V is a component variety of $\mathfrak{x} \cap \mathfrak{y}$, then $i(V, \mathfrak{x} \cap \mathfrak{y}, S)$ is a positive integer.*

Proof. In the proof of Theorem 3.4 it has been shown that

$$i(\Lambda_j, \mathfrak{X} \cap \mathfrak{Y}, S_K) = e(Q(\Lambda_j/S_K); \xi, \eta),$$

so that Λ is an integral effective cycle. But then $i(V, \mathfrak{x} \cap \mathfrak{y}, S)$ is an integer because it is the multiplicity of V in $\Lambda\{v\}^*$ (v having the same meaning as in the proof of Theorem 3.4), Q.E.D.

From Lemma 3.2, Theorems 3.2 and 3.4, and Lemma 2.3 of [2], it is now possible to see that a cycle $\mathfrak{z}$ is a complete intersection at V on S, and has the set of representatives $\{\zeta\}$ at V on S, if and only if $\mathfrak{z}$ coincides locally at V with $\mathfrak{x}_1 \cap \mathfrak{x}_2 \cap \cdots$, where $\mathfrak{x}_i$ is the $(n-1)$-dimensional cycle

$$\mathfrak{x}_i = \sum_j v_{ij}(\zeta_i)\,[K_{v_{ij}} : k(\mathfrak{C}(v_{ij}/S))]\,\mathfrak{C}(v_{ij}/S);$$

here v_{ij} ($j = 1, 2, \cdots$) are all the discrete normalized valuations of $k(S)$ over k of rank 1 and dimension $n-1$ such that $v_{ij}(\zeta_i) > 0$.

THEOREM 4.2. *Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of S of dimensions r, s such that $r + s - n \geq 0$; let V be a component variety of $\mathfrak{y} \cap \mathfrak{z}$. Let k' be an extension of k, and $\mathfrak{y}'$, $\mathfrak{z}'$ the modified extensions of $\mathfrak{y}$, $\mathfrak{z}$ over k'. Then each component V_j of $V_{k'}$ is a component variety of $\mathfrak{y}' \cap \mathfrak{z}'$, and*

$$\sum_j i(V_j, \mathfrak{y}' \cap \mathfrak{z}', S_{k'}) V_j$$

is the modified extension over k' of $i(V, \mathfrak{y} \cap \mathfrak{z}, S) V$.

Proof. The first assertion is evidently true. In order to prove the second statement, let $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ be the general homographic transforms of $\mathfrak{y}$, $\mathfrak{z}$ respectively, constructed with two independent sets of indeterminates $\{u_{ij}\}$, $\{v_{ij}\}$. Set

$$K = k(\cdots, u_{ij} u_{00}^{-1}, \cdots, \cdots, v_{ij} v_{00}^{-1}, \cdots),$$

and let $\mathfrak{Y}$, $\mathfrak{Z}$ be the modified extensions of $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ respectively over K. Then $\Lambda = \mathfrak{Y} \cap \mathfrak{Z}$ exists, and if v is a place of K over k such that

$$K_v = k, \; u_{ij}(v) = v_{ij}(v) = 0 \text{ if } i \neq j, \; u_{ii}(v) = v_{ii}(v) = 1,$$

then $i(V, \mathfrak{y} \cap \mathfrak{z}, S) V$ is part of $\Lambda\{v\}^*$ by Theorem 3.2. Now let $\mathfrak{Y}'$, $\mathfrak{Z}'$, K' be obtained from $\mathfrak{y}'$, $\mathfrak{z}'$ as $\mathfrak{Y}$, $\mathfrak{Z}$, K are from $\mathfrak{y}$, $\mathfrak{z}$; then $\mathfrak{Y}'$, $\mathfrak{Z}'$ are the modified extensions of $\mathfrak{Y}$, $\mathfrak{Z}$ respectively over K'. If $\Lambda' = \mathfrak{Y}' \cap \mathfrak{Z}'$, assume for a moment Λ' to be the modified extension of Λ over k'. If v' is any extension of v to K' over k' such that $K_{v'} = k'$, then $\Lambda'\{v'\}^*$ is the modified extension of $\Lambda\{v\}^*$ over k', and therefore

$$\sum_j i(V_j, \mathfrak{y}' \cap \mathfrak{z}', S_{k'}) V_j$$

is the modified extension over k' of $i(V, \mathfrak{y} \cap \mathfrak{z}, S) V$, as claimed. We conclude that the theorem is true if it is true when applied to $\mathfrak{Y}$, $\mathfrak{Z}$, or also, *a fortiori*, if it is true under the additional assumption that $\mathfrak{y}$, $\mathfrak{z}$ are complete intersections at V. This, in turn, is equivalent, by Lemma 3.2, to the following assertion: Let A be an irreducible subvariety of S, $\{\zeta\}$ a set of parameters of $R = Q(A/S)$; let A' be the modified extension of $1A$ over k', $A_1, \cdots, A_m$ its component varieties, and set

$$R_i = Q(A_i/S_{k'}).$$

Then

$$A' = \sum_i e(R_i; \zeta)\, e(R; \zeta)^{-1} A_i.$$

Now, if k' is an algebraic function field over k the proof of this statement is implicitly contained in the proof of Lemma 1.2; otherwise, it can be obtained by a well-known limiting process, Q.E.D.

THEOREM 4.3 (BEZOUT'S THEOREM). *Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of S such that $\mathfrak{y} \cap \mathfrak{z}$ exists. Then*

$$\text{ord}\,(\mathfrak{y} \cap \mathfrak{z}) = (\text{ord}\,\mathfrak{y})\,(\text{ord}\,\mathfrak{z}).$$

Proof. By Theorem 3.1, we may assume without loss of generality that $\mathfrak{y}$ and $\mathfrak{z}$ are irreducible; and, by Theorem 4.2, we may assume k to be algebraically closed. Let $\mathfrak{Y}$, $\mathfrak{Z}$, K, Λ have the same meanings as in the proof of Theorem 4.2. Then

$$\text{ord}\,(\mathfrak{y} \cap \mathfrak{z}) = \text{ord}\,\Lambda.$$

Since k is algebraically closed, $\mathfrak{Y}$ and $\mathfrak{Z}$ are the modified extensions over K of the general elements of the homographic systems $\mathfrak{H}$, $\mathfrak{K}$ of $\mathfrak{y}$, $\mathfrak{z}$ respectively. According to Lemma 2.4, $\mathfrak{H}$ and $\mathfrak{K}$ contain two degenerate cycles $\mathfrak{y}'$, $\mathfrak{z}'$, and therefore they contain the homographic systems $\mathfrak{H}'$, $\mathfrak{K}'$ of $\mathfrak{y}'$, $\mathfrak{z}'$ respectively (Lemma 2.2). Two cycles $\mathfrak{y}''$, $\mathfrak{z}''$ of $\mathfrak{H}'$, $\mathfrak{K}'$, respectively, can be found in such a way that $\mathfrak{y}'' \cap \mathfrak{z}''$ exists; we have then that $\mathfrak{y}'' \cap \mathfrak{z}'' = \Lambda\{v\}^*$ for some $v \in M(K)$ (Theorem 3.2), and therefore

$$\text{ord}\,(\mathfrak{y}'' \cap \mathfrak{z}'') = \text{ord}\,\Lambda = \text{ord}\,(\mathfrak{y} \cap \mathfrak{z}).$$

If $r = \text{ord}\,\mathfrak{y}$, $s = \text{ord}\,\mathfrak{z}$, we have

$$\mathfrak{y}'' = \sum_{i=1}^{r} 1\mathfrak{y}_i, \quad \mathfrak{z}'' = \sum_{i=1}^{s} 1\mathfrak{z}_i,$$

the $\mathfrak{y}_i$'s and $\mathfrak{z}_i$'s being linear varieties. Lemma 3.2 gives that for each i, j the intersection $1\mathfrak{y}_i \cap 1\mathfrak{z}_j$ is an irreducible cycle whose radical is a linear variety. Hence Theorem 3.1 implies that $\text{ord}\,(\mathfrak{y} \cap \mathfrak{z}) = rs$, Q.E.D.

THEOREM 4.4 (CRITERION FOR SIMPLE INTERSECTIONS). *Let $\mathfrak{y}$, $\mathfrak{z}$ be irreducible cycles of S, of dimensions r, s respectively such that $r + s - n \geq 0$. Let V be a component of* rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$. *Then the following four statements are equivalent*:

(1) *$i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ exists and equals* 1;

(2) *let $\{X\}$ be the* h.g.p. *of S; let*

$$\{f_1(X), f_2(X), \cdots\} \text{ and } \{g_1(X), g_2(X), \cdots\}$$

be bases of $\wp(\text{rad}\,\mathfrak{y}/S)$ and $\wp(\text{rad}\,\mathfrak{z}/S)$ respectively. Let $\{x\}$ be the h.g.p. *of V. Then the Jacobian matrix $J(f(X), g(X); X, t)$ acquires the rank $2n - r - s$ when $\{X\}$ is replaced by $\{x\}$. Here $\{t\}$ is a p-independent basis of k over k^p if p is the characteristic of k*;

(3) *there are regular sets of parameters* $\{\zeta\}$, $\{\eta\}$ *of* $Q(\text{rad } \mathfrak{z}/S)$, $Q(\text{rad } \mathfrak{y}/S)$ *respectively such that* $\{\zeta, \eta\}$ *is a regular set of parameters of* $Q(V/S)$;

(4) $\wp(V/S)$ *is an isolated primary component of*

$$\wp(\text{rad } \mathfrak{y}/S) + \wp(\text{rad } \mathfrak{z}/S).$$

If ins $V = 1$, *then* $J(f(X), g(X); X, t)$ *in Statement* 2 *can be replaced by* $J(f(X), g(X); X)$.

Proof. Let $\mathfrak{Y}$, $\mathfrak{Z}$, Λ, K have the same meaning as in the proof of Theorem 4.2. Let S_1, S_2 be the projective spaces whose h.g.p. are $\{u\}$, $\{v\}$ respectively. Set

$$R = S_1 \times S_2, Z = D_{\mathfrak{Z},R}, Y = D_{\mathfrak{Y},R}, Z^* = D_{\mathfrak{Z}^*, S_2}, Y^* = D_{\mathfrak{Y}^*, S_1},$$

and let P, Q be points of S_1, S_2 such that

$$Y^*\{P\}^* = \mathfrak{y}, Z^*\{Q\}^* = \mathfrak{z};$$

set also $G = P \times Q$. Then the ideal whose basis is the set of the

$$F_i(X, u) = f_i(\cdots, \textstyle\sum_j u_{lj} u_{00}^{-1} X_j, \cdots)$$

has

$$\wp(\text{rad } Y/k[X, \cdots, u_{ij} u_{00}^{-1}, \cdots, \cdots, v_{ij} v_{00}^{-1}, \cdots])$$

as an isolated primary component, and the ideal whose basis is the set of the

$$G_i(X, v) = g_i(\cdots, \textstyle\sum_j v_{lj} v_{00}^{-1} X_j, \cdots)$$

has

$$\wp(\text{rad } Z/k[X, \cdots, u_{ij} u_{00}^{-1}, \cdots, \cdots, v_{ij} v_{00}^{-1}, \cdots])$$

as an isolated primary component. If assertion 1 is true, then only one component Λ' of Λ has the property that $L' = \text{rad } D_{\Lambda',R}$ contains $V \times G$; besides, rad Λ' has in Λ the multiplicity 1. Therefore, by Lemma 3.2, $\{F_1, F_2, \cdots, G_1, G_2, \cdots\}$ is the basis of an ideal of which

$$\wp(L'/k[X, \cdots, u_{ij} u_{00}^{-1}, \cdots, \cdots, v_{ij} v_{00}^{-1}, \cdots])$$

is an isolated primary component. Since

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(V \times G/1L'; S, R)^*,$$

and since upon replacing the u_{ij}'s, v_{ij}'s by their values at G the $F_i(X, u)$, $G_i(X, v)$ are replaced by the $f_i(X)$, $g_i(X)$, Theorem 5.6 of [2] or its corollary implies that Statement 2 is true.

Assume now Statement 2 to be true; then Theorem 10 of [6] implies that Statement 3 is true. Finally, if Statement 3 is true, then $\mathfrak{y}$ and $\mathfrak{z}$ are complete intersections at V on S, and Lemma 3.2, together with Theorem 2.1 of [2], yields the result that Statement 1 is true. Statement 4 is clearly a consequence of Statement 3, and it implies Statement 2, Q.E.D.

COROLLARY. *With notations as in Theorem* 4.4, *if*

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = 1,$$

then V is simple on rad $\mathfrak{y}$ *and* rad $\mathfrak{z}$.

Proof. This is a consequence of Statement 3 of the theorem and of a well-known result on regular local rings, Q.E.D.

5. **Intersection of cycles of an algebraic irreducible variety.** Let V be an irreducible variety over the field k, U a subvariety of V, S the *ambient space of* V. By this expression we mean to express the fact that if $\{X\}$ is the h.g.p. of S, then the h.g.p. $\{x\}$ of V is a homomorphic image of $\{X\}$; of course S is not the only projective space of which V is a subvariety. Let $\mathfrak{z}$ be an unmixed cycle of V. We say that $\mathfrak{z}$ is a *section of V at U* if there exists an unmixed cycle $\mathfrak{Z}$ of S such that $1V \cap \mathfrak{Z}$ and $\mathfrak{z}$ coincide locally at U. We shall develop in this section a theory of intersections of cycles of V which will be valid when the cycles are sections of V at some U; before we do so, however, it is important to show that this is the case under the customary conditions. Namely, we have:

THEOREM 5.1. *Let V be an irreducible variety over the algebraically closed field k, S the ambient space of V, U a nonempty irreducible subvariety of V, simple on V, $\mathfrak{z}$ an irreducible cycle of V; then there exists an irreducible cycle $\mathfrak{Z}$ of S such that $1V \cap \mathfrak{Z}$ coincides with $\mathfrak{z}$ locally at V.*

Proof. Since, by Theorem 3 of [6], each U simple on V contains a point P simple on V, the theorem will be proved in general if it is proved under the assumption that U is a point. Let $\{x\}$ be a n.h.g.p. of V for which U is a finite distance, $R = Q(U/V)$. If

$$m = \dim S,\ n = \dim V,\ r = \dim \mathfrak{z},$$

let $\{y_1, \cdots, y_n\}$ be a set of regular parameters of R contained in $k[x]$; then $y_1, \cdots, y_n$ are algebraically independent over k. Let F be the projective space over k whose n.h.g.p. is $\{y\}$, and set

$$U' = \wp(\wp(U/k[x]) \cap k[y]),\quad Z' = \wp(\wp(\mathrm{rad}\ \mathfrak{z}/k[x]) \cap k[y]).$$

Then U' is a point, and $\dim Z' \leq r$. The embedding of $k[y]$ in $k[x]$ gives an irreducible algebraic correspondence D between F and V, such that rad D is birationally equivalent to V in a birational correspondence which is regular[2] at finite distance; we shall therefore denote subvarieties of V and D which correspond to each other with the same symbol. Since $Q(U'/F)$ contains a set of regular parameters of R, from the corrollary to Lemma 1.1 we obtain

$$e(U/D;\ U',\ V)^* = 1.$$

Let Z be a component of $[D;\ Z',\ V]$ containing U; then Theorem 1.1 implies that $\dim Z = \dim Z'$; since among the Z's there is one which contains rad $\mathfrak{z}$, and which therefore has dimension $\geq r$, we conclude that $\dim Z' = r$, so that $\dim Z = r$ for each Z. Now, by Theorem 1.1, we have

$$1 = e(U/D;\ U',\ V)^* = e(U/\textstyle\sum_Z e(Z/D;\ Z',\ V)^* Z;\ U',\ V)^*.$$

Since Z' is simple on F, according to a remark preceding Theorem 5.5 of [2] we have that each $e(Z/D;\ Z',\ V)^*$ is an integer; we cannot state that $e(U/1Z;\ U',\ V)^*$ exists for each Z; however, according to Lemma 1.1, we may operate in the following way: Replace, in Lemma 1.1, D, D^*, V, F, G, k respectively by

$$\textstyle\sum_Z e(Z/D;\ Z',\ V)^* Z,\ U,\ V,\ Z',\ U',\ k,$$

and select correspondingly Z^*, G', D', U_1^*, U_2^*, $\cdots$ to replace F', G', D', D_1^*, D_2^*, $\cdots$ in Lemma 1.1; impose upon Z^* the additional condition that $\{1D_j';\ V,\ G'\}^*$ exists for each component variety D_j' of D'; for each Z, set

$$\alpha(Z) = \textstyle\sum_j e(U_j^*/D_l';\ V,\ G')^* \operatorname{ord}(1U_j^*)[G'],$$

where l is such that

$$(1D_l')\{Z^*\}^* = (1Z)\{Z'\}^*.$$

Since $\{1D_l';\ V,\ G'\}^*$ exists, and $h(U_j^*) = 1$, we deduce that $\alpha(Z)$ is an integer. The α of Lemma 1.1 is given by

$$\alpha = \textstyle\sum_Z e(Z/D;\ Z',\ V)^*\ \alpha(Z),$$

and therefore. since $\operatorname{ord} U = 1$ in this case, Lemma 1.1 itself gives that

[2] T is regular at U if for each $U' = T(U)$ it is true that $Q(U/V) = Q(U'/V')$; in this case U' is unique.

$$1 = e(U/\sum_Z e(Z/D;\, Z',\, V)^* Z;\, V,\, U')^* = \sum_Z e(Z/D;\, Z',\, V)^*\ \alpha(Z).$$

Since we have seen that each $e(Z/D;\, Z',\, V)^*$ and each $\alpha(Z)$ is an integer, it follows that there is exactly one Z, namely rad $\mathfrak{z}$, and that $e(Z/D;\, Z',\, V)^* = 1$.

Now, the set $\{y\}$ can be identified with a subset of $k[X]$, $\{X\}$ being the n.h.g.p. of S which corresponds to $\{x\}$. Set

$$\mathfrak{Z} = 1\wp(\wp(Z'/k[y])\, k[X]),$$

so that $\mathfrak{Z}$ is an irreducible cycle of S of dimension $r + m - n$. The fact that the only Z is rad $\mathfrak{z}$ means that rad $\mathfrak{z}$ is the only component of $V \cap$ rad $\mathfrak{Z}$ containing U; since

$$r = n + \dim \mathfrak{Z} - m,$$

we also have that rad $\mathfrak{z}$ is a component variety of $1V \cap \mathfrak{Z}$. Finally, since

$$e(Z/D;\, Z',\, V)^* = 1,$$

a regular set of parameters of $Q(Z'/F)$ is a regular set of parameters of $Q(Z/V)$, and this means that $\wp(\text{rad }\mathfrak{z}/k[X])$ is an isolated primary component of

$$\wp(V/k[X]) + \wp(\text{rad }\mathfrak{Z}/k[X]).$$

This, in turn, by Statement 4 of Theorem 4.4 shows that $i(\text{rad }\mathfrak{z},\, 1V \cap \mathfrak{Z},\, S) = 1$, Q.E.D.

Let V be an irreducible n-dimensional variety over the (arbitrary) field k, and let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of V of dimension r, s respectively; if U is an irreducible subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$, we say that U is a *component variety of* $(\mathfrak{y} \cap \mathfrak{z},\, V)$ if $\dim U = r + s - n$. If $\mathfrak{z}$ is a section of V at U, let $\mathfrak{Z}$ be an unmixed cycle of the m-dimensional ambient space S of V, such that $\mathfrak{z}$ coincides locally at U with $\mathfrak{Z} \cap 1V$. If U is also a subvariety of rad $\mathfrak{y}$, then it is a subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{Z}$. Since

$$\dim \mathfrak{Z} = s + m - n,$$

by Theorem 2.2 we have

$$\dim U \geq r + s - n.$$

Assume U to have exactly the dimension $r + s - n$, so that it is a component variety of $(\mathfrak{y} \cap \mathfrak{z},\, V)$ and of $\mathfrak{y} \cap \mathfrak{Z}$. Assume also $\mathfrak{y}$ to be a section of V at U, and let $\mathfrak{Y}$ be related to $\mathfrak{y}$ as $\mathfrak{Z}$ is to $\mathfrak{z}$. The number $i(U,\, \mathfrak{y} \cap \mathfrak{Z},\, S)$ exists, and by Theorem 3.4 it equals

$$i(U, \mathfrak{Y} \cap 1V \cap \mathfrak{Z}, S) = i(U, \mathfrak{Y} \cap \mathfrak{z}, S).$$

This proves that

$$i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U, \mathfrak{Y} \cap \mathfrak{z}, S)$$

does not depend on the choice of $\mathfrak{Y}$, $\mathfrak{Z}$, but depends only on U, $\mathfrak{y}$, $\mathfrak{z}$, V; accordingly, it will be denoted by $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$. We shall put $i(U, \mathfrak{y} \cap \mathfrak{z}, V) = 0$ if $\dim U = r + s - n$ but $U \not\subseteq \operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$. A generalization of the meaning of this symbol will be given after Theorem 5.9; the remark following Theorem 5.9 contains comments on the validity of most results of this section for the generalized symbol. Theorem 3.3 yields:

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V) = i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U, \mathfrak{Z} \cap \mathfrak{y}, S) = i(U, \mathfrak{z} \cap \mathfrak{y}, V);$$

that is, we have the following result:

THEOREM 5.2 (COMMUTATIVITY LAW). *If one of the symbols*

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V), i(U, \mathfrak{z} \cap \mathfrak{y}, V)$$

has a meaning, the other also has a meaning, and their values are equal.

The number $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ is called the *intersection multiplicity of* $\mathfrak{y}$ *and* $\mathfrak{z}$ *at* U *on* V. Assume that $\mathfrak{y}$, $\mathfrak{z}$ are such that each component U_j of $\operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$ is a component variety of $(\mathfrak{y} \cap \mathfrak{z}, V)$, and that $i(U_j, \mathfrak{y} \cap \mathfrak{z}, V)$ is defined for each j; in this case we shall set

$$(\mathfrak{y} \cap \mathfrak{z}, V) = \sum_j i(U_j, \mathfrak{y} \cap \mathfrak{z}, V)\, U_j;$$

the cycle $(\mathfrak{y} \cap \mathfrak{z}, V)$ is called the *intersection of* $\mathfrak{y}$ *and* $\mathfrak{z}$ *on* V. The locutions "to be part of $(\mathfrak{y} \cap \mathfrak{z}, V)$", "to coincide locally at ⋯ with $(\mathfrak{y} \cap \mathfrak{z}, V)$", "to exist locally at ⋯ ", and "the local part of $(\mathfrak{y} \cap \mathfrak{z}, V)$ at ⋯ " shall have a meaning even if $(\mathfrak{y} \cap \mathfrak{z}, V)$ does not exist, in exactly the same way as the similar locutions in § 3 have a meaning even if $\mathfrak{y} \cap \mathfrak{z}$ does not exist. Obviously, in the special case in which $V = S$, the symbols $i(U, \mathfrak{y} \cap \mathfrak{z}, S)$ as defined here or in § 3 have the same meaning; accordingly, the symbol $\mathfrak{y} \cap \mathfrak{z}$ of § 3 shall be denoted from now on by $(\mathfrak{y} \cap \mathfrak{z}, S)$.

From Theorem 4.2 we obtain:

THEOREM 5.3. *Let* V *be an irreducible variety over* k, $\mathfrak{y}$ *and* $\mathfrak{z}$ *two unmixed cycles of* V *such that*

$$\dim \mathfrak{y} + \dim \mathfrak{z} - \dim V \geq 0,$$

and let $\mathfrak{x}$ *be a part of* $(\mathfrak{y} \cap \mathfrak{z}, V)$. *Let* k' *be an extension of* k; V' *the extension of* V *over* k'; $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ *the modified extensions of* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *respectively over* k'. *Assume* V' *to be irreducible. Then* $\operatorname{ins} V\ (\operatorname{ins} V')^{-1}\mathfrak{x}'$ *is part of* $(\mathfrak{y}' \cap \mathfrak{z}', V')$.

From the definition, and from Theorem 3.1, we obtain:

THEOREM 5.4 (DISTRIBUTIVITY LAW). *If* U, $\mathfrak{y}_j$, $\mathfrak{z}_l$, V *are such that* $i(U, \mathfrak{y}_j \cap \mathfrak{z}_l, V)$ *has a meaning for* $j, l = 1, 2$, *and if*

$$\dim \mathfrak{y}_1 = \dim \mathfrak{y}_2,\ \dim \mathfrak{z}_1 = \dim \mathfrak{z}_2,$$

then

$$i(U, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), V)$$

has a meaning and equals

$$\sum_{j,\, l=1}^{2} i(U, \mathfrak{y}_j \cap \mathfrak{z}_l, V).$$

THEOREM 5.5 (ASSOCIATIVITY LAW). *Let* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *be three unmixed cycles of the n-dimensional irreducible variety* V *over* k, *of dimensions* r, s, t *respectively. Let* U *be a component of* $\operatorname{rad} \mathfrak{x} \cap \operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$ *of dimensions* $r + s + t - 2n$; *assume* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *to be sections of* V *at* U; *let* $\mathfrak{x}'$, $\mathfrak{y}'$ *be the local parts, at* U, *of* $(\mathfrak{x} \cap \mathfrak{y}, V)$, $(\mathfrak{y} \cap \mathfrak{z}, V)$ *respectively. Then* $i(U, \mathfrak{x}' \cap \mathfrak{z}, V)$ *and* $i(U, \mathfrak{x} \cap \mathfrak{z}', V)$ *exist and are equal.* Their common value is denoted by $i(U, \mathfrak{x} \cap \mathfrak{y} \cap \mathfrak{z}, V)$, and a similar notation is used when more than three cycles are involved.

Proof. Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be unmixed cycles of S (the ambient space of V) such that $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ coincide locally at U with $(\mathfrak{X} \cap 1V, S)$, $(\mathfrak{Y} \cap 1V, S)$, $(\mathfrak{Z} \cap 1V, S)$ respectively. Then $(\mathfrak{X} \cap \mathfrak{Y}, S)$ and $(\mathfrak{Y} \cap \mathfrak{Z}, S)$ exist locally at U; let $\mathfrak{X}'$, $\mathfrak{Z}'$ be the local parts of $(\mathfrak{X} \cap \mathfrak{Y}, S)$, $(\mathfrak{Y} \cap \mathfrak{Z}, S)$, respectively, at U. Theorem 3.4 implies that

$$i(U, \mathfrak{X}' \cap \mathfrak{z}, S) = i(U, \mathfrak{x} \cap \mathfrak{Z}', S);$$

on the other hand, again by Theorem 3.4, $(\mathfrak{X}' \cap 1V, S)$ coincides locally at U with $(\mathfrak{X} \cap \mathfrak{y}, S)$, and therefore with $(\mathfrak{x} \cap \mathfrak{y}, V)$ and with $\mathfrak{x}'$; this proves that

$$i(U, \mathfrak{X}' \cap \mathfrak{z}, S) = i(U, \mathfrak{x}' \cap \mathfrak{z}, V).$$

In the same way we obtain

$$i(U, \mathfrak{x} \cap \mathfrak{Z}', S) = i(U, \mathfrak{x} \cap \mathfrak{z}', V), \text{ Q.E.D.}$$

THEOREM 5.6 (TRANSITIVITY LAW). *Let V be an irreducible variety over k, W an irreducible subvariety of V. Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of W, and U a component variety of* $(\mathfrak{y} \cap \mathfrak{z}, W)$. *Let $\mathfrak{Y}$, $\mathfrak{Z}$ be unmixed cycles of V such that* $(\mathfrak{Y} \cap 1W, V)$, $(\mathfrak{Z} \cap 1W, V)$ *exist locally at U and coincide at U with $\mathfrak{y}$, $\mathfrak{z}$ respectively. Then* $(\mathfrak{Y} \cap \mathfrak{Z}, V)$ *exists locally at U; let $\mathfrak{X}$ be the local part of* $(\mathfrak{Y} \cap \mathfrak{Z}, V)$ *at U. Then* $(\mathfrak{X} \cap 1W, V)$ *and* $(\mathfrak{y} \cap \mathfrak{z}, W)$ *both locally exist and coincide at U.*

Proof. Let $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ be unmixed cycles of the ambient space S of V such that $(\mathfrak{Y}^* \cap 1V, S)$, $(\mathfrak{Z}^* \cap 1V, S)$ coincide locally at U with $\mathfrak{Y}$, $\mathfrak{Z}$ respectively. Then $(\mathfrak{Y}^* \cap 1W, S)$, $(\mathfrak{Z}^* \cap 1W, S)$ coincide locally at U with $\mathfrak{y}$, $\mathfrak{z}$ respectively by definition. Let $\mathfrak{X}^*$ be the local part of $(\mathfrak{Y}^* \cap \mathfrak{Z}^*, S)$ at U; $\mathfrak{X}^*$ exists because the dimensions fulfill the correct relations. Then $(\mathfrak{X}^* \cap 1V, S)$ coincides locally at U with $\mathfrak{X}$, so that $(\mathfrak{X}^* \cap 1W, S)$ coincides locally at U with $(\mathfrak{X} \cap 1W, V)$. On the other hand, $(\mathfrak{y} \cap \mathfrak{z}, W)$ coincides locally at U with $(\mathfrak{Y}^* \cap \mathfrak{Z}^* \cap 1W, S)$, Q.E.D.

Theorem 5.6 also shows that in the definition of $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$, the ambient space S could be replaced by any space S containing V as a subvariety.

THEOREM 5.7 (LAW OF THE CONSERVATION OF THE NUMBER). *Let A be an irreducible variety over k, and K an algebraic function field over k; let $\mathfrak{B}$ be an irreducible algebraic correspondence between K and A, and let $\mathfrak{X}$, $\mathfrak{Y}$ be unmixed cycles of* rad $\mathfrak{B}$. *Let $v \in M(K)$ be such that $K_v = k$, and that $\mathfrak{B}\{v\}^*$ is irreducible, say $\mathfrak{B}\{v\}^* = 1V$, where V is an irreducible variety over k. Set*

$$\mathfrak{x} = \mathfrak{X}\{v\}^*, \quad \mathfrak{y} = \mathfrak{Y}\{v\}^*,$$

so that $\mathfrak{x}$, $\mathfrak{y}$ are unmixed cycles of V; let U be a component variety of $(\mathfrak{x} \cap \mathfrak{y}, V)$. *Among the components of* rad $\mathfrak{X} \cap$ rad $\mathfrak{Y}$, *let $\mathfrak{U}_j$ $(j = 1, 2, \cdots)$ be those such that* rad $(1\mathfrak{U}_j)\{v\}^*$ *contains U; then*

$$\dim \mathfrak{U}_j = \dim \mathfrak{X} + \dim \mathfrak{Y} - \dim \mathfrak{B}$$

for each j. Assume $\mathfrak{X}$, $\mathfrak{Y}$ to be sections of rad $\mathfrak{B}$ *at $\mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \cdots$. Then* (1) $\alpha_j = i(\mathfrak{U}_j, \mathfrak{X} \cap \mathfrak{Y}, \text{rad } \mathfrak{B})$ *exists for each j, so that $\mathfrak{U} = \sum_j \alpha_j \mathfrak{U}_j$ exists,* (2) *U is a component variety of each $(1\mathfrak{U}_j)\{v\}^*$, and* (3) *$i(U, \mathfrak{x} \cap \mathfrak{y}, V)$ exists and equals the multiplicity of U in $\mathfrak{U}\{v\}^*$.*

Proof. We need to prove only the last statement, since the others are an immediate consequence of the relations between the dimensions. Let $\mathfrak{S}$ be the ambient space of $\mathfrak{B}$, $\mathfrak{X}'$ an unmixed cycle of $\mathfrak{S}$ such that $(\mathfrak{X}' \cap \mathfrak{B}, \mathfrak{S})$ and $\mathfrak{X}$ coincide locally at $\mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \cdots$. Then $\mathfrak{U}$ coincides locally at each $\mathfrak{U}_j$ with $(\mathfrak{X}' \cap \mathfrak{Y}, \mathfrak{S})$ by definition. Let $S = \mathfrak{S}\{v\}^*$ be the ambient space of V; the application of Theorem 3.2 to the two algebraic correspondences $\mathfrak{X}'$, $\mathfrak{V}$ between K and S proves that $(\mathfrak{X}'\{v\}^* \cap 1V, S)$ coincides locally with $\mathfrak{x}$ at U. The same theorem, applied to $\mathfrak{X}'$ and $\mathfrak{Y}$, yields the result that $(\mathfrak{X}'\{v\}^* \cap \mathfrak{y}, S)$ coincides locally at U with $\mathfrak{U}\{v\}^*$; therefore $\mathfrak{U}\{v\}^*$ coincides locally at U with $(\mathfrak{x} \cap \mathfrak{y}, V)$, Q.E.D.

THEOREM 5.8. *Let V be an n-dimensional irreducible variety over k, $\mathfrak{y}$ an r-dimensional irreducible cycle of V, U an irreducible subvariety of* rad $\mathfrak{y}$, *$\mathfrak{z}$ an s-dimensional cycle of V which is a complete intersection at U on V, and such that*

$$r + s - n = \dim U.$$

Assume $\mathfrak{y}$ to be a section of V at U. Let $\{\zeta\}$ be a set of representatives of $\mathfrak{z}$ at U on V, and set

$$\mathfrak{p} = \mathfrak{P}(\text{rad } \mathfrak{y}/V) \cap Q(U/V).$$

If σ is the homomorphic mapping of $Q(U/V)$ whose kernel is $\mathfrak{p}$, then $\{\sigma\zeta\}$ is a set of parameters of $Q(U/\text{rad } \mathfrak{y})$, and $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists and equals

$$e(Q(U/\text{rad } \mathfrak{y}); \sigma\zeta).$$

Proof. If $\{x\}$ is a n.h.g.p. of V for which U is at finite distance, we may assume $\zeta_j \in k[x]$ for each j. Let $\{X\}$ be the correspondent n.h.g.p. of the ambient space S of V, τ the homomorphic mapping of $k[X]$ onto $k[x]$ such that $\tau X_j = x_j$. Let z_j $(j = 1, 2, \cdots)$ be elements of $k[X]$ such that $\tau z_j = \zeta_j$; then the set $\{z\}$ is a subset of a set of parameters of $Q(U/S)$, and, if $m = \dim S$, there exists a cycle $\mathfrak{Z}$ of S, of dimension $s + m - n$, such that $\mathfrak{Z}$ is a complete intersection at U on S, and has $\{z\}$ as a set of representatives at U on S. Therefore, by Lemma 3.2, $\mathfrak{z}$ coincides locally at U with $(\mathfrak{Z} \cap 1V, S)$, so that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists and coincides with $i(U, \mathfrak{y} \cap \mathfrak{Z}, S)$ locally at U; this, in turn, by Lemma 3.2, equals $e(Q(U/\text{rad } \mathfrak{y}); \sigma\tau z)$, Q.E.D.

THEOREM 5.9 (RELATIVE INVARIANCE OF THE INTERSECTION MULTIPLICITY). *Let V, V' be irreducible varieties over k, T a birational correspondence between V and V'; let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of V, U a component variety of $(\mathfrak{y} \cap \mathfrak{z}, V)$ such that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists. Assume T to be regular*[2] *at U, so that T is also regular at each component variety $\mathfrak{y}_j$ of $\mathfrak{y}$ containing U and at each component variety $\mathfrak{z}_l$ of $\mathfrak{z}$ containing U. Let a_j, b_l be the multiplicities of $\mathfrak{y}_j$, $\mathfrak{z}_l$, respectively, in $\mathfrak{y}$, $\mathfrak{z}$; set*

$$U' = T(U),\ \mathfrak{y}_j' = T(\mathfrak{y}_j),\ \mathfrak{z}_l' = T(\mathfrak{z}_l),\ \mathfrak{y}' = \sum_j a_j\, \mathfrak{y}_j',\ \mathfrak{z}' = \sum_l b_l\, \mathfrak{z}_l'.$$

Then if $i(U', \mathfrak{y}' \cap \mathfrak{z}', V')$ *exists it equals* $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$.

The cycle $\mathfrak{y}'$ is called a *transform of* $\mathfrak{y}$ *at* U *in* (or *with respect to*) T.

Proof. By considering the composite variety of V and V', we may clearly reduce the proof to the following simpler case: There exists a n.h.g.p. $\{x\}$ of V for which U is a finite distance, and there exist elements $x_1', x_2', \cdots \in k(V)$ such that $\{x, x'\}$ is a n.h.g.p. of V' for which U' is a finite distance. In this case let S be the ambient space of V, and let $\{X\}$ be the n.h.g.p. of S corresponding to $\{x\}$; if S' is similarly related to V', we may assume that a n.h.g.p. of S' has the form $\{X, X'\}$ $\{X'\}$ being a set of indeterminates. The correspondence between V' and V is now visualized as a "projection" of V' on $S \subseteq S'$.

If A (resp. A') is an irreducible subvariety of S (resp. of V') containing U (resp. U'), whose n.h.g.p. is $\{\xi\}$ (resp. $\{\xi, \xi'\}$), we shall denote by A^* the irreducible subvariety of S' whose n.h.g.p. is $\{\xi, X'\}$; therefore we have

$$A^* \cap S = A \qquad (\text{resp. } A^* \cap V' = A').$$

This correspondence generates in an obvious way a correspondence

$$\mathfrak{Z} \longrightarrow \mathfrak{Z}^* \qquad (\text{resp. } \mathfrak{Z}' \longrightarrow \mathfrak{Z}^*)$$

among cycles. Now, let $\mathfrak{Z}$ be a cycle of S such that $(1V \cap \mathfrak{Z}, S)$ coincides locally at U with $\mathfrak{z}$; then

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V) = i(U, \mathfrak{y} \cap \mathfrak{Z}, S)$$

by definition. Theorem 4.4 readily shows that $\mathfrak{y}$, $\mathfrak{Z}$, $1V$ coincide, respectively, with $(\mathfrak{y}^* \cap 1S, S')$, $(\mathfrak{Z}^* \cap 1S, S)$, $(1V^* \cap 1S, S')$ locally at U, and then an immediate application of Theorem 5.6 yields that

$$i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U^*, \mathfrak{y}^* \cap \mathfrak{Z}^*, S').$$

In like manner, we obtain that $\mathfrak{z}^*$ coincides locally at U^* with $(\mathfrak{Z}^* \cap 1V^*, S')$, and therefore also that

$$i(U^*, \mathfrak{y}^* \cap \mathfrak{Z}^*, S') = i(U^*, \mathfrak{y}^* \cap \mathfrak{z}^*, V^*).$$

We now wish to show that $1V'$ is a complete intersection on V^* at each irreducible subvariety A' of V' which contains U' (and which is therefore regular for the birational correspondence between V and V'). Let in fact A be the transform of A' in V; since

$$Q(A/V) = Q(A'/V'),$$

there exists a $p \in k[x] - \wp(A/k[x])$ such that $px'_j \in k[x]$ for each j. We also have $p \in k[x, X'] - \wp(A'/k[x, X'])$, and therefore

$$X'_j - x'_j = p^{-1}(pX'_j - x'_j) \in Q(A'/V^*)$$

for each j. The set $\{\cdots, X'_j - x'_j, \cdots\}$ is a regular set of parameters of $Q(V'/V^*)$, hence a subset of a set of parameters of $Q(A'/V^*)$, hence also a set of representatives of $1V'$ at A' on V^*, as announced.

This being established, we apply Theorem 5.8 to the varieties V^*, U' and and the irreducible cycles $1V'$, $1U^*$, obtaining the result that $i(U', 1U^* \cap 1V', V^*)$ exists and equals

$$e(Q(U'/U^*); \cdots, X'_j - \xi'_j, \cdots),$$

where we have denoted by $\{\xi, \xi'\}$ the n.h.g.p. of U'; but, as before,

$$\{\cdots, X'_j - \xi'_j, \cdots\}$$

is a regular set of parameters of $Q(U'/U^*)$, and therefore

$$(1U^* \cap 1V', V^*) = 1U'.$$

Likewise, we obtain that $(\mathfrak{y}^* \cap 1V', V^*)$ and $(\mathfrak{z}^* \cap 1V', V^*)$ coincide locally at U' with $\mathfrak{y}'$, $\mathfrak{z}'$ respectively. Now, Theorem 5.6 applied to V^*, V', $\mathfrak{y}'$, $\mathfrak{z}'$, U', $\mathfrak{y}^*$, $\mathfrak{z}^*$ yields the result that $(\mathfrak{y}' \cap \mathfrak{z}', V')$ exists locally at U' and coincides locally at U' with

$$(i(U^*, \mathfrak{y}^* \cap \mathfrak{z}^*, V^*)\, U^* \cap 1V', V^*).$$

In view of the previous equalities, this amounts to saying that

$$i(U', \mathfrak{y}' \cap \mathfrak{z}', V') = i(U, \mathfrak{y} \cap \mathfrak{z}, V), \text{ Q.E.D.}$$

Theorem 5.9 implies that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ depends only on $Q(U/V)$, on the quotient rings in V of those component varieties of $\mathfrak{y}$, $\mathfrak{z}$ which contain U, and on the multiplicities of such component varieties in $\mathfrak{y}$, $\mathfrak{z}$ respectively. Accordingly, in the notations of Theorem 5.9, if $\mathfrak{y}'$, $\mathfrak{z}'$ are not both sections of V' at U', but $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists, we shall define $i(U', \mathfrak{y}' \cap \mathfrak{z}', V')$ to be equal to $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$; Theorem 5.9 itself shows that this is a good difinition, that is, that it is independent of the choice of V'. This enables us to define $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ also when V is an irreducible pseudovariety (see [1]), since each irreducible pseudovariety is regularly equivalent to an irreducible variety. The question is now raised as to whether all the results of this section remain true for the present extended definition of the meaning of the symbol $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$. The answer is as follows:

REMARK. Theorems 5.2 to 5.9 remain true after we replace the word "vaiety" by the word "pseudovariety", and the sentence "$\mathfrak{y}$ is a section of V at U" (or a logically equivalent one) by the sentence "there exists an irreducible variety V', birationally equivalent to V in a correspondence T which is regular at each component of U, such that a transform of $\mathfrak{y}$ at U in T is a section of V' at $T(U)$" (or by a logically equivalent one). The question is not even raised, however, when U is simple on V and the ground field is algebraically closed (see Theorem 5.1).

A comparison between Theorem 5.8 and the corollary to Lemma 1.1 shows the *a posteriori* connection between the theory of intersections and the theory of algebraic correspondences, namely:

THEOREM 5.10. *Let D be an unmixed algebraic correspondence between the irreducible variety F over k and the projective space S over k, and assume each component of D to operate on the whole F. Let G be an irreducible subvariety of F, D^* a component of $[D; S, G]$. Then if*

$$e(D^*/D; S, G)^* \text{ and } i(D^*, D \cap 1(S \times G), S \times F)$$

both exist, they are equal.

From Theorems 5.1, 4.1, and 4.4 we obtain:

THEOREM 5.11. *Let U be a simple irreducible subvariety of the irreducible variety V over the algebraically closed field k. If $\mathfrak{y}$, $\mathfrak{z}$ are irreducible cycles of V such that U is a component variety of $(\mathfrak{y} \cap \mathfrak{z}, V)$, then $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists and is a positive integer. A necessary and sufficient condition in order that*

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V) = 1$$

is that $\wp(U/V)$ be an isolated primary component of

$$\wp(\text{rad } \mathfrak{y}/V) + \wp(\text{rad } \mathfrak{z}/V).$$

Let finally U, V be irreducible subvarieties of a projective space S over an algebraically closed field k; let S' be a "copy" of S over k, U' a copy of U in S', M a component variety of $(1U \cap 1V, S)$. Let Δ be the identical algebraic correspondence between S and S', and set

$$M_\Delta = [\Delta; M, S'], \; U_\Delta = [\Delta; U, S'], \; V_\Delta = [\Delta; V, S'].$$

From the results of the present section, the following equalities are easily established:

$$1(U' \times V) = (1(U' \times S) \cap 1(S' \times V), S \times S');$$

$$1U_\Delta = (\Delta \cap 1(U' \times S), S \times S'), \quad 1V_\Delta = (\Delta \cap 1(V \times S'), S \times S'),$$

$$i(M, 1U \cap 1V, S) = i(M_\Delta, 1U_\Delta \cap 1V_\Delta, \text{rad}\,\Delta)$$

$$= i(M_\Delta, \Delta \cap 1(U' \times S) \cap 1(V \times S'), S \times S')$$

$$= i(M_\Delta, \Delta \cap 1(U' \times V), S \times S'),$$

and this, by Theorem 5.8, proves that our definition of intersection multiplicities coincides with the one given in [3] for the case of algebraic varieties, when the latter is defined.

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TWO EXISTENCE THEOREMS FOR SYSTEMS OF LINEAR INEQUALITIES

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1. Introduction. In a previous paper [1], the writer initiated the development of the theory of linear inequalities by means of metric methods. This program is continued in the present note to obtain existence theorems for the solutions of two types of (finite) homogeneous systems of inequalities; existence criteria for such systems, different from those established in this paper, are given in the fundamental work of Theodore Motzkin [4]; see also [3].

If A denotes an $m \times n$ matrix of real elements, and x a column matrix of n indeterminates, then the matrix Ax gives rise to the two systems of m homogeneous linear inequalities in n unknowns,

$$Ax \geq 0 \tag{1}$$

and

$$Ax \geqq 0, \tag{2}$$

where the notation ≥ 0 is interpreted to demand that at least one of the left members in (1) be positive. In this note necessary and sufficient conditions are found in order that these systems have a solution, which is nontrivial in the case of system (2). These conditions are expressed in terms of the signs of certain minors of the symmetric positive semi-definite matrix of order m formed upon multiplying the matrix A by its transpose A^T. They follow easily from a lemma concerning the distribution of $n+2$ points of the convexly metrized unit n-sphere S_n; this lemma is stated without proof in [2].

2. The Lemma. Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of the S_n and denote the geodesic distance of p_i, p_j by p_ip_j; that is, p_ip_j is the length of a shorter great circle arc that joins p_i and p_j. Denoting the determinant

$$|\cos p_ip_j| \qquad (i, j = 0, 1, \cdots, n+1)$$

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by

$$\Delta(p_0, p_1, \cdots, p_{n+1}),$$

we recall the well-known result that

$$\Delta(p_0, p_1, \cdots, p_{n+1}) = 0,$$

while *each principal minor of the determinant is nonnegative.* If, moreover, a principal minor satisfies

$$\Delta(p_{i_0}, p_{i_1}, \cdots, p_{i_k}) \neq 0,$$

then the points $p_{i_0}, p_{i_1}, \cdots, p_{i_k}$ are contained irreducibly in a k-dimensional (great) hypersphere S_k, and conversely. Clearly each $(m+1)$-tuple of such a set of $k+1$ points is contained irreducibly in an S_m.

LEMMA. *If*

$$p_0, p_1, \cdots, p_n, p_{n+1} \in S_n,$$

with

$$\Delta(p_0, p_1, \cdots, p_n) \neq 0,$$

then (*i*) *the points* $p_0, p_1, \cdots, p_{n-1}$ *determine uniquely an* $(n-1)$*-dimensional great hypersphere* S_{n-1}, *and* (*ii*) *the points* p_n, p_{n+1} *lie on the same or on opposite sides of the hypersphere* S_{n-1} *if and only if the cofactor* $[\cos p_n p_{n+1}]$ *of the element* $\cos p_n p_{n+1}$ *in* $\Delta(p_0, p_1, \cdots, p_{n+1})$ *be negative or positive, respectively.*

Proof. Since $\Delta(p_0, p_1, \cdots, p_n) \neq 0$ (and consequently is positive), the points $p_0, p_1, \cdots, p_n$ are irreducibly contained in S_n, and so $p_0, p_1, \cdots, p_{n-1}$ are irreducibly contained in a great hypersphere $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$ which they determine uniquely.

Let s be any element of S_n, $s \neq p_0, p_1, \cdots, p_n$. Now

$$(1) \quad \Delta(p_0, p_1, \cdots, p_{n-1})\, \Delta(p_0, p_1, \cdots, p_n, s) = \Delta(p_0, p_1, \cdots, p_n)\, \Delta(p_0, p_1, \cdots, p_{n-1}, s) - [\cos p_n s]^2,$$

and the vanishing of $\Delta(p_0, p_1, \cdots, p_n, s)$, together with the nonvanishing of $\Delta(p_0, p_1, \cdots, p_n)$, implies that $[\cos p_n s] = 0$ if and only if

$$\Delta(p_0, p_1, \cdots, p_{n-1}, s) = 0;$$

that is, if and only if $s \in S_{n-1}(p_0, p_1, \cdots, p_{n-1})$.

It follows at once that *if* p, q *are any elements of* S_n *which are on the same side of* $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, *then*

$$\operatorname{sgn}[\cos p_n p] = \operatorname{sgn}[\cos p_n q],$$

where $[\cos p_n p]$, $[\cos p_n q]$ are cofactors of the indicated elements in the determinants $\Delta(p_0, p_1, \cdots, p_n, p)$, $\Delta(p_0, p_1, \cdots, p_n, q)$, respectively. For in the contrary case, the continuous function $[\cos p_n s]$ changes sign for $s = p$ and $s = q$, and consequently it vanishes for some point of the geodesic (shorter) arc joining p and q. But by the above, this point belongs to $S_{n-1}(p_0, p_1, \cdots, p_n)$, and so p, q are on opposite sides of this great hypersphere, contrary to assumption.

If, therefore, p_n and p_{n+1} are on the same side of $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, then

$$\operatorname{sgn}[\cos p_n p_{n+1}] = \operatorname{sgn}[\cos p_n p_n] = -\operatorname{sgn}\Delta(p_0, p_1, \cdots, p_n),$$

and consequently $[\cos p_n p_{n+1}] < 0$.

Suppose, now, that p_n and p_{n+1} are on opposite sides of

$$S_{n-1}(p_0, p_1, \cdots, p_{n-1}),$$

and denote the reflection of p_n in this hypersphere by p_n^*. Then p_n^*, p_{n+1} are on the same side of the hypersphere, and so

$$\operatorname{sgn}[\cos p_n p_{n+1}] = \operatorname{sgn}[\cos p_n p_n^*].$$

From the vanishing of $\Delta(p_0, p_1, \cdots, p_n, p_n^*)$ and the relations

$$p_i p_n^* = p_i p_n \qquad (i = 0, 1, \cdots, n-1),$$

which follow from the definition of p_n^*, (1) yields

$$[\cos p_n p_n^*] = \pm\Delta(p_0, p_1, \cdots, p_n).$$

To determine the sign, we have, first,

$$[\cos p_n p_n^*] = - \begin{vmatrix} 1 & \cos p_0 p_1 & \cdots & & \cos p_0 p_n \\ \cos p_0 p_1 & 1 & \cdots & & \cos p_1 p_n \\ \cdot & \cdot & \cdots & & \cdot \\ \cos p_0 p_{n-1} & \cos p_1 p_{n-1} & \cdots & 1 & \cos p_{n-1} p_n \\ \cos p_0 p_n^* & \cos p_1 p_n^* & \cdots & & \cos p_n p_n^* \end{vmatrix}.$$

Taking account of the relations $p_i p_n^* = p_i p_n$ ($i = 0, 1, \cdots, n-1$), and writing the determinant as the sum of two determinants whose last rows are $\cos p_0 p_n$, $\cos p_1 p_n, \cdots, \cos p_{n-1} p_n, 1$ and $0, 0, \cdots, 0, \cos p_n p_n^* - 1$, respectively, we easily obtain

(2) $[\cos p_n p_n^*]$

$$= - \Delta(p_0, p_1, \cdots, p_n) + (1 - \cos p_n p_n^*)\, \Delta(p_0, p_1, \cdots, p_{n-1}).$$

Then clearly

$$[\cos p_n p_n^*] = \Delta(p_0, p_1, \cdots, p_n) > 0,$$

for if the negative sign were valid, substitution in (2) would give

$$(1 - \cos p_n p_n^*)\, \Delta(p_0, p_1, \cdots, p_{n-1}) = 0.$$

But

$$\Delta(p_0, p_1, \cdots, p_{n-1}) \neq 0$$

because $p_0, p_1, \cdots, p_{n-1}$ are irreducibly contained in S_{n-1}, while, since p_0, $p_1, \cdots, p_{n-1}, p_n$ are irreducibly contained in S_n, $p_n \notin S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, and so p_n is distinct from its reflection p_n^* in that hypersphere; that is,

$$1 - \cos p_n p_n^* \neq 0.$$

Hence if p_n, p_{n+1} are on opposite sides of $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, then $[\cos p_n p_{n+1}] > 0$. To complete the proof, it suffices to observe that if

$$[\cos p_n p_{n+1}] \neq 0$$

then $p_{n+1} \notin S_{n-1}(p_0, p_1, \cdots, p_{n-1})$. This is evident upon substituting p_{n+1} for s in (1).

COROLLARY. *Let $p_0, p_1, \cdots, p_{n+1}$ be pairwise distinct points of S_n, no $n+1$ of which are in a great hypersphere. If $\epsilon_{ij} = 1$ or -1 according as p_i and p_j are on opposite sides or on the same side, respectively, of the great hypersphere*

$$S_{n-1}(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}) \qquad (i, j = 0, 1, \cdots, n+1;\ i \neq j)$$

and $\epsilon_{ii} = 1$ $(i = 0, 1, \cdots, n+1)$, then the matrix (ϵ_{ij}) $(i, j = 0, 1, \cdots, n+1)$ has rank 1.

REMARK. In a manner similar to that employed above, companion theorems that characterize in a *purely metric* way the sides of hyperplanes in n-dimensional euclidean and hyperbolic spaces that are determined by a given set of n points may be obtained. We state the euclidean theorem, which may be exploited to obtain existence theorems for systems of linear inequalities in much the same way as the lemma just proved will be used in the next section.

THEOREM 1. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of euclidean n-space E_n with $p_0, p_1, \cdots, p_n$ irreducibly contained in E_n. Then $p_0, p_1, \cdots, p_{n-1}$ determine a unique hyperplane E_{n-1}, and p_n, p_{n+1} are on the same side or on opposite sides of this hyperplane if and only if*

$$\operatorname{sgn}[p_n p_{n+1}^2] = (-1)^n \quad or \quad (-1)^{n+1},$$

respectively, where $[p_n p_{n+1}^2]$ denotes the cofactor of $p_n p_{n+1}^2$ in the determinant

$$D(p_0, p_1, \cdots, p_{n+1}) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_0p_1^{\ 2} & \cdots & p_0p_{n+1}^2 \\ 1 & p_0p_1^{\ 2} & 0 & \cdots & p_1p_{n+1}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_0p_{n+1}^2 & p_1p_{n+1}^2 & \cdots & 0 \end{vmatrix}$$

We observe, moreover, that for $p_0, p_1, \cdots, p_n$ irreducibly contained in E_n it may be shown that

$$[p_{n-1}p_n^2]$$
$$= (-1)^{n-1} 2^{n-1} [(n-1)!]^2 V(p_0, \cdots, p_{n-2}, p_{n-1}) V(p_0, p_1, \cdots, p_{n-2}, p_n)$$
$$\times \cos \measuredangle (p_0, p_1, \cdots, p_{n-2} : p_{n-1}, p_n),$$

where $[p_{n-1}p_n{}^2]$ is the cofactor of $p_{n-1}p_n{}^2$ in $D(p_0, p_1, \cdots, p_n)$, V is the volume of the $(n-1)$-dimensional simplex determined by the points indicated, and $\measuredangle(p_0, p_1, \cdots, p_{n-2}: p_{n-1}, p_n)$ denotes the "dihedral" angle with $(n-2)$-dimensional edge $E_{n-2}(p_0, p_1, \cdots, p_{n-2})$ of the simplex with vertices p_0, $p_1, \cdots, p_{n-1}, p_n$.

Hence $[p_{n-1}p_n{}^2] = 0$ if and only if $\measuredangle(p_0, p_1, \cdots, p_{n-2}: p_{n-1}, p_n) = \pi/2$, and $\operatorname{sgn}[p_{n-1}p_n{}^2] = (-1)^{n-1}$ if and only if the dihedral angle is acute.

It is, perhaps, worth pointing out that Theorem 1 yields a *purely metric characterization* of a nondegenerate simplex (interior and boundary) of E_n. For if $p_0, p_1, \cdots, p_n$ are the vertices of such a simplex, a point p of E_n evidently belongs to its interior or boundary if and only if p and p_i are not on opposite sides of the hyperplane $E_{n-1}(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n)$ $(i = 0, 1, \cdots, n)$; that is, according to the theorem, if and only if

$$\operatorname{sgn}[p_i p^2] = (-1)^n \text{ or } 0 \qquad (i = 0, 1, \cdots, n),$$

where $[p_i p^2]$ is the cofactor of $p_i p^2$ in the determinant $D(p_0, p_1, \cdots, p_n, p)$.

Since a point of E_n is contained in the convex extension of a k-tuple of E_n (not of E_{n-1}) if and only if it belongs to the simplex determined by some $n+1$ points of the k-tuple, the above observation yields a metric characterization of such convex extensions.

3. The theorems. We are now in position to prove the two existence theorems.

THEOREM 2. *Let $Ax \geq 0$ be a system of m linear inequalities in n indeterminates with rank $r+1$, and let B denote the determinant of the matrix AA^T. The system has a solution if and only if a shifting of rows and corresponding columns of A exists such that*

(*i*) *the upper left principal minor M of B of order $r+1$ does not vanish,*

(*ii*) *each minor of B formed from M by replacing its last row with that part of the j-th row of B contained in the first $r+1$ columns $(j = r+2, r+3, \cdots, m)$ is positive or zero.*

Proof. Each row of A gives, after normalization, a point of the unit n-sphere S_n, and since the rank of A is $r+1$ it follows that the m "row points" are contained irreducibly in an r-dimensional hypersphere S_r of the S_n. Denoting by p_i the point corresponding to the i-th row of A $(i = 1, 2, \cdots, m)$ after a shifting of rows and columns of A in conformity with the hypotheses has been carried out, we see that $p_1, p_2, \cdots, p_r, p_{r+1}$ lie irreducibly in S_r, and that $p_1, p_2, \cdots, p_r$ determine a unique $(r-1)$-dimensional great hypersphere $S_{r-1}(p_1, p_2, \cdots, p_r)$.

Now the cofactor $[\cos p_{r+1} p_j]$ of the element $\cos p_{r+1} p_j$ in the vanishing determinant $\Delta(p_1, p_2, \cdots, p_{r+1}, p_j)$ $(j = r + 2, r + 3, \cdots, m)$ has the sign *opposite* to that of the minor of that element which, in turn, has the same sign as the minor of B described in hypothesis (*ii*). Hence

$$[\cos p_{r+1} p_j] \leqq 0 \qquad (j = r + 2, \cdots, m)$$

and so, by the lemma, each of the points $p_{r+2}, p_{r+3}, \cdots, p_m$ lies on the same side of $S_{r-1}(p_1, p_2, \cdots, p_r)$ as p_{r+1}, or is contained in that hypersphere.

Hence the m points are contained in a hemi-S_r with at least one of the m points, p_{r+1}, not on the S_{r-1} forming its "rim". The center of this hemi-S_r is evidently a solution of the system of inequalities, and so the conditions stated in the theorem are sufficient. We have, indeed, found that the S_r itself contains a solution of the system.

To extablish the necessity, we remark that if the system has a solution in S_n, then it has a solution in the S_r containing the m points $p_1, p_2, \cdots, p_m$ irreducibly. For if $p \in S_n$ which is a solution of the system, then p is the center of a hemi-S_n which contains $p_1, p_2, \cdots, p_m$ and has at least one of these points, say, p_m, in its interior. But if the S_r has no solution of the system, the (spherical) convex extension of the m-tuple is the whole S_r, which must be contained in the hemi-S_n on which the m points lie. But this is impossible since the point $p_m{}^*$ diametral to p_m lies in the S_r but it clearly does not belong to the hemi-S_n.

If, therefore, the system has a solution, there is a point of the S_r containing $p_1, p_2, \cdots, p_m$ which is the center of a hemi-S_r that contains $p_1, p_2, \cdots, p_m$, *with at least one of these points in its interior.* It is easily seen that any such hemi-S_r may be rotated so as to retain this property and have some r of the m points, say $p_1, p_2, \cdots, p_r$, on the S_{r-1} forming its rim. If p_{r+1} is in the interior of the hemi-S_r so obtained, then clearly each of the remaining points is either on its rim or on the same side of the rim as p_{r+1}. Invoking, now, the lemma, we see that the conditions of the theorem are satisfied.

REMARK. The direction-cosines of the normal to the hyperplane E_r determined by the points $p_1, p_2, \cdots, p_r$ (and the origin) give a solution of the system of inequalities; but since these numbers are found by evaluating determinants, a solution method based on the theorem is probably not suitable for computing machines.

THEOREM 3. *Let $Ax \geqq 0$ be a system of m linear inequalities in n indeterminates. The system possesses a nontrivial solution if and only if whenever the* *rank of A equals n, a shifting of rows and corresponding columns of the determinant B of AA^T exists such that* (*i*) *the* nth *order upper left principal minor M of B is not zero, while* (*ii*) *each* nth *order minor of B obtained from M by replacing the last row of M with that part of the j-th row contained in the first n columns of B is positive or zero for $j = n+1, n+2, \cdots, m$.*

Proof. The rank of A is, of course, at most n; and if it is less than n then the m row points lie in an E_{n-1} containing the origin, the coefficients of which annul all the members of the system of inequalities and hence form a solution.

If the rank of A equals n then the row points are not contained in any E_{n-1} passing through the origin, and so the system has a nontrivial solution if and only if the system $Ax \geq 0$ has a solution; that is (by virtue of Theorem 3), if and only if conditions (*i*) and (*ii*) are satisfied.

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TRANSLATION INVARIANT MEASURE OVER SEPARABLE HILBERT SPACE AND OTHER TRANSLATION SPACES

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1. Introduction. We consider the problem of defining a nontrivial, translation-invariant Borel measure over real separable Hilbert space. As noted by Loewner [4], this is not possible; but instead of relinquishing as he does the real number system for a non-Archimedean ordered field for the values of a "measure," we shall consider several topological subspaces of Hilbert space arising frequently in analysis. These are locally compact; and using either the Kolmogoroff stochastic processes construction [2], or else following the Haar measure construction [1] or [5], we can get a nontrivial, essentially translation-invariant Borel measure. However, since the special subspaces considered are not groups under translation, and do not even contain a group germ, the usual Haar measure construction must be modified in a special fashion, and the precise translation invariance obtained is somewhat restrictive. Actually we carry through this modified Haar measure construction for the more general situation of a locally compact translation space, which is defined as an appropriate subspace of an Abelian topological group. The results are collected in a summary at the end.

2. Formulation of the problem. Let

$$\ell_2 = \left\{ x = \{x_n\} \;\middle|\; \sum_{n=1}^{\infty} (x_n)^2 < +\infty\,,\ x_n \text{ real} \right\},$$

the square summable real sequences and thus the real separable Hilbert space prototype. Since ℓ_2 is a subset of R_∞, the countably infinite Cartesian product of the real line $(-\infty, \infty)$, we have available on ℓ_2 as well as the ℓ_2 norm metric topology also the product topology defined relatively from R_∞. Under these two topologies we shall consider the ℓ_2-subsets

$$X = \{ x \in \ell_2 \mid |x_n| \le h(n) \text{ for all } n \},$$

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$$Y = \{ x \in \ell_2 \mid \sum_{j=n}^{\infty} |x_j|^2 \le f(n) \text{ for all } n \},$$

where $f(n)$ and $h(n)$ are specified functions defined over the integers $n \ge 1$ with values real or $+\infty$ having $h(n) > 0$ and $f(n) \ge f(n+1) > 0$.

Let $Z = X$ or Y; we want to define the Borel class of subsets of Z. The open intervals of Z are defined relatively from the elementary open intervals of R_∞, and so we can define $\mathfrak{B}_1$ as the σ-algebra of subsets of Z generated by the open intervals, $\mathfrak{B}_2$ as that generated by the product-topology open sets, $\mathfrak{B}_3$ by the metric spheres, and $\mathfrak{B}_4$ by the metricly open sets. Actually $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = \mathfrak{B}_4$, and will be denoted by $\mathfrak{B}$ and called the class of Borel subsets of Z. To see this we note first by using the rationals that R_∞ and hence Z has a countable basis of open intervals, so $\mathfrak{B}_1 = \mathfrak{B}_2$. Similarly $\mathfrak{B}_3 = \mathfrak{B}_4$, since ℓ_2 and hence Z is a separable metric space and thus has a countable basis of spheres. Since any product-topology open set is clearly open metricly, $\mathfrak{B}_2 \subseteq \mathfrak{B}_4$. Now it is easy to see that any closed sphere

$$S = \{ x \in Z \mid \|x - y\| \le \rho \}$$

is actually closed in the product topology. Since any open sphere is a countable union of closed ones, $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$. Thus $\mathfrak{B}_3 = \mathfrak{B}_4$ makes $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = \mathfrak{B}_4$, as desired.

Define

$$[A + u] = \{ x \in R_\infty \mid (x - u) \in A \}$$

for $u \in R_\infty$ and for any subset A of R_∞. We note that $u \in Z$ and $A \subseteq Z$ do not always make $[A + u] \subseteq Z$ if $Z \ne \ell_2$. However, if $A \in \mathfrak{B}$ and $u \in R_\infty$ then $[A + u] \cap Z \in \mathfrak{B}$. For

$$\mathfrak{J} = \{ A \mid [A + u] \cap Z \in \mathfrak{B} \}$$

is easily seen to be a σ-algebra containing the intervals of Z, so $\mathfrak{B} = \mathfrak{B}_1 \subseteq \mathfrak{J}$, which gives the result.

Our problem is to find a Borel measure ϕ, that is, a nonnegative extended real set function defined and countably additive over $\mathfrak{B}$, which is nontrivial (Condition I) and translation-invariant (Condition II or II′) according to a specified topology.

CONDITION I. $\phi(Z) > 0$ and $\phi(V) < +\infty$ for some nonempty V open in the specified topology;

CONDITION II. $\phi([A+u]) = \phi(A)$ if $A \in \mathcal{B}$, $u \in \ell_2$, and $[A+u] \subseteq Z$;

CONDITION II′. a) $\phi([A+u]) = \phi(A)$ if $A \in \mathcal{B}$, $u \in \ell_2$, $A \subseteq V$, where V and $[V+u]$ are both open subsets of Z.

b) $\phi([A+u] \cap Z) \leq \phi(A)$ if $u \in \ell_2$ and both A and $[A+u] \cap Z$ are open subsets of Z.

Condition II clearly implies II′, and hence is a stronger requirement.

3. **Negative results.** We shall start with a few preliminary lemmas. First define

$$S(Z, x, \rho) = \{y \in Z \mid \|x-y\| < \rho\},$$

the ρ-radius open Z-sphere about x.

LEMMA 1. *For any real $r > 0$ there exists no nonnegative, finitely additive set function ϕ over the Borel subsets of*

$$Z = Y = \overline{S(\ell_2, 0, r)},$$

satisfying II′, (*or thus* II *also*), *under the metric topology such that*

$$0 < \phi(S(\ell_2, 0, \rho)) < +\infty \quad \textit{for} \quad 0 < \rho \leq r.$$

Proof. Let

$${}_p x = \{{}_p x_j\} \in S(\ell_2, 0, r)$$

by defining ${}_p x_j = 0$ if $j \neq p$ and ${}_p x_p = r/2$ for integer $p \geq 1$. Let

$$V_p = S\left(\ell_2, {}_p x, \frac{1}{4} r\right),$$

so that $V_p \subseteq S(\ell_2, 0, r)$; and $V_p \cap V_q = \emptyset$ for $p \neq q$ follows from

$$\|y - y'\| \geq \|{}_p x - {}_q x\| - 2\left(\frac{1}{4} r\right) = \frac{\sqrt{2}-1}{2} r > 0$$

for $y \in V_p$ and $y' \in V_q$. But II′ under the metric topology makes

$$\phi(V_p) = \phi\left(S\left(\ell_2, 0, \frac{1}{4} r\right)\right) = b$$

with $0 < b < +\infty$. Thus

$$S(\ell_2, 0, r) \supset \bigcup_{p=1}^{N} V_p ,$$

and finite additivity of ϕ yields the contradiction

$$0 < Nb = \sum_{p=1}^{N} \phi(V_p) \leq \phi(S(\ell_2, 0, r)) < +\infty$$

for arbitrary integer N. Thus such ϕ cannot exist.

LEMMA 2. *If*

$$0 < \inf_{n \geq 1} h(n) \text{ for } Z = X,$$

or if

$$0 < \inf_{n \geq 1} f(n) \text{ for } Z = Y,$$

then for any $x \in Z$ and $\rho > 0$ there exists some $z \in Z$ and $\rho' > 0$ such that $S(\ell_2, z, \rho') \subseteq S(Z, x, \rho)$.

Proof. For the given $x \in Z$ choose some $N \geq 1$ so that

$$\sum_{j=N+1}^{\infty} (x_j)^2 \leq \left(\frac{1}{3}\rho\right)^2 ,$$

possible since $x \in \ell_2$. Define

$$y' = (y_1, \cdots, y_N) = P(y) \in E_N$$

as the projection of ℓ_2 onto Euclidean N space E_N. Clearly $P(Z)$ is a convex set with a nonvoid interior in E_N including the origin; so we can find an interior point z' on the line-segment from $x' = P(x)$ to the origin so that

$$\sum_{n=1}^{N} (z_n - x_n)^2 < \left(\frac{1}{3}\rho\right)^2 .$$

Define $z \in \ell_2$ so that $z' = P(z)$ by taking $z_n = 0$ for $n \geq N + 1$. Thus

$$||x - z|| = \left[\sum_{n=1}^{N} (z_n - x_n)^2 + \sum_{n=N+1}^{\infty} x_n^2 \right]^{1/2} < \frac{\sqrt{2}}{3} \rho .$$

Let

$$b_0 = \inf_{n \geq 1} h(n) > 0 \text{ for } Z = X,$$

or

$$b_0 = \left[\inf_{n \geq 1} f(n) \right]^{1/2} > 0 \text{ for } Y.$$

Now if $Z = X$, by choosing $\rho'' > 0$ so that $\rho'' < b_0$ and

$$S(E_N, z', \rho'') \subseteq P(Z),$$

as we may since $z' \in \operatorname{int} P(Z)$, we get

$$S(\ell_2, z, \rho'') \subseteq Z.$$

If $Z = Y$, then $z' \in \operatorname{int} P(Z)$ makes

$$\sum_{j=n}^{N} (z_j)^2 < f(n)$$

for $1 \leq n \leq N$, so here we choose $0 < \rho'' < b_0$ and

$$\rho'' < \min_{1 \leq n \leq N} \left([f(n)]^{1/2} - \left[\sum_{j=n}^{N} (z_j)^2 \right]^{1/2} \right).$$

Thus

$$\left[\sum_{j=n}^{\infty} (y_j)^2 \right]^{1/2} \leq ||y - z|| + \left[\sum_{j=n}^{N} (z_j)^2 \right]^{1/2} < f(n) \text{ for } 1 \leq n \leq N,$$

and

$$\left[\sum_{j=n}^{\infty} (y_j)^2 \right]^{1/2} \leq ||y - z|| < b_0 \leq f(n) \text{ for } n \geq N + 1,$$

makes $S(\ell_2, z, \rho'') \subseteq Y = Z$.

Thus

$$\rho' = \min\left(\rho'', \frac{3-\sqrt{2}}{3}\rho\right) > 0$$

yields

$$S(\ell_2, z, \rho') \subseteq Z \cap S(\ell_2, x, \rho) = S(Z, x, \rho)$$

as desired, since

$$\|y - z\| < \frac{3-\sqrt{2}}{3}\rho$$

makes

$$\|x - y\| \leq \|y - z\| + \|x - z\| < \rho$$

because $\|x - z\| < (\sqrt{2}/3)\rho$.

THEOREM 3. *If*

$$0 < \liminf_{n \to \infty} h(n) \textit{ with } Z = X,$$

or if

$$0 < \liminf_{n \to \infty} f(n) \textit{ with } Z = Y,$$

then there exists no Borel measure ϕ on such Z which is nontrivial (I) *and translation-invariant* (II′) *under the norm-metric topology.*

Proof. Set

$$b_0 = \inf_{n \geq 1} h(n) \text{ if } Z = X,$$

or

$$b_0 = \left[\inf_{n \geq 1} f(n)\right]^{1/2} \text{ if } Z = Y;$$

thus clearly $b_0 > 0$ is required by hypothesis. Obviously

$$S(Z, 0, \rho) = S(\ell_2, 0, \rho)$$

for $0 < \rho \leq b_0$, so the metricly open set

$$S(Z, x, \rho) = [S(\ell_2, 0, \rho) + x] \cap Z = [S(Z, 0, \rho) + x] \cap Z$$

for such ρ. Hence if ϕ exists, then $\phi(S(Z, x, \rho)) \leq \phi(S(Z, 0, \rho))$ by Condition II′b) for $x \in \ell_2$, $0 < \rho \leq b_0$.

Now set

$$b_1 = \inf\{\text{all } \rho > 0 \text{ such that } \phi(S(Z, 0, \rho)) > 0\},$$

so $\phi(S(Z, 0, \rho)) > 0$ for $\rho > b_1$, and $= 0$ for $0 < \rho < b_1$ if $b_1 > 0$. Actually $b_1 = 0$. For if not set $\delta = (\min b_0, b_1/2)$; then Z, being separable, is a countable union of spheres of radius $\rho \leq \delta$. But such spheres have

$$\phi(S(Z, x, \rho)) \leq \phi(S(Z, 0, \rho)) = 0,$$

implying $\phi(Z) = 0$ by countable additivity, which contradicts Condition I. Thus $b_1 = 0$ and $\phi(S(Z, 0, \rho)) > 0$ for all $\rho > 0$.

We want to show that $\phi(S(Z, 0, r)) < +\infty$ for some $r > 0$. By Condition I under the metric topology and Lemma 2 it is clear that there exists some $r > 0$ and $z \in Z$ such that

$$S(\ell_2, z, r) \subseteq Z \text{ and } \phi(S(\ell_2, z, r)) < +\infty.$$

Since $S(\ell_2, z, r) \subseteq Z$, it is easily seen for either $X = Z$ or $Y = Z$ that we must have $r \leq b_0$, and hence

$$Z \supseteq S(\ell_2, 0, r) = S(Z, 0, r).$$

Thus $[S(Z, 0, r) + z] = S(\ell_2, z, r)$, an open subset of Z, so Condition II′a) makes

$$\phi(S(Z, 0, r)) = \phi(S(\ell_2, z, r)) < +\infty.$$

Thus

$$0 < \phi(S(Z, 0, \rho)) < +\infty$$

with $S(Z, 0, \rho) = S(\ell_2, 0, \rho)$ for $0 < \rho \leq r$ for some r, $0 < r < b_0$, which is impossible by Lemma 1. Thus the stated ϕ cannot exist.

We also easily get the following considerably weaker result for the product topology.

THEOREM 4. *If* $\{n \mid h(n) = +\infty\}$ *is an infinite set, then there exists no Borel measure* ϕ *on* X *which is nontrivial* (I) *and translation-invariant* (II′) *under the product topology.*

Proof. Let V be any nonempty open interval of X. It is clear that by translating along each of the finite set of coordinates given in the definition of the interval V, we can find a finite or countable set of ${}_px \in \ell_2$ such that

$$[V + {}_px] \subseteq X \text{ and } X = \bigcup_{p=1} [V + {}_px].$$

Also Condition II′a) makes $\phi(V + {}_px) = \phi(V)$ if ϕ exists. Thus $\phi(X) > 0$ for nontriviality yields by countable additivity $\phi(V) > 0$ for any open intervall $V \neq \phi$.

Now Condition I under the product topology implies that some open interval $V_0 \neq \phi$ has $\phi(V_0) < +\infty$, so $0 < \phi(V_0) < +\infty$. Since V_0 is defined in terms of only a finite number of coordinates, and $\{n \mid h(n) = +\infty\}$ is infinite, there must exist some p so that $x \in V_0$ imposes no restriction on the pth coordinate of x. Let

$$W_0 = \{y \in V_0 \mid |y_p| < 1\},$$

a nonvoid open X interval, so $\phi(W_0) > 0$. Let ${}_0z_j = 0$ if $j \neq p$, ${}_0z_p = 1$, so clearly $\{[W_0 + m\, {}_0z]\}$ form a disjoint union of sets $\subseteq V_0$ for different integer m, with

$$\phi([W_0 + m\, {}_0z]) = \phi(W_0)$$

by Condition II′a). Thus

$$+\infty = \sum_{m=1}^{\infty} \phi(W_0) = \phi\left(\bigcup_{m=1}^{\infty} [W_0 + m\, {}_0z]\right) \leq \phi(V_0) < +\infty,$$

which is a contradiction. Thus ϕ cannot exist.

We remark that $\ell_2 = X$ by taking $h(n) \equiv +\infty$, so Theorems 3 and 4 show that there exists no Borel measure ϕ on ℓ_2 which is nontrivial and translation-invariant under either the norm metric or product topologies.

4. Positive results via Kolmogoroff. We want to give conditions under which an invariant measure does exist on X or Y, getting a converse of Theorem 3. For X we shall use the construction of Kolmogoroff [2, p. 27] of a probability measure P over real product spaces, in our case R_∞. Here we need a family Q of real set functions, each member $Q_{n_1,\cdots,n_k}$ being nonnegative and countably additive over the intervals of E_k, with coordinates indexed $n_1, \cdots, n_k$, and having $Q_{n_1,\cdots,n_k}(E_k) = 1$. The family Q is assumed to satisfy Kolmogoroff's two consistency conditions:

$$Q_{n_1,\cdots,n_k}(-\infty, +\infty; a_2, b_2; \cdots; a_k, b_k) = Q_{n_2,\cdots,n_k}(a_2, b_2; \cdots; a_k, b_k),$$

$$Q_{n_1,\cdots,n_k}(a_1, b_1; \cdots; a_k, b_k) = Q_{n_1',\cdots,n_k'}(a_1', b_1'; \cdots; a_k', b_k'),$$

where $n_i' = n_j$, $a_i' = a_j$, $b_i' = b_j$ for $n_1', \cdots, n_k'$ a reordering of $n_1, \cdots, n_k$. The resulting P has $P(I) = Q(\tilde{I})$ if the interval I is the cylinder set by $n_1, \cdots, n_k$ of the interval $\tilde{I}$ of E_k, P being the Borel-Hopf extension [1, p. 54] of Q from the intervals to the Borel sets.

THEOREM 5. *If*

$$\sum_{n=N+1}^{\infty} [h(n)]^2 < +\infty$$

for some finite N, then for X the product and metric topologies coincide, X being locally compact; there exists a Borel measure ϕ which is nontrivial (I) *and translation-invariant* (II) *on X; and such a measure is unique up to constant factors.*

Proof. The stated condition on $h(n)$ makes the equivalence of the topologies over X obvious, as well as local compactness. Let X', ℓ_2', and R_∞' be defined like X, ℓ_2, and R_∞, except only with coordinates of $n \geq N+1$, so clearly

$$X = A_N \times X',$$

where A_N is an interval of E_N. Construct the Borel measure P^* on R_∞' by the Kolmogoroff construction from

$$Q_{n_1,\cdots,n_k}(a_1, b_1; \cdots; a_k, b_k) = \prod_{j=1}^{k} \frac{1}{2h(n_j)} E(n_j, a_j, b_j),$$

where $E(n, a, b)$ is the length, possibly zero, of the interval of intersection of $[-h(n), h(n)]$ and $[a, b]$. This Q-function family has $Q_{n_1, \cdots, n_k}(E_k) = 1$, has Q countably additive since it is a multiple of k dimensional Lebesque measure, and satisfies Kolmogoroff's consistency conditions as needed.

Let

$$V_p = \{x \in R'_\infty \mid |x_p| > h(p)\}$$

open in R'_∞; clearly

$$P^*(V_p) = Q(\tilde{V}_p) = \frac{1}{2h(p)} [E(p, -\infty, -h(p)) + E(p, h(p), +\infty)] = 0.$$

Now

$$X' = \{x \in \ell'_2 \mid |x_n| \leq h(n) \text{ for } n \geq N+1\};$$

and the given condition on $h(n)$ makes it possible to replace ℓ'_2 by R'_∞ in this formula, so that

$$X' = R'_\infty - \bigcup_{p=N+1}^{\infty} V_p,$$

which is in the Borel family $\mathfrak{B}^*$ of R'_∞. Thus $P^*(X') = P^*(R'_\infty) = 1$ follows from $P^*(V_p) = 0$, and X' is thick in R'_∞ (see [1, p. 74]). Hence $P(A \cap X') = P^*(A)$ defines P uniquely over sets $A \cap X'$, $A \in \mathfrak{B}^*$, which form the Borel family $\mathfrak{B}$ of X', so P is a Borel probability measure on X' with $P(I \cap X') = Q(\tilde{I})$.

Of μ_N is N-dimensional Lebesque measure, $\phi = \mu_N \times P$ is a Borel measure on $A_N \times X' = X$. Also

$$\phi(X) = \mu_N(A_N) > 0,$$

and we obtain

$$\phi(B \times X') = \mu_N(B) < +\infty$$

for open bounded E_N intervals $B \subseteq A_N$ by using $P(X') = 1$, and thus ϕ is nontrivial (I) on X.

We want to show ϕ to be translation-invariant (II) on X. If W is any X-interval, then $W = X \cap I$ with I an R_∞-interval, and if $u \in \ell_2$, set

$$B_p = \{x \in R_\infty \mid |x_p| \leq h(p)\},$$

$$C_n = I \cap \left(\bigcap_{p=1}^{n} [B_p - u] \right) \cap X,$$

and

$$D_n = [I + u] \cap X \cap \left(\bigcap_{p=1}^{n} [B_p + u] \right),$$

so that

$$\phi(W \cap [X - u]) = \phi(I \cap [X - u] \cap X) = \lim_{n \to \infty} \phi(C_n)$$

and

$$\phi([W + u] \cap X) = \phi([I + u] \cap X \cap [X + u]) = \lim_{n \to \infty} \phi(D_n).$$

Now the first n coordinate edges of D_n are those of C_n translated by the corresponding u coordinates. Thus taking $n >$ the greatest of the finite number of coordinate indices involved in I, from $\phi = \mu_N \times P$ and $P(X' \cap J) = Q(\tilde{J})$ we get $\phi(C_n) = \phi(D_n)$, both being the product of a normalization factor and the first n coordinate edge lengths. Thus we have

$$\phi(W \cap [X - u]) = \lim_{n \to \infty} \phi(C_n) = \lim_{n \to \infty} \phi(D_n) = \phi([W + u] \cap X),$$

as desired.

Now let $[A + u] \subseteq X$ be given for some Borel subset A of X. If $\{W_i\}$ is a countable disjoint X-interval family covering A, then also

$$A \subseteq \bigcup_i (W_i \cap [X - u]) \subseteq \bigcup_i W_i .$$

Since

$$\phi(A) = \inf_{A \subseteq \bigcup_i W_i} [\Sigma_i \phi(W_i)]$$

as the unique Borel-Hopf extension [1, p. 54] of ϕ from the intervals to the Borel sets, we have

$$\begin{aligned}\phi(A) &= \inf_{A \subseteq \bigcup_i W_i} (\Sigma_i \phi(W_i \cap [X - u])) \\ &= \inf_{A \subseteq \bigcup_i W_i} (\Sigma_i \phi([W_i + u] \cap X)) \geq \phi([A + u])\end{aligned}$$

from

$$\phi(W_i \cap [X - u]) = \phi([W_i + u] \cap X).$$

Thus $\phi(A) \geq \phi([A + u])$, and symmetrically $\phi([A + u]) \geq \phi(A)$, so that $\phi(A) = \phi([A + u])$ for Condition II of translation-invariance.

Finally for the uniqueness of ϕ it is easy to see by division of intervals into large numbers of equal subintervals that any nontrivial, translation-invariant ψ will have $\psi(I)$, I being an interval of X, proportional to the length of each of the edges of I. By our definition of μ_N and Q, this makes $\psi(I) = K\phi(I)$, with $0 < K < +\infty$ and K independent of I. The extension to all Borel sets thus gives $\psi(A) = K\phi(A)$, $A \in \mathcal{B}$, as desired.

5. **Haar measure and translation spaces.** For the space Y our positive result is a complete converse of Theorem 3. We shall get the result by considering a considerably more general situation. Let the Hausdorff space R be an Abelian topological group, and as before define

$$[A + u] = \{x \in R \mid (x - u) \in A\}$$

under R-group addition for $A \subseteq R$ and $u \in R$. Consider a fixed closed subset Z of R, which becomes a Hausdorff space under the relative topology from R, but not in general a group under R-group addition. Such a space containing the zero of R is said to be a translation space if it satisfies the following condition:

i) If V is any open subset of Z containing zero, then Z is covered by the open interiors in Z of the sets of the collection $\{Z \cap [V + u] \mid u \in R\}$.

LEMMA 6. *X is a translation space for $R = \ell_2$ under the metric topology.*

Proof. Let V be the given neighborhood of zero, so that we have some small $\rho > 0$ with $S(Z, 0, \rho) \subseteq V$. Then for any given $z \in Z = X$ we will find

$u \in Z$ and $\rho' > 0$ so that

$$S(Z, z, \rho') \subseteq Z \cap [S(Z, 0, \rho) + u] \subseteq Z \cap [V + u],$$

which makes $z \in \text{int}\,(Z \cap [V + u])$ for Condition i). First since the given $z \in \ell_2$, we can find finite N so that

$$\left(\sum_{n=N+1}^{\infty} z_n^2 \right)^{1/2} < \frac{1}{2} \rho,$$

and then define $u \in Z = X$ by $u_n = z_n$ for $1 \leq n \leq N$ and $u_n = 0$ for $n > N$. Then set

$$\rho' = \min \left(\frac{1}{2}\rho,\ h(n) \ \text{ for } \ n = 1, 2, \cdots, N \right) > 0,$$

so any $x \in S(Z, z, \rho')$ has

$$||x - u|| \leq ||x - z|| + ||z - u|| < \rho' + \frac{1}{2}\rho \leq \rho.$$

Any such x also has

$$|x_n - u_n| = |x_n - z_n| < \rho' \leq h(n)$$

for $1 \leq n \leq N$, and

$$|x_n - u_n| = |x_n| \leq h(n)$$

for $n > N$, so that $x \in [S(Z, 0, \rho) + u]$. Thus

$$S(Z, z, \rho') \subseteq Z \cap [S(Z, 0, \rho) + u],$$

as desired.

LEMMA 7. *Y is a translation space for $R = \ell_2$ under the metric topology.*

Proof. If V is the given neighborhood of zero in $Z = Y$, we can find $\rho > 0$ with $\rho^2 < f(1)$ and $S(Z, 0, \rho) \subseteq V$. Now either $\rho^2 \leq f(n)$ for all n, or else by the definition of Y there is a unique finite N with

$$f(N) \geq \rho^2 > f(N+1).$$

In the first case for the given $z \in Z$ we take $u = z$, and since now $S(Y, 0, \rho) = S(\ell_2, 0, \rho)$ by $\rho^2 \leq f(n)$, we have

$$S(Z, z, \rho) = Z \cap [S(\ell_2, 0, \rho) + u] \subseteq Z \cap [V + u]$$

for $z \in \operatorname{int}(Z \cap [V + u])$ as desired for Condition i).

In the second case for the given $z \in Z = Y$ we define $u \in Z$ by $u_n = z_n$ for $1 \leq n \leq N$, and $u_n = 0$ for $n > N$. In this case also we have

$$S(Z, u, \rho) = Z \cap [S(Z, 0, \rho) + u].$$

For the left side clearly includes the right side, while if $y \in S(Z, u, \rho)$, then for $1 \leq n \leq N$ we have

$$\sum_{j=n}^{\infty} (y_j - u_j)^2 \leq \sum_{j=1}^{\infty} (y_j - u_j)^2 < \rho^2 \leq f(n).$$

For $n > N$ we have

$$\sum_{j=n}^{\infty} (y_j - u_j)^2 = \sum_{j=n}^{\infty} y_j^2 \leq f(n),$$

so that

$$y \in Z \cap [S(Z, 0, \rho) + u],$$

and hence

$$S(Z, u, \rho) \subseteq Z \cap [S(Z, 0, \rho) + u]$$

for equality. Finally since $z \in S(Z, u\ \rho)$ by

$$||z - u|| = \left(\sum_{j=N+1}^{\infty} z_j^2 \right)^{1/2} \leq \sqrt{f(N+1)} < \rho,$$

we have

$$z \in S(Z, u, \rho) \subseteq Z \cap [V + u],$$

so that

$$z \in \operatorname{int}(Z \cap [V + u]),$$

$S(Z, u, \rho)$ being open, for Condition i).

Thus X and Y are special translation spaces, so the result we shall obtain for translation spaces applies to them. For the general translation space Z we define the Borel class $\mathfrak{B}$ as the σ-algebra generated by the open subsets of Z, given by the relative topology from R. For a Borel measure ϕ defined over $\mathfrak{B}$ we note that Condition I of nontriviality and II′ of translation-invariance still make perfect sense in this more general context, if $u \in \ell_2$ in II′ is replaced by $u \in R$. We shall now establish that a locally compact translation space does possess something like a Haar measure, that is a nontrivial, translation-invariant, regular Borel measure. First we need a few more lemmas.

LEMMA 8. *If $V \subseteq W$ are both open subsets of the translation space Z and if $[W + u] \cap Z$ is open in Z for some $u \in R$, then so also is $[V + u] \cap Z$.*

Proof. Since Z is a translation space, it is closed in R, so $Z - W$ and $Z - V$ are both closed in R as well as in Z. Since open and closed subsets of the topological group R remain such under translation, $B = [(Z - W) + u] \cap Z$ and $C = [(Z - V) + u] \cap Z$ are both closed in R, and hence in Z. Defining $A = (R - [Z + u]) \cap Z$, we have

$$A \cup B = Z - ([W + u] \cap Z),$$

known closed in Z, so that $\bar{A} - A \subseteq B$ must follow. We obtain $B \subseteq C$ from $V \subseteq W$, and this makes $\bar{A} - A \subseteq C$; thus $Z - ([V + u] \cap Z) = A \cup C$ is closed in Z, or $[V + u] \cap Z$ is open, as desired.

Let $[B + C] = \{x + y \mid x \in B \text{ and } y \in C\}$ and $B^- = \{x \mid -x \in B\}$ for the following lemma.

LEMMA 9. *If the translation space Z has compact subsets B and C with $B \cap C = \phi$, then there exists some Z-neighborhood V of zero so that*

$$[B + V^-] \cap [C + V^-] = \phi.$$

Moreover, both $[V + z] \cap B \neq \phi$ and $[V + z] \cap C \neq \phi$ are not simultaneously possible for any $z \in R$.

Proof. Since B and C are compact subsets of Z, they are also such of the topological group R. Thus there exists an R-neighborhood W of zero so that

$$[B + W^-] \cap [C + W^-] = \phi.$$

Hence $V = Z \cap W$, so $V^- \subseteq W^-$, gives the first result. If $[V + z] \cap B \neq \phi$ and $[V + z] \cap C \neq \phi$, then $z \in [B + V^-] \cap [C + V^-] = \phi$, a contradiction, which gives the last.

Following Halmos [1, p. 252], if B and C are subsets of the translation space Z, we let $(C:B)$ denote the least cardinal (thus $\aleph_0$ or an integer ≥ 0) of sets P of $z \in R$ such that

$$C \subseteq \bigcup_{z \in P} [B + z].$$

LEMMA 10. *If C is a compact subset of the translation space Z and V is an open Z-subset containing zero, then $(C:V) < +\infty$.*

Proof. By Condition i) we have

$$C \subseteq \bigcup_{u \in R} \text{int}\,(Z \cap [V + u]),$$

an open covering of compact C. Thus there exists a finite set A of such u with

$$C \subseteq \bigcup_{u \in A} \text{int}\,(Z \cap [V + u]) \subseteq \bigcup_{u \in A} [V + u],$$

and hence

$$(C:V) \leq (\text{card}\, A) < +\infty.$$

This lemma is the only place where Condition i) is used to get our following main result on the existence of a Haar measure.

THEOREM 11. *If Z is a locally compact translation space, then there exists a regular Borel measure ϕ on Z which is nontrivial* (I) *and translation-invariant* (II′).

Proof. Since Z is locally compact, it possesses a neighborhood V_1 of zero such that $\overline{V}_1$ is compact, so $0 < (\overline{V}_1 : V) < +\infty$ for any other Z-neighborhood V of zero, by Lemma 10. Also clearly

$$(C:V) \leq (C:\overline{V}_1)\,(\overline{V}_1:V) \leq (C:V_1)\,(\overline{V}_1:V),$$

so we may define

$$\lambda_v(C) = (\overline{V}_1 : V)^{-1}\ (C : V)$$

and have

$$0 \leq \lambda_v(C) \leq (C : V_1) < +\infty$$

for any compact subset C of Z and any Z-neighborhood V of zero. Following Halmos [1, pp. 254-256], we construct a content λ from λ_v. Let Ω be the Cartesian product of the bounded closed intervals $[0, (C : V_1)]$ over all compact subsets C of Z; Ω is compact by Tychonoff's theorem, and each $\lambda_v \in \Omega$. Setting

$$\Lambda(V) = \{\lambda_w \mid W \subseteq V,\ W \text{ a } Z\text{-neighborhood of zero}\},$$

we see that Ω contains by compactness some $\lambda \in \bigcap_V \overline{\Lambda(V)}$, the intersection being over all Z-neighborhoods V of zero. As in [1], this function $\lambda(C)$ defined over compact Z-subsets C is a content; that is, for subsets B, C, and D compact we have

$$0 \leq \lambda\ (C) \leq \lambda\ (B) < +\infty$$

if $C \subseteq B$, and

$$\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$$

with equality if $C \cap D = \phi$ by use of Lemma 9. Also $\lambda(\overline{V}_1) = 1$ since $\lambda_v(\overline{V}_1) = 1$ for any V. For translation invariance we note that if $[C + z] \subseteq Z$ for a compact Z-subset C and $z \in R$, then $[C + z]$ is also compact, since translation by z is a homeomorphism of R onto R; $([C + z] : V) = (C : V)$, obviously; and thus $\lambda_v([C + z]) = \lambda_v(C)$ for any neighborhood V makes $\lambda([C + z]) = \lambda(C)$.

Let W be any subset of Z, define the inner content

$$\lambda_*(W) = \sup \lambda(C)$$

over compact $C \subseteq W$, and for any subset E define

$$\phi(E) = \inf \lambda_*(W)$$

over open Z subsets $W \supseteq E$. Restricting ϕ to $\mathfrak{B}$, we see that ϕ is a regular Borel measure on Z; ϕ is nontrivial (I) by

$$\phi(Z) \geq \phi(\overline{V}_1) \geq \lambda(\overline{V}_1) = 1 \text{ and } \phi(V_1) \leq \lambda(\overline{V}_1) = 1,$$

(see [1, 53 C and E, p. 234]).

It remains only to show that ϕ is translation-invariant (II$'$). First

$$\lambda_*([W + z]) = \lambda_*(W)$$

for $z \in R$ and any Z-subset W having $[W + z] \subseteq Z$. For then compact $C \subseteq W$ has $[C + z] \subseteq Z$ and compact, so $\lambda([C + z]) \equiv \lambda(C)$ and thus $\lambda_*([W + z]) \geq \lambda_*(W)$. The opposite inequality follows symmetrically to give the result, since any compact $C' \subseteq [W + z]$ has $C = [C' - z]$ compact with

$$C \subseteq W \subseteq Z \text{ and } \lambda(C) = \lambda(C').$$

Now if V is an open Z-subset then $\phi(V) = \lambda_*(V)$ since λ_* is monotone. Thus If V and $[V + u] \cap Z$ are both open in Z, and $u \in R$, then $W \subseteq V$ and $[W + u] = [V + u] \cap Z$, where $W = [([V + u] \cap Z) - u]$ so that

$$\phi([V + u] \cap Z) = \lambda_*([W + u] = \lambda_*(W) \leq \lambda_*(V) = \phi(V)$$

for part b) of Condition II$'$.

For part a), assume $A \in \mathfrak{B}$, $u \in R$, and $A \subseteq V_0$, where V_0 and $[V_0 + u]$ are both open Z-subsets. Then for any open Z-subset $V \supseteq A$, Lemma 8 with $V' = V \cap V_0$ and $W' = V_0$ both open makes $[V \cap V_0 + u]$ open also, and we note that

$$[A + u] \subseteq [V \cap V_0 + u] \subseteq [V_0 + u] \subseteq Z.$$

Hence

$$\lambda_*([V \cap V_0 + u]) = \lambda_*(V \cap V_0)$$

makes

$$\phi(A) = \inf_{\text{open } V \supseteq A} \lambda_*(V) = \inf_{\text{open } V \supseteq A} \lambda_*(V \cap V_0)$$

$$= \inf_{\text{open } V \supseteq A} \lambda_*([V \cap V_0 + u]) \geq \inf_{\text{open } W \supseteq [A+u]} \lambda_*(W) = \phi([A + u]).$$

Symmetrically, $\phi([A + u]) \geq \phi(A)$ gives $\phi([A + u]) = \phi(A)$ for our result.

Presumably results similar to Theorem 11 are true for similar subspaces of non-Abelian topological groups. We have considered only the Abelian case for simplicity and because the interesting examples in analysis are Abelian.

COROLLARY 12. *If*

$$\liminf_{n \to \infty} f(n) = 0,$$

then the space Y *is locally compact under coincident metric and product topologies, and* Y *possesses a regular Borel measure nontrivial* (I) *and translation invariant* (II′) *under this topology.*

Proof. The coincidence of the topologies and local compactness of Y is trivial from $f(n) \downarrow 0$; and Lemma 7 and Theorem 11 give the rest.

6. **Another translation space example.** In addition to X and Y, we want to give another example of a translation space, still with $R = \ell_2$. Let

$$Z_1 = \left\{ x \in \ell_2 \;\middle|\; \sum_{n=1}^{\infty} n^{2r} (x_n)^2 \leq M \right\}$$

for some fixed real $r > 0$ and $M > 0$, so that clearly Z_1 is actually compact. Such a space would arise by using Fourier analysis on L_2-function-spaces in which the rth derivative was subjected to a fixed bound in norm. We shall now show that Z_1 is a translation space, though our proof seems unnecessarily long.

LEMMA 13. *If* $u \in Z_1$ *has* $u_n = 0$ *for* $n > N$ *for some finite* N, *and*

$$\rho N^r \leq \frac{1}{2} \left\{ \sum_{n=1}^{N} n^{2r} (u_n)^2 \right\}^{1/2}$$

for some $\rho > 0$, *then*

$$Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

open in Z_1.

Proof. We only need to show that

$$S(Z_1, u, \rho) \subseteq Z_1 \cap [S(Z_1, 0, \rho) + u],$$

the opposite inclusion being obvious. Consider any $z \in S(Z_1, u, \rho)$; we need only show $(z - u) \in Z_1$. Here $\| z - u \| < \rho$, so

$$\sum_{n=1}^{N} n^{2r} (z_n - u_n)^2 < N^{2r} \rho^2,$$

and thus from

$$\rho N^r \leq \frac{1}{2}\left\{\sum_{n=1}^{N} n^{2r}(u_n)^2\right\}^{1/2}$$

we obtain, by Minkowski's inequality,

$$0 < \left\{\sum_{n=1}^{N} n^{2r}(u_n)^2\right\}^{1/2} - \rho N^r < \left\{\sum_{n=1}^{N} n^{2r}(z_n)^2\right\}^{1/2}.$$

Thus $u_n = 0$ for $n > N$ and $z \in Z_1$ yields

$$\sum_{n=1}^{\infty} n^{2r}(z_n - u_n)^2 = \sum_{n=1}^{N} n^{2r}(z_n - u_n)^2 + \sum_{n=N+1}^{\infty} n^{2r}(z_n)^2$$

$$< \rho^2 N^{2r} + \sum_{n=1}^{\infty} n^{2r}(z_n)^2 - \sum_{n=1}^{N} n^{2r}(z_n)^2$$

$$< \rho^2 N^{2r} + M - \left(\left\{\sum_{n=1}^{N} n^{2r}(u_n)^2\right\}^{1/2} - \rho N^r\right)^2$$

$$= M - \left(\left\{\sum_{n=1}^{N} n^{2r}(u_n)^2\right\}^{1/2} - 2\rho N^r\right)\left\{\sum_{n=1}^{N} n^{2r}(u_n)^2\right\}^{1/2} \leq M.$$

Thus we have shown that

$$\sum_{n=1}^{\infty} n^{2r}(z_n - u_n)^2 < M,$$

so $(z - u) \in Z_1$ as desired.

THEOREM 14. *Z_1 satisfies Condition* i), *and hence is a compact translation space possessing a Haar measure in the sense of Theorem* 11.

Proof. We merely need to verify Condition i) for Z_1. Thus given any open Z_1-subset V containing zero and any $z \in Z_1$, we shall find some $u \in Z_1$ and $\rho > 0$ so that $S(Z_1, 0, \rho) \subseteq V$ and

$$z \in Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

open in Z_1, which makes $z \in \text{int } (Z_1 \cap [V + u])$, as desired. Here we need consider only $z \neq 0$, since $u = 0$ makes $0 \in V = \text{int } (Z_1 \cap [V + u])$ for the result if $z = 0$. Since $z \neq 0$, we may choose N sufficiently large so that

$$\beta = \left(\sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right)^{-1} \left(\sum_{n=N+1}^{\infty} n^{2r} (z_n)^2 \right)$$

has $0 \leq \beta < 1/5$, and so that

$$\frac{\sqrt{M}}{2N^r} < \rho_1$$

for some ρ_1 such that $S(Z_1, 0, \rho_1) \subseteq V$. Let

$$\rho = \frac{1}{2N^r} \left(\sum_{n=1}^{N} N^{2r} (z_n)^2 \right)^{1/2},$$

so

$$\rho \leq \frac{\sqrt{M}}{2N^r} < \rho_1 \quad \text{and} \quad S(Z_1, 0, \rho) \subseteq V.$$

Define $u \in Z_1$ by $u_n = z_n$ for $1 \leq n \leq N$ and $u_n = 0$ for $n > N$. By Lemma 13, we have

$$Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

open in Z_1. Finally to complete the proof we have $z \in S(Z_1, u, \rho)$, for

$$\begin{aligned} ||z - u||^2 &= \sum_{n=N+1}^{\infty} (z_n)^2 \leq \frac{1}{N^{2r}} \left(\sum_{n=N+1}^{\infty} n^{2r} (z_n)^2 \right) \\ &= \frac{\beta}{N^{2r}} \left(\sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right) < \frac{1}{N^{2r}} \left(\frac{1-\beta}{4} \right) \left(\sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right) \\ &= \frac{1}{(2N^r)^2} \left(\sum_{n=1}^{N} n^{2r} (z_n)^2 \right) = \rho^2, \end{aligned}$$

or $||z-u|| < \rho$, as desired, since $\beta < (1-\beta)/4$ from $0 \leq \beta < 1/5$.

7. Summary of results. We have discussed here the translation spaces

$$X = \{x \in \ell_2 \mid |x_n| \leq h(n)\}$$

and

$$Y = \{x \in \ell_2 \mid \sum_{j=n}^{\infty} x_j^2 \leq f(n)\},$$

and also

$$Z_1 = \{x \in \ell_2 \mid \sum_{n=1}^{\infty} n^{2r}(x_n)^2 \leq M\}$$

in §6, all being subspaces of real separable Hilbert space. For X under the metric topology we have found (Theorem 3) that there exists no nontrivial, translation-invariant (II or II′) Borel measure if

$$\liminf_{n \to \infty} h(n) > 0;$$

under the product topology we have the same conclusion if $h(n) = +\infty$ infinitely often (Theorem 4). If

$$\sum^{\infty} [h(n)]^2 < +\infty,$$

which is equivalent to local compactness, then under the metric topology X has a nontrivial, translation-invariant (II) Borel measure which is unique up to constant factors (Theorem 5). For Y under the metric topology

$$\liminf_{n \to \infty} f(n) = 0,$$

or thus $f(n) \downarrow 0$, is equivalent to local compactness, and necessary and sufficient for the existence of a nontrivial, translation-invariant (II′) Borel measure (Theorem 3 and Corollary 12). Also we found (Theorem 12) that any locally compact translation space possesses a nontrivial, translation-invariant (II′) Borel measure; thus so does Z_1 (Theorem 14).

It is clear from the foregoing results that local compactness is in general the crucial condition for the existence of a nontrivial, translation-invariant Borel measure. This is well known for topological groups [5, p. 144], and conjectured for spaces with a group germ (a neighborhood of zero in which group addition is always possible). However, it is to be noted that neither X nor Y, when locally compact, nor Z_1 has a group germ. Thus our results seem to be new, and the concept of a translation-space a fruitful one. In fact the idea of a group germ cannot lead to anything here; for it is not difficult to see that any convex metric subspace of ℓ_2, which is locally compact and contains a group germ under ℓ_2-vector-addition, must be finite dimensional, hence a subspace of E_N and thus trivial. In connection with local compactness it should be noted that our results are not complete for X; here if $\sum^{\infty}[h(n)]^2 = +\infty$ the space is not locally compact under the metric topology and presumably no nontrivial, invariant Borel measure exists. We could only show this if

$$\liminf_{n\to\infty} h(n) > 0,$$

which assumes more.

The construction of an invariant measure on subspaces of real separable Hilbert space suggests an attempt to carry over vector analysis from E_N. In particular, in a later paper the author investigates the relationship between ℓ_2-vector-differentiation [6, p. 72] and Fourier transforms over X. Here X is a modification of Jessen's torus space [3] and can be made into a group, so standard Fourier theory applies [7 or 5].

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ON A PAPER OF NIVEN AND ZUCKERMAN

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1. **Introduction.** Let 'digit' mean an integer in the range $0 \leq a < 10$. For digits $a_1, a_2, \cdots, a_r$; $b_1, b_2, \cdots, b_s$ $(s \geq r)$ and integer m, denote by

$$R_m(a_1, \cdots, a_r, b_1, \cdots, b_s)$$

the number of solutions of

$$b_n = a_1, b_{n+1} = a_2, \cdots, b_{n+r-1} = a_r \qquad (0 < n < n + r \leq s;\ n \equiv m \bmod r),$$

so that

$$(1) \qquad 0 \leq R_m(a_1, \cdots, a_r;\ b_1, \cdots, b_s) \leq s - r + 1.$$

Suppose that

$$x_1, x_2, \cdots$$

is an infinite sequence of digits. It has been shown [2] that if

$$(2) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{r} R_m(a_1, \cdots, a_r; x_1, \cdots, x_N) = 10^{-r}$$

for all integers r and digits $a_1, \cdots, a_r$, then

$$(3) \qquad \lim_{N\to\infty} \frac{1}{N} R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) = r^{-1}\ 10^{-r}$$

for all integers r, m, and digits $a_1, \cdots, a_r$. A possibly simpler proof is as follows.

2. **Proof.** Let $\epsilon > 0$ and digits $a_1, \cdots, a_r$ be given. The simple argument of Hardy-Wright [1] shows that if the integer s is fixed large enough, then

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$$(4)\qquad \operatorname*{Max}_{\mu}\left|R_{\mu}(a_1, \cdots, a_r;\ b_1, \cdots, b_s) - \frac{s-r+1}{r\,10^r}\right| < \epsilon(s-r+1)$$

except for at most $\epsilon 10^s$ sets of digits $b_1, \cdots, b_s$. ('Exceptional' sets.) Thus, by (2) with $b_1, \cdots, b_s$ for $a_1, \cdots, a_r$, the number of exceptional sets

$$(5)\qquad x_t, x_{t+1}, \cdots, x_{t+s-1} \qquad (1 \leq t \leq N-s+1)$$

is at most $2\epsilon N$ for all large enough N.

On the other hand,

$$(6)\qquad (s-r+1)\,R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N)$$

differs from

$$(7)\qquad \sum_{t=1}^{N-s+1} R_{m-t+1}(a_1, \cdots, a_r;\ x_t, \cdots, x_{t+s-1})$$

by at most $2s^2$, since each solution of

$$a_1 = x_n,\ a_2 = x_{n+1}, \cdots,\ a_r = x_{n+r-1} \quad (s \leq n \leq N-s;\ n \equiv m \bmod r)$$

contributes exactly $s-r+1$ both to (6) and to (7). Hence, using the estimate (3) for the at most $2\epsilon N$ exceptional sets (5), and the estimate (4) for the others, we have

$$\left|R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) - \frac{N-s+1}{r\,10^r}\right|$$
$$\leq \frac{2s^2}{s-r+1} + \epsilon(N-s+1) + 2\epsilon N,$$

and so

$$\limsup_{N}\left|\frac{1}{N} R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) - r^{-1}\,10^{-r}\right| \leq 3\epsilon.$$

Since ϵ is arbitrarily small, this proves (3) as required.

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SPECTRAL THEORY II. RESOLUTIONS OF THE IDENTITY

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Introduction

In attempting to extend elementary divisor theory to the case of a linear operator on a complex Banach space one is naturally led to a consideration of the various equivalent definitions of the multiplicity $\nu(\lambda)$ of a complex number λ as a root of the minimal equation of a finite matrix T. Of the numerous equivalent definitions of this integer we have found only one which seems to have some virtue when applied to the infinite dimensional case. That one is as follows: $\nu(\lambda)$ is the smallest positive integer or zero for which

$$|\xi - \lambda|^{\nu(\lambda)} \; |(\xi - T)^{-1}|$$

is bounded for ξ near λ. Thus the rate of growth of the resolvent

$$T(\xi) = (\xi - T)^{-1}$$

for ξ near λ determines $\nu(\lambda)$. In this paper we consider the problem of determining conditions on the rate of growth and the mean rate of growth of the resolvent which are necessary and sufficient for a complete reduction of a linear operator on a complex Banach space. What is to be meant by a "complete" reduction? There are several apparent meanings that might be given to the notion of the resolution of the identity for an operator, all reducing to the classical one in the case of a finite matrix. For example, are we to require that E_σ be defined for all Borel sets σ or for σ in some sufficiently large subalgebra; should it be countably or just finitely additive; should it be bounded or not? All problems are legitimate and in this paper we have chosen the most restrictive of all the obvious interpretations. Consequently the conditions found on $T(\xi)$ are restrictive and the corresponding class of operators is small. On the other hand, such operators have many important properties not shared by operators outside this class. Other meanings for the notion of resolution of the identity will be

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considered in another report.

Before stating what is to be meant by a resolution of the identity for T, let us recall that if T is a linear operator in the finite dimensional linear vector space X over the field of complex numbers, and $\Pi_i (\lambda - \lambda_i)^{\nu_i}$ (λ_i distinct) is its minimal polynomial, then there are projections E_{λ_i} with

$$E_{\lambda_i} X = [x \mid (T - \lambda_i)^{\nu_i} x = 0]$$

and such that

$$I = E_{\lambda_1} + \cdots + E_{\lambda_k}.$$

If, for a Borel set σ in the complex plane, E_σ is defined to be the sum of those E_{λ_i} for which $\lambda_i \in \sigma$, then E_σ is a resolution of the identity for T in the sense that it has the properties (i) below:

$$\text{(i)} \begin{cases} E_\sigma E_\delta = E_{\sigma\delta}, \quad E_{\sigma'} = I - E_\sigma, \quad TE_\sigma = E_\sigma T \\ E_\sigma x \text{ is completely additive in } \sigma,\ x \in X \\ \text{the spectrum of } T \text{ when considered as an operator in } E_\sigma X \text{ is contained} \\ \text{in } \bar{\sigma}, \text{ the closure of } \sigma. \end{cases}$$

If, for a given linear operator T in a complex Banach space, there exists a family E_σ (σ a Borel set) of operators in X satisfying (i), then E_σ is called a *resolution of the identity for T*. Such operators will be called *spectral operators*. If T is a spectral operator its resolution of the identity is unique, and operators $f(T)$ corresponding to scalar functions analytic and single valued on the spectrum $\sigma(T)$ are given by the formula

$$\text{(ii)} \qquad f(T) = \sum_{n=0}^{\infty} \int_{\sigma(T)} \frac{f^{(n)}(\lambda)}{n!} (T - \lambda)^n \, dE_\lambda ,$$

where the integral exists as a Riemann integral in the uniform topology of operators and the series is convergent in the uniform topology of operators.

The main problem is, however, to determine when T is a spectral operator. We have endeavored to state conditions on the rate of growth and the mean rate of growth of the resolvent

$$T(\xi) = (\xi - T)^{-1}$$

which are sufficient and in some cases necessary and sufficient for the existence of a resolution of the identity. In order to do this, we have had to restrict ourselves to the case where the spectrum $\sigma(T)$ lies in a sufficiently smooth Jordan curve. To describe briefly in this introduction the nature of the results obtained in this direction, suppose that T has its spectrum in the interval $[0, 1]$. The underlying assumption is then that for each $\lambda \in [0, 1]$ there is a positive integer $\nu(\lambda)$ and a positive number $M(\lambda)$ such that

$$\text{(iii)} \qquad |\mu^{\nu(\lambda)}\, T(\lambda + i\mu)| \leq M(\lambda), \quad 0 < |\mu| < 1.$$

This alone is far from sufficient to ensure that T is a spectral operator, even in case

$$\nu(\lambda) = M(\lambda) = 1.$$

An obvious necessary condition may be stated in terms of the following notion of residue. Let C be a rectifiable Jordan curve contained in the set where $x^* T(\xi)x$ is analytic. Let σ be the set of all singularities of $x^* T(\xi)x$ which are inside C. Then

$$(x^*, x)_\sigma = \frac{1}{2\pi i} \int_C x^* T(\xi) x d\xi$$

is called a residue of $x^*T(\xi)x$. It is clear that if T has a resolution of the identity then

$$(x^*, x)_\sigma = x^* E_\sigma x,$$

and hence

$$\text{(iv)} \qquad |(x^*, x)_\sigma| \leq K|x^*|\ |x|, \quad x \in X, \ x^* \in X^*.$$

Conditions (iii) and (iv) are very nearly sufficient to ensure that T is a spectral operator. In reflexive spaces they are sufficient. In general though there are operators satisfying (iii) and (iv) with

$$\nu(\lambda) = M(\lambda) = 1$$

and not possessing a resolution of the identity. A final condition which in the case of a weakly complete space X makes the set of (iii), (iv), (v) sufficient for the existence of a resolution of the identity is the following. Let M_λ, N_λ be zeros and the range of $(T - \lambda)^{\nu(\lambda)}$, respectively. The condition is:

(v) For every λ in a set dense in $[0, 1]$, $M_\lambda + N_\lambda$ is dense in X.

In case $\nu(\lambda) = 1$, the condition (iv) may be stated in the equivalent form:

$$\text{(iv)}' \qquad \underset{0<\mu<1}{\text{l.u.b.}} \int_0^1 |x^*\{T(\lambda + i\mu) - T(\lambda - i\mu)\}x|\,d\lambda < \infty .$$

Unless $\nu(\lambda) = 1$ the condition (iv)$'$ is more restrictive than (iv). However there is a condition analogous to (iv)$'$ which is equivalent to (iv). It may be stated in terms of a decomposition of the resolvent. It turns out that for a spectral T there are two operators $U(\xi)$ and $V(\xi)$ such that

$$T(\xi) = U(\xi) + V(\xi)$$

and such that $x^*V(\xi)x$ is the derivative of a single valued analytic function at every point ξ where $x^*T(\xi)x$ is analytic, and $U(\xi)$ satisfies the condition (iv)$'$. The condition (iv) may be replaced by:

(iv)$''$ The resolvent $T(\xi)$ has a decomposition as described above.

In any one of the following situations the conditions (iii) and (iv) (or (iv)$'$ or (iv)$''$) are sufficient for the existence of a resolution of the identity since *in these cases* (v) *will automatically be satisfied*:

(a) The union of the resolvent set and the continuous spectrum is dense on $[0, 1]$.

(b) There is no interval of positive length consisting entirely of points in the point spectrum of the adjoint.

(c) X is reflexive.

(d) T is completely continuous.

Let $d(\xi)$ be the distance from ξ to the spectrum $\sigma(T)$; then a condition more restrictive than (iii) is

$$\text{(iii)}' \qquad |d^m(\xi)\,T(\xi)| \le M, \text{ near } \sigma(T).$$

This condition is necessary and sufficient for the simplification of (ii) to

$$\text{(ii)}' \qquad f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\lambda)}{n!}\,(T - \lambda)^n\,dE_\lambda .$$

Thus, in a weakly complete space, (iii)$'$, (iv), (v) imply that T is a spectral operator satisfying (ii)′. In a reflexive space, (iii)′ and (iv) are equivalent to the statement that T is a spectral operator satisfying (ii)′.

In case X is not weakly complete, the above statements remain valid providing the notion of the resolution of the identity is weakened in the following manner. Instead of requiring that E_σ be defined for all Borel sets, we demand that it be defined and countably additive on the Boolean algebra determined by the real intervals. This enables one to define the integral occurring in (ii)′. Thus in this extended sense we may say that for an arbitrary complex Banach space the conditions (iii)′, (iv), (v) imply that T is a spectral operator satisfying (ii)′.

Although this is the second in a series of articles on spectral theory, not much knowledge of the contents of the first [1] paper is assumed or used. We collect here the terminology, notation, and results from that paper that are used in the present one. An *admissible* domain is an open set bounded by a finite number of rectifiable Jordan curves. It is called a *T-admissible* domain in case its boundary is contained in the resolvent set $\rho(T)$ of T. The class of complex valued functions analytic and single valued on some T-admissible domain containing the spectrum $\sigma(T)$ is denoted by $F(T)$ or $F(\sigma(T))$. For $f \in F(T)$, the operator $f(T)$ is defined by the formula

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda) T(\lambda)\, d\lambda,$$

where C is the boundary of some T-admissible domain containing the spectrum of T. The mapping, given by the above formula, of the algebra of analytic functions into an algebra of operators is a homomorphism which assigns the operators I, T to the functions 1, λ, respectively.

I. Operators with nondense spectra and preliminary lemmas

In this section we consider an operator T whose spectrum $\sigma(T)$ is nondense in the complex plane. Two conditions concerning the singularities of the analytic function $(\xi - T)^{-1}x$ are introduced (these are 1.7 and 1.14 below). As we show later, these are necessary conditions for the existence of a resolution of the identity regardless of the operator T or the character of the space X. The main purpose of §1 is to show how near these two conditions come to being sufficient. Later, in §2, we shall determine the meaning of these two conditions in terms of the rate of growth and the mean rate of growth of the resolvent $T(\xi)$ for ξ near the spectrum. The basic assumption for §1 is then:

1.1. ASSUMPTION. *The spectrum $\sigma(T)$ of T is nondense in the complex* *plane.*

This means that the resolvent set $\rho(T)$ of T is dense in the plane. The chief purpose of this assumption is to prove the following lemma which asserts that the analytic function $(\xi - T)^{-1}x$ is single valued; if this fact is already known then Assumption 1.1 may easily, in most of what follows, be discarded. Since 1.1 is the underlying assumption for practically all of § 1, *it will not be explicitly stated in the lemmas to follow.* The other assumptions 1.7 and 1.14, and others in § 2, will however be indicated parenthetically when they are used.

1.2. LEMMA. *For each $x \in X$ the analytic function $T(\xi)x$ defined on $\rho(T)$ has a unique maximal single valued analytic extension.*

Let f, g be two vector-valued analytic functions defined on open sets $D(f)$, $D(g)$, respectively. We suppose that $D(f)D(g) \supset \rho(T)$ and that

$$f(\xi) = T(\xi)\,x = g(\xi) \text{ for } \xi \in \rho(T).$$

Let $\xi_0 \in D(f)\,D(g)$. By 1.1, there is a sequence of points $\xi_n \in \rho(T)$ with $\xi_n \longrightarrow \xi_0$, and so $f(\xi_0) = g(\xi_0)$. Thus, if $\rho(x)$ is the union of all open sets containing $\rho(T)$ upon which $T(\xi)x$ has an analytic extension, we have uniquely defined upon $\rho(x)$ an analytic extension of $T(\xi)x$.

1.3. DEFINITIONS. By $x(\xi)$ we shall mean the unique maximal single valued analytic extension of $T(\xi)x$ whose existence is established in 1.2. The symbol $\rho(x)$ will be used for the domain of definition of $x(\xi)$, and the symbol $\sigma(x)$ will be used for the set of singularities of $x(\xi)$. Thus $\sigma(x)$ is the complement of $\rho(x)$, and $\rho(x) \supset \rho(T)$, $\sigma(x) \subset \sigma(T)$.

1.4. DEFINITION. By $[x]$ we shall mean the smallest closed linear manifold containing all of the vectors $T(\xi)x$, $\xi \in \rho(T)$.

1.5. LEMMA. *For every $x \in X$ we have:*

1.5.1. $x \in [x]$;

1.5.2. $f(T)\,[x] \subset [x]$, $f \in F(\sigma(T))$;

1.5.3. $x(\xi) \in [x]$, $\xi \in \rho(x)$;

1.5.4. $[y] \subset [x]$, $y \in [x]$.

Let C be a large circle such that

$$x = \frac{1}{2\pi i} \int_C T(\xi)\,x\,d\xi \in [x];$$

this proves 1.5.1. Let $y \in [x]$ and $f \in F(\sigma(T))$. Since y may be approximated by sums of the form $\sum \alpha_j T(\xi_j)x$, $f(T)y$ may be approximated by sums of the form

$$\sum \alpha_j T(\xi_j) f(T)x = \sum \alpha_j \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi) T(\xi) x}{\xi_j - \xi} d\xi,$$

where Γ is chosen, in the domain of regularity of f, to include $\sigma(T)$ and exclude the points ξ_j. Thus $f(T)y \in [x]$, and 1.5.2 is proved. Next let $\xi_0 \in \rho(x)$ and, using (1.1), choose a sequence $\xi_n \in \rho(T)$ with $\xi_n \longrightarrow \xi_0$. Thus

$$T(\xi_n)x \longrightarrow x(\xi_0),$$

and since $[x]$ is closed we have $x(\xi_0) \in [x]$. Finally if $y \in [x]$ we have, by 1.5.2, $T(\xi)y \in [x]$, $\xi \in \rho(T)$, and thus $[y] \subset [x]$. This completes the proof of 1.5.

1.6. LEMMA. *For $x, y \in X$ we have*

$$\sigma(x + y) \subset \sigma(x) \cup \sigma(y),$$

and for $\xi \in \rho(x)\rho(y)$ we have

$$x(\xi) + y(\xi) = (x + y)(\xi).$$

On the open set $\rho(x)\rho(y)$, the function $x(\xi) + y(\xi)$ is an analytic extension of

$$T(\xi)x + T(\xi)y = T(\xi)(x + y), \quad \xi \in \rho(T).$$

Thus $\rho(x + y) \supset \rho(x)\rho(y)$, and for $\xi \in \rho(x)\rho(y)$ we have, by 1.2,

$$x(\xi) + y(\xi) = (x + y)(\xi).$$

The second assumption which is needed in most of § 1 is:

1.7. ASSUMPTION. *If σ is a closed set of complex numbers, then the set $[\sigma]$ of all vectors x with $\sigma(x) \subset \sigma$ is also closed.*

1.8. LEMMA. (*Assumption* 1.7.) *If σ is a closed set of complex numbers, then $[\sigma]$ is a closed linear manifold, $T[\sigma] \subset [\sigma]$, and the spectrum of T when considered as an operator in $[\sigma]$ is contained in σ.*

That $[\sigma]$ is a closed linear manifold follows from 1.6 and 1.7. Since

$$Tx(\mu) = T(\mu)Tx,$$

for μ in $\rho(T)$ we have $\rho(x) \subset \rho(Tx)$ or $\sigma(Tx) \subset \sigma(x)$, and thus $T[\sigma] \subset [\sigma]$. Now let $x \in [\sigma]$, $\xi \in \sigma'$ (the complement of σ in the whole plane), $\xi_n \in \rho(T)$, and $\xi_n \longrightarrow \xi$. Since for $\mu \in \rho(T)$ we have

$$T(\mu)x = (\xi_n - T)T(\mu)T(\xi_n)x,$$

it follows that $\rho(x) \subset \rho(T(\xi_n)x)$ and thus $T(\xi_n)x \in [\sigma]$. Since

$$T(\xi_n)x = x(\xi_n) \longrightarrow x(\xi)$$

and $[\sigma]$ is closed, we have $x(\xi) \in [\sigma]$. Thus, since

$$(\xi - T)x(\xi) = x,$$

it follows that

$$(\xi - T)[\sigma] = [\sigma].$$

To see that $\xi - T$ is one-to-one on $[\sigma]$, suppose that

$$(\xi - T)y = 0, \ y \in [\sigma].$$

Then

$$y(\lambda) = \frac{y}{(\lambda - \xi)} \text{ and } \sigma(y) \subset (\xi) \cap \sigma = \phi,$$

the void set. This means that $y(\lambda)$ is analytic for all λ and thus that $y = 0$. Hence, if $\xi \in \sigma'$ then $\xi - T$ is a one-to-one map of $[\sigma]$ into all of itself.

1.9. LEMMA. (*Assumption* 1.7.) *For every pair σ_1, σ_2 of disjoint closed sets, there is a constant $K(\sigma_1, \sigma_2)$ such that*

$$|x(\xi)| \leq K(\sigma_1, \sigma_2)\,|x|, \quad \xi \in \sigma_1, \ x \in [\sigma_2].$$

By 1.8, σ_1 is contained in the resolvent set of T when considered as an operator in $[\sigma_2]$. Since $x(\xi)$ is the value of this resolvent at the point $\xi \in \sigma_1$ when operating on $x \in [\sigma_2]$, the present lemma follows from the preceeding one.

1.10. LEMMA. (*Assumption* 1.7.) *For every $x \in X$ we have $T[x] \subset [x]$, and when T is regarded as an operator in the space $[x]$ it has $\sigma(x)$ for its* *spectrum and* $\rho(x)$ *for its resolvent set.*

It was proved in 1.5.2 that $T[x] \subset [x]$. Let ρ_1 be the resolvent set of T as an operator in $[x]$. Using 1.5.2 again, we readily show that $\rho(T) \subset \rho_1$; and since $T(\xi)x$ is analytic on ρ_1 (since $x \in [x]$, by 1.5.1), we have $\rho_1 \subset \rho(x)$. We shall now show that for every $y \in [x]$ we have $\rho(y) \supset \rho(x)$, which means that for every $y \in [x]$ the function $T(\xi)y$ defined for $\xi \in \rho(T)$ has an analytic extension to $\rho(x)$. Elements of the form

$$(*) \qquad y = \sum \alpha_j T(\xi_j)x, \quad \xi_j \in \rho(T)$$

are dense in $[x]$, and for such y we have, for $\mu \in \rho(T)$,

$$T(\mu)y = \sum \alpha_j T(\xi_j)\ T(\mu)x.$$

Thus $T(\mu)y$ has the analytic extension $\sum \alpha_j T(\xi_j)x(\mu)$, $\mu \in \rho(x)$, and so, for y of the form (*), we have $\rho(y) \supset \rho(x)$, $\sigma(y) \subset \sigma(x)$. Let $y \in [x]$, and let y_n be a sequence of vectors of the form (*) with $y_n \longrightarrow y$. Since $y_n - y_m$ has the form (*), we have $\sigma(y_n - y_m) \subset \sigma(x)$. Let N be a neighborhood whose closure $\overline{N} \subset \rho(x)$, so that $\overline{N}$ and $\sigma(x)$ are closed disjoint sets. By 1.9, then,

$$|y_n(\xi) - y_m(\xi)| = |(y_n - y_m)(\xi)| \leq K(\overline{N}, \sigma(x))\ |y_n - y_m| \longrightarrow 0$$

uniformly for $\xi \in N$. The function

$$f(\xi) = \lim_n y_n(\xi)$$

is analytic on N, and for every $\xi \in \rho(T)N$ we have

$$f(\xi) = \lim_n y_n(\xi) = \lim_n (\xi - T)^{-1} y_n = (\xi - T)^{-1} y.$$

Hence $f(\xi) = y(\xi)$, $\xi \in N$, and $\rho(y) \supset \rho(x)$. Finally we let $\xi_0 \in \rho(x)$ and show that $\xi_0 - T$ is a one-to-one map of $[x]$ into all of itself. Let $y \in [x]$; then since $\xi_0 \in \rho(x) \subset \rho(y)$ we have, by 1.5.3 and 1.5.4, $y(\xi_0) \in [y] \subset [x]$. Since

$$(\xi - T)y(\xi) = y$$

for $\xi \in \rho(T)$, this same equation must hold for $\xi \in \rho(y)$; in particular,

$$(\xi_0 - T)y(\xi_0) = y.$$

Thus $(\xi_0 - T)[x] = [x]$. To see that $\xi_0 - T$ is a one-to-one map on $[x]$, let $y \in [x]$ and $(\xi_0 - T)y = 0$. For large ξ, we have the expansion

$$T(\xi) = \sum_{n=0}^{\infty} \frac{(T-\xi_0)^n}{(\xi-\xi_0)^{n+1}};$$

hence $T(\xi)\, y = y/(\xi - \xi_0)$. Thus if $y \neq 0$ we have $\sigma(y)$ consisting of the single point $\xi_0 \in \rho(x) \subset \rho(y)$, a contradiction since $\rho(y)$ and $\sigma(y)$ are disjoint. Thus it has been proved that for every $\xi \in \rho(x)$ the operator $\xi - T$ is a one-to-one map of $[x]$ into all of itself, and hence $\rho(x) \subset \rho_1 \subset \rho(x)$.

1.11. LEMMA. (*Assumption* 1.7.) *If $y \in [x]$ then $\sigma(y) \subset \sigma(x)$.*

This was proved (in the form $\rho(y) \supset \rho(x)$) during the course of the proof of Lemma 1.10.

1.12. LEMMA. (*Assumption* 1.7.) *The set $\sigma(x)$ is void if and only if $x = 0$.*

If $x = 0$ it follows from Definition 1.3 that $\sigma(x)$ is void. Conversely, if $\sigma(x)$ is void then by (7) the spectrum of T as an operator in the space $[x]$ is void. This, according to Taylor's result [3], implies that $[x]$ consists of the zero vector alone. Hence $x = 0$.

1.13. LEMMA. (*Assumption* 1.7.) *Let σ be a set of complex numbers, and σ' its complement. If $x + y = x_1 + y_1$, where $\sigma(x), \sigma(x_1) \subset \sigma$ and $\sigma(y), \sigma(y_1) \subset \sigma'$, then $x = x_1$, $y = y_1$.*

The sets

$$\sigma_1 = \sigma(x) \cup \sigma(x_1), \quad \sigma_2 = \sigma(y) \cup \sigma(y_1)$$

are bounded, closed, and disjoint. Since, by 1.6, $\sigma(x+y) \subset \sigma_1 \cup \sigma_2$, there is an admissible contour C containing σ_1 and excluding σ_2 which lies in $\rho(x+y)$. Thus

$$\frac{1}{2\pi i}\int_C (x+y)(\xi)\,d\xi = \frac{1}{2\pi i}\int_C x(\xi)\,d\xi + \frac{1}{2\pi i}\int_C y(\xi)\,d\xi$$

Since $y(\xi)$ is regular in the closed domain bounded by C, the second integral on the right side of the above equality is zero. Since $\sigma(x)$ is contained within the domain bounded by C we see, from 1.10, that the first integral on the right of the above equality is equal to x. Hence

$$\frac{1}{2\pi i} \int_C (x + y)(\xi)d\xi = x,$$

and similarly

$$\frac{1}{2\pi i} \int_C (x_1 + y_1)(\xi)d\xi = x_1 .$$

Thus $x = x_1$, $y = y_1$.

In most of what follows we shall need besides Assumption 1.7 the following:

1.14. ASSUMPTION. *There is a constant K, depending only upon T, such that for every pair x, y of vectors with $\sigma(x)$, $\sigma(y)$ disjoint we have*

$$|x| \leq K|x + y|.$$

1.15. DEFINITION. By s_1 we shall mean the family of all sets σ with the property that vectors of the form $x + y$ with $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ are dense in X. Clearly, if $\sigma \in s_1$ then the complement $\sigma' \in s_1$.

1.16. LEMMA. (*Assumptions* 1.7, 1.14.) *For $\sigma \in s_1$ there is one and only one bounded projection E_σ on X with the properties $E_\sigma x = x$ if $\sigma(x) \subset \sigma$; $E_\sigma x = 0$ if $\sigma(x) \subset \sigma'$. This projection has the further properties that*

$$E_\sigma + E_{\sigma'} = I, \; E_\sigma E_{\sigma'} = 0, \; |E_\sigma| \leq K.$$

Vectors of the form $z = x + y$ with $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ are dense in X. In view of 1.13 it is permissible to define, on this dense set, $E_\sigma z = x$. From 1.14 it follows that $|E_\sigma z| \leq K|z|$. Now if

$$z_1 = x_1 + y_1 \text{ with } \sigma(x_1) \subset \sigma, \; \sigma(y_1) \subset \sigma',$$

then

$$z + z_1 = x + x_1 + y + y_1 ,$$

and, by 1.6, $\sigma(x + x_1) \subset \sigma$, $\sigma(y + y_1) \subset \sigma'$. Thus

$$E_\sigma(z + z_1) = E_\sigma z + E_\sigma z_1 ,$$

and E_σ is additive and continuous on a dense linear set. Thus $E_\sigma z$ is uniquely defined for $z \in X$ by the requirements that $E_\sigma z$ is continuous in z. For elements z of the original dense set we have

$$E_\sigma^2 z = E_\sigma x = x = E_\sigma z, \text{ where } z = x + y, \ \sigma(x) \subset \sigma, \ \sigma(y) \subset \sigma'.$$

Thus $E_\sigma^2 = E_\sigma$. It is also clear that

$$E_\sigma E_{\sigma'} = 0 \text{ and } E_\sigma + E_{\sigma'} = I.$$

If A_σ is another bounded projection with the properties

$$A_\sigma x = x \text{ if } \sigma(x) \subset \sigma \text{ and } A_\sigma x = 0 \text{ if } \sigma(x) \subset \sigma',$$

then for $z = x + y$, where $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ we have $A_\sigma z = x = E_\sigma z$, and hence $A_\sigma z = E_\sigma z$ for every $z \in X$.

1.17. LEMMA. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$ *and* $f \in F(\sigma(T))$, *then* $f(T)E_\sigma = E_\sigma f(T)$.

Let $z = x + y$, $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$. Then

$$f(T)z = f(T)x + f(T)y.$$

By 1.5.1 and 1.5.2, $f(T)x \in [x]$; and by 1.11, $\sigma(f(T)x) \subset \sigma(x) \subset \sigma$. Similarly, $\sigma(f(T)y) \subset \sigma(y) \subset \sigma'$. So

$$E_\sigma f(T)z = f(T)x = f(T)E_\sigma z.$$

Since the vectors z are dense, the lemma is proved.

1.18. LEMMA. (*Assumptions* 1.7, 1.14.) *We have* $\sigma(E_\sigma x) \subset \sigma(x)$, $\sigma \in s_1$, $x \in X$.

We have, by 1.17,

$$T(\xi)E_\sigma x = E_\sigma T(\xi)x, \quad \xi \in \rho(T),$$

and hence the analytic function $T(\xi)E_\sigma x$ has the analytic extension $E_\sigma x(\xi)$ for $\xi \in \rho(x)$. Thus $\rho(E_\sigma x) \supset \rho(x)$ and $\sigma(E_\sigma x) \subset \sigma(x)$.

1.19. DEFINITION. For $\sigma \in s_1$, define $X_\sigma = E_\sigma X$.

1.20. DEFINITION. If M is a closed linear manifold in X for which $TM \subset M$, we use the symbol $\sigma(M)$ for the spectrum of T when considered as an operator in M, and the symbol $\rho(M)$ for the resolvent set of T as an operator in M.

1.21. THEOREM. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$, *then* $TX_\sigma \subset X_\sigma$ *and* $\sigma(X_\sigma) \subset \bar{\sigma}$, *where* $\bar{\sigma}$ *is the closure of* σ.

It follows from 1.17 that $TX_\sigma \subset X_\sigma$. Let $\xi \notin \bar{\sigma}$. We shall first show that $\xi - T$ is one-to-one on X_σ. If $x \in X_\sigma$, $(\xi - T)x = 0$, then $x(\lambda) = x/(\lambda - \xi)$ since for all large λ we have

$$T(\lambda) = \sum_{n=0}^{\infty} \frac{(T - \xi)^n}{(\lambda - \xi)^{n+1}}.$$

Since $x \in X_\sigma$, we have $x = E_\sigma x$; and since $x(\lambda)$ is everywhere regular except possibly at the point $\xi \in \sigma'$, we have $\sigma(x) \subset \sigma'$, from which it follows that $E_\sigma x = 0$. Thus $\xi - T$ is one-to-one on X_σ. We next show that $(\xi - T)X_\sigma = X_\sigma$. Let

$$x \in X_\sigma \text{ and } x_n + y_n \longrightarrow x, \ \sigma(x_n) \subset \sigma, \ \sigma(y_n) \subset \sigma'.$$

Then

$$x = E_\sigma x = \lim_n E_\sigma(x_n + y_n) = \lim_n E_\sigma x_n = \lim_n x_n.$$

Let $y_n = x_n(\xi)$, so that

$$y_n - y_m = x_n(\xi) - x_m(\xi) = (x_n - x_m)(\xi),$$

and hence, by 1.9,

$$|y_n - y_m| \le K_1 \, |x_n - x_m| \longrightarrow 0.$$

Let $y = \lim_n y_n$, so that

$$x = \lim_n x_n = \lim (\xi - T)y_n = (\xi - T)y.$$

It remains to be shown that $y \in X_\sigma$. Since $\xi \in \rho(x_n)$, we see from 1.5.3 that

$$y_n = x_n(\xi) \in [x_n],$$

and thus 1.11 gives $\sigma(y_n) \subset \sigma(x_n) \subset \sigma$. Thus $y_n = E_\sigma y_n \in X_\sigma$ and $y \in X_\sigma$. We have shown that if $\xi \notin \bar{\sigma}$ then $\xi - T$ is a one-to-one map of X_σ into all of itself; that is, $\sigma(X_\sigma) \subset \bar{\sigma}$.

1.22. LEMMA. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$, *then* $\sigma(E_\sigma x) \subset \bar{\sigma}\,\sigma(x)$

for every $x \in X$.

In view of 1.18 it will suffice to show that $\sigma(E_\sigma x) \subset \overline{\sigma}$. From 1.21 it follows that $\rho(E_\sigma x) \supset \overline{\sigma'}$, and thus $\sigma(E_\sigma x) \subset \overline{\sigma}$.

1.23. DEFINITION. The symbol s_2 will be used for the family of all sets σ having the following property. For every $x \in X$ and every $\epsilon > 0$ there are vectors x_1, x_1' with $\sigma(x_1) \subset \sigma(x)\sigma$, $\sigma(x_1') \subset \sigma(x)\sigma'$ and $|x_1 + x_1' - x| < \epsilon$.

1.24. LEMMA. *The family* s_2 *is a Boolean algebra and* $s_2 \subset s_1$.

That $s_2 \subset s_1$ is clear from the definition of these classes. Let $\sigma_1,\ \sigma_2 \in s_2$, $x \in X$ and $\epsilon > 0$. We then have

$$x = x_1 + x_1' + u_1,\quad \sigma(x_1) \subset \sigma(x)\sigma_1,\quad \sigma(x_1') \subset \sigma(x)\sigma_1',\quad |u_1| < \epsilon/2;$$

$$x_1' = x_2 + x_2' + u_2,\quad \sigma(x_2) \subset \sigma(x_1')\sigma_2,\quad \sigma(x_2') \subset \sigma(x_1')\sigma_2',\quad |u_2| < \epsilon/2;$$

$$x = x_1 + x_2 + x_2' + u_1 + u_2.$$

Using 1.6 we see that

$$\begin{aligned}\sigma(x_1 + x_2) \subset \sigma(x_1) \cup \sigma(x_2) &\subset (\sigma(x)\sigma_1) \cup (\sigma(x_1')\sigma_2)\\ &\subset (\sigma(x)\sigma_1) \cup (\sigma(x)\sigma_2)\\ &= \sigma(x)\ (\sigma_1 \cup \sigma_2),\end{aligned}$$

and

$$\sigma(x_2') \subset \sigma(x_1')\sigma_2' \subset \sigma(x)\sigma_1'\ \sigma_2' = \sigma(x)\ (\sigma_1 \cup \sigma_2)'.$$

Thus $\sigma_1 \cup \sigma_2 \in s_2$. It is clear from 1.23 that s_2 is closed under complementation and that the void set and the whole plane are in s_2. Thus s_2 is closed under crosscut; that is, $\sigma_1 \sigma_2 \in s_2$ if $\sigma_1, \sigma_2 \in s_2$, and s_2 is a Boolean algebra.

1.25. THEOREM. (*Assumptions* 1.7, 1.14.) *On the Boolean algebra* s_2 *the projections* E_σ *have the following properties*:

$$E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1 \cup \sigma_2},\quad E_{\sigma_1} E_{\sigma_2} = E_{\sigma_1 \sigma_2},\quad E_\sigma' = E_{\sigma'};$$

$$E_{\sigma(T)} = I,\quad E_\phi = 0,\ \textit{where } \phi \textit{ is the void set.}$$

If the projections E_σ, $\sigma \in s_2$ are ordered in the usual fashion (that is, $E_{\sigma_1} \subset E_{\sigma_2}$ means $E_{\sigma_2} E_{\sigma_1} = E_{\sigma_1}$ or equivalently $X_{\sigma_1} \subset X_{\sigma_2}$) then, by definition, $E_{\sigma_1} \cup E_{\sigma_2}$ is the smallest projection containing E_{σ_1} and E_{σ_2}. It may be given by the formula

$$E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1} + E_{\sigma_2} - E_{\sigma_1} E_{\sigma_2}.$$

This formula is readily derived from the relation

$$E_{\sigma_1} E_{\sigma_2} = E_{\sigma_2} E_{\sigma_1},$$

which of course will be established as soon as we have shown that

$$E_{\sigma_1} E_{\sigma_2} = E_{\sigma_1 \sigma_2}.$$

Now let $x \in X$, $\epsilon > 0$, σ_1, $\sigma_2 \in s_2$. We have

$$x = x_1 + x_1' + u, \quad x_1' = x_2 + x_2' + v, \quad x_1 = y_2 + y_2' + w,$$

where $|u|$, $|v|$, $|w| < \epsilon$, and

$$\sigma(x_1) \subset \sigma(x)\sigma_1, \quad \sigma(x_1') \subset \sigma(x)\sigma_1',$$

$$\sigma(x_2) \subset \sigma(x_1')\sigma_2 \subset \sigma(x)\sigma_1' \sigma_2 \subset (\sigma_1 \sigma_2)',$$

$$\sigma(x_2') \subset \sigma(x_1')\sigma_2' \subset \sigma(x)\sigma_1' \sigma_2' \subset (\sigma_1 \cup \sigma_2)' \subset (\sigma_1 \sigma_2)',$$

$$\sigma(y_2) \subset \sigma(x_1)\sigma_2 \subset \sigma(x)\sigma_1 \sigma_2,$$

$$\sigma(y_2') \subset \sigma(x_1)\sigma_2' \subset \sigma(x)\sigma_1 \sigma_2' \subset (\sigma_1 \sigma_2)'\sigma_1.$$

Place $z = y_2 + y_2' + x_2 + x_2'$, $y = u + v + w$, so that $x = z + y$ and $|y| < 3\epsilon$. Remembering that $E_\sigma x = x$ for every x with $\sigma(x) \subset \sigma$ and $E_\sigma x = 0$ if $\sigma(x) \subset \sigma'$, we see from the above inclusion relations that

$$E_{\sigma_1} z = y_2 + y_2', \quad E_{\sigma_2} z = y_2 + x_2,$$

$$E_{\sigma_2} E_{\sigma_1} z = E_{\sigma_1} E_{\sigma_2} z = E_{\sigma_1 \sigma_2} z = y_2,$$

$$(E_{\sigma_1} + E_{\sigma_2} - E_{\sigma_1} E_{\sigma_2}) z = y_2 + y_2' + x_2 = E_{\sigma_1 \cup \sigma_2} z.$$

Hence

$$|(E_{\sigma_1} E_{\sigma_2} - E_{\sigma_1 \sigma_2})x| = |(E_{\sigma_1} E_{\sigma_2} - E_{\sigma_1 \sigma_2})y| \leq 3K(K+1)\epsilon.$$

Since ϵ is independent of x, we have

$$E_{\sigma_1} E_{\sigma_2} = E_{\sigma_1 \sigma_2} = E_{\sigma_2} E_{\sigma_1}.$$

Also,

$$|((E_{\sigma_1} \cup E_{\sigma_2}) - E_{\sigma_1 \cup \sigma_2})x| \leq 4K|y| < 12K\epsilon,$$

so that $E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1 \cup \sigma_2}$. The remaining conclusions have been proved in 1.16.

1.26. DEFINITION. (Assumptions 1.7, 1.14.) The symbol s_3 will be used for those sets $\sigma \in s_1$ for which there exist closed sets $\mu_n, \nu_n \in s_2$ with $\nu_n \subset \sigma$, $\mu_n \subset \sigma'$, $n = 1, 2, \cdots$ and

$$x = \lim_n (E_{\nu_n} + E_{\mu_n})x, \quad x \in X.$$

1.27. LEMMA. (*Assumptions* 1.7, 1.14.) *The family* s_3 *is a Boolean algebra and* $s_3 \subset s_2$.

If $\sigma \in s_3$ and μ_n, ν_n are as in 1.26, then by 1.22 we have

$$\sigma(E_{\nu_n}x) \subset \nu_n\, \sigma(x) \subset \sigma\, \sigma(x), \quad \sigma(E_{\mu_n}x) \subset \mu_n\, \sigma(x) \subset \sigma'\, \sigma(x),$$

and so $\sigma \in s_2$; that is, $s_3 \subset s_2$. It is clear that s_3 is closed under complementation; hence, in order to show that s_3 is a Boolean algebra, it will suffice to show that it is closed under the operation of forming unions. Let $\sigma_1, \sigma_2 \in s_3$ and $\nu(i, n)$, $\mu(i, n)$, $(i = 1, 2;\ n = 1, 2, \cdots)$ be closed s_2 sets with $\nu(i, n) \subset \sigma_i$, $\mu(i, n) \subset \sigma_i'$ $(i = 1, 2)$ and

$$x = \lim_n \left(E_{\nu(i,n)} + E_{\mu(i,n)}\right) x \qquad (i = 1, 2).$$

Then

$$x = E_{\nu(1,n)}x + E_{\mu(1,n)}x + u_n, \quad \text{and } u_n \longrightarrow 0.$$

Thus

$$E_{\sigma_1'}x = E_{\mu(1,n)}x + E_{\sigma_1'}u_n \quad \text{and} \quad E_{\mu(1,n)}x \longrightarrow E_{\sigma_1'}x.$$

This last fact shows that the sequence v_n defined by the next equation has the property that $v_n \longrightarrow 0$:

$$E_{\mu(1,n)}x = E_{\nu(2,n)}E_{\mu(1,n)}x + E_{\mu(2,n)}E_{\mu(1,n)}x + v_n.$$

Upon substituting the above expression for $E_{\mu(1,n)}x$ into the formula defining u_n, we see by using 1.25 that

$$x = E_{\nu_n}x + E_{\mu_n}x + u_n + v_n,$$

where

$$\nu_n = \nu(1,n) \cup \nu(2,n)\mu(1,n) \text{ and } \mu_n = \mu(2,n)\mu(1,n).$$

Since ν_n, μ_n are closed s_2 sets (by 1.24) with $\nu_n \subset \sigma_1 \cup \sigma_2$, and $\mu_n \subset \sigma_1' \sigma_2' = (\sigma_1 \cup \sigma_2)'$, it follows that $\sigma_1 \cup \sigma_2$ is an s_3 set, and the lemma is established.

1.28. LEMMA. (*Assumptions* 1.7, 1.14.) *Let* $\sigma \in s_3$, $x \in X$, $\epsilon > 0$. *Then there are sets* $\mu, \nu \in s_2$ *with* μ *open,* ν *closed,* $\mu \supset \sigma \supset \nu$, *and such that*

$$|E_\omega x| < \epsilon, \quad \omega \subset \mu - \nu, \quad \omega \in s_2;$$

$$|E_\sigma x - E_{\sigma_1}x| < \epsilon, \quad \mu \supset \sigma_1 \supset \nu, \quad \sigma_1 \in s_2.$$

Since σ is an s_3 set, there are closed sets $\nu, \mu' \in s_2$ with

$$\nu \subset \sigma, \quad \mu' \subset \sigma', \quad x = E_\nu x + E_{\mu'}x + u, \text{ and } |u| < \epsilon.$$

Then

$$|E_\mu x - E_\nu x| = |E_\mu u| < K\epsilon.$$

Let $\omega \subset \mu - \nu$, $\omega \in s_2$. Then

$$|E_\omega x| = |E_\omega E_{\mu-\nu}x| = |E_\omega(E_\mu - E_\nu)x| < K^2\epsilon;$$

this proves the first conclusion. Now

$$E_\sigma - E_{\sigma_1} = E_{\sigma-\sigma\sigma_1} + E_{\sigma\sigma_1} - E_{\sigma_1-\sigma\sigma_1} - E_{\sigma\sigma_1} = E_{\sigma-\sigma\sigma_1} - E_{\sigma_1-\sigma\sigma_1},$$

and since $\sigma - \sigma\sigma_1 \subset \mu - \nu$, $\sigma_1 - \sigma\sigma_1 \subset \mu - \nu$, the second conclusion follows from the first conclusion.

1.29. LEMMA. (*Assumption* 1.7, 1.14.) *If* σ_m, $\sigma \in s_3$; $\sigma_m \subset \sigma_{m+1}$ $(m = 1, 2, \cdots)$; *and* $\sigma_m \longrightarrow \sigma$; *then* $E_{\sigma_m}x \longrightarrow E_{\sigma}x$, $x \in X$.

Let $x \in X$, $\epsilon > 0$ be arbitrary. Let $\epsilon_n > 0$ and $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. Using 1.28, pick open sets $\mu_n \in s_2$ with $\mu_n \supset \sigma_n$ and

$$|E_{\omega}x| < \epsilon_n, \quad \omega \subset \mu_n - \sigma_n, \quad \omega \in s_2.$$

We shall now show that

$$(*) \qquad |E_{\omega}x| < \sum_{j=1}^{n} \epsilon_j, \quad \omega \subset (\mu_1 + \cdots + \mu_n) - \sigma_n, \quad \omega \in s_2.$$

The statement $(*)$ is true for $n = 1$. Assume that it is true for n and let

$$\omega \subset (\mu_1 + \cdots + \mu_{n+1}) - \sigma_{n+1}, \quad \omega \in s_2,$$

so that

$$\omega = \omega[(\mu_1 + \cdots + \mu_n) - \sigma_n] + \omega(\mu_{n+1} - \sigma_{n+1})$$

and $\omega = \omega_1 + \omega_2$, where

$$\omega_1 = \omega[(\mu_1 + \cdots + \mu_n) - \sigma_n](\mu_{n+1} - \sigma_{n+1})', \quad \omega_2 = \omega(\mu_{n+1} - \sigma_{n+1}).$$

By our induction assumption we have

$$|E_{\omega_1}x| < \sum_{j=1}^{n} \epsilon_j,$$

and since $\omega_2 \subset \mu_{n+1} - \sigma_{n+1}$ we have

$$|E_{\omega_2}x| < \epsilon_{n+1}.$$

Since ω_1 and ω_2 are disjoint, it follows that

$$|E_{\omega}x| \leq |E_{\omega_1}x| + |E_{\omega_2}x| < \sum_{j=1}^{n+1} \epsilon_j;$$

this proves $(*)$. Now let $\xi_n = \mu_1 + \cdots + \mu_n$, so that ξ_n is open, increasing with n, $\xi_n \supset \sigma_n$, and $|E_{\omega}x| < \epsilon$ for every $\omega \in s_2$ with $\omega \subset \xi_n - \sigma_n$. Using 1.28, let

μ, $\nu \in s_2$, μ open, ν closed, $\mu \supset \sigma \supset \nu$, and $|E_\sigma x - E_{\sigma_1} x| < \epsilon$ for every $\sigma_1 \in s_2$ with $\mu \supset \sigma_1 \supset \nu$. We have

$$\mu \supset \bigcup_{m=1}^{\infty} \xi_m \mu \supset \mu \bigcup_{m=1}^{\infty} \sigma_m = \mu \sigma \supset \sigma \supset \nu .$$

Since ν is closed, there is an integer k_0 such that

$$\mu \supset \bigcup_{m=1}^{k} \xi_m \mu = \xi_k \mu \supset \nu, \qquad (k \geq k_0).$$

Thus

$$|E_\sigma x - E_{\xi_k \mu} x| < \epsilon, \qquad (k \geq k_0),$$

and since $\xi_k \mu - \sigma_k \subset \xi_k - \sigma_k$ we have, from (*),

$$|E_{\xi_k \mu} x - E_{\sigma_k} x| = |E_{\xi_k \mu - \sigma_k} x| < \epsilon, \qquad (k = 1, 2, \cdots).$$

Hence

$$|E_\sigma x - E_{\sigma_k} x| < \epsilon, \qquad (k \geq k_0);$$

this proves the lemma.

1.30. THEOREM. (*Assumptions* 1.7, 1.14.) *For each $x \in X$ the set function $E_\sigma x$ is countably additive on the Boolean algebra s_3.*

The conclusion of the theorem means that if σ, $\sigma_n \in s_3$, $\sigma_n \sigma_m$ is void for $n \neq m$, $\sigma = \bigcup_1^\infty \sigma_n$, then

$$\sum_{n=1}^{\infty} E_{\sigma_n} x = E_\sigma x .$$

The lemma is an immediate consequence of 1.25, 1.27, and 1.29.

1.31. DEFINITION. By a *Borel algebra* of sets we shall mean a Boolean algebra of sets which is closed under the operation of taking denumerable unions.

1.32. DEFINITION. The smallest Borel algebra of sets containing the Boolean algebra s_3 will be called the family of sets *measurable* T and will be denoted by $m(T)$.

In part of what follows we shall assume:

1.33. ASSUMPTION. *The space X is weakly complete.*

1.34. THEOREM. (*Assumptions* 1.7, 1.14, 1.33.) *The function E_e defined on s_3 to the set of bounded projections on X has a unique extension to $m(T)$ with the properties* (*the first of which ensures uniqueness*):

(i) $E_e x$ *is countably additive on* $m(T)$, $x \in X$;

(ii) $|E_e| \leq K$, $e \in m(T)$;

(iii) $E_{e_1} E_{e_2} = E_{e_1 e_2}$, $e_1, e_2 \in m(T)$;

(iv) $E_{e_1 \cup e_2} = E_{e_1} \cup E_{e_2}$, $e_1, e_2 \in m(T)$;

(v) $E_e' = E_{e'}$, $E_{\sigma(T)} = I$, $E_\phi = 0$, $e \in m(T)$, ϕ *void*;

(vi) $f(T)E_e = E_e f(T)$, $e \in m(T)$, $f \in F(\sigma(T))$.

For point sets e_n, e we mean by $e_n \longrightarrow e$ or $\lim_n e_n = e$ that

$$e = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} e_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} e_n,$$

and we recall that if

$$a_n \longrightarrow a, \; b_n \longrightarrow b$$

then

$$a_n b_n \longrightarrow ab, \; a_n \cup b_n \longrightarrow a \cup b, \; a_n' \longrightarrow a'.$$

We define a transfinite sequence of Boolean algebras $\beta_0, \beta_1, \cdots$ as follows: $\beta_0 = s_3$ and β_α consists of all e such that there exists a sequence

$$e_n \in \bigcup_{\gamma < \alpha} \beta_\gamma$$

with $e_n \longrightarrow e$. Thus $m(T) = \bigcup_{\gamma < \omega} \beta_\gamma$, where ω is the first ordinal whose cardinal is that of a nonenumerable class. For each $x \in X$ and $x^* \in X^*$ there is, according to a well-known theorem of Hahn, a uniquely defined countably additive numerical set function $m(e, x, x^*)$ on $m(T)$ such that

$$m(e, x, x^*) = x^*E_e x, \quad e \in s_3 = \beta_0, \quad x \in X, \quad x^* \in X^*.$$

We first show that for every $x \in X$ and $e \in m(T)$ there is a unique vector $x_e \in X$ such that

$$x^* x_e = m(e, x, x^*), \quad x^* \in X^*.$$

This is true for $e \in \beta_0$. Assume that it is true for $e \in \bigcup_{\gamma<\alpha} \beta_\gamma$ and let $e \in \beta_\alpha$, $e_n \in \bigcup_{\gamma<\alpha} \beta_\gamma$, $e_n \longrightarrow e$. Then

$$x^* x_{e_n} = m(e_n, x, x^*) \longrightarrow m(e, x, x^*), \quad x^* \in X^*.$$

Since X is weakly complete, there is a vector x_e with

$$x^* x_e = m(e, x, x^*), \quad x^* \in X^*.$$

This last equation shows that x_e is independent of the sequence $e_n \longrightarrow e$ and also is uniquely defined. Next consider the statements:

$$|m(e, x, x^*)| \leq K|x| \; |x^*|;$$

$$m(e, x_1 + x_2, x^*) = m(e, x_1, x^*) + m(e, x_2, x^*);$$

$$m(e, \alpha x, x^*) = \alpha m(e, x, x^*), \quad \alpha \text{ scalar.}$$

These relations hold for $e \in \beta_0$, and since $m(e, x, x^*)$ is continuous on $m(T)$ in the topology $e_n \longrightarrow e$ it is seen by induction that they hold for any $e \in m(T)$. They show that for fixed $e \in m(T)$ the vector x_e is linear and continuous in x; that is, for $e \in m(T)$ there is a bounded linear operator E_e on X with $E_e x = x_e$. Hence we have

$$x^*E_e x = m(e, x, x^*), \; |E_e| \leq K, \; e \in m(T), \; x \in X, \; x^* \in X^*.$$

The uniqueness of E_e follows from the uniqueness of $m(e, x, x^*)$ asserted by Hahn's theorem. That $E_e x$ is countably additive on $m(E)$ in the strong topology of operators and not merely in the weak topology follows from a theorem of Orlicz concerning weakly complete spaces. Banach has restated the theorem of Orlicz in a form to hold on any Banach space and it reads as follows [2]:

ORLICZ-BANACH THEOREM. *If all the partial sums of* $\sum x_n$ *converge weakly to an element, then the series* $\sum x_n$ *is unconditionally convergent.*

The countable additivity of $E_e x$ is a corollary. For let

$$e_n e_m = \phi,\ n \neq m,\ e_n \in m(T),\ e = \bigcup_{n=1}^{\infty} e_n.$$

For every set π of integers, let

$$e_\pi = \bigcup_{n \in \pi} e_n.$$

Then we have the weak series convergence:

$$\sum_{n \in \pi} E_{e_n} x = E_{e_\pi} x.$$

Thus, according to the Orlicz-Banach theorem, $\sum E_{e_n} x$ converges unconditionally in the strong vector topology. The sum is, of course, $E_e x$ since

$$x^* E_e x = \sum x^* E_{e_n} x,\ x^* \in X^*.$$

Thus we have proved statements (i) and (ii). Statement (iii) holds for e_1, $e_2 \in \beta_0$. We suppose that

$$E_a E_b = E_{ab},\ a, b \in \bigcup_{\gamma < \alpha} \beta_\gamma,$$

and let

$$a, b_n \in \bigcup_{\gamma < \alpha} \beta_\gamma \text{ with } b_n \longrightarrow b \in \beta_\alpha.$$

Then

$$x^* E_a E_b x = m(b, x, x^* E_a) = \lim_n (b_n, x, x^* E_a) = \lim_n x^* E_a E_{b_n} x$$

$$= \lim_n x^* E_{ab_n} x = \lim_n m(ab_n, x, x^*) = m(ab, x, x^*) = x^* E_{ab} x.$$

Thus

$$E_a E_b = E_{ab} \text{ for } a \in \bigcup_{\gamma < \alpha} \beta_\gamma,\ b \in \beta_\alpha.$$

Next choose

$$a_n \in \bigcup_{\gamma < a} \beta_\gamma \text{ with } a_n \longrightarrow a \in \beta_a.$$

Then

$$x^*E_aE_bx = m(a, E_bx, x^*) = \lim_n m(a_n, E_bx, x^*) = \lim_n x^*E_{a_n}E_bx$$

$$= \lim_n x^*E_{a_nb}x = \lim_n m(a_nb, x, x^*) = m(ab, x, x^*) = x^*E_{ab}x .$$

This proves (iii). Statements (v) and (vi) are readily proved by induction, and (iv) follows from (iii) and (v).

1.35. DEFINITION. If for each $e \in B$, the Borel sets in the complex plane, there is a bounded linear operator E_e on X, then the function E_e on B to the ring of operators on X is called a *resolution of the identity* in case $E_{e'} = I - E_e$, $E_{e_1}E_{e_2} = E_{e_1 e_2}$ for $e, e_1, e_2 \in B$, and x^*E_ex is countably additive on B for every $x \in X$, $x^* \in X^*$.

1.36. LEMMA. *A resolution of the identity has the further properties*

(i) *E_ex is countably additive on B, $x \in X$;*

(ii) $\sup_{e \in B} |E_e| < \infty$;

(iii) $E_{e_1}E_{e_2} = E_{e_1e_2}$, $E_{e_1 \cup e_2} = E_{e_1} \cup E_{e_2}$, $e_1, e_2 \in B$;

(iv) $E_e' = E_{e'}$, $E_\phi = 0$, $E_p = I$, $e \in B$, *ϕ void, p = the whole plane.*

Statement (i) follows from the Orlicz-Banach theorem, and (ii) from the principle of uniform boundedness. $E_\phi = 0$ since E_e is additive in e; hence $I = E_\phi' = E_p$. The second part of (iii) follows from the first part and (iv).

1.37. DEFINITION. A resolution of the identity E_e is called a *resolution of the identity for the linear operator T* in case

$$TE_e = E_eT \text{ and } \sigma(E_eX) \subset \bar{e}, \; e \in B.$$

1.38. LEMMA. *Let X be weakly complete, and let T be a bounded linear operator in X whose spectrum is nondense. Then T has a resolution of the identity if and only if it satisfies the conditions* 1.7, 1.14 *and*:

1.39. *For every complex number λ and every $\epsilon > 0$ there is an s_3 set of diameter $< \epsilon$ and containing λ as an interior point.*

Furthermore, when T has a resolution of the identity E_e it is unique and has *the following properties*:

(i) *if* σ *is closed, then* $E_\sigma x = x$ *if and only if* $\sigma(x) \subset \sigma$;

(ii) $\sigma(E_\sigma x) \subset \overline{\sigma}\,\sigma(x)$, $\sigma \in B$, $x \in X$.

To prove the sufficiency of the conditions it is, in view of 1.34, sufficient to show that $\sigma(X_e) \subset \overline{e}$, $e \in B$, where $X_e = E_e X$. Let ξ be a complex number not in $\overline{e}\sigma(T)$, and with each $\lambda \in \overline{e}\sigma(T)$ associate an s_3 set σ_λ whose diameter is less than 1/2 the distance from ξ to $\overline{e}\sigma(T)$ and such that λ is in the interior of σ_λ. A finite number $\sigma_1, \cdots, \sigma_n$ of these sets σ_λ covers $\overline{e}\sigma(T)$, and since s_3 is a Boolean algebra the set $\sigma = \bigcup_{i=1}^n \sigma_i \in s_3$. Since $s_3 \subset s_1$, we see from 1.21 that $\sigma(X_\sigma) \subset \overline{\sigma}$. But since

$$E_e = E_e E_{\sigma(T)} = E_{e\sigma(T)} \subset E_{\overline{e}\sigma(T)} \subset E_\sigma ,$$

we have $X_e \subset X_\sigma$. Since $\xi \notin \overline{\sigma}$ and $\sigma(X_\sigma) \subset \overline{\sigma}$, the operator $\xi - T$ is one-to-one on X_σ to all of X_σ, and hence likewise on the invariant subspace X_e. Thus

$$\xi \in \rho(X_e),\ (\overline{e}\sigma(T))' \subset \rho(X_e),\ \ \overline{e} \supset \overline{e}\sigma(T) \supset \sigma(X_e).$$

It will now be shown that if T has a resolution of the identity E_e, then it is unique. Let A_e also be a resolution of the identity for T. Let σ, σ_1 be disjoint closed sets of complex numbers. Since $\sigma(E_{\sigma_1}X) \subset \sigma_1$, the function $T(\xi)\,E_{\sigma_1}x$ analytic on $\rho(T)$ has an analytic extension to σ_1'. Hence also the function

$$T(\xi)\,A_\sigma E_{\sigma_1}x = A_\sigma\, T(\xi)\, E_{\sigma_1}x$$

analytic on $\rho(T)$ has an analytic extension to σ_1'. Since $\sigma(A_\sigma X) \subset \sigma$, the function $T(\xi)\,A_\sigma E_{\sigma_1}x$ has an analytic extension to σ'. Thus

$$\rho(A_\sigma E_{\sigma_1}x) \supset \sigma_1' \cup \sigma' = (\sigma_1\sigma)' = \text{the whole plane};$$

that is, $\sigma(A_\sigma E_{\sigma_1}x)$ is void. By 1.12 we have $A_\sigma E_{\sigma_1} = 0$. Likewise $E_{\sigma_1}A_\sigma = 0$. Now there are closed sets $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma_1'$, and hence $A_{\sigma_n}x \longrightarrow A_{\sigma_1'}x = A_{\sigma_1}'x$. Then

$$0 = A_{\sigma_n}E_{\sigma_1}x = E_{\sigma_1}A_{\sigma_n}x \longrightarrow E_{\sigma_1}A_{\sigma_1}'x = A_{\sigma_1}'E_{\sigma_1}x.$$

So

$$0 = E_{\sigma_1}(I - A_{\sigma_1}) = (I - A_{\sigma_1})E_{\sigma_1},\ E_{\sigma_1} = E_{\sigma_1}A_{\sigma_1} = A_{\sigma_1}E_{\sigma_1}.$$

Similarly,

$$A_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1}.$$

Since $A_\sigma x$ and $E_\sigma x$ are both countably additive on B and coincide for closed sets σ, they must coincide for all $\sigma \in B$. We shall now show that if T has a resolution of the identity E_e, and σ is closed, then $E_\sigma x = x$ if and only if $\sigma(x) \subset \sigma$. Let σ be closed and $E_\sigma x = x$. Since $\sigma(X_\sigma) \subset \sigma$, the function

$$T(\xi)E_\sigma x = T(\xi)x$$

analytic on $\rho(T)$ has an analytic extension to σ'. Thus $\rho(x) \supset \sigma'$, $\sigma(x) \subset \sigma$. Conversely, let $\sigma(x) \subset \sigma$, where σ is closed. Let σ_n be closed, $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma'$, so that

$$x = E_\sigma x + E'_\sigma x = E_\sigma x + E_{\sigma'} x = E_\sigma x + \lim_n E_{\sigma_n} x.$$

Since $\sigma_n \sigma(T)$ and σ are disjoint, closed, and $\sigma_n \sigma(T)$ is bounded, there is an admissible contour C_n surrounding $\sigma_n \sigma(T)$ and excluding σ. Also, since

$$\sigma(X_{\sigma_n}) \subset \sigma_n \sigma(T),$$

we have

$$E_{\sigma_n} x = \frac{1}{2\pi i} \int_{C_n} (E_{\sigma_n} x)(\xi)\, d\xi = \frac{1}{2\pi i} \int_{C_n} E_{\sigma_n} x(\xi)\, d\xi.$$

However, since $\sigma(x) \subset \sigma$, the function $x(\xi)$ is analytic on and within C_n. Thus

$$E_{\sigma_n} x = 0 \text{ and } x = E_\sigma x.$$

In this proof we have used the equality

$$(E_{\sigma_n} x)(\xi) = E_{\sigma_n} x(\xi),$$

which is clear from 1.2 since both functions are analytic extensions of

$$T(\xi)\, E_{\sigma_n} x = E_{\sigma_n} T(\xi) x$$

on $\rho(T)$. We shall next show that $\sigma(E_\sigma x) \subset \overline{\sigma}\sigma(x)$ for every $\sigma \in B$ and $x \in X$. Since $\sigma(X_\sigma) \subset \overline{\sigma}$, it is clear that $\sigma(E_\sigma x) \subset \overline{\sigma}$. Since

$$(E_\sigma x)(\xi) = E_\sigma x(\xi) \text{ for } \xi \in \rho(T),$$

we have $\rho(E_\sigma x) \supset \rho(x)$, $\sigma(E_\sigma x) \subset \sigma(x)$. Thus $\sigma(E_\sigma x) \subset \bar{\sigma}\sigma(x)$. The necessity of 1.7 follows immediately from (i). Next we shall show the necessity of 1.14. As pointed out in 1.36 (ii) it follows from the principle of uniform boundedness that

$$\sup_{e \in B} |E_e| \equiv K < \infty.$$

Let $\sigma = \sigma(x)$, $\sigma_1 = \sigma(y)$ be disjoint. Then, by (i), $E_\sigma x = x$; and, by (ii) and 1.12, $E_\sigma y = 0$. Thus

$$|x| = |E_\sigma(x + y)| \leq K|x + y|.$$

Finally we prove the necessity of 1.39. Let σ be a circle and its interior. Let σ_n be closed and $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma'$. Then $E_\sigma x + E_{\sigma_n} x \longrightarrow x$. Since

$$\sigma(E_\sigma x) \subset \sigma\sigma(x) \quad \text{and} \quad \sigma(E_{\sigma_n} x) \subset \sigma_n \sigma(x) \subset \sigma'\sigma(x),$$

we see that σ is an s_3 set.

II. Operators whose spectra lie in a rectifiable Jordan curve

In order to apply the final lemma of §1 we find it necessary to restrict further the nature of the spectrum $\sigma(T)$. Later we shall be interested in specific cases where the spectrum lies either in a straight line segment or in a circle, and these two cases may be treated simultaneously by restricting the spectrum in the manner described in the next paragraph. When this is done and a rate of growth is imposed upon the resolvent (Assumption 2.1), it is possible to give conditions, of a nature much more applicable than those of the preceding lemma, which will ensure the existence of a resolution of the identity. This may be accomplished in a variety of ways, and some of the sets of conditions given are necessary as well as sufficient.

Throughout §2 it is assumed that the spectrum $\sigma(T)$ is contained in a closed rectifiable Jordan curve Γ_0. In order to be able to manipulate in a fairly simple fashion the analytical operations involved, we suppose further that Γ_0 is embedded in a one parameter family $\Gamma_\delta(-\delta_0 \leq \delta \leq \delta_0,\ 0 < \delta_0 \leq 1/2)$ of closed rectifiable Jordan curves, with Γ_{δ_1} interior to Γ_{δ_2} for $-\delta_0 \leq \delta_1 < \delta_2 \leq \delta_0$. The curve Γ_δ is defined by a function

$$\xi = \xi(\lambda, \delta),\ -1 \leq \lambda \leq 1, \text{ with } \xi(-1, \delta) = \xi(1, \delta).$$

We suppose that the parameter δ has been chosen in such a way that $|\delta|$ is the distance measured along the arc $\xi(\lambda, \delta)$ from the point $\xi(\lambda, 0)$ to the point $\xi(\lambda, \delta)$, and that the arcs $\xi(\lambda, \delta)$, $-\delta_0 \leq \delta \leq \delta_0$, for different values of λ, do not intersect. Finally we suppose that for each $\delta \in [-\delta_0, \delta_0]$ the point $\xi(\lambda, \delta)$ traces, as λ increases from -1 to $+1$, the curve Γ_δ in a counterclockwise direction.

2.1. ASSUMPTION. *The spectrum $\sigma(T)$ of T is contained in the rectifiable Jordan curve Γ_0 described above, and the rate of growth of the resolvent $T(\xi)$ for $\xi = \xi(\lambda, \delta)$ near the spectrum is restricted by the condition*

$$\limsup_{\delta \to 0} \; |\delta^{\nu(\lambda)} T(\xi)| < \infty, \; -1 \leq \lambda \leq 1,$$

where $\nu(\lambda)$ is a nonnegative function defined for $-1 \leq \lambda \leq 1$.

Since the function $\nu(\lambda)$ may be increased without destroying the above property, and since $\delta_0 \leq 1/2$, every operator T satisfying 2.1 has an index function $\nu(\lambda)$ according to the following definition.

2.2. DEFINITION. Any nonnegative integer-valued function $\nu(\lambda)$ satisfying the condition

$$|\delta^{\nu(\lambda)} \; T(\xi)| \leq 1, \; 0 < |\delta| < \delta_0, \; \lambda \in [-1, 1],$$

will be called an *index function* for T.

It might be pointed out that if $\nu(\lambda)$ is defined only on the set $\Lambda \subset [-1, 1]$ consisting of all those λ for which $\xi(\lambda, 0) \in \sigma(T)$, and the above inequality is valid for $\lambda \in \Lambda$, then T has an index function. It is not assumed that $\nu(\lambda)$ is bounded, and it is erroneous to conclude that T has a bounded index function providing $\nu(\lambda)$ is bounded on Λ. Elementary operators exist for which every index function is unbounded and at the same time every index function is bounded on Λ.

2.3. LEMMA. (*Assumption* 2.1.) *There is an index function $\nu(\lambda)$ for T with the property that every interval of positive length contains an interval of positive length upon which $\nu(\lambda)$ is constant.*

Let Δ be a closed subinterval of $[-1, 1]$. Let Δ_n be the set of all $\lambda \in \Delta$ such that

$$|\delta^n \; T(\xi)| \leq 1, \; 0 < |\delta| \leq \delta_0.$$

Since for fixed $\delta \neq 0$ the point $\xi = \xi(\lambda, \delta)$, and thus $T(\xi)$, is continuous in λ, it follows that Δ_n is closed. By 2.1 we see that $\Delta = \bigcup \Delta_n$, and thus the desired conclusion follows from the category theorem of Baire.

2.4. LEMMA. (*Assumption* 2.1.) *Let* $\nu(\lambda)$ *be an index function for* T: $0 < \delta \leq \delta_0$; $\lambda_1, \lambda_2 \in [-1, 1]$; $\xi_1 = \xi(\lambda_1, 0)$, $\xi_2 = \xi(\lambda_2, 0)$ *distinct points of* Γ_0; *and* $\nu_1 = \nu(\lambda_1)$, $\nu_2 = \nu(\lambda_2)$. *Let* $C(\lambda_1, \lambda_2)$ *be the rectifiable contour (oriented counterclockwise) composed of the following four arcs. There are two cases: If* $\lambda_1 < \lambda_2$ *the arcs are*

$$\xi(\lambda_1, \mu),\ -\delta \leq \mu \leq \delta; \quad \xi(\lambda_2, \mu),\ -\delta \leq \mu \leq \delta;$$

$$\xi(\lambda, \delta),\ \lambda_1 \leq \lambda \leq \lambda_2; \quad and \quad \xi(\lambda, -\delta),\ \lambda_1 \leq \lambda \leq \lambda_2;$$

whereas if $\lambda_2 < \lambda_1$ *we use the arcs*

$$\xi(\lambda_1, \mu),\ -\delta \leq \mu \leq \delta; \quad \xi(\lambda_2, \mu),\ -\delta \leq \mu \leq \delta;$$

$$\xi(\lambda, \delta),\quad \lambda \not\in (\lambda_2, \lambda_1); \quad and \quad \xi(\lambda, \delta),\quad \lambda \not\in (\lambda_2, \lambda_1).$$

Let $P(\xi)$ *be a polynomial in* ξ. *Then*

$$I(\lambda, \lambda_2) = \frac{1}{2\pi i} \int_{C(\lambda_1, \lambda_2)} P(\xi)(\xi - \xi_1)^{\nu_1} (\xi - \xi_2)^{\nu_2}\, T(\xi)\, d\xi$$

exists as a Riemann integral, is independent of δ, *and has the properties*

$$\lim_{0 < |\lambda_2 - \lambda_1| \to 0} I(\lambda_1, \lambda_2) = 0, \quad \sigma(I(\lambda_1, \lambda_2)x) \subset [\xi_1, \xi_2],$$

where $[\xi_1, \xi_2]$ *is the closed subarc of* Γ_0 *consisting of all points of* Γ_0 *which are inside or on the contour* $C(\lambda_1, \lambda_2)$.

The integrand is defined and continuous at every point of the contour $C(\lambda_1, \lambda_2)$ except at the points ξ_1 and ξ_2. Since $\nu(\lambda)$ is an index function for T, the integrand is bounded on the curve $C(\lambda_1, \lambda_2)$. Hence $I(\lambda_1, \lambda_2)$ exists. It is clearly independent of δ since the integrand has its only singularities on the curve Γ_0. Now let $\lambda_1 < \lambda_2$ so that $C(\lambda_1, \lambda_2)$ consists of the arcs AB, BC, CD, DA, where A, B, C, D are given by the complex numbers

$$\xi(\lambda_2, -\delta),\ \xi(\lambda_1, -\delta),\ \xi(\lambda_1, \delta),\ \xi(\lambda_2, \delta).$$

Let

$$K = \text{l.u.b.}\ |P(\xi)|,\ \xi \in \Gamma_{\delta_0};\quad K_\delta = \text{l.u.b.}\ |T(\xi)|,\ \xi \in \Gamma_\delta.$$

Let $0 < \epsilon < \delta_0$, $\delta = \epsilon/2$, and let γ_ϵ be such that for $0 < \lambda_2 - \lambda_1 < \gamma_\epsilon$ we have the lengths of the arcs AB, CD, and $\xi_1 \xi_2$ (on Γ_0) all less than δ and also less than $\epsilon/K(\delta)$. Then the integrand $f(\xi)$ defining $I(\lambda_1, \lambda_2)$ for $\xi = \xi(\lambda, \mu)$ satisfies

$$|f(\xi)| \leq K|\xi - \xi_2|^{\nu_2}\ |\mu|^{\nu_1}\ |T(\xi)| \leq K\ |\xi - \xi_2|^{\nu_2},\ \xi \in BC.$$

Since, for $\xi \in BC$, we have

$$|\xi - \xi_2| \leq |\xi_2 - \xi_1| + |\xi_1 - \xi| \leq \delta + \mu \leq 2\delta < 1,$$

it follows that for $\xi \in BC$, and likewise for $\xi \in DA$, we have the bound

$$|f(\xi)| \leq K.$$

For ξ on AB or CD, we have

$$|f(\xi)| \leq KK_\delta\ |\xi - \xi_2|^{\nu_2}\ |\xi - \xi_1|^{\nu_1} \leq KK_\delta.$$

Thus if $0 < \lambda_2 - \lambda_1 < \gamma_\epsilon$, then

$$|I(\lambda_1, \lambda_2)| \leq \frac{1}{2\pi}[4\delta K + 2KK_\delta\ \epsilon/K_\delta] = 2K\epsilon/\pi.$$

Now let $x \in X$, $\eta \in \rho(T)$, and η outside of $C(\lambda_1, \lambda_2)$. Then

$$\begin{aligned} T(\eta)&I(\lambda_1, \lambda_2)x \\ &= \frac{1}{2\pi i}\int_{C(\lambda_1, \lambda_2)} P(\xi)(\xi - \xi_1)^{\nu_1}(\xi - \xi_2)^{\nu_2}(\eta - \xi)^{-1}\, T(\xi)x\, d\xi \\ &\quad + \frac{1}{2\pi i}\, T(\eta)x \int_{C(\lambda_1, \lambda_2)} P(\xi)(\xi - \xi_1)^{\nu_1}(\xi - \xi_2)^{\nu_2}(\eta - \xi)^{-1} d\xi \\ &= \frac{1}{2\pi i}\int_{C(\lambda_1, \lambda_2)} P(\xi)(\xi - \xi_1)^{\nu_1}(\xi - \xi_2)^{\nu_2}(\eta - \xi)^{-1}\, T(\xi)x\, d\xi, \end{aligned}$$

and the last integral gives an analytic extension of $T(\eta)\ I(\lambda_1, \lambda_2)x$ for all η outside $C(\lambda_1, \lambda_2)$.

2.5. LEMMA. (*Assumption* 2.1.) *The operator T satisfies the conditions* 1.1 *and* 1.7.

Condition 1.1 clearly is satisfied since a rectifiable Jordan curve is non-dense in the plane. To prove 1.7, let σ be a closed subset of the spectrum, $x_n \in [\sigma]$, $x_n \longrightarrow x$. We make an indirect proof by supposing that there is a point $\xi \in \sigma(x)\sigma'$. According to the Heine-Borel theorem there are closed disjoint subarcs $\Delta_1, \cdots, \Delta_p$ of Γ_0 with $\sigma \subset \Delta = \Delta_1 \cup \cdots \cup \Delta_p$ and $\xi \in \Delta'$. Let

$$-1 \leq \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_p < \mu_p < 1$$

be such that the arc Δ_j, $(j = 1, \cdots, p)$ is defined by $\xi(\lambda, 0)$, $\lambda_j \leq \lambda \leq \mu_j$. Let $C_j = C(\lambda_j, \mu_j)$ as defined in 2.4. Since $x_n(\xi)$ has its singularities in the set Δ, we see that

$$\int_C \prod_{j=1}^{p} (\xi - \xi_j)^{\nu(\lambda_j)} (\xi - \zeta_j)^{\nu(\mu_j)} x_n(\xi) d\xi = 0,$$

where

$$\xi_j = \xi(\lambda_j, 0), \quad \zeta_j = \xi(\mu_j, 0),$$

and C is any contour of the form $C = C(\lambda_1, \lambda_2)$ providing $[\lambda_1, \lambda_2]$ is disjoint with Δ. Since, by 2.4,

$$I\left(\lambda_j - \frac{1}{n}, \lambda_j\right) \longrightarrow 0, \quad I\left(\mu_j, \mu_j + \frac{1}{n}\right) \longrightarrow 0,$$

we have,

$$\prod_{j=1}^{p} (T - \xi_j)^{\nu(\lambda_j)} (T - \zeta_j)^{\nu(\mu_j)} x_n$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{p} \int_{C_j} \prod_{k=1}^{p} (\xi - \xi_k)^{\nu(\lambda_k)} (\xi - \zeta_k)^{\nu(\mu_k)} x_n(\xi) d\xi.$$

Since the convergence, as $n \longrightarrow \infty$, of the integrands on the right side of the above equality is bounded, we may in this equation replace x_n by x and conclude from 2.4 and 1.6 that

$$\sigma\left(\prod_{j=1}^{p} (T - \xi_j)^{\nu(\lambda_j)} (T - \zeta_j)^{\nu(\mu_j)} x\right) \subset \Delta.$$

The desired contradiction will be obtained as soon as we show that the above inclusion implies that $\sigma(x) \subset \Delta$. But this implication follows immediately by induction from the statement

$$\sigma(z) \subset \sigma((\xi - T)z) \cup (\xi),$$

which is verified as follows. Since

$$T(\mu)z = (\xi - \mu)^{-1} \{T(\mu)(\xi - T)z - z\}, \quad \mu \neq \xi,$$

any point μ other than $\mu = \xi$ to which $T(\mu)(\xi - T)z$ has an analytic extension must be in $\rho(z)$; that is,

$$\rho((\xi - T)z) \subset \rho(z) \cup (\xi).$$

Thus

$$\sigma((\xi - T)z) \supset \sigma(z) \cap (\xi)'.$$

2.6. LEMMA. (*Assumption* 2.1.) *Let* $\nu(\lambda)$ *be an index function for* T. *For every complex number* ξ *and every nonnegative integer* n, *define*

$$\mathfrak{M}_\xi^n = \mathop{\mathcal{E}}_x [(T - \xi)^n x = 0], \quad \mathfrak{N}_\xi^n = (T - \xi)^n X.$$

Then for $\xi = \xi(\lambda, 0)$ *we have*

$$\mathfrak{M}_\xi^{\nu(\lambda)} = \mathfrak{M}_\xi^n, \quad \overline{\mathfrak{N}}_\xi^{\nu(\lambda)} = \overline{\mathfrak{N}}_\xi^n, \qquad n \geq \nu(\lambda).$$

Let $\xi_1 = \xi(\lambda, \delta)$; then

$$(\xi - T)\, T(\xi_1) = (\xi - \xi_1)\, T(\xi_1) + I.$$

Now assume, for the purpose of induction, that (the preceeding identity is the case when $j = 1$)

$$(\xi - T)^j\, T(\xi_1) = (\xi - \xi_1)^j\, T(\xi_1) + (\xi - \xi_1)^{j-1} + (\xi - \xi_1)^{j-2} (\xi - T)$$
$$+ \cdots + (\xi - \xi_1)(\xi - T)^{j-2} + (\xi - T)^{j-1}.$$

Multiplying this by $\xi - T$, we have

$$(\xi - T)^{j+1} T(\xi_1) = (\xi - \xi_1)^j [(\xi - \xi_1)T(\xi_1) + I] + (\xi - \xi_1)^{j-1} (\xi - T) + \cdots + (\xi - \xi_1)(\xi - T)^{j-1} + (\xi - T)^j,$$

and hence

$$(*) \quad (\xi - T)^{j+1} T(\xi_1) = (\xi - \xi_1)^{j+1} T(\xi_1) + (\xi - \xi_1)^j + (\xi - \xi_1)^{j-1} (\xi - T) + \cdots + (\xi - \xi_1)(\xi - T)^{j-1} + (\xi - T)^j.$$

Now in $(*)$ put $j = \nu(\lambda)$, and operate on a vector $x \in \mathfrak{M}_\xi^{\nu(\lambda)+1}$. We get

$$0 = (\xi - \xi_1)^{\nu(\lambda)+1} T(\xi_1)x + (\xi - \xi_1)^{\nu(\lambda)}x + \cdots + (\xi - \xi_1)(\xi - T)^{\nu(\lambda)-1}x + (\xi - T)^{\nu(\lambda)}x.$$

If we let $\delta \longrightarrow 0$, then $\xi_1 \longrightarrow \xi$; and since δ measures the arc $\xi\xi_1$, we have

$$|(\xi - \xi_1)^{\nu(\lambda)+1} T(\xi_1)x| = \left| \delta^{\nu(\lambda)+1} T(\xi_1)\left(\frac{\xi - \xi_1}{\delta}\right)^{\nu(\lambda)+1} \right| \leq \delta \longrightarrow 0.$$

This shows that $x \in \mathfrak{M}_\xi^{\nu(\lambda)}$. Thus

$$\mathfrak{M}_\xi^{\nu(\lambda)+1} \subset \mathfrak{M}_\xi^{\nu(\lambda)},$$

and hence

$$\mathfrak{M}_\xi^{\nu(\lambda)} = \mathfrak{M}_\xi^n, \quad n \geq \nu(\lambda).$$

Using $(*)$ again except now with x arbitrary, we may write

$$(\xi - T)^{\nu(\lambda)} x = (\xi - T)^{\nu(\lambda)+1} T(\xi_1)x + O(\delta),$$

where $O(\delta)$ is a vector which approaches zero with δ. Thus

$$\mathfrak{N}_\xi^{\nu(\lambda)} \subset \overline{\mathfrak{N}}_\xi^{\nu(\lambda)+1} \subset \overline{\mathfrak{N}}_\xi^{\nu(\lambda)},$$

and so

$$\overline{\mathfrak{N}}_{\xi}^{\nu(\lambda)} = \overline{\mathfrak{N}}_{\xi}^{n}, \ n \geq \nu(\lambda).$$

2.7. DEFINITION. (Assumption 2.1.) Let $\nu(\lambda)$ be an index function for T. As shown in 2.6, the manifolds $\mathfrak{M}_{\xi}^{\nu(\lambda)}$ and $\overline{\mathfrak{N}}_{\xi}^{\nu(\lambda)}$ are independent of the index function $\nu(\lambda)$. They will henceforth be designated by the symbols $\mathfrak{M}_{\xi}$, $\mathfrak{N}_{\xi}$, respectively.

Of the three principal conditions of Lemma 1.38, namely 1.7, 1.14, and 1.39, the condition 1.7 has already been shown (by 2.5) to be a consequence of 2.1. Neither one of the other two conditions is a consequence of 2.1 alone, and we concentrate our attention now on restating 1.39 in a more applicable form. The following assumption 2.8, which may be used to replace 1.39, turns out to be necessary as well as sufficient. It has the disadvantage of not always being easily applicable. *It should be noted, however, that in the case of operators with only continuous spectra it is trivially satisfied. Or more generally, if every subarc of* Γ_0 *contains points either in the resolvent set or in the continuous spectrum then* 2.8 *is automatically satisfied.*

2.8. ASSUMPTION. (*See Assumption* 2.1.) *For every* ξ *in a dense set on* Γ_0,

$$\mathfrak{M}_{\xi} + \mathfrak{N}_{\xi} \text{ is dense in } X.$$

2.9. LEMMA. (*Assumptions* 2.1, 1.14.) *The set of points* ξ *on the curve* Γ_0 *for which* $\mathfrak{M}_{\xi}\ \mathfrak{N}_{\xi} \neq 0$ *is nondense on* Γ_0. *Moreover,* $\mathfrak{M}_{\xi}\ \mathfrak{N}_{\xi} = 0$ *for every* ξ, *interior to a subarc of* Γ_0 *upon which some index function is constant. For such* ξ, *the set* $\mathfrak{M}_{\xi} \oplus \mathfrak{N}_{\xi}$ *is closed. Thus if* 2.8 *is satisfied then*

$$\mathfrak{M}_{\xi} \oplus \mathfrak{N}_{\xi} = X,$$

for every ξ *in a set dense on* Γ_0.

In view of 2.3, the first statement is a consequence of the second. Accordingly, let $\nu(\lambda)$ be an index function which is constant on the interval $[\lambda_1, \lambda_2]$, and let $\xi = \xi(\lambda, 0)$ where $\lambda_1 < \lambda < \lambda_2$. Let

$$\nu = \nu(\lambda),\ \xi_n = \xi(\lambda_n, 0),\ \zeta_n = \xi(\mu_n, 0),$$

where

$$\lambda_1 < \lambda_n < \lambda < \mu_n < \lambda_2 \ \text{ and } \ \lambda_n \longrightarrow \lambda, \quad \mu_n \longrightarrow \lambda.$$

Take $P(\xi) \equiv 1$ in 2.4, so that

$$I(\lambda_n, \mu_n) + I(\mu_n, \lambda_n) = (T - \xi_n)^{\nu} (T - \zeta_n)^{\nu} \longrightarrow (T - \xi)^{2\nu}.$$

Thus, it is seen from 2.4 that the vector $(T - \xi)^{2\nu} x$ is the limit of vectors $x_n = I(\mu_n, \lambda_n) x$ and $\xi \in \rho(x_n)$. Thus 2.6 and 2.7 show that every vector $x \in \mathfrak{N}_\xi$ is the limit of a sequence $\{x_n\}$ with $\xi \in \rho(x_n)$. Now if $x \in \mathfrak{M}_\xi \mathfrak{N}_\xi$ we have $(T - \xi)^{\nu} x = 0$, and hence $\sigma(x) \subset (\xi)$. If $\xi \in \rho(x_n)$ and $x_n \longrightarrow x$, then $\xi \in \rho(-x_n)$; and, by 1.14,

$$|x| \leq K\, |x - x_n| \longrightarrow 0, \quad x = 0, \text{ and } \mathfrak{M}_\xi \mathfrak{N}_\xi = 0.$$

Finally we show that $\mathfrak{M}_\xi \oplus \mathfrak{N}_\xi$ is closed. Let

$$x_n + y_n \longrightarrow z, \quad \text{where } x_n \in \mathfrak{M}_\xi, \; y_n \in \mathfrak{N}_\xi .$$

Since $y_n - y_m \in \mathfrak{N}_\xi$, there are vectors u_{nm} with

$$|y_n - y_m - u_{nm}| < (nm)^{-1}, \quad \xi \in \rho(u_{nm}).$$

By 1.14, then,

$$|x_n - x_m| \leq K\, |x_n - x_m - u_{nm}| \leq K\, \{|x_n - x_m + y_n - y_m| + (nm)^{-1}\} \longrightarrow 0.$$

2.10. Lemma. *(Assumptions 2.1, 1.14, 2.8.) For every pair ξ, ξ_1 of distinct points in a set dense on Γ_0 we have*

$$\mathfrak{M}_\xi \oplus \mathfrak{M}_{\xi_1} \oplus \mathfrak{N}_\xi \mathfrak{N}_{\xi_1} = X;$$

and $\mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$ is the closure of the manifold $(T - \xi)^{\nu} (T - \xi_1)^{\nu_1} X$, where $\nu = \nu(\lambda)$, $\nu_1 = \nu(\lambda_1)$, $\xi = \xi(\lambda, 0)$, $\xi_1 = \xi(\lambda_1, 0)$, and $\nu(\lambda)$ is an index function for T.

For every ξ in a set Γ dense on Γ_0 we have, by 2.9, projections A_ξ and A_ξ' with

$$A_\xi + A_\xi' = I, \quad A_\xi X = \mathfrak{M}_\xi, \quad A_\xi' X = \mathfrak{N}_\xi .$$

Since for $\mu \in \rho(T)$, we have $T(\mu)\mathfrak{N}_\xi \subset \mathfrak{N}_\xi$ and $T(\mu)\mathfrak{M}_\xi \subset \mathfrak{M}_\xi$, it follows that $T(\mu) A_\xi x = A_\xi T(\mu) x$, and thus $\rho(x) \subset \rho(A_\xi x)$. Since

$$(T - \xi_1)^{\nu_1} A_{\xi_1} = 0,$$

we have $\sigma(A_{\xi_1} x) \subset (\xi_1)$, and hence $\xi \in \rho(A_{\xi_1} x) \subset \rho(A_\xi A_{\xi_1} x)$. Since $\sigma(A_\xi y) \subset (\xi)$, we have $y \in X$; then $A_\xi A_{\xi_1} = 0$ by 1.12 and 2.5. Similarly $A_{\xi_1} A_\xi = 0$. Thus

$$I = (A_\xi + A'_\xi)(A_{\xi_1} + A'_{\xi_1}) = A_\xi + A'_\xi + A'_\xi A'_{\xi_1},$$

and this proves the first statement of the lemma. Now let $x \in \mathfrak{N}_\xi^\nu \mathfrak{N}_{\xi_1}^{\nu_1}$, so that

$$x = (T - \xi)^\nu y, \quad y = A_{\xi_1} y + A'_{\xi_1} y, \quad x = A'_{\xi_1} x.$$

By 2.7 there are vectors v_n with $(T - \xi_1)^{\nu_1} v_n \longrightarrow A'_{\xi_1} y$ and

$$x = \lim_n (T - \xi)^\nu [A_{\xi_1} y + (T - \xi_1)^{\nu_1} v_n],$$

$$x = A'_{\xi_1} x = \lim_n (T - \xi)^\nu (T - \xi_1)^{\nu_1} v_n.$$

Thus $\mathfrak{N}_\xi^\nu \mathfrak{N}_{\xi_1}^{\nu_1}$ as well as its closure $\mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$ is contained in the closure

$$(T - \xi)^\nu (T - \xi_1)^{\nu_1} X.$$

Obviously $(T - \xi)^\nu (T - \xi_1)^{\nu_1} X \subset \mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$, and so the proof is complete.

2.11. THEOREM. (*Assumptions* 1.14, 2.1, 2.8.) *Every Borel set is measurable T; and if X is weakly complete then T has a resolution of the identity.*

Let $-1 \leq \alpha < \beta < \gamma < \delta < 1$, and choose $\lambda_1, \lambda_2, \mu_1, \mu_2$ so that

$$\alpha < \lambda_1 < \lambda_2 < \beta, \quad \gamma < \mu_1 < \mu_2 < \delta,$$

and such that there is an index function $\nu(\lambda)$ which is constant on the intervals $[\lambda_1, \lambda_2]$, $[\mu_1, \mu_2]$. This is possible in view of 2.3. Since an index function may be increased without destroying the property of being an index function, we shall suppose that $\nu(\lambda)$ has the constant value ν on both of the intervals $[\lambda_1, \lambda_2]$, $[\mu_1, \mu_2]$. Let

$$\xi_1 = \xi(\lambda_1, 0),\ \xi_2 = \xi(\lambda_2, 0),\ \zeta_1 = \xi(\mu_1, 0),\ \zeta_2 = \xi(\mu_2, 0),$$

and

$$f(\xi) = (\xi - \xi_1)^\nu (\xi - \xi_2)^\nu (\xi - \zeta_1)^\nu (\xi - \zeta_2)^\nu.$$

Then for appropriate choices of the polynomial $P(\xi)$ in 2.4 we have

$$f(T) = I(\lambda_1, \lambda_2) + I(\mu_1, \mu_2) + I(\mu_2, \lambda_1) + I(\lambda_2, \mu_1).$$

By 2.10 there are points λ, μ with $\lambda_1 < \lambda < \lambda_2$, $\mu_1 < \mu < \mu_2$, and such that for $\xi = \xi(\lambda, 0)$, $\zeta = \xi(\mu, 0)$ we have

$$(*) \qquad \mathfrak{M}_\xi \oplus \mathfrak{M}_\zeta \oplus \mathfrak{N}_\xi\ \mathfrak{N}_\zeta = X.$$

Now if we let $\lambda_1 \longrightarrow \lambda$, $\lambda_2 \longrightarrow \lambda$, $\mu_1 \longrightarrow \mu$, $\mu_2 \longrightarrow \mu$ then by 2.4 we have

$$I(\lambda_1, \lambda_2) \longrightarrow 0,\ I(\mu_1, \mu_2) \longrightarrow 0,$$

and

$$(T - \xi)^{2\nu} (T - \zeta)^{2\nu} x = \lim f(T)x = \lim \{I(\mu_2, \lambda_1)x + I(\lambda_2, \mu_1)x\}.$$

Also, by 2.4, we have $\sigma(I(\mu_2, \lambda_1)x) \subset [\zeta_2, \xi_1]$ and $\sigma(I(\lambda_2, \mu_1)x) \subset [\xi_2, \zeta_1]$, where, for two points $\xi' = \xi(\lambda', 0)$, $\xi'' = \xi(\lambda'', 0)$ on Γ_0, the symbol $[\xi', \xi'']$ means the closed subarc of Γ_0 defined by $\xi(\lambda, 0)$, $\lambda' \le \lambda \le \lambda''$, if $\lambda' < \lambda''$ and closed subarc $\xi(\lambda, 0)$, $\lambda \not\in (\lambda'', \lambda')$, if $\lambda'' < \lambda'$.

Since $2\nu(\lambda)$ is also an index function, it is seen from 2.10 that $\mathfrak{N}_\xi\ \mathfrak{N}_\zeta$ is the closure of $(T - \xi)^{2\nu} (T - \zeta)^{2\nu} X$, and hence every vector in $\mathfrak{N}_\xi\ \mathfrak{N}_\zeta$ is the limit of a sequence of vectors of the form $x + y$ with $\sigma(x) \subset [\xi, \zeta]$, $\sigma(y) \subset [\xi, \zeta]'$. Since $\sigma(x) \subset (\xi)$ if $x \in \mathfrak{M}_\xi$, we have, from 1.6 and $(*)$ above, the fact that every vector in X is the limit of a sequence of vectors of the form $x + y$ with $\sigma(x) \subset [\xi, \zeta]$, $\sigma(y) \subset [\xi, \zeta]'$. This shows that $[\xi, \zeta]$ is an s_1 set for T (see Definition 1.15). The above argument shows also that $[\zeta, \xi]$ is an s_1 set for T. We shall next show that $\sigma = [\xi, \zeta]$ is an s_2 set for T. If the intervals (α, β) and (α, δ) that we started with above are replaced by the intervals $(\lambda - 1/n, \lambda)$ and $(\mu, \mu + 1/n)$, we see that there are points λ_n, μ_n, with $\lambda - 1/n < \lambda_n < \lambda$, $\mu < \mu_n < \mu + 1/n$, such that $\sigma_n \equiv [\zeta_n, \xi_n]$, where

$$\zeta = \xi(\mu_n, 0),\ \xi_n = \xi(\lambda_n, 0),$$

is an s_1 set. Now let

$$y = (T - \xi)^{2\nu} (T - \zeta)^{2\nu} x, \quad y_n = (T - \xi)^{\nu} (T - \xi_n)^{\nu} (T - \zeta)^{\nu} (T - \zeta_n)^{\nu} x,$$

so that $y_n \longrightarrow y$, and for appropriate choices of $P(\xi)$ in 2.4 we have

$$y_n = I(\lambda_n, \lambda)x + I(\lambda, \mu)x + I(\mu, \mu_n)x + I(\mu_n, \lambda_n)x.$$

Thus by 2.4 we may write

$$y = I(\lambda, \mu)x + I(\mu_n, \lambda_n)x + z_n, \text{ where } z_n \longrightarrow 0.$$

Now from 2.4, 2.5, and 1.16 we see that

$$E_\sigma I(\lambda, \mu) = I(\lambda, \mu), \quad E_\sigma I(\mu_n, \lambda_n) = 0,$$

$$E_{\sigma_n} I(\lambda, \mu) = 0, \quad E_{\sigma_n} I(\mu_n, \lambda_n) = I(\mu_n, \lambda_n);$$

and since $|E_{\sigma_n}| \leq K$ we may write $y = E_\sigma y + E_{\sigma_n} y + v_n$ where $v_n \longrightarrow 0$. Since y is an arbitrary point in the manifold $(T - \xi)^{2\nu} (T - \zeta)^{2\nu} X$ whose closure is $\mathfrak{N}_\xi \; \mathfrak{N}_\zeta$, and since

$$|E_\sigma + E_{\sigma_n}| \leq 2K,$$

we have

$$y = E_\sigma y + \lim_n E_{\sigma_n} y$$

for every $y \in \mathfrak{N}_\xi \; \mathfrak{N}_\zeta$. For $x \in \mathfrak{M}_\xi \oplus \mathfrak{M}_\zeta$ we have $\sigma(x) \subset \sigma$; and so, by 1.16, we have $E_\sigma x = x$, $E_{\sigma_n} x = 0$. Hence, it follows from (*) that

$$(**) \qquad x = E_\sigma x + \lim_n E_{\sigma_n} x, \quad x \in X.$$

Now 2.5 and 1.22 show that $\sigma(E_\sigma x) \subset \sigma\sigma(x)$, $\sigma(E_{\sigma_n} x) \subset \sigma_n \sigma(x) \subset \sigma' \sigma(x)$, so that σ is an s_2 set. The same argument shows that $[\zeta_n, \xi_n] = \sigma_n$ is an s_2 set and hence (**) shows that σ is an s_3 set for T. Thus we have proved that if $-1 \leq \alpha < \beta < \gamma < \delta < 1$ there are points λ, μ with $\alpha < \lambda < \beta$, $\gamma < \mu < \delta$, such that $\sigma = [\xi(\lambda, 0), \xi(\mu, 0)]$ is an s_3 set. This clearly implies the statement 1.39; hence every Borel set is measurable T. Theorem 2.11 then follows from 1.38.

III. The operational calculus

3.1. DEFINITION OF $\int_\sigma f(\lambda)\,dE_\lambda$. In what follows we shall be concerned with an integral, $\int f(\lambda)\,dE_\lambda$, where E_e is the resolution of the identity for an operator T. The functions f to be integrated are either scalar- or operator-valued; they will always be continuous, so that the Riemann integral will suffice. Although the applications to be made are to operators satisfying the preceding restrictions, it seems desirable to word the definitions and elementary properties of $\int f(\lambda)\,dE_\lambda$ in terms of an arbitrary operator T on an arbitrary space X subject to the single restriction that T has a resolution of the identity. Since $\sigma(T)$ is bounded, it may for any $\delta > 0$ be partitioned into disjoint, nonvoid Borel sets $\Delta_1, \cdots, \Delta_n$ whose diameters are less than δ. The norm $|\pi|$ of such a partitioning $\pi = (\Delta_1, \cdots, \Delta_n)$ is $|\pi| = \max_i \operatorname{diam} \Delta_i$. If for a scalar- or operator-valued function f defined on $\sigma(T)$ we have the sums $\sum_\pi f(\lambda_i) E_{\Delta_i}$ converging, as $|\pi| \longrightarrow 0$, to a limit independent of the choice of $\lambda_i \in \Delta_i$, the function f is said to be integrable. Of course the convergence of $\sum_\pi f(\lambda_i) E_{\Delta_i}$ as $|\pi| \longrightarrow 0$ may be in the weak, strong, or uniform topology of operators; but *for the functions we shall integrate, it is always in the uniform topology of operators*, so we need not concern ourselves here with the other cases. The integral is defined by

$$\int f(\lambda)\,dE_\lambda \equiv \lim_{|\pi| \to 0} \sum_\pi f(\lambda_i) E_{\Delta_i};$$

and for any Borel set σ in the plane we define $\int_\sigma f(\lambda)\,dE_\lambda$ to be $E_\sigma \int f(\lambda)\,dE_\lambda$.

3.2. LEMMA. *If for each e in a Borel algebra β there is a bounded linear operator A_e in the space X such that $x^* A_e x$ is countably additive on β for each $x \in X$, $x^* \in X^*$, then there is a constant $v(A)$ such that*

$$\sum_i |x^* A_{e_i} x| \leq v(A)\,|x|\,|x^*|, \quad e_i e_j \text{ void for } i \neq j, \quad e_i \in \beta.$$

Let $\pi = \{e_i\}$ be a finite or enumerable sequence of disjoint elements in β. For each $x \in X$, $x^* \in X^*$, define $U_\pi(x, x^*)$ as the point in the complex Banach space ℓ_1 (the space of absolutely convergent sequences) given by the sequence $\{x^* A_{e_i} x\}$ (if the sequence $\{e_i\}$ is finite we extend it to an infinite sequence by taking e_n to be the void set for all large n). For fixed x, π, the function $U_\pi(x, x^*)$ is additive, homogeneous, and closed; hence $U_\pi(x, x^*)$ is continuous in x^*. Similarly, $U_\pi(x, x^*)$ is continuous in x for fixed x^*, π. Thus for each π, $U_\pi(x, x^*)$ is simultaneously continuous in x, x^*. Since the numerical function

$x^*A_e x$ is countably additive on β, we have

$$(*)\qquad \sup_{\pi} |U_\pi(x, x^*)| < \infty, \quad x \in X, \ x^* \in X^*.$$

Let Z_n be the set of points (x, x^*) in the Cartesian product space $Z = X \times X^*$, where $|U_\pi(x, x^*)| \leq n$ for every π. Since $U_\pi(x, x^*)$ is continuous in (x, x^*), Z_n is closed. From $(*)$ we have $Z = \cup Z_n$, and so the Baire catagory theorem gives an integer n_0, a point $(x_0, x_0^*) \in Z$, and an $r_0 > 0$, such that

$$|U_\pi(x, x^*)| \leq n_0, \ \pi, \ |x - x_0| < r_0, \ |x^* - x_0^*| < r_0.$$

Now if $y \in X$, $y^* \in X^*$, and $|y|$, $|y^*| < r_0$, we have

$$U_\pi(y, y^*) = U_\pi(x_0 - y, x_0^* - y^*) = U_\pi(x_0 - y, x_0^*)$$

$$= U_\pi(x_0, x_0^* - y^*) + U_\pi(x_0, x_0^*),$$

and $|U_\pi(y, y'') \leq 4n_0$. Thus if $v(A) = 5n_0/r_0^2$, we have

$$|U_\pi(y, y^*)| \leq v(A)\,|y|\,|y^*|.$$

3.3. LEMMA. *Every continuous scalar function f on $\sigma(T)$ is integrable, and*

$$|\int f(\lambda)\,dE_\lambda| \leq \max_{\lambda \in \sigma(T)} |f(\lambda)|\,v(E),$$

$$|\int_\sigma f(\lambda)\,dE_\lambda| \leq \sup_{\lambda \in \sigma} |f(\lambda)|\,v(E),$$

where $v(E)$ is the constant of 3.2. Also if α is an arbitrary parameter and $f(\alpha, \lambda)$ is continuous for $\lambda \in \sigma(T)$ uniformly with respect to α, then the sums $\sum_\pi f(\alpha, \lambda_i)E_{\Delta_i}$ for a partition $\pi = (\Delta_i, \cdots, \Delta_n)$, $\lambda_i \in \Delta_i$, converge, as $|\pi| \longrightarrow 0$, uniformly with respect to α.

For two partitions $\pi = (\Delta_i, \cdots, \Delta_n)$, $\pi' = (\Delta_1', \cdots, \Delta_m')$ of $\sigma(T)$, and for $\lambda_i \in \Delta_i$, $\lambda_j' \in \Delta_j'$ $(i = 1, \cdots, n, \ j = 1, \cdots, m)$, we have

$$\sum_{i=1}^{n} f(\alpha, \lambda_i)E_{\Delta_i} - \sum_{j=1}^{m} f(\alpha, \lambda_j')E_{\Delta_j'} = \sum_{i=1}^{n}\sum_{j=1}^{m} \{f(\alpha, \lambda_i) - f(\alpha, \lambda_j')\}E_{\Delta_i\Delta_j'}.$$

If for $\epsilon > 0$, $\delta(\epsilon) > 0$ is chosen so that

$$|f(\alpha, \lambda) - f(\alpha, \lambda')| < \epsilon \text{ for } |\lambda - \lambda'| < \delta(\epsilon),$$

we have, by 3.2,

$$\left| \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} - \sum_{j=1}^{m} f(\alpha, \lambda_j') E_{\Delta_j} \right| < \epsilon\, v(E), \ \alpha, \ |\pi|, \ |\pi'| < \delta(\epsilon)$$

$$\left| \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} \right| \le \max_{\lambda \in \sigma(T)} |f(\alpha, \lambda)|\, v(E),$$

and

$$\left| E_\sigma \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} \right| \le \sup_{\lambda \in \sigma} |f(\alpha, \lambda)|\, v(E);$$

this proves the lemma.

While in our final results the only operator-valued functions we shall have to integrate are of the form $(T - \lambda I)^n f(\lambda)$, where f is a scalar function, it will, during the course of the proofs to follow, be necessary to integrate functions in a more extensive class. Accordingly, we consider functions of the following type. Let D_1 be an open set containing the closure $\overline{D}$ of an admissible open set $D \supset \sigma(T)$. Let C be the boundary of D. Let $f(\alpha, \lambda)$ be a scalar function defined for $\alpha \in D_1$, $\lambda \in \sigma(T)$, continuous similtaneously in both variables over their respective ranges, and analytic for $\alpha \in D_1$ uniformly with respect to $\lambda \in \sigma(T)$; that is,

$$\frac{f(\alpha + \eta, \lambda) - f(\alpha, \lambda)}{\eta} \longrightarrow \frac{\partial f(\alpha, \lambda)}{\partial \alpha}$$

uniformly with respect to $\lambda \in \sigma(T)$. Because the continuity in λ is uniform with respect to α on C, the operator-valued function

$$f(T, \lambda) = \frac{1}{2\pi i} \int_C f(\alpha, \lambda) T(\alpha)\, d\alpha$$

depends continuously on λ. It is this type of continuous operator-valued function defined by a scalar function $f(\alpha, \lambda)$ whose singularities in α stay uniformly away from $\sigma(T)$ as λ varies over $\sigma(T)$ that we shall be integrating. For the sake of brevity we shall call functions $f(\alpha, \lambda)$ of the above type *T-uniform.*

3.4. LEMMA. *Let $f(\alpha, \lambda)$ be T-uniform. Then $f(T, \lambda)$ is integrable, and for every Borel set σ we have*

$$\int_\sigma f(T, \lambda)dE_\lambda = \frac{1}{2\pi i} \int_C \left(\int_\sigma f(\alpha, \lambda)dE_\lambda \right) T(\alpha)d\alpha.$$

Let $\pi = (\Delta_i, \cdots, \Delta_n)$, $\lambda_i \in \Delta_i$, be a partition of $\sigma(T)$. Then

$$\sum_i f(T, \lambda_j)E_{\Delta_j} = \sum_j \left(\frac{1}{2\pi i} \int_C f(\alpha, \lambda_j)T(\alpha)d\alpha\right)E_{\Delta_j}$$

$$= \frac{1}{2\pi i} \int_C \left(\sum_j f(\alpha, \lambda_j)E_{\Delta_j}\right) T(\alpha)d\alpha .$$

The desired results follow from 3.3.

3.5. LEMMA. *Let $f(\alpha, \lambda)$, $g(\alpha, \lambda)$ be T-uniform. Then $f(\alpha, \lambda)g(\alpha, \lambda)$ is T-uniform, and for a partition $\pi = (\Delta_i, \cdots, \Delta_n)$ of $\sigma(T)$ and points $\lambda_j, \lambda_j' \in \overline{\Delta}_j$ we have*

$$\lim_{|\pi| \to 0} \sum_\pi f(T, \lambda_j)\, g(T, \lambda_j')E_{\Delta_j} = \int f(T, \lambda)\, g(T, \lambda)dE_\lambda .$$

It is clear that there is a common domain of uniform analyticity. Let C be its boundary. For $\epsilon > 0$, fix $\delta > 0$ such for every pair $\lambda, \lambda' \in \sigma(T)$ with $|\lambda - \lambda'| < \delta$ we have

$$|f(\alpha, \lambda)[g(\alpha, \lambda') - g(\alpha, \lambda)]| \leq \epsilon, \quad \alpha \in C.$$

From 3.2, if $|\pi| < \delta$ then

$$\sum_\pi f(\alpha, \lambda_i)[g(\alpha, \lambda_j') - g(\alpha, \lambda_i)]E_{\Delta_j}| \leq \epsilon\ v(E), \alpha \in C.$$

Now

$$f(T, \lambda_j)\ g(T, \lambda_j') = f(T, \lambda_j)\ g(T, \lambda_j) + f(T, \lambda_j)[g(T, \lambda_j'] - g(T, \lambda_j)]$$

and

$$|\sum f(T, \lambda_j)[g(T, \lambda_j') - g(T, \lambda_j)] E_{\Delta_j}|$$

$$= \left| \frac{1}{2\pi i} \int_C T(\alpha) d\alpha \ (\sum f(\alpha, \lambda_j) [g(\alpha, \lambda_j') - g(\alpha, \lambda_j)] E_{\Delta_j}) \right|$$

$$= (\epsilon/2\pi) \left(\max_{\alpha \in C} |T(\alpha)| \right) (\text{length } C) \ v(E).$$

Thus in order to prove the lemma, it suffices to show that

$$\lim_{\pi} \sum f(T, \lambda_j) \ g(T, \lambda_j) E_{\Delta_j} = \int f(T, \lambda) \ g(T, \lambda) dE_\lambda .$$

(Note that if $\lambda_j \in \Delta_j$ and not merely in $\overline{\Delta}_j$, there is nothing left to prove.) But this is clear since the function $f(T, \lambda) \, g(T, \lambda)$ is continuous in λ.

3.6. LEMMA. *If f is integrable (scaler- or operator-valued) and U is a bounded linear operator in X which commutes with E_e, $e \in B$, then Uf is integrable and*

$$U \int_\sigma f(\lambda) dE_\lambda = \int_\sigma Uf(\lambda) dE_\lambda, \ \sigma \in B.$$

The proof is clear from the definitions.

3.7. THEOREM. *Let X be arbitrary and T a bounded linear operator in X with the resolution of the identity E_e. For any closed set of points $\rho \in \rho(T)$, we have*

$$(\xi - T)^{-1} = \sum_{n=0}^{\infty} \int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} \, dE_\lambda ,$$

where the sum converges in the uniform topology of operators and uniformly with respect to $\xi \in \rho$.

In view of the elementary identity

$$(\xi - T) \sum_{n=0}^{p} \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} = I - \frac{(T - \lambda)^{p+1}}{(\xi - \lambda)^{p+1}}$$

and 3.6, we have

$$(\xi - T) \sum_{n=0}^{p} \frac{(T-\lambda)^n}{(\xi-\lambda)^{n+1}} dE_\lambda = I - \int \frac{(T-\lambda)^{p+1}}{(\xi-\lambda)^{p+1}} dE_\lambda .$$

Now let δ = the distance from ρ to $\sigma(T)$. This is positive since $\sigma(T)$ is bounded and closed. Break $\sigma(T)$ into disjoint measurable parts $\Delta_1, \cdots, \Delta_n$ of which the diameters satisfy

$$\text{diameter } \Delta_j \leq \delta/4 \qquad (j = 1, \cdots, n).$$

Let C_j be a circle of diameter $\delta/2$ containing $\bar{\Delta}_j$ in its interior, so that

$$|(\alpha-\lambda)(\xi-\lambda)^{-1}| \leq 1/2, \quad \xi \in \rho, \ \lambda \in \Delta_j, \quad \alpha \in C_j .$$

Let $T_\alpha(\Delta_j)$ be the resolvent of T when considered as an operator in $X_{\bar{\Delta}_j}$. Since $\sigma(X_{\bar{\Delta}_j}) \subset \bar{\Delta}_j$ and

$$\left(\int_{\Delta_j} \frac{(T-\lambda)^p}{(\xi-\lambda)^p} dE_\lambda \right) X \subset X_{\bar{\Delta}_j} ,$$

we have, from 3.3,

$$\left| \int_{\Delta_j} \frac{(T-\lambda)^p}{(\xi-\lambda)^p} dE_\lambda \right|$$

$$= \left| \int_{\Delta_j} \left(\frac{1}{2\pi i} \int_{C_j} \frac{(\alpha-\lambda)^p}{(\xi-\lambda)^p} T_\alpha(\Delta_j)\, d\alpha \right) dE_\lambda \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_j} T_\alpha(\Delta_j)\, d\alpha \int_{\Delta_j} \frac{(\alpha-\lambda)^p}{(\xi-\lambda)^p} dE_\lambda \right|$$

$$\leq \frac{1}{2\pi} \max_{\alpha \in C_j} |T_\alpha(\Delta_j)| \ v(E)/2^p, \quad \xi \in \rho.$$

Since $\int = \int_{\Delta_1} + \cdots + \int_{\Delta_n}$, we have

$$(\xi - T) \sum_{n=0}^{p} \int \frac{(T-\lambda)^n}{(\xi-\lambda)^{n+1}} dE_\lambda \longrightarrow I$$

uniformly for $\xi \in \rho$. This proves the theorem.

3.8. DEFINITION. An operator is called a *spectral operator* if it has a resolution of the identity.

3.9. THEOREM. *If T is a bounded spectral operator, the resolution of the the identity E_e is unique, and for every $f \in F(T)$ we have*

$$f(T) = \sum_{n=0}^{\infty} \int \frac{f^{(n)}(\lambda)}{n!} (T - \lambda)^n \, dE_\lambda ,$$

where the integral exists as a Riemann integral in the uniform topology of operators and the sum converges in the uniform topology of operators.

We shall first show uniqueness. Let A_e, E_e be resolutions of the identity for T. Let σ_1, σ_2 be disjoint and each consist of a circle and its interior. Let $T^1(\xi)$ be the resolvent of T when considered as an operator on $E_{\sigma_1}X$. Let $T^2(\xi)$ be the resolvent of T when considered as an operator in $A_{\sigma_2}X$. Then for $\xi \not\subset \sigma_1$, $T^1(\xi) E_{\sigma_1} A_{\sigma_2}$ is a bounded linear operator in X and analytic for $\xi \in \sigma_1'$. Likewise for $\xi \in \sigma_2'$, $E_{\sigma_1} T^2(\xi) A_{\sigma_2}$ is a bounded linear operator in X and analytic for $\xi \in \sigma_2'$. Let $\xi \in (\sigma_1 \cup \sigma_2)'$. Since $\xi - T$ commutes with E_{σ_1}, we have

$$(\xi - T) E_{\sigma_1} T^2(\xi) A_{\sigma_2} = E_{\sigma_1} A_{\sigma_2} ,$$

and operating on the left with $T^1(\xi)$ we have

$$E_{\sigma_1} T^2(\xi) A_{\sigma_2} = T'(\xi) E_{\sigma_1} A_{\sigma_2} .$$

If $f(\xi)$ is defined to be

$$\begin{aligned} f(\xi) &= E_{\sigma_1} T^2(\xi) A_{\sigma_2} = T^1(\xi) E_{\sigma_1} A_{\sigma_2}, \quad \xi \in (\sigma_1 \cup \sigma_2)' \\ &= E_{\sigma_1} T^2(\xi) A_{\sigma_2}, \quad \xi \in \sigma_1 \\ &= T^1(\xi) E_{\sigma_1} A_{\sigma_2}, \quad \xi \in \sigma_2 , \end{aligned}$$

then f is an entire function. Since for large ξ, we have

$$f(\xi) = T(\xi) E_{\sigma_1} A_{\sigma_2} \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty .$$

it follows that $f(\xi)=0$ and $E_{\sigma_1} A_{\sigma_2}=0$. By symmetry, $A_{\sigma_2} E_{\sigma_1}=0$. Let σ_n, $(n=3, 4, \cdots)$ be the set of those points of $\sigma(T)$ whose distance from σ_1 is $\geq 1/n$. Let $\delta_1, \cdots, \delta_{p(n)}$ each consist of a circle and its interior and be such that the set $\delta^n=\cup\delta_i$ is disjoint with σ_1 and covers σ_n. Then, since $E_{\sigma_1} A_{\delta_i}=0$, we have $A_{\delta_i} \subset E_{\sigma_1'} X$ and

$$A_{\sigma_n} X \subset A_{\delta^n} X = (\bigcup_i A_{\delta_i})\, X = \bigcup_i (A_{\delta_i} X) \subset E_{\sigma_1'} X .$$

Hence

$$E_{\sigma_1}' A_{\sigma_n} = A_{\sigma_n},\ E_{\sigma_1} A_{\sigma_n} = 0 .$$

But $A_{\sigma_n} x \longrightarrow A_{\sigma_1'} x$, $x \in X$, and so

$$E_{\sigma_1} A_{\sigma_1}' = 0,\ E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

Also, since $A_{\delta_i} E_{\sigma_1} = 0$, we have $E_{\sigma_1} X \subset A_{\delta_i'} X$ and

$$E_{\sigma_1} X \subset \bigcap_i (A_{\delta_i'} X) = (\bigcap_i A_{\delta_i'}) X = A_{\delta^n}' X.$$

Thus

$$A_{\delta^n}' E_{\sigma_1} = E_{\sigma_1},\ A_{\delta^n} E_{\sigma_1} = 0,\ A_{\sigma_1}' E_{\sigma_1} = 0,\ A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1},$$

and therefore

$$E_{\sigma_1} = A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

By symmetry,

$$A_{\sigma_1} = A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

From this it readily follows that $A_\sigma = E_\sigma$ for any Borel set σ. Now let $f \in F(T)$. Let C be an admissible contour upon which f is analytic and such that

$$f(T) = \frac{1}{2\pi i} \int_C f(\xi) T(\xi) d\xi .$$

Now, by 3.7, we have

$$T(\xi) = \sum_{n=0}^{\infty} \int (T - \lambda)^n \, (\xi - \lambda)^{-n-1} \, dE_\lambda \, ,$$

and the series converges in the uniform topology of operators and uniformly for $\xi \in C$. So

$$f(T) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C f(\xi) d\xi \left(\int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} \, dE_\lambda \right).$$

Since

$$\int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} \, dE_\lambda = \sum_{n=0}^{n} \binom{n}{r} T^r \int \frac{\lambda^{(n-r)}}{(\xi - \lambda)^{n+1}} \, dE_\lambda \, ,$$

it follows from 3.3, that the Riemann sums approximating

$$\int (T - \lambda)^n \, (\xi - \lambda)^{-n-1} \, dE_\lambda$$

converge uniformly with respect to $\xi \in C$.

Hence

$$f(T) = \sum_{n=0}^{\infty} \int \frac{f^{(n)}(\lambda)}{n!} \, (T - \lambda)^n \, dE_\lambda .$$

From this point on we shall again restrict our attention to the case of an operator T whose spectrum lies in a rectifiable Jordan curve Γ_0 and whose resolvent satisfies the growth condition 2.1. It will be convenient to state the condition 1.14 in terms of residues as defined in the following:

3.10. DEFINITION. Let $\sigma(x)$, $\sigma(x^*x)$ be the sets of singularities of the functions $x(\xi) = (\xi - T)^{-1}x$, $x^*x(\xi)$, respectively. Let σ be open and closed in $\sigma(x)$, and

$$x_\sigma = \frac{1}{2\pi i} \int_C x(\xi) \, d\xi \, ,$$

where C is a rectifiable Jordan curve containing σ in its interior and having $\sigma(x)\sigma'$ in its exterior. Then the vector x_σ is called the σ – *residue* of $x(\xi)$.

Similarly if σ is open and closed in $\sigma(x^*x)$ then the scalar

$$(x^*x)_\sigma = \frac{1}{2\pi i} \int_C x^*x(\xi)\, d\xi$$

is called the σ – residue of $x^*x(\xi)$.

With this terminology, condition 1.14 asserts the existence of a constant K such that $|x_\sigma| \leq K\,|x|$.

3.11. THEOREM. *Let X be weakly complete, and let T be a bounded linear operator in X whose spectrum lies in the rectifiable Jordan curve Γ_0 and whose resolvent is restricted in its growth by Assumption 2.1. Then T is a spectral operator and satisfies the equation*

$$f(T) = \sum_{n=0}^{\infty} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!} (T-\xi)^n\, dE_\xi\,, \quad f \in F(T),$$

providing:

(i) (*The density condition.*) *For every ξ in a set dense on Γ_0 $\mathfrak{M}_\xi + \mathfrak{N}_\xi$ is dense in X.*

(ii) (*The boundedness condition.*) *There is a constant K such that all residues x_σ satisfy the inequality*

$$|x_\sigma| \leq K\,|x|, \quad x \in X.$$

This theorem is an immediate corollary of 2.11 and 3.8.

Conditions will now be given which are of a nature more applicable than (i) and (ii) of 3.11 and which are sufficient (and in some cases necessary) to imply (i) and (ii). We shall begin with a brief analysis of some conditions which are sufficient to imply the density condition (i).

3.12. THEOREM. *The operator T of Theorem 3.11 satisfies the density condition* (i) *of that theorem in case any one of the following is true:*

(i) *Every subarc* (*of positive length*) *of Γ_0 contains points either in the continuous spectrum or in the resolvent set.*

(ii) *No subarc* (*of positive length*) *of Γ_0 consists entirely of points in the* *point spectrum of the adjoint T^* of T.*

(iii) *The space X is reflexive and $\nu(\lambda) = 1$ is an index function for T.*

(iv) *The space X is reflexive and the adjoint T^* satisfies the boundedness condition* (ii) *of* 3.11.

(v) *The operator T is completely continuous.*

The first statement is obvious since if ξ is in either the resolvent set or the continuous spectrum we have $\mathfrak{R}_\xi = X$. The second statement is equally clear since it is seen from the Hahn-Banach theorem that ξ is in the point spectrum of the adjoint if and only if $\mathfrak{R}_\xi \neq X$. Next, if (iii) holds, let

$$\xi = (\lambda, \delta), \ \xi_0 = \xi(\lambda, 0),$$

so that

$$|(\xi - \xi_0) T(\xi)| \leq |\delta T(\xi)| \leq 1.$$

Now

$$(\xi - \xi_0) T(\xi)(\xi_0 - T) = \xi - \xi_0 - (\xi - \xi_0)^2 T(\xi),$$

and hence

$$(*) \qquad \lim_{\delta \to 0} (\xi - \xi_0) T(\xi)(\xi_0 - T) = 0.$$

Now let x be an arbitrary vector in X. Since X is reflexive, the set

$$(\xi - \xi_0) T(\xi)x, \quad 0 < \delta \leq \delta_0,$$

is weakly compact, and there is a vector $y \in X$ and a sequence $\delta_n \longrightarrow 0$ such that for $\xi_n = \xi(\lambda, \delta_n)$ we have

$$(\xi_n - \xi_0) T(\xi_n) x \longrightarrow y \text{ weakly}.$$

The equation (*) shows that $y \in \mathfrak{M}_{\xi_0}$. To see that $x - y \in \mathfrak{R}_{\xi_0}$, let $x^*\mathfrak{R}_{\xi_0} = 0$. Then

$$x^*(\xi - \xi_0) T(\xi) = x^*,$$

and so $x^*(x-y)=0$. Hence $\mathfrak{M}_{\xi_0}+\mathfrak{N}_{\xi_0}=X$. To prove the fourth statement we note first that, since $\sigma(T)=\sigma(T^*)$ and $|T(\xi)|=|T^*(\xi)|$, the adjoint T^* satisfies 2.1, and any index function $\nu(\lambda)$ for T is also an index function for T^*. By 2.9, then, $\mathfrak{M}_\xi(T^*)\,\mathfrak{N}_\xi(T^*)=0$ for ξ interior to any interval upon which an index function is constant. Such ξ are by 2.3 dense on Γ_0. Let ξ be such a point on Γ_0, and let

$$x^*\mathfrak{M}_\xi = 0 = x^*\mathfrak{N}_\xi .$$

It will suffice to prove that $x^*=0$. Since $x^*\mathfrak{N}_\xi=0$, we have $x^*\in\mathfrak{M}_\xi(T^*)$. To see that $x^*\in\mathfrak{N}_\xi(T^*)$ (which will prove $x^*=0$), it will suffice, since X is reflexive, to show that $x^*x_0=0$ for every x_0 with $\mathfrak{N}_\xi(T^*)x_0=0$, that is, for every x_0 with

$$y^*(\xi-T)^\nu x_0 = 0,\ y^*\in Y^*.$$

But such an x_0 is in $\mathfrak{M}_\xi$, and so $x^*x_0=0$. The final statement (v) follows from the fact that the spectrum of a completely continuous operator is at most denumerable.

N. B. As the above proof shows, the condition that X be reflexive (in (iv)) may be replaced by the statement that, for ξ in a dense set on Γ_0, the manifold $\mathfrak{N}_\xi(T^*)$ is regularly closed. Also in (iii) the condition of reflexivity may be replaced by the assumption that the set of vectors $(\xi-\xi_0)T(\xi)x$, $0<\delta<\delta_0$, is weakly compact.

3.13. THEOREM. *Let X be a reflexive space and T a bounded linear operator in X whose spectrum lies in the rectifiable Jordan curve Γ_0 and whose resolvent satisfies the growth condition* 2.1. *Then T is a spectral operator and satisfies the equation*

$$f(T) = \sum_{n=0}^{\infty} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!}(T-\xi)^n\, dE_\xi ,\quad f\in F(T),$$

if and only if there is a constant K such that all the residues $(x^*x)_\sigma$ satisfy the inequality.

$$|(x^*,x)_\sigma| \le K|x|\,|x^*| .$$

The residue condition is clearly necessary since

$$(x^*x)_\sigma = x^*E_\sigma x.$$

To see that it is sufficient we note first that it implies the condition (ii) of Theorem 3.11. Since X is reflexive, the residue $(x^*, x)_\sigma$ is equal to the residue $(x^{**}, x^*)_\sigma$ calculated for the adjoint T^* (here $x^{**}x^* = x^*x$, $x^* \in X^*$) and so the residue condition of 3.13 implies that the adjoint T^* satisfies (ii) of 3.11. The present theorem then follows from 3.11 and 3.12 (iv).

We now turn our attention to stating the requirement (of 3.11 or 3.13) that the residues be bounded, in a form which, in some instances, is more readily applied.

3.14. THEOREM. *Let $\xi = \xi(\lambda, \delta)$ have continuous first partial derivatives, and let $\xi^- = \xi(\lambda, -\delta)$. Then the residues $(x^*, x)_\sigma$ and x_σ will have a bound of the form $K|x|\,|x^*|$ in case*

$$\underset{0<\delta<\delta_0}{\text{l.u.b.}} \int_{-1}^{+1} \left| x^*\left\{ T(\xi)\frac{\partial \xi}{\partial \lambda} - T(\xi^-)\frac{\partial \xi^-}{\partial \lambda} \right\} x \right| d\lambda \le M\,|x|\,|x^*|.$$

Let $0 \le \lambda_j < \lambda_j' \le 1$, let $C_j = C(\lambda_j, \lambda_j')$ be as in 2.4, and let C be the set C_j, $(j = 1, \cdots n)$. Suppose that C lies in the domain of analyticity of $x^*x(\xi)$. Then

$$\int_C x^*x(\xi)\,d\xi = \sum_{j=1}^{n} \int_{\lambda_j}^{\lambda_j'} x^*\left\{ x(\xi)\frac{\partial \xi}{\partial \lambda} - x(\xi^-)\frac{\partial \xi^-}{\partial \lambda} \right\} d\lambda + I(\delta),$$

where $I(\delta)$ is a sum of integrals $\int x^*x(\xi)\,d\xi$ taken along the arcs $\xi(\lambda_j, \mu)$, $\xi(\lambda_j', \mu)$, $-\delta \le \mu \le \delta$. Thus $\lim_{\delta\to 0} I(\delta) = 0$, and

$$\limsup_{\delta\to 0} \left| \int_C x^*x(\xi)\,d\xi \right|$$

$$\le \underset{0<\delta<\delta_0}{\text{l.u.b.}} \int_{-1}^{+1} \left| x^*\left\{ T(\xi)\frac{\partial \xi}{\partial \lambda} - T(\xi^-)\frac{\partial \xi}{\partial \lambda} \right\} x \right| d\lambda \le M\,|x|\,|x^*|.$$

The condition of 3.13 is far from necessary, and is not satisfied by the resolvent $T(\xi)$ if its rate of growth for ξ near $\sigma(T)$ is not that of the inverse of the distance from ξ to $\sigma(T)$. To avoid this objection a similar condition, as is evident from the above proof, may be stated.

3.15. THEOREM. *Let* $T(\xi) = U(\xi) + V(\xi)$, *where* $x^*V(\xi)x$ *is the derivative of a single valued analytic function at each point* ξ *where* $x^*T(\xi)x$ *is analytic. Then the residues* $(x^*, x^*)_\sigma$ *and* x_σ *will have a bound of the forms* $K|x|\ |x^*|$ *provided that* $U(\xi)$ *satisfies the condition of* 3.14.

Operators of finite type

As is to be expected from the analogy with the elementary divisor theory for a finite matrix, certain spectral operators should satisfy the formula

$$f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!} (T - \xi)^n \, dE_\xi , \ f \in F(T).$$

One might expect this to be true if the spectrum $\sigma(T)$ is nowhere dense and if the resolvent $T(\xi)$ has for ξ near $\sigma(T)$ the same rate of growth as

$$[\operatorname{dis}(\xi, \sigma(T)]^{-m}.$$

We have been able to prove this only in the case where $\sigma(T)$ is restricted to lie in a sufficiently smooth Jordan curve.

We shall assume throughout the following discussion that the function $\xi(\lambda, \mu)$ *defining the net described in* 2.1 *has continuous second partial derivatives.* The purpose of this assumption is to assure that the length of the contour $C(\lambda_1, \lambda_2)$ of 2.4 is at most $K\delta$, provided that $\lambda_1 < \lambda_2$ and $\delta = \lambda_2 - \lambda_1$. Also the diameter of $C(\lambda_1, \lambda_2)$ is at most $K\delta$ for $\delta = \lambda_2 - \lambda_1$.

3.16. LEMMA. (*Assumption* 2.1.) *Let* $d(\xi)$ *be the distance from* ξ *to the spectrum* $\sigma(T)$. *If* $|d^m(\xi)\ T(\xi)|$ *is bounded for* ξ *near* $\sigma(T)$, *then*

$$\int_{\sigma(T)} f(T, \xi)\ (T - \xi)^{2m}\ dE_\xi = 0$$

for every T-uniform $f(\alpha, \xi)$.

We may and shall assume that $\nu(\lambda) = m$ is an index function for T, so that

$$|\delta^m T_{(\xi)}| \le 1, \ 0 < |\delta| < \delta_0, \ \lambda \in [-1, 1].$$

Let $\lambda_1 < \lambda_2$, $\lambda_2 - \lambda_1 < \delta_0$. Let $C(\lambda_1, \lambda_2)$ be the contour defined in 2.4 with $\delta = \lambda_2 - \lambda_1$. Let Δ be the closed subarc of Γ_0 defined by $\xi(\lambda, 0)$, $\lambda_1 \le \lambda \le \lambda_2$.

Let $I(\lambda_1, \lambda_2)$ be the integral defined in 2.4 with $\nu_1 = \nu_2 = m$ and $P(\xi) = 1$. Let $\lambda_n < \lambda_1$, $\lambda_2 < \mu_n$, $\lambda_n \longrightarrow \lambda_1$, $\mu_n \longrightarrow \lambda_2$. By 2.4, we have $I(\lambda_n, \lambda_1) \longrightarrow 0$, $I(\lambda_2, \mu_n) \longrightarrow 0$. Also, by 2.4, 1.39 (ii), 2.5, and 1.12, we have $E_\Delta I(\mu_n, \lambda_n) = 0$. If $\xi_j = \xi(\lambda_j, 0)$ $(j = 1, 2)$, then

$$(T - \xi_1)^m (T - \xi_2)^m = I(\lambda_n, \lambda_1) + I(\lambda_1, \lambda_2) + I(\lambda_2, \mu_n) + I(\mu_n, \lambda_n);$$

and so we have

$$E_\Delta (T - \xi_1)^m \ (T - \xi_2)^m = E_\Delta I(\lambda_1, \lambda_2).$$

But by 2.4, we have $\sigma(I(\lambda_1, \lambda_2)x) \subset \Delta$; hence by 1.39 (i) it is seen that $I(\lambda_1, \lambda_2) = E_\Delta I(\lambda_1, \lambda_2)$. Thus

$$(*) \qquad E_\Delta (T - \xi_1)^m \ (T - \xi_2)^m = I(\lambda_1, \lambda_2).$$

Now, since $\delta = \lambda_2 - \lambda_1$, there are constants K_1, K_2 such that

$$\max_{\lambda_1 \leq \lambda \leq \lambda_2} |\xi(\lambda, \pm\delta) - \xi(\lambda_1, 0)| \leq K_1 \delta,$$

and

$$\text{length } C(\lambda_1, \lambda_2) \leq K_2 \delta.$$

It follows from the definition of $I(\lambda_1, \lambda_2)$ therefore that

$$(\dagger) \qquad |I(\lambda_1, \lambda_2)| \leq K_3 \delta^{m+1}.$$

Let the interval $[-1, 1]$ be partitioned into n intervals $[\lambda_{j-1}, \lambda_j]$ each of length $2/n$, and let Δ_j be the corresponding subarcs of Γ_0 with end points ξ_{j-1}, ξ_j. Statements $(*)$ and $(\dagger)$ then give

$$\left| \sum_{j=1}^{n} f(T, \xi_j) \ (T - \xi_j)^m \ (T - \xi_{j-1})^m \ E_{\Delta_j} \right| \leq K_4 n^{-m}.$$

Hence, by 3.5, we have

$$\int_{\sigma(T)} F(T, \xi) \ (T - \xi)^{2m} \ dE_\xi = 0.$$

3.17. LEMMA. *Under the hypothesis of* 3.16, *we have*

$$(T-\xi)^m E_{(\xi)} = 0,$$

where (ξ) *is the set consisting of the single point* ξ.

From 3.16, we get

$$(T-\xi)^{2m} E_{(\xi)} = E_{(\xi)} \int_{\sigma(T)} (T-\mu)^{2m}\, dE_\mu = 0.$$

Thus

$$T(\alpha)E_{(\xi)} = \sum_{j=0}^{2m-1} \frac{(T-\xi)^j E}{(\alpha-\xi)^{j+1}} (\xi)$$

has a pole of order $\leq 2m$ at $\alpha = \xi$. Since $|d^m(\alpha)\, T(\alpha)|$ is bounded for α near ξ, the pole must be of order $\leq m$; that is, $(T-\xi)^m E_{(\xi)} = 0$.

3.18. LEMMA. *Under the hypothesis of* 3.16, *we have*

$$\int_{\sigma(T)} f(T, \xi)\ (T-\xi)^j\ dE_\xi = 0, \quad j \geq m$$

for every T-uniform function $f(\alpha, \xi)$.

For $\xi = \xi(\lambda, 0) \in \Gamma_0$ and $0 < |\delta| < \delta_0$, let $\xi_\delta = \xi(\lambda, \delta)$. Then

$$(\xi - T)\ T(\xi_\delta) = (\xi - \xi_\delta)\ T(\xi_\delta) + I.$$

Now assume for the purposes of induction (the above equality is the case $j = 1$,) that

$$(\xi - T)^j\ T(\xi_\delta) = (\xi - \xi_\delta)^j\ T(\xi_\delta) + (\xi - \xi_\delta)^{j-1} + (\xi - \xi_\delta)^{j-2}(\xi - T)$$
$$+ \cdots + (\xi - \xi_\delta)(\xi - T)^{j-2} + (\xi - T)^{j-1}.$$

Multiplying by $(\xi - T)$, we have

$$(\xi - T)^{j+1}\ T(\xi_\delta) = (\xi - T)^j\ (\xi - T)\ T(\xi_\delta) = (\xi - T)^j\,[(\xi - \xi_\delta)\ T(\xi_\delta) + I]$$

$$= (\xi - \xi_\delta)^{j+1}\ T(\xi_\delta) + (\xi - \xi_\delta)^j + (\xi - \xi_\delta)^{j-1}\ (\xi - T)$$

$$+ \cdots + (\xi - \xi_\delta)\ (\xi - T)^{j-1} + (\xi - T)^j\,.$$

Hence

$$f(T, \xi)\ [(\xi - T)^{j+1}\ T(\xi_\delta) - (\xi - T)^j]$$

$$= f(T, \xi)\ [(\xi - \xi_\delta)^{j+1}\ T(\xi_\delta) + (\xi - \xi_\delta)^j + \cdots + (\xi - \xi_\delta)\ (\xi - T)^{j-1}].$$

Thus we may state:

(*) If for some $j = 1, 2, \cdots$ we have

$$\int_{\sigma(T)} f(T, \xi)\ (\xi - T)^{j+1}\ T(\xi_\delta)\, dE_\xi = 0,\ \ 0 < |\delta| < \delta_0\,,$$

then

$$\lim_{\delta \to 0} \int_{\sigma(T)} f(T, \xi)\ (\xi - \xi_\delta)^{j+1}\ T(\xi_\delta)\, dE_\xi = - \int_{\sigma(T)} f(T, \xi)\, (\xi - T)^j\, dE_\xi\,.$$

Now let $0 < |\delta_i| < \delta_0$, $(i = 1, 2, \cdots, m)$. By 3.16, then,

$$\int_{\sigma(T)} f(T, \xi)\ (T - \xi)^{2m}\ T(\xi_{\delta_1})\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\, dE_\xi = 0.$$

To this equation we may apply (*) with $\delta = \delta_1$, and with $f(T, \xi)$ replaced by $f(T, \xi)\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})$. Thus,

$$\lim_{\delta_1 \to 0} \int_{\sigma(T)} f(T, \xi)\ (\xi - \xi_{\delta_1})^{2m}\ T(\xi_{\delta_1}) \cdots T(\xi_{\delta_m})\, dE_\xi$$

$$= - \int_{\sigma(T)} f(T, \xi)\ (\xi - T)^{2m-1}\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\, dE_\xi\,.$$

Since

$$|(\xi - \xi_{\delta_1})^{2m}\ T(\xi_{\delta_1})| \leq K\, |\delta_1|^m,$$

the integrand on the right side of the preceeding equation approaches zero with δ_1 and uniformly with respect to $\xi \in \sigma(T)$. Thus

$$\int_{\sigma(T)} f(T, \xi)\,(\xi - T)^{2m-1}\, T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\, dE_\xi = 0.$$

A repetition of this process clearly yields the desired result:

$$\int_{\sigma(T)} f(T, \xi)\,(\xi - T)^j\, dE_\xi = 0, \quad j \geq m.$$

3.19. DEFINITION. Let m be a positive integer. A spectral operator T is said to be *of type m* in case

$$f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!}\,(T - \xi)^n\, dE_\xi$$

for every f single valued and analytic on $\sigma(T)$, that is, for $f \in F(T)$.

Let us recall that for the case in hand (that is, $\nu(\lambda) = m$ is an index function), the manifolds $\mathfrak{M}_\xi$, $\mathfrak{N}_\xi$ are respectively the zeros and the closure of the range $(T - \xi)^m$. Then if $d(\xi)$ is the distance from ξ to the spectrum $\sigma(T)$ we may state:

3.20. THEOREM. *If X is weakly complete, T will be a spectral operator and of type m providing*

(i) *$d^m(\xi)\, T(\xi)$ is bounded for ξ near $\sigma(T)$,*

(ii) *for ξ in a set dense in Γ_0 the manifold $\mathfrak{M}_\xi + \mathfrak{N}_\xi$ is dense in X.*

(iii) *all residues x_σ have a bound of the form $K\,|x|$.*

This theorem follows immediately from 3.11 and 3.18.

N. B. 1. As before, the condition (ii) is automatically satisfied if T enjoys any one of the properties listed in 3.12. Also (iii) is satisfied if the resolvent $T(\xi)$ satisfies the mean rate of growth condition of 3.14 or 3.15.

N. B. 2. *In case X is not weakly complete* it is still true that E_Δ is defined for every closed subarc of Γ_0 (see proof of 2.11), and E_Δ is completely additive, in the strong topology of operators, on the Boolean algebra determined by such arcs. Thus the integral

$$\int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!} (T-\xi)^n \, dE_\xi$$

may be defined and the operational calculus developed even though E_e may not be defined as an operator in X for every Borel set e.

An immediate corollary is (see 3.13):

3.21. THEOREM. *If X is reflexive, then T will be a spectral operator of type m if and only if*

(i) *$d^m(\xi)\, T(\xi)$ is bounded for ξ near $\sigma(T)$,*

(ii) *all residues $(x^*, x)_\sigma$ have a bound of the form $K|x|\ |x^*|$.*

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ON ANALYTIC CHARACTERISTIC FUNCTIONS

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1. **Introduction.** In this paper we discuss certain properties of characteristic functions. Theorem 1 gives a sufficient condition on the characteristic function of a distribution in order that the moments of the distribution should exist. The existence of the moments is usually proven under the assumption that the characteristic function is differentiable [4]. The condition of Theorem 1 is somewhat more general and the proof shorter and more elementary. The remaining theorems deal with analytic characteristic functions, and again some known results are proved in a simple manner. Some applications are discussed; in particular, it is shown that an analytic characteristic function of an infinitely divisible law can have no zeros inside its strip of convergence. This property is used to construct an example where an infinitely divisible law (the Laplace distribution) is factored into two noninfinitely divisible factors.

2. **An existence theorem.** Let $F(x)$ be a probability distribution, that is, a never-decreasing, right-continuous function such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The Fourier transform of $F(x)$, that is, the function

$$\phi(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x), \tag{1.1}$$

is called the *characteristic function* of the distribution $F(x)$. The characteristic function exists for real values of t for any distribution, but the integral (1.1) does not always exist for complex t. This paper deals mostly with characteristic functions which are analytic in a neighborhood of the origin.

For an arbitrary function $f(y)$, we denote in the following the first difference by

$$\Delta_1 f(y;\, t) = \Delta f(y;\, t) = f(y + t) - f(y - t),$$

and define the higher differences by

$$\Delta_{k+1} f(y;\, t) = \Delta\Delta_k f(y;\, t)$$

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for $k = 1, 2, \cdots$. It can then easily be shown that

$$\Delta_n f(y; t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f[y + (n - 2k)t].$$

In particular, for the function $f(y) = e^{imy}$ we have

$$\Delta_n f(y; t) = e^{imy}(e^{imt} - e^{-imt})^n = e^{imy}[2i \sin mt]^n.$$

We first prove two lemmas.

LEMMA 1. *Let $\phi(t)$ be the characteristic function of a probability distribution $F(x)$, and let $\Delta_{2k}\phi(0; t)/(2t)^{2k}$ be the* $2k$th *difference quotient of $\phi(t)$ at the origin. Assume that*

$$\liminf_{t \to \infty} \left| \frac{\Delta_{2k}\phi(0; t)}{(2t)^{2k}} \right| < \infty.$$

Then the $2k$th *moment m_{2k} of the distribution $F(x)$ exists, as do all the moments m_r of order $r < 2k$.*

LEMMA 2. *Under the assumptions of Lemma* 1, *the derivatives $\phi^{(r)}(t)$ exist for all t and for $r = 1, 2, \cdots, 2k$; and*

$$\phi^{(r)}(t) = i^r \int_{-\infty}^{+\infty} x^r e^{itx} dF(x).$$

Moreover, $|\phi^{(2r)}(t)| \leq |\phi^{(2r)}(0)| = m_{2r}$ *for* $r = 1, 2, \cdots, k$.

Proof. The assumption of Lemma 1 means that there is a constant $M < \infty$ such that

$$\liminf_{t \to 0} \left| \frac{\Delta_{2k}\phi(0; t)}{(2t)^{2k}} \right| = M. \tag{1.2}$$

From (1.1) it is seen that for

$$\phi(y) = \int_{-\infty}^{+\infty} e^{iyx} dF(x)$$

we have

$$\Delta_{2k}\phi(y;\,t) = \int_{-\infty}^{+\infty} e^{ixy}\,[2i\ \sin\ xt]^{2k}\,dF(x)$$

and

$$\left|\frac{\Delta_{2k}\phi(0;\,t)}{(2t)^{2k}}\right| = \int_{-\infty}^{+\infty}\left(\frac{\sin xt}{t}\right)^{2k} dF(x).$$

We see therefore from (1.2) that

$$M = \liminf_{t\to 0} \int_{-\infty}^{+\infty}\left(\frac{\sin xt}{t}\right)^{2k} dF(x),$$

and hence that

$$M \geq \liminf_{t\to 0} \int_{-a}^{+a}\left(\frac{\sin xt}{t}\right)^{2k} dF(x) = \int_{-a}^{+a} x^{2k}\,dF(x)$$

for any finite a. It follows then that the $2k$th moment

$$m_{2k} = \int_{-\infty}^{+\infty} x^{2k}\,dF(x)$$

exists and that $M \geq m_{2k}$. Let next r be a positive integer such that $r < k$; then $x^{2k} > x^{2r}$ if $|x| > 1$, and

$$m_{2k} > \int_{|x|>1} x^{2k}\,dF(x) > \int_{|x|>1} x^{2r}dF(x),$$

so that the moments of even order m_{2r} $[r = 1, 2, \cdots, (k-1)]$ exist also. Moreover,

$$\int_a^b |x^{2r-1}|\,dF(x) \leq \frac{1}{2}\int_a^b (x^{2r} + x^{2r-2})\,dF(x) \leq \frac{1}{2}\,[m_{2r} + m_{2r-2}]$$

for any a and b. This shows that the absolute moments of odd order not exceeding $2k$, and therefore also the moments m_{2r-1} $(r = 1, 2, \cdots, k)$, exist. This proves Lemma 1.

From the existence of the moments m_r $(r = 1, 2, \cdots, 2k)$ we see immediately that $\int_{-\infty}^{+\infty} x^r e^{itx}\,dF(x)$ exists and converges absolutely and uniformly for all real t and $r \leq 2k$. It follows then from a well-known theorem (see for instance

[2, pp. 67-68]) that all derivatives exist and are obtained by differentiating under the integral sign. This proves Lemma 2.

From Lemma 1 and 2 we obtain immediately:

THEOREM 1. *Let $\phi(t)$ be the characteristic function of a distribution $F(x)$, and assume that, for an infinite sequence of even integers $\{2n_k\}$,*

$$\liminf_{t\to 0}\left|\frac{\Delta_{2n_k}\phi(0;\,t)}{(2t)^{2n_k}}\right| = M_k \tag{1.3}$$

is finite (not necessarily bounded) for $k = 1, 2, \cdots$. Then all the moments m_r of the distribution $F(x)$ exist; and $\phi(t)$ can be differentiated for all real t any number of times, with

$$\phi^{(r)}(t) = i^r \int_{-\infty}^{+\infty} x^r e^{itx}\, dF(x).$$

COROLLARY 1. *If all the derivatives of the characteristic function exist at the origin, then all the moments of the distribution exist.*

This corollary was proved by R. Fortet [4]; it is also stated in some text books of probability [2, 5], as well as in a paper by P. Lévy [7]. Theorem 1 is somewhat more general; the proof given here is similar to the proof indicated for the corollary by H. Cramér [2, p. 89].

3. **Analytic characteristic functions.** From now on we assume that the characteristic function $\phi(t)$ coincides with an analytic function in some neighborhood of the origin. Then the assumptions of the corollary are satisfied, all the moments exist, and the characteristic function has the expansion

$$\phi(z) = \sum_{k=0}^{\infty} \frac{i^k m_k}{k!} z^k \qquad \text{for } |z| < \rho, \tag{2.1}$$

where $\rho > 0$ is the radius of convergence of the series.

We write

$$\phi_0(z) = \frac{1}{2}[\phi(z) + \phi(-z)]$$

for the even part of $\phi(z)$, and

$$\phi_1(z) = \frac{1}{2}[\phi(z) - \phi(-z)]$$

for the odd part of $\phi(z)$, then the two series

$$(2.2) \qquad \begin{cases} \phi_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k m_{2k}}{(2k)!} z^{2k}, \\ \phi_1(z) = \sum_{k=1}^{\infty} \frac{i^{2k-1} m_{2k-1}}{(2k-1)!} z^{2k-1} \end{cases}$$

converge also in circles about the origin. Denote the radii of convergence of these series by ρ_0 and ρ_1.

If we denote the kth absolute moment of $F(x)$ by

$$\beta_k = \int_{-\infty}^{+\infty} |x|^k \, dF(x),$$

and observe that

$$|x^{2k-1}| \leq \frac{1}{2}(x^{2k} + x^{2k-2}),$$

we see that

$$(2.3) \qquad \frac{m_{2k-1}}{(2k-1)!} \leq \frac{\beta_{2k-1}}{(2k-1)!} \leq \frac{1}{2}\left[\frac{m_{2k}}{(2k)!}(2k) + \frac{m_{2k-2}}{(2k-2)!}\right].$$

This shows that

$$\rho_1 \geq \rho_0 \geq \rho.$$

We see further from (2.3) and $\beta_{2k} = m_{2k}$ that the series $\sum_{k=0}^{\infty} \beta_k z^k/k!$ converges for $|z| < \rho_0$. From Lemma 2 we see, for any real ξ, that

$$|\phi^{(2k)}(\xi)| \leq m_{2k}.$$

Hence if we denote the radius of convergence of the Taylor series of $\phi_0(z)$ around ξ by $\rho_0(\xi)$, then

$$\rho_0(\xi) \geq \rho_0(0) = \rho_0.$$

Similarly it follows from

$$|\phi^{(2k-1)}(\xi)| \leq \beta_{2k-1}$$

and (2.3) that

$$\rho_1(\xi) \geq \rho_1(0) = \rho_1 \geq \rho.$$

We have thus shown that the Taylor series of $\phi_0(z)$ and also that of $\phi_1(z)$ around ξ converge in circles of radii at least equal to ρ. The same is therefore true for the expansion of $\phi(z)$ around ξ; thus we conclude that the function $\phi(z)$ is analytic at least in the strip

$$-\rho < \mathfrak{I}(z) < +\rho.$$

The analyticity of $\phi(z)$ in a horizontal strip follows also from a result of R. P. Boas [1]. Boas showed that the Fourier-Stieltjes transform of a bounded and never-decreasing function is analytic in a horizontal strip provided that it is analytic in a neighborhood of the origin.

We show next that the representation of the characteristic function by the Fourier integral (1.1) is valid in the strip $-\rho < \mathfrak{I}(z) < +\rho$.

We saw above that the series $\sum_{v=0}^{\infty} |y|^v \beta_v/v!$ converges for $|y| < \rho$. Clearly,

$$\sum_{v=0}^{\infty} \frac{|y|^v \beta_v}{v!} \geq \sum_{v=0}^{\infty} \frac{|y|^v}{v!} \int_{-A}^{+A} |x|^v \, dF(x) = \int_{-A}^{+A} e^{|yx|} \, dF(x)$$

for any A. Therefore the integral $\int_{-\infty}^{+\infty} e^{|yx|} \, dF(x)$ exists, and hence the integral $\int_{-\infty}^{+\infty} e^{izx} \, dF(x)$ is convergent whenever $|e^{izx}| \leq e^{|yx|}$, where $z = \zeta + iy$. Thus for any ζ and $|y| < \rho$ the integral is convergent. This integral is an analytic function in its strip of convergence and agrees with $\phi(z)$ for real z; therefore it must agree with $\phi(z)$ also for complex values $z = \zeta + iy$ provided $|y| < \rho$.

We are now in a position to formulate our main result.

THEOREM 2. *If a characteristic function $\phi(z)$ is analytic in a neighborhood of the origin, then it is also analytic in a horizontal strip and can be represented in this strip by a Fourier integral. Either this strip is the whole plane, or it has one or two horizontal boundary lines. The purely imaginary points on the boundary* *of the strip of convergence (if it is not the whole plane) are singular points of* $\phi(z)$.

The first part of Theorem 2 was established above; we have to prove the statement concerning the singular points of $\phi(z)$.

The integral

$$\phi(z) = \int_{-\infty}^{+\infty} e^{izx}\, dF(x)$$

converges in a strip $-\alpha < \mathfrak{J}(z) < +\beta$, where $\alpha \geq \rho$ and $\beta \geq \rho$, and is analytic inside this strip. To carry out the proof concerning the singular points of $\phi(z)$, we use the decomposition

$$\phi(z) = \int_{0}^{\infty} e^{izx}\, dF(x) + \int_{-\infty}^{0} e^{izx}\, dF(x) = \mathfrak{L}_1(z) + \mathfrak{L}_2(z) \quad \text{(say)}.$$

Now $\mathfrak{L}_1(z)$ and $\mathfrak{L}_2(z)$ are Laplace integrals, convergent in the half-planes $\mathfrak{J}(z) > -\alpha$ and $\mathfrak{J}(z) < \beta$, respectively. Let $z = iw$; then $iz = -w$. If $z = \zeta + iy$, then $w = -i\zeta + y$; thus

$$\mathfrak{L}_1(iw) = \int_{0}^{\infty} e^{-wx}\, dF(x) \equiv \Phi(w)$$

is convergent for $\mathfrak{R}(w) > -\alpha$.

It is known that the Laplace transform

$$g(s) = \int_{0}^{\infty} e^{-st}\, dG(t)$$

of a monotonic function $G(t)$ has a singularity at the real point of its axis of convergence. For a proof the reader is referred to [9, p. 58]. This theorem is similar to well-known theorems in power series and Dirichlet series. From the fact that $F(x)$ is nondecreasing we conclude therefore that $-\alpha$ is a singular point of $\Phi(w)$. Thus $-i\alpha$ is a singular point of $\phi(z)$. In the same way it is also seen that $i\beta$ is a singular point of $\phi(z)$.

Theorem 2 was stated without proof in a recent paper by D. Dugué [3], and is indicated in a footnote of an earlier paper by P. Lévy [7].

An immediate consequence of the preceding result is this:

COROLLARY 2. *A necessary condition that a function analytic in some* *neighborhood of the origin be a characteristic function is that in either half-plane the singularity nearest to the real axis be located on the imaginary axis.*

4. **Applications**. In the following we discuss some applications of our new results.

The corollary to Theorem 2 can sometimes be used to decide whether the the quotient of two characteristic functions is again a characteristic function. We illustrate this by an example. Let

$$\phi_1(t) = \left[\left(1 - \frac{it}{a}\right)\left(1 - \frac{it}{a+ib}\right)\left(1 - \frac{it}{a-ib}\right)\right]^{-1}$$

and

$$\phi_2(t) = \left[1 - \frac{it}{a}\right]^{-1},$$

with $a^2 \geq b^2 > 0$. It is easy to see that both these functions are characteristic functions. Their quotient

$$\phi(t) = \frac{\phi_1(t)}{\phi_2(t)}$$

satisfies the elementary necessary conditions for characteristic functions, namely $\phi(-t) = \overline{\phi(t)}$, $\phi(0) = 1$, $|\phi(t)| \leq 1$ for real t. However, the condition of the corollary to Theorem 2 is violated since $\phi(t)$ has no singularity on the imaginary axis while it has a pair of conjugate complex poles $\pm b - ia$. Therefore $\phi(t)$ can not be a characteristic function.

Theorem 2 can also be used to establish the following property of analytic characteristic functions.

THEOREM 3. *Let $\phi(z)$ be an analytic characteristic function. Then for any horizontal line in the strip, the function $\phi(z)$ and its derivatives $\phi^{(k)}(z)$ all attain their absolute maxima on the imaginary axis.*

Proof. By Theorem 2 we have

$$\phi(z) = \int_{-\infty}^{+\infty} e^{ixz}\, dF(x)$$

in the strip of convergence. Let $z = ia + \eta$; then

$$\max_{-\infty<\eta<\infty} |\phi(ia+\eta)| \leq \int_{-\infty}^{+\infty} e^{-ax}\, dF(x) = \phi(ia).$$

This result is due to Dugué [3].

We also obtain:

COROLLARY 3. *An analytic characteristic function has no zeros on the segment of the imaginary axis inside the strip of analyticity. The zeros and the singular points of $\phi(z)$ are located symmetrically with respect to the imaginary axis.*

The first part of the corollary follows immediately from Theorem 3; we obtain the statement about the location of the zeros and singularities of $\phi(z)$ if we observe that the functional relation

$$\phi(z) = \overline{\phi(-\bar{z})}$$

holds not only in the strip of convergence of the Fourier integral but in the entire domain of regularity of $\phi(z)$.

An important theorem on analytic characteristic functions is due to P. Lévy [6] and D. Raikov [8]:

THEOREM OF LÉVY AND RAIKOV. *Let $\phi(t)$ be an analytic characteristic function, and assume that $\phi(t) = \phi_1(t)\,\phi_2(t)$, where $\phi_1(t)$ and $\phi_2(t)$ are both characteristic functions. Then the factors $\phi_1(t)$ and $\phi_2(t)$ are also analytic functions, and their representations as Fourier integrals converge at least in the strip of convergence of $\phi(t)$.*

This theorem was originally proven by P. Lévy [6; 7] only for entire characteristic functions; a simple proof may be found in [3].

From the foregoing theorem we easily deduce:

THEOREM 4. *Let $\phi(t)$ be the characteristic function of an infinitely divisible law, and assume that $\phi(t)$ is an analytic function. Then $\phi(t)$ has no zeros inside its strip of convergence.*

If $\phi(z)$ is the characteristic function of an infinitely divisible law, then the function $[\phi(z)]^{1/n}$ must be a characteristic function for any n, and also a factor of $\phi(z)$. If furthermore $\phi(z)$ is also assumed to be analytic, then, by the Lévy-Raikov theorem, $[\phi(z)]^{1/n}$ must be analytic at least in the strip of convergence of $\phi(z)$. If $\phi(z)$ were to have a zero at some point z_0, then $[\phi(z)]^{1/n}$

would have a singularity at z_0 for sufficiently large n, which is impossible.

We can use Theorem 4 in the construction of an example which shows that an infinitely divisible law can be obtained as the product of two noninfinitely divisible laws. Let

$$\phi(t) = \left[\left(1+\frac{it}{v}\right)\left(1+\frac{it}{\bar{v}}\right)\right] \Big/ \left[\left(1-\frac{it}{a}\right)\left(1-\frac{it}{v}\right)\left(1-\frac{it}{\bar{v}}\right)\right], \qquad v = a + ib .$$

A simple computation shows that $\phi(t)$ is a characteristic function if

$$b \geq 2\sqrt{2}\, a .$$

Then also $\phi(-t)$ is a characteristic function, as is

$$\psi(t) = \phi(t)\,\phi(-t) = \frac{1}{1+\dfrac{t^2}{a^2}} .$$

The characteristic functions $\phi(t)$ and $\phi(-t)$ are analytic characteristic functions with zeros in their strip of convergence: hence they are not characteristic functions of infinitely divisible laws. Their product $\psi(t)$ is the characteristic function of the Laplace distribution, which is known to be infinitely divisible.

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EVALUATION OF AN INTEGRAL OCCURRING IN SERVOMECHANISM THEORY

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1. Introduction. In the study of dynamical systems in general, and servomechanisms in particular, it is often required to determine the (constant) coefficients in a linear, ordinary, differential equation in such a way as to minimize an integral involving the square of the difference between the solution of the equation and a known function. The latter may be given in either analytical or numerical form. In the design of a servomechanism the known function is the "input"; the solution of the equation is the "output"; and the coefficients of the equation are the circuit constants to be determined. A similar problem arises in the study of aircraft flight records, in which the known function is any of the dynamic variables used to describe the motion, and the coefficients are the so-called aerodynamic derivatives, the determination of which is the purpose of the flight.

Mathematically similar problems also arise in the analysis of a mixture of radioactive substances or of bacteria. The known function is, say, the total weight of the mixture as a function of time, and the unknown coefficients are the relative weights of the different substances initially present.

All such problems can be solved by the method of least squares, and the procedure always leads, at a certain stage, to the evaluation of an integral of a particular type. This integral has been studied by R. S. Phillips [3, Chap. 7, §7.9], who has given a procedure for its evaluation and a short table of results. The purpose of the present note is to derive a simple, explicit formula for this integral.

2, Evaluation of the integral. The integral to be evaluated is

$$I = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{g(x)}{h(x)\,h(-x)}\,dx, \tag{1}$$

where

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$$i = \sqrt{-1},$$

$$g(x) = \sum_{k=1}^{n} g_k \, x^{2(n-k)},$$

$$h(x) = \sum_{k=0}^{n} a_k \, x^{n-k}, \; a_k \text{ real}, \; a_0 \neq 0.$$

There is no loss of generality in restricting $g(x)$ to contain only even powers, since odd powers would make no contribution to the value of the integral. It is assumed that the zeros of $h(x)$ are all distinct and have their real parts negative. Then the integration can be performed immediately by means of the theory of residues [4, Chap. 6], and the result is

$$(2) \qquad I = \sum_{k=1}^{n} A_k,$$

where A_k is the residue of the integrand at x_k, and $h(x_k) = 0$. This expression can be evaluated in terms of the coefficients g_k and a_k by starting with the obvious identity

$$\frac{g(x)}{h(x)\,h(-x)} \equiv \sum_{k=1}^{n} A_k \left(\frac{1}{x - x_k} - \frac{1}{x + x_k} \right).$$

Clearing fractions gives

$$(3) \qquad g(x) \equiv \sum_{k=1}^{n} A_k \left[\frac{h(x)}{x - x_k} \, h(-x) + \frac{h(-x)}{-x - x_k} \, h(x) \right].$$

Since x_k is a zero of $h(x)$, the quantity $h(x)/(x - x_k)$ is a polynomial; in fact,

$$\frac{h(x)}{x - x_k} = \sum_{j=0}^{n-1} x^{n-1-j} \sum_{i=0}^{j} a_i \, x_k^{j-i}.$$

Substitution in (3) gives an identity between two polynomials. Equating coefficients of like powers of x gives a set of simultaneous, linear, algebraic equations for the A_k:

$$(4) \qquad \sum_{k=1}^{n} \alpha_{lk} A_k = (-1)^n g_l/2 \qquad (l = 1, 2, \cdots, n),$$

where

$$\alpha_{lk} = \sum_{i=1}^{n} b_{li} x_k^{i-1} \qquad (l, k = 1, 2, \cdots n),$$

$$b_{li} = \sum_{j=1}^{n} c_{lj} d_{ji} \qquad (l, i = 1, 2, \cdots n),$$

$$c_{lj} = a_{2l-j} \qquad (l, j = 1, 2, \cdots, n),$$

$$d_{ji} = (-1)^j a_{j-i} \qquad (j, i = 1, 2, \cdots, n),$$

with the convention that $a_k = 0$ if $k < 0$ or $k > n$. With $|\alpha_{lk}|$ for the determinant with n rows and n columns having α_{lk} in the lth row and kth column, the rule for multiplying determinants [1, Chap. 8] gives

$$|\alpha_{lk}| = |c_{lj}| \cdot |d_{ji}| \cdot |x_k^{i-1}| .$$

Now,

$$|d_{ji}| = \begin{vmatrix} -a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ -a_2 & -a_1 & -a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \pm a_{n-1} & \pm a_{n-2} & \pm a_{n-3} & \cdots & \pm a_0 \end{vmatrix} = (-1)^{n(n+1)/2} a_0^n$$

and

$$x_k^{i-1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \equiv V_n,$$

where V_n is the well-known Vandermonde determinant.

Hence, writing $C_n \equiv |c_{lj}|$, we have

$$|\alpha_{lk}| = (-1)^{n(n+1)/2} a_0^n C_n V_n,$$

In equation (4), write $\beta_l = (-1)^n g_l/2$ for convenience, and subtract $\beta_l I$ from both sides. Recalling equation (2), we see that the resulting system can be put in the form

$$(5) \qquad \begin{cases} I - \sum_{k=1}^{n} A_k = 0 \\ \beta_l I + \sum_{k=1}^{n} (\alpha_{lk} - \beta_l) A_k = \beta_l, & (1 \le l \le n), \end{cases}$$

a system of $n+1$ equations in the $n+1$ unknowns $I, A_1, A_2, \cdots, A_n$ that can be solved directly for I. First consider the determinant, D, of the coefficients in the left members of (5):

$$D = \begin{vmatrix} 1 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} - \beta_1 & \cdots & \alpha_{1n} - \beta_1 \\ \beta_2 & \alpha_{21} - \beta_2 & \cdots & \alpha_{2n} - \beta_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} - \beta_n & \cdots & \alpha_{nn} - \beta_n \end{vmatrix}.$$

Adding the first column to each of the succeeding columns immediately gives the result

$$D = |\alpha_{ij}| = (-1)^{n(n+1)/2} a_0^n C_n V_n .$$

Now $V_n \neq 0$, since all the zeros, x_k, of $h(x)$ were assumed to be distinct; and C_n does not vanish, since it is precisely the Hurwitz determinant [2, p. 163] of the polynomial $h(x)$, all the roots of which lie in the left half-plane. Hence $D \neq 0$, and the system (5) can be solved for I directly by Cramer's rule [1, Chap. 8]

$$DI = \begin{vmatrix} 0 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} - \beta_1 & \cdots & \alpha_{1n} - \beta_1 \\ \beta_2 & \alpha_{21} - \beta_2 & \cdots & \alpha_{2n} - \beta_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} - \beta_n & \cdots & \alpha_{nn} - \beta_n \end{vmatrix} .$$

Again adding the first column to each succeeding column gives

$$DI = \begin{vmatrix} 0 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} & \cdots & \alpha_{1n} \\ \beta_2 & \alpha_{21} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix} .$$

By the definition of α_{ij}, this can be factored twice to give

$$DI = \frac{M}{a_0} \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \beta_1 & C_{11} & C_{12} & \cdots & C_{1n} \\ \beta_2 & C_{21} & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \beta_n & C_{n1} & C_{n2} & \cdots & C_{nn} \end{vmatrix} ,$$

where

$$M = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & d_{11} & \cdots & d_{1n} \\ 0 & d_{21} & \cdots & d_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & d_{n1} & \cdots & d_{nn} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ 0 & x_1 & \cdots & x_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & x_1^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} .$$

Thus,

$$(-1)^{n(n+1)/2}\, a_0^n\, C_n\, V_n\, I = \frac{-1}{a_0} \begin{vmatrix} \beta_1 & C_{12} & \cdots & C_{1n} \\ \beta_2 & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & C_{n2} & \cdots & C_{nn} \end{vmatrix} \cdot (-1)^{n(n+1)/2}\, a_0^n\, V_n\, .$$

The relation $\beta_l = (-1)^n\, g_l/2$ gives, finally, the desired formula:

$$I \equiv \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{g(x)\,dx}{h(x)\,h(-x)} = \frac{(-1)^{n+1}}{2\,a_0} \cdot \frac{G_n}{C_n}\,, \tag{6}$$

where

$$G_n = |g_{ij}|,\ C_n = |c_{ij}| \qquad (1 \le i, j, n),$$

$$c_{ij} = a_{2i-j},\ g_{ij} = \begin{cases} g_i & \text{if } j = 1 \\ c_{ij} & \text{if } j > 1\,, \end{cases}$$

Since I is a continuous function of the coefficients of $h(x)$, and hence of the zeros, equation (6) remains true when two zeros coincide.

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CAPACITY, VIRTUAL MASS, AND GENERALIZED SYMMETRIZATION

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1. **Introduction.** A body of revolution B can be symmetrized with respect to its axis of symmetry in a number of ways. One of these is the Schwarz symmetrization, which preserves the volume of B. Another is the Steiner symmetrization of the meridian section of B, which preserves the area of this section but in general decreases the volume. The influence of the Schwarz symmetrization on the capacity has been investigated by G. Pólya and G. Szegő, [1]. More recently P. R. Garabedian and D. C. Spencer [2] discussed the same question for the virtual mass of bodies of revolution. In the present paper we shall study by a different and simpler method the behavior of the capacity and virtual mass under a more general type of symmetrization, which includes the Schwarz and Steiner symmetrizations as particular cases.

2. **Definitions.** Let the (x, y)-plane be the meridian plane of B, the x-axis being the axis of symmetry. The part of the meridian section of B which lies in the upper half plane $y \geq 0$ is denoted by D. The complement of D in the half plane is designated as E. We assume that D is simply connected and that E is a connected domain. The boundary of D consists in general of a segment of the x-axis and a line L. We exclude the case where L is a closed curve and lies entirely above the x-axis, as is the case in which B is a torus. We assume L to have at most a finite number ot angular points.

We shall use in this paper some recent results of axially symmetric potential theory in n-dimensional space. This theory which is of mathematical interest in itself will be used here mainly as a tool to obtain results for bodies of revolution in three dimensions.

Let us henceforth consider our (x, y)-plane as the meridian plane of a body of revolution $B[n]$ in n-dimensions, $n = 3, 4, 5, \cdots$. We assume that $B[n]$ has the same meridian section D as our three-dimensional body $B = B[3]$. All quantities considered hereafter are defined in the meridian plane and therefore are functions of x and y only. Actually we shall never use $B[n]$ but only its meridian section.

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Let $\phi(x, y)$ be an axially symmetric potential function defined for $y \geq 0$ and let $\psi(x, y)$ be the corresponding stream function. We have then the generalized Stokes-Beltrami equations

$$(1) \qquad y^{n-2} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad y^{n-2} \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x}.$$

To emphasize the dependence of ϕ and ψ on n we shall often use the notations $\phi[n]$ and $\psi[n]$.

The volume $V[n]$ of $B[n]$ is given by

$$(2) \qquad V[n] = \omega_{n-1} \iint_D y^{n-2}\, dx\, dy,$$

where $\omega_h = 2\pi^{h/2}/\Gamma(h/2)$. We introduce the capacity $C[n]$ of $B[n]$ by the formula

$$(3) \qquad C[n] = \frac{\omega_{n-1}}{\omega_n} \iint_E y^{n-2} (\operatorname{grad} \phi[n])^2\, dx\, dy,$$

where $\phi[n]$ is a potential which assumes the value unity on L and vanishes as $(x^2 + y^2)^{-(n-2)/2}$ at infinity. It is obvious that (3) reduces to the classical definition of the capacity for $n = 3$.

We define also the virtual mass $M[n]$ of $B[n]$ by the formula

$$(4) \qquad M[n] = \omega_{n-1} \iint_E y^{-(n-2)} (\operatorname{grad} \psi[n])^2\, dx\, dy.$$

The function $\psi[n]$ in (4) assumes the value $y^{n-1}/(n-1)$ on L and vanishes at infinity like $y^{n-1} (x^2 + y^2)^{-n/2}$. Our definition of the virtual mass generalizes that of P. R. Garabedian and D. C. Spencer [2].

3. The correspondence principle and the fundamental formula. We use here a relationship due to A. Weinstein [4],

$$(5) \qquad \psi[n] = y^{n-1} (n-1)^{-1} \phi[n+2].$$

This equation shows that to each stream function $\psi[n]$ corresponds a well-defined potential $\phi[n+2]$. In particular to the stream function $\psi[n]$ in formula (4) corresponds a potential $\phi[n+2]$ which assumes the value unity on L and vanishes as $(x^2 + y^2)^{-n/2}$ at infinity. In other words $\phi[n+2]$ is the electrostatic potential of $B[n+2]$. The substitution of (5) into (4) leads after an elementary integration by parts to the fundamental formula

$$(6)\qquad M[n] + V[n] = \pi^{n/2}\left[(n-1)\,\Gamma\left(\frac{n}{2}+1\right)\right]^{-1} C[n+2],$$

which we shall employ here in the study of the behavior of $M[n]$.

4. **Generalized symmetrization.** A line $x =$ constant, $y \geq 0$, intersects L in m points $y_1(x) > y_2(x) > y_3(x) > \cdots > y_m(x) > 0$. The number of intersections m usually depends on x. Let us consider the line L_q defined by the equation

$$(7)\qquad y^q(x) = \sum_{k=1}^{m} (-1)^{k-1}\, y_k^q(x),$$

where q is a positive constant not necessarily an integer. The body of revolution $B_q[n]$ with section D_q defined by its profile L_q is said to be obtained by a symmetrization S_q. Let us note that S_{n-1} can be considered as a Schwarz symmetrization of $B[n]$. On the other hand, under S_1 the meridian profile of $B[n]$ undergoes a Steiner symmetrization. Our main results are embodied in the following theorems:

I. $V[n]$ does not increase under S_q for $0 < q \leq n-1$ and does not decrease under S_q for $q \geq n-1$. In particular, $V[n]$ remains invariant under S_{n-1}.

II. $C[n]$ does not increase under S_q for $0 < q \leq n-1$.

III. $M[n]$ does not increase under S_q for $n-1 \leq q \leq n+1$.

Let us observe that by (6) Theorem III follows immediately from I and II. In order to prove Theorems I and II we shall first establish some useful inequalities.

5. **Fundamental inequalities.** Let $y_1 > y_2 > \cdots > y_m > 0$ and let q and s be two positive numbers. We have then

$$(8)\qquad \left[\sum_{k=1}^{m} (-1)^{k-1}\, y_k^q\right]^{1/q} \leq \left[\sum_{k=1}^{m} (-1)^{k-1}\, y_k^{q+s}\right]^{1/(q+s)} \leq \left[\sum_{k=1}^{m} y_k^q\right]^{1/q}.$$

To prove the second inequality of (8) let us observe that it is sufficient to show that

$$(9)\qquad \left[\sum_{k=1}^{m} y_k^{q+s}\right]^{1/(q+s)} \leq \left[\sum_{k=1}^{m} y_k^{q}\right]^{1/q}.$$

Let us put $y_k^q = a_k$ and $(q+s)/q = r > 1$. Then we need only show that

$$(10)\qquad a_1^r + a_2^r + \cdots + a_m^r \leq (a_1 + a_2 + \cdots + a_m)^r.$$

But this is a classical inequality [5, p. 32]. As to the first part* of (8) we give here a proof communicated to us by H. F. Weinberger [7]. This inequality does not seem to be mentioned in the available literature. Using again the notations in (10) and putting

$$(11)\qquad F(a_1, a_2, \cdots, a_m) = \sum_{k=1}^{m} (-1)^{k-1} a_k^r - \left[\sum_{k=1}^{m} (-1)^{k-1} a_k\right]^r,$$

we have to prove that, for $a_1 > a_2 > \cdots > a_m > 0$ and $r > 1$,

$$(12)\qquad F(a_1, a_2, \cdots, a_m) \geq 0.$$

This inequality is obviously true for $m = 1$ and follows immediately if $m = 2$ from inequality (10). Let us therefore assume that (12) holds if we replace m by $m - 2$; this is equivalent to assuming the inequality

$$(13)\qquad F(a_2, a_2, a_3, \cdots, a_m) \geq 0.$$

We have also

$$(14)\qquad F(a_1, a_2, \cdots, a_m) = F(a_2, a_2, a_3, \cdots, a_m) + \int_{a_2}^{a_1} \frac{\partial F}{\partial a_1}\, da_1.$$

But from (11) we observe that

$$(15)\qquad \frac{1}{r}\frac{\partial F}{\partial a_1} = a_1^{r-1} - \left[\sum_{k=1}^{m} (-1)^{k-1} a_k\right]^{r-1},$$

which shows that $\partial F/\partial a_1$ is nonnegative. Since the same holds by assumption for $F(a_2, a_2, a_3, \cdots, a_m)$ we obtain at once the required inequality (12).

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*R. Bellman has pointed out that this inequality holds more generally with y^r replaced by an arbitrary continuous convex function $f(y)$ defined for $y \geq 0$.

6. **The effect of the generalized symmetrization on** $V[n]$. It follows immediately from (2) that

$$V[n] = \omega_{n-1} (n-1)^{-1} \int_{\alpha}^{\beta} \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1} \right] dx, \tag{16}$$

where the integral is taken over the interval (α, β) bounded by the greatest and smallest values of x on L. Let us apply the symmetrization S_q defined by (7). The volume $V_q[n]$ is then given by

$$V_q[n] = \omega_{n-1} (n-1)^{-1} \int_{\alpha}^{\beta} \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{q} \right]^{n-1/q} dx. \tag{17}$$

By (8) we see that for $q \leq n-1$ we have

$$V_q[n] \leq \omega_{n-1} (n-1)^{-1} \int_{\alpha}^{\beta} \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1} \right] dx = V[n]. \tag{18}$$

On the other hand for $q \geq n-1$ we have again by (8)

$$V_q[n] \geq \omega_{n-1} (n-1)^{-1} \int_{\alpha}^{\beta} \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1} \right] dx = V[n]. \tag{19}$$

The formulas (18) and (19) establish the proof of Theorem I of § 4.

7. **The effect of generalized symmetrization on** $C[n]$. In studying the behavior of $C[n]$ under the symmetrization S_q we shall generalize to a certain extent the procedure given by Pólya and Szegö for the Steiner symmetrization [1, p. 182]. Let us introduce a Cartesian system (x, y, z) and consider a surface $z(x, y)$ defined in a large half circle A enclosing D. We assume $z(x, y)$ to be a function positive throughout A and vanishing on the circular portion of its boundary. The particular function z which we shall consider will assume a constant positive value z_0 in the subdomain D of A. This value will be the maximum of $z(x, y)$ in A. We further assume that $z(x, y)$ is analytic outside D. The surface $z = z(x, y)$ except for its flat portion may also be defined as a surface $y = y(x, z)$ in a certain domain G of the (x, z)-plane. However, $y(x, z)$ may not be a single-valued function of x and z. For this reason we must consider as in [1] the surfaces $y_k(x, z)$ $(k = 1, 2, \cdots, m)$, where

$$y_1(x, z) > y_2(x, z) > \cdots > y_m(x, z) > 0.$$

These surfaces taken together with the flat portion constitute the surface $z(x, y)$.

Let us consider the integral

$$(20) \qquad I = \iint_G \sum_{k=1}^{m} y_k^{n-2}(x,z) \left[1 + \left(\frac{\partial y_k}{\partial x}\right)^2 + \left(\frac{\partial y_k}{\partial z}\right)^2\right]^{1/2} dx\,dz.$$

Let us first apply the symmetrization S_{n-1} by putting

$$(21) \qquad y_*^{n-1} = \sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1},$$

and consider the integral

$$(22) \qquad I_* = \iint_G y_*^{n-2} \left[1 + \left(\frac{\partial y_*}{\partial x}\right)^2 + \left(\frac{\partial y_*}{\partial z}\right)^2\right]^{1/2} dx\,dz.$$

We prove now that

$$(23) \qquad I \geq I_*.$$

In fact by substituting (21) into (22) and computing $\partial y_*/\partial x$ and $\partial y_*/\partial z$ we obtain the formula

$$(24) \quad I_* = \iint_G \left\{\left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1}\right]^{2(n-2)/(n-1)} + \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-2} \frac{\partial y_k}{\partial x}\right]^2 + \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-2} \frac{\partial y_k}{\partial z}\right]^2\right\}^{1/2} dx\,dz.$$

According to the inequality (8), I_* will not diminish if we replace the first square bracket in (24) by $[\sum_{k=1}^{m} y_k^{n-2}]^2$. Upon applying the Minkowski inequality we find that the integrand in I_* is not greater than the integrand in I; this proves formula (23).

Let us observe that

$$(25) \qquad I = \iint_A y^{n-2}\left[1 + z_x^2 + z_y^2\right]^{1/2} dx\,dy - \iint_D y^{n-2}\,dx\,dy,$$

the last integral being the contribution from the flat part of the z surface. We now insert into (25) the expression $z(x, y) = \epsilon\Phi(x, y)$, ϵ being a small positive number and Φ satisfying the same conditions as z. This substitution yields

$$(26) \qquad I = \iint_{A-D} y^{n-2}\, dx\, dy + (\epsilon^2/2) \iint_{A} y^{n-2} (\Phi_x^2 + \Phi_y^2)\, dx\, dy + O(\epsilon^4).$$

According to inequality (23), I does not increase under S_{n-1}. The first integral in (26) is obviously equal to the same integral taken over the symmetrized domain $A_* - D_*$, where $A_* = A$. By letting ϵ tend to zero we conclude in the usual way [1] that the integral

$$\iint_{A} y^{n-2} (\Phi_x^2 + \Phi_y^2)\, dx\, dy$$

does not increase under S_{n-1}. If we let the radius of the half circle bounding A tend to infinity we obtain the same statement for a function Φ which vanishes at infinity, providing that the integral converges. In particular if we take for Φ a function which is equal to unity in D and equal to the electrostatic potential $\phi[n]$ in E we find that $C[n]$ does not increase under S_{n-1}.

In order to prove that $C[n]$ does not increase under S_q for $0 < q \leq n-1$ let us observe that under S_q the line L bounding $D[n]$ goes into a line L_q which has by the inequalities (8) the following property: if $q_1 < q_2$ then the domain $D_{q_1}[n]$ bounded by L_{q_1} has no points outside the domain $D_{q_2}[n]$ bounded by L_{q_2}. We denote the capacities corresponding to these domains by $C_{q_1}[n]$ and $C_{q_2}[n]$, respectively. It is a well known property of the ordinary three-dimensional capacity that if one body contains another body the former has the larger capacity. The proof of this statement is based essentially on the variational definition of the capacity. The same property holds obviously for all values of n. We therefore have $C_{q_1}[n] \leq C_{q_2}[n]$. In particular $C_q[n] \leq C_{n-1}[n]$. As we have already proved $C_{n-1}[n] \leq C[n]$ we obtain the result

$$(27) \qquad C_q[n] \leq C[n], \qquad 0 < q \leq n-1,$$

which concludes the proof of Theorem II of § 4. As already mentioned in § 4, Theorem III follows immediately as a corollary of I and II.

8. **Steiner's Symmetrization of the meridian section with respect to the y-axis.** We shall consider briefly a symmetrization of the domain D with respect to the y-axis defined by the classical equation

(28) $$2x = \sum_{k=1}^{m} (-1)^{k-1} x_k .$$

In a manner similar to that used in §7 we find that $V[n]$ remains invariant and $C[n]$ and $M[n]$ do not increase under such a symmetrization.

9. **Concluding remarks.** All results of §4 can be extended to the case of two dimensional bodies which are symmetric with respect to the x-axis. It should be noted that these results hold for $C[2]$ as long as the radius of A remains finite. It has already been proven [1, 2] that $C[2]$ and $M[2]$ do not increase under S_1 and also that $C[3]$ and $M[3]$ do not increase under S_2. These cases are included in our Theorems II and III. We note also that formula (6) appears in an equivalent form for $n = 2$ and $n = 3$ in papers by G. I. Taylor [6] and M. Schiffer and G. Szegö [3], where $C[4]$ and $C[5]$ are (up to a constant factor) called dipole coefficients. No attempt was made in these papers to study the behavior of the dipole coefficients under symmetrization. However, it was recognized in [3] that they are increasing set functions, a fact which becomes almost obvious in our theory of generalized electrostatics (see §7). Finally let us remark that in §2 we have introduced the (x, y)-plane as the meridian plane of an n-dimensional space. But since all quantities are defined in terms of x and y, the index n appearing in our formulas need not be restricted to integral values. In fact it can easily be seen that all our formulas and results remain valid for all real positive values of n greater than two. For such values of n our results are mathematical statements about certain integrals such as $V[n]$, $C[n]$, and $M[n]$ which are associated with the generalized Stokes-Beltrami equations.

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THE BOUNDEDNESS OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION IN THE COMPLEX DOMAIN

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1. Introduction. Let $Q(z)$ be an analytic function of the complex variable z in a domain. In the following we shall be concerned with the differential equation

$$\frac{d^2W}{dz^2} + Q(z)\,W = 0. \tag{1}$$

Only those solutions $W(z)$ of (1) which are distinct from the trivial solution $(\equiv 0)$ shall be considered.

For a real-valued continuous solution $y(x) \not\equiv 0$ of the differential equation

$$\frac{d^2y}{dx^2} + f(x)\,y = 0, \tag{2}$$

where $f(x)$ is a real-valued piecewise continuous function of the real variable x for $0 \le x < \infty$, N. Levinson [1] has shown that the rapidity with which $y(x)$ can grow, and the rapidity with which it can tend to zero, both depend on the growth of $\alpha(x)$, where

$$\alpha(x) = \int_0^x |f(x) - a|\,dx, \tag{3}$$

and a is a real positive constant. More precisely, he showed that

$$y(x) = O\left(\exp\left[\frac{1}{2}\,a^{-1/2}\,\alpha(x)\right]\right), \tag{4}$$

and that if $\alpha(x) = O(x)$ as $x \longrightarrow \infty$, then

$$\limsup_{x\to\infty} |y(x)| \exp\left[\frac{1}{2}\,a^{-1/2}\,\alpha(x)\right] > 0. \tag{5}$$

If there exists a positive constant a such that $\alpha(x)$ converges as $x \longrightarrow \infty$, then

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from (4) it follows that every solution $y(x)$ of (2) is bounded as $x \longrightarrow \infty$. Levinson also showed that (4) and (5) are the best possible results of their types.

Along any line of the z-plane, for instance the real axis, the differential equation (1) has the form

$$\frac{d^2W}{dx^2} + Q(x)\, W = 0, \tag{6}$$

where x is real. Along a line, the growth of the solutions $W(x)$ of (6) also depends on the growth of a function similar to that in (3), and they also satisfy two relations like (4) and (5). These relations will be established in §2. From these results, we can obtain sufficient conditions for the boundedness of the solutions of (1) on a line, or on certain regions of the z-plane.

In §3 we shall investigate the asymptotic behavior of the solutions of (6) when they are bounded. In §6 we shall give a relation of the boundedness of the solutions of a self-adjoint differential equation of the third order and a differential equation of the second order.

2. Growth of the solutions along the real axis. We now consider equation (6) where x is real. Let $q_1(x)$ and $q_2(x)$ be, respectively, the real and imaginary parts of $Q(x)$. If

$$\phi(x) = \int_0^x [\,|a - q_1(x)| + |q_2(x)|\,]\, dx, \tag{7}$$

where a is a positive constant, then $\phi(x)$ determines not only how large a solution $W(x)$ of (6) can become, but also determines how small it can become. These results are contained in the following two theorems.

THEOREM 1. *If*

(a) $W(x)$ *is a solution of* (6),
(b) $\phi(x)$ *is defined as in* (7),

then

$$W(x) = O\left(\exp\left[\frac{1}{2}\; a^{-1/2}\; \phi(x)\right]\right),\quad W'(x) = O\left(\exp\left[\frac{1}{2}\; a^{-1/2}\; \phi(x)\right]\right). \tag{8}$$

An immediate consequence of this theorem is the following corollary.

COROLLARY 1.1. *Every solution $W(x)$ of the equation* (6) *and its derivative $W'(x)$ are bounded as $x \longrightarrow \infty$ provided there exists a positive constant a such that $\phi(x)$ converges as $x \longrightarrow \infty$.*

In Theorem 1 we cannot expect to replace $\phi(x)$ by a more symmetric form

$$\int_0^x [|a - q_1(x)| + |b - q_2(x)|]\, dx,$$

where $b \neq 0$ and is real, and $a > 0$. A counter-example is the differential equation

$$\frac{d^2W}{dx^2} + (1 + i)\, W = 0,$$

which has solutions unbounded as $x \longrightarrow \infty$.

THEOREM 2. *If*

(a) $W(x)$ *is a solution of* (6),

(b) $\phi(x) = O(x)$ *as* $x \longrightarrow \infty$, *where* $\phi(x)$ *is defined as in* (7),

then

$$\limsup_{x \to \infty} |W(x)| \exp\left[\frac{1}{2}\, a^{-1/2}\, \phi(x)\right] > 0. \tag{9}$$

Clearly $\limsup |W(x)| > 0$ as $x \longrightarrow \infty$ if $\phi(x)$ is convergent.

That (8) and (9) are the best possible results follows from the fact that (4) and (5) are the best possible results.

We shall now prove Theorem 1 and 2.

Proof of Theorem 1. Let the real and imaginary parts of a solution $W(x)$ of (6) be $u(x)$ and $v(x)$, respectively. Separating the real and imaginary part of (6), we obtain

$$u'' + q_1(x)\, u - q_2(x)\, v = 0, \tag{10}$$

$$v'' + q_2(x)\, u + q_1(x)\, v = 0. \tag{11}$$

Suppose $a > 0$, and let

$$H(x) = |W'(x)|^2 + a|W(x)|^2 = u'^2(x) + v'^2(x) + a[u^2(x) + v^2(x)]. \tag{12}$$

Then using (10) and (11), we have

$$\frac{dH}{dx} = 2(u'u'' + v'v'') + 2a(uu' + vv') \tag{13}$$

$$= 2[a - q_1(x)]\,(uu' + vv') + 2q_2(x)\,(u'v - uv').$$

Using the following inequalities,

$$2uu' \leq a^{-1/2}(au^2 + u'^2), \qquad 2vv' \leq a^{-1/2}(av^2 + v'^2),$$

$$2u'v \le a^{-1/2}(u'^2 + av^2), \qquad 2uv' \le a^{-1/2}(v'^2 + au^2),$$

and (13), we see that

$$\frac{dH}{dx} \le a^{-1/2}[|a - q_1(x)| + |q_2(x)|](u'^2 + v'^2 + au^2 + av^2) \tag{14}$$

$$= a^{-1/2}(|a - q_1(x)| + |q_2(x)|)\, H.$$

Since $H > 0$, we have

$$\frac{1}{H}\frac{dH}{dx} \le a^{-1/2}[|a - q_1(x)| + |q_2(x)|] \tag{15}$$

Integrating (15) from 0 to x, we obtain

$$H(x) \le H(0)\exp[a^{-1/2}\phi(x)]. \tag{16}$$

In view of the definition of $H(x)$, the expression in (6) is equivalent to the two in (8). This completes the proof of Theorem 1.

Proof of Theorem 2. In much the same way as in the proof of Theorem 1, it is easy to show that

$$\frac{1}{H}\frac{dH}{dx} \ge -a^{-1/2}[|a - q_1(x)| + |q_2(x)|]$$

Consequently, we have

$$H(x) = |W'(x)|^2 + a|W(x)|^2 \ge C\exp[-a^{-1/2}\phi(x)]. \tag{17}$$

For each positive integer n, let x_n, x_n', x_n'' be points in the interval $n \le x \le n+1$ such that

$$|W(x_n)| = \max |W(x)|, \quad |u'(x_n')| = \min |u'(x)|, \quad |v'(x_n'')| = \min |v'(x)|$$

in the interval $n \le x \le n+1$. Integrating (10) from x_n' to x_n and (11) from x_n'' to x_n, we obtain

$$u'(x_n) = u'(x_n') + \int_{x_n'}^{x_n}[-q_1(x)\,u(x) + q_2(x)\,v(x)]\,dx \tag{18}$$

$$\le |u'(x_n')| + |W(x_n)|\int_n^{n+1}(|q_1(x)| + |q_2(x)|)\,dx,$$

$$v'(x_n) = v'(x_n'') + \int_{x_n''}^{x_n}[-q_2(x)\,u(x) - q_1(x)\,v(x)]\,dx \tag{19}$$

$$\le |v'(x_n'')| + |W(x_n)|\int_n^{n+1}[|q_1(x)| + |q_2(x)|]\,dx.$$

Since

$$|W'(x_n)| \le |u'(x_n)| + |v'(x_n)|,$$

(18) and (19) yield

$$(20) \qquad |W'(x_n)| \le |u'(x_n')| + |v'(x_n'')| + 2|W(x_n)| \int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx.$$

Clearly either $|u'(x_n')| = 0$ or $u'(x)$ does not change sign in $n \le x \le n+1$. If $u'(x)$ does not change sign in $n \le x \le n+1$, we have

$$(21) \quad 2 \max_{n \le x \le n+1} |u(x)| \ge |u(n+1) - u(n)| = \left| \int_n^{n+1} u'(x)\, dx \right| \ge |u'(x_n')|.$$

Obviously (21) holds if $|u'(x_n')| = 0$. So (21) is always true. Hence

$$(22) \qquad 2|W(x_n)| \ge 2 \max_{n \le x \le n+1} |u(x)| \ge |u'(x_n')|.$$

Similarly,

$$(23) \qquad 2|W(x_n)| \ge |v'(x_n'')|.$$

Substitution of (22) and (23) into (20) yields

$$(24) \qquad |W'(x_n)| \le |W(x_n)| \left\{4 + 2 \int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx\right\}.$$

From (17) and (24), we obtain

$$(25) \qquad |W(x_n)|^2 \left\{\left(4 + 2 \int_n^{n+1} [|q_1(x)| + |q_2(x)|)\, dx\right]^2 + a\right\} \ge C \exp[-a^{-1/2} \phi(x_n)].$$

Since $\phi(x) = O(x)$ as $x \longrightarrow \infty$, it is easy to show that, for an infinite number ot n,

$$\int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx$$

is bounded. Thus for an infinite number of n, we have the inequality

$$(26) \qquad |W(x_n)|^2 \exp[a^{-1/2} \phi(x_n)] \ge C_1$$

for some positive constant C_1. Consequently (26) yields the result

$$\limsup_{x \to \infty} |W(x)| \exp\left[\frac{1}{2} a^{-1/2} \phi(x)\right] > 0.$$

This completes the proof of Theorem 2.

3. **Asymptotic behavior of the solutions.** If $\phi(x)$ converges as $x \longrightarrow \infty$, the solutions $W(x)$ of (6) are not only bounded, but also resemble the solutions of the differential equation

$$\frac{d^2 W}{dx^2} + aW = 0. \tag{27}$$

This result is proved in the following theorem.

THEOREM 3. *If*

(a) $W(x)$ *is a solution of* (6),

(b) $\phi(x)$, *defined as in* (7), *converges as* $x \longrightarrow \infty$,

then for some complex constants A *and* B,

$$\lim_{x \to \infty} [W(x) - (A \sin \sqrt{a}\, x + B \cos \sqrt{a}\, x)] = 0. \tag{28}$$

Proof of Theorem 3. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the equation (27) such that

$$y_1(0) = 0, \quad y_1'(0) = 1; \quad y_2(0) = 1, \quad y_2'(0) = 0. \tag{29}$$

Rewrite (6) in the form

$$\frac{d^2 W}{dx^2} + aW = [a - Q(x)]\, W. \tag{30}$$

Then a solution $W(x)$ of (30) can be expressed as

$$\begin{aligned} W(x) &= A\, y_1(x) + B\, y_2(x) \\ &\quad + \int_x^\infty [a - Q(t)]\, W(t)\, [y_2(x)\, y_1(t) - y_1(x)\, y_2(t)]\, dt \end{aligned} \tag{31}$$

for some complex constants A and B, where the integral is convergent since $\phi(x)$ is convergent, $W(x)$ is bounded, and

$$y_1(x) = a^{-1/2} \sin \sqrt{a}\, x, \quad y_2(x) = \cos \sqrt{a}\, x;$$

(31) can be obtained by the method of variation of constants. Hence the absolute value of the integral in (31) can be arbitrarily small if x is large enough. In other

words,

$$\lim_{x \to \infty} \{ W(x) - [A\ y_1(x) + B\ y_2(x)] \} = 0.$$

This completes the proof.

Differentiating (31) clearly yields

$$\lim_{x \to \infty} \{ W'(x) - [A\ y_1'(x) + B\ y_2'(x)] \} = 0.$$

4. **Boundedness of the solutions in certain regions.** In this section we shall apply the results of Theorem 1 to obtain sufficient conditions for the boundedness of the solutions of the equation (1) in certain regions of the z-plane.

Let R be the region

$$z = x + iy, \quad 0 \le x < \infty, \quad \alpha \le y \le \beta. \tag{32}$$

On a half line $L(y_0)$, $z = x + iy_0$, in R, the differential equation (1) becomes

$$\frac{d^2 W}{dx^2} + Q(x + iy_0)\, W = 0. \tag{33}$$

Denoting the real and imaginary part of $Q(x + iy_0)$ by $q_1(x, y_0)$ and $q_2(x, y_0)$, respectively, we see that according to Theorem 1, the growth of a solution $W(x)$ of (1) on $L(y_0)$ depends on the growth of

$$\phi(x, y_0) = \int_0^x [\,|a - q_1(x, y_0)| + |q_2(x, y_0)|\,]\, dx, \tag{34}$$

where a is a positive constant. If $\phi(x, y_0)$ is convergent for some positive constant a, then $W(z)$ and $W'(z)$ are bounded on $L(y_0)$, and

$$\lim_{x \to \infty} [W(x + iy_0) - (A \sin \sqrt{a}\, x + B \cos \sqrt{a}\, x)] = 0$$

for some complex constants A and B. Let

$$\Phi(x, y_0) = \int_0^x |a - Q(x + iy_0)|\, dx. \tag{35}$$

Clearly the convergence of $\Phi(x, y_0)$ implies the convergence of $\phi(x, y_0)$. Let $\Phi(x, y_0)$ be uniformly bounded in R in the sense that for each y_0 ($\alpha \le y_0 \le \beta$), there exists a positive constant a such that sup a is finite and inf a is positive, and $\Phi(x, y_0) \le M$, M being some constant, for all x in $0 \le x < \infty$ and all y_0 in $\alpha \le y \le \beta$; then by applying (16) on each $L(y_0)$, it is easy to see that $W(z)$ and $W'(z)$ are bounded in R. If the condition that sup a is finite is removed, clearly we still have $W(z)$ bounded in R. This proves the following theorem.

THEOREM 4. *If*

(a) *R is a region defined as in* (32),
(b) *$\Phi(x, y_0)$, defined as in* (35), *is uniformly bounded in R in the sense defined above,*

then each solution $W(z)$ of (1) *and its derivative $W'(z)$ are bounded in R.*

Consider another region R,

$$z = x + re^{i\theta_0}, \qquad 0 \leq r < \infty, \ \alpha \leq x \leq \beta, \tag{36}$$

where θ_0 is a real constant. On a half line $L(x_0)$, $z = x_0 + r \exp(i\theta_0)$, in R, the equation (1) reduces to

$$\frac{d^2 W}{dr^2} + P(r, x_0)\, W = 0, \tag{37}$$

where $P(r, x_0) = Q[x_0 + r \exp(i\theta_0)] \exp(2i\theta_0)$.

THEOREM 5. *If*

(a) *R is a region defined as in* (36),
(b) *for each x_0, $\alpha \leq x_0 \leq \beta$, there exists a positive constant a such that* sup *a is finite and* inf *a is positive and*

$$\int_0^r |a - P(r, x_0)|\, dr \leq M,$$

M being some constant, for all r in $0 \leq r < \infty$ and all x_0 in $\alpha \leq x \leq \beta$,

then each solution $W(z)$ of (1) *and its derivative $W'(z)$ are bounded in R.*

The proofs of this theorem and of the following Theorem 6 are similar to that of Theorem 4.

Denote by S the sector

$$z = re^{i\theta}, \qquad 0 \leq r < \infty, \ \alpha \leq \theta \leq \beta. \tag{38}$$

On a fixed ray $\theta = \theta_0$ in S, equation (1) reduces to

$$\frac{d^2 W}{dr^2} + T(r, \theta_0)\, W = 0, \tag{39}$$

where $T(r, \theta_0) = Q(r \exp(i\theta_0)) \exp(2i\theta_0)$. We have the following result.

THEOREM 6. *If*

(a) S *is a region defined as in* (38),

(b) *for each* θ_0, $\alpha \le \theta_0 \le \beta$, *there exists a positive constant* a *such that* $\sup a$ *is finite and* $\inf a$ *is positive and*

$$\int_0^r |a - T(r, \theta_0)| \, dr \le M,$$

M *being some constant, for all* r *in* $0 \le r < \infty$ *and all* θ_0 *in* $\alpha \le \theta \le \beta$,

then each solution $W(z)$ *of* (1) *and its derivative* $W'(z)$ *are bounded in* S.

5. **Extension.** Let C be an analytic curve [2, p. 702]

$$x = f(t), \; y = g(t), \tag{40}$$

where t is real. Along C the equation (1) has the form

$$\frac{d^2 W}{dt^2} + A(t) \frac{dW}{dt} + B(t) W = 0. \tag{41}$$

It is well known that equation (41) can be reduced to the form of (6). It follows that our results apply to the solutions along a line or in regions bounded by lines as well as to the solutions along an analytic curve or in regions bounded by analytic curves.

6. **A self-adjoint differential equation of the third order.** Let $Y(z)$ be a solution of the self-adjoint differential equation

$$\frac{d^2 Y}{dz^3} + Q(z) \frac{dY}{dz} + \frac{1}{2} \frac{dQ(z)}{dz} Y = 0, \tag{42}$$

where $Q(z)$ is analytic in a region R. Let $W(z)$ be a solution of

$$\frac{d^2 W}{dz^2} + \frac{1}{4} Q(z) W = 0. \tag{43}$$

In Theorem 7 we shall prove that every solution $Y(z)$ of (42) is bounded in R if and only if every solution $W(z)$ of (43) is bounded in R. In fact the growth of the solutions of (43) determines and is determined by the growth of the solutions of (42).

THEOREM 7. *Every solution* $Y(z)$ *of* (42) *is bounded in* R *if and only if every solution* $W(z)$ *of* (43) *is bounded in* R.

Proof. Let $W_1(z)$ and $W_2(z)$ be any two linearly independent solutions of

(43). The theorem follows from the fact that $W_1^2(z)$, $W_1(z)\,W_2(z)$ and $W_2^2(z)$ are three linearly independent solutions of (42). That $W_1^2(z)$, $W_1(z)\,W_2(z)$ and $W_2^2(z)$ are solutions of (42) can be verified by substitution. We now show that they are linearly independent. If A, B, C are constants, and if

$$AW_1^2(z) + BW_1(z)\,W_2(z) + CW_2^2(z) \equiv 0, \tag{44}$$

then by factoring (44) we get

$$[AW_1(z) + bW_2(z)]\,[cW_1(z) + dW_2(z)] \equiv 0, \tag{45}$$

where a, b, c and d depend on A, B and C. Hence at least on the factors in (45) is identically zero. It follows that either $a = b = 0$ or $c = d = 0$. Consequently $A = B = C = 0$. This completes the proof.

7. Added in proof. With the aid of the Phragmén-Lindelöf theorems [see 3], the results of §4 can be greatly improved.

For example, let R be the region defined as in (32), with $\beta - \alpha = \pi h^{-1}$. Let there be a positive constant a such that as $x \longrightarrow \infty$,

$$\phi(x, y) = O(e^{kx}), \tag{46}$$

where $k < h$, uniformly for y in $\alpha \leq \beta$, and that

$$\phi(x, \alpha) = O(1), \quad \phi(x, \beta) = O(1). \tag{47}$$

Then, by Corollary 1.1, any solution $W(z)$ of (1) is bounded on $L(\alpha)$ and on $L(\beta)$, and so is bounded on these lines and on the segment $x = 0$ in R. From (46) and Theorem 1, we have

$$W(z) = O(e^{Me^{kx}})$$

uniformly in y, where M is some positive constant. By a theorem of Phragmén-Lindelöf, $W(z)$ is then bounded in R. Similarly $W'(z)$ is bounded in R.

Using Theorem 3, from (47), we see that

$$W(z) - (A_1 \sin a^{1/2} z + B_1 \cos a^{1/2} z) \tag{48}$$

tends to zero as $z \longrightarrow \infty$ on $L(\alpha)$ for some constants A_1 and B_1. Similarly (48) tends to zero on $L(\beta)$ if A_1 and B_1 are replaced, respectively, by some constants A_2 and B_2. Write

$$F_i(z) = A_i \sin a^{1/2} z + B_i \cos a^{1/2} z, \qquad (i = 1, 2).$$

Then

$$[W(z) - F_1(z)][W(z) - F_2(z)]$$

tends to zero as $z \longrightarrow \infty$ on $L(\alpha)$ and on $L(\beta)$; and since it is bounded in R, by another theorem of Phragmén-Lindelöf, it tends uniformly to zero as $z \longrightarrow \infty$. Thus to any ϵ there corresponds a segment $z = x_0 + iy$ in R on which

$$(49) \qquad |W(z) - F_1(z)|\ |W(z) - F_2(z)| \leq \epsilon.$$

At every point of this segment either

$$|W(z) - F_1(z)| \leq \epsilon^{1/2} \text{ or } |W(z) - F_2(z)| \leq \epsilon^{1/2} \text{ (or both)},$$

and we may suppose that the former inequality holds at $y = \alpha$, the latter at $y = \beta$; let y_0 be the upper bound of values of y for which the former holds; then y_0 is either a point where the latter holds, or a limit of such points; hence, since both factors on the left side of (49) are continuous, both inequalities hold at y_0. At $z = x_0 + iy_0$, we then have

$$(50) \qquad |F_1(z) - F_2(z)| \leq |W(z) - F_1(z)| + |W(z) - F_2(z)| \leq 2\epsilon^{1/2}.$$

On the other hand, (49) holds on every segment $z = x_1 + iy$ if x_1 is large enough, and there is a point $z = x_1 + iy_1$ at which (50) holds. Consider an arbitary segment $z = x_2 + iy$. Since $F_1(z) - F_2(z)$ is a periodic function in x, there is a point on this segment at which (50) holds. But $F_1(z) - F_2(z)$ is continuous and ϵ is arbitary, so that $F_1(z) - F_2(z) = 0$ at some point on this segment, and therefore on every segment. If these points have a limit-point inside R, then $F_1(z) = F_2(z)$ in R; otherwise there is a segment on $y = \alpha$ or $y = \beta$ in which $F_1(z) - F_2(z) = 0$, then $A_1 = A_2$, $B_1 = B_2$, and hence $F_1(z) = F_2(z)$ in R. Thus as $z \longrightarrow \infty$ the function (48) tends to zero on $L(\alpha)$ and on $L(\beta)$, and since it is bounded in R, by a theorem of Phragmén-Lindelöf, it tends to zero uniformly in $\alpha \leq y \leq \beta$.

Similarly, as $z \longrightarrow \infty$, we see that

$$W'(z) - a^{1/2}(A_1 \cos a^{1/2} z - B_1 \sin a^{1/2} z)$$

tends to zero uniformly in $\alpha \leq y \leq \beta$.

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ERRATA

J. L. Brenner, *Matrices of quaternions*,
Vol. I, p. 329 line 18, read "*sfield*" for "*fields*".

P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*,
Vol. II, p. 71 line 11, read $A \geq 3^{1/2}\ \alpha^2/2$ for $A \geq 3^1/_2\ \alpha^2/_2$.

Everett Larguier, *Homology bases with applications to local connectedness*,
Vol. II, p. 193 line 15, read $1/k$ for i/k.
Vol. II, p. 198 line 7, read $x \in S$ for $x \quad S$.

Lars V. Ahlfors, *Remarks on the Neumann-Poincaré integral equation*,
Vol. II, p. 271 last line should read *Pacific J. Math.* 2 (1952), 271-280.

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