

Pacific Journal of Mathematics

ON A PAPER OF NIVEN AND ZUCKERMAN

J. W. S. CASSELS

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1. **Introduction.** Let 'digit' mean an integer in the range $0 \leq a < 10$. For digits $a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s$ ($s \geq r$) and integer m , denote by

$$R_m(a_1, \dots, a_r, b_1, \dots, b_s)$$

the number of solutions of

$$b_n = a_1, b_{n+1} = a_2, \dots, b_{n+r-1} = a_r \quad (0 < n < n+r \leq s; n \equiv m \pmod{r}),$$

so that

$$(1) \quad 0 \leq R_m(a_1, \dots, a_r; b_1, \dots, b_s) \leq s - r + 1.$$

Suppose that

$$x_1, x_2, \dots$$

is an infinite sequence of digits. It has been shown [2] that if

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^r R_m(a_1, \dots, a_r; x_1, \dots, x_N) = 10^{-r}$$

for all integers r and digits a_1, \dots, a_r , then

$$(3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} R_m(a_1, \dots, a_r; x_1, \dots, x_N) = r^{-1} 10^{-r}$$

for all integers r, m , and digits a_1, \dots, a_r . A possibly simpler proof is as follows.

2. **Proof.** Let $\epsilon > 0$ and digits a_1, \dots, a_r be given. The simple argument of Hardy-Wright [1] shows that if the integer s is fixed large enough, then

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$$(4) \quad \text{Max}_{\mu} \left| R_{\mu}(a_1, \dots, a_r; b_1, \dots, b_s) - \frac{s-r+1}{r 10^r} \right| < \epsilon(s-r+1)$$

except for at most $\epsilon 10^s$ sets of digits b_1, \dots, b_s . ('Exceptional' sets.) Thus, by (2) with b_1, \dots, b_s for a_1, \dots, a_r , the number of exceptional sets

$$(5) \quad x_t, x_{t+1}, \dots, x_{t+s-1} \quad (1 \leq t \leq N-s+1)$$

is at most $2\epsilon N$ for all large enough N .

On the other hand,

$$(6) \quad (s-r+1) R_m(a_1, \dots, a_r; x_1, \dots, x_N)$$

differs from

$$(7) \quad \sum_{t=1}^{N-s+1} R_{m-t+1}(a_1, \dots, a_r; x_t, \dots, x_{t+s-1})$$

by at most $2s^2$, since each solution of

$$a_1 = x_n, a_2 = x_{n+1}, \dots, a_r = x_{n+r-1} \quad (s \leq n \leq N-s; n \equiv m \pmod r)$$

contributes exactly $s-r+1$ both to (6) and to (7). Hence, using the estimate (3) for the at most $2\epsilon N$ exceptional sets (5), and the estimate (4) for the others, we have

$$\begin{aligned} & \left| R_m(a_1, \dots, a_r; x_1, \dots, x_N) - \frac{N-s+1}{r 10^r} \right| \\ & \leq \frac{2s^2}{s-r+1} + \epsilon(N-s+1) + 2\epsilon N, \end{aligned}$$

and so

$$\limsup_N \left| \frac{1}{N} R_m(a_1, \dots, a_r; x_1, \dots, x_N) - r^{-1} 10^{-r} \right| \leq 3\epsilon.$$

Since ϵ is arbitrarily small, this proves (3) as required.

REFERENCES

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, First Edition, Oxford, 1938, §9.13.
2. I. Niven and H. S. Zuckerman, *On the definition of normal numbers*, Pacific J. Math. **1** (1951), 103-110.

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