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VOLUME IN TERMS OF CONCURRENT CROSS-SECTIONS

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1. Of the two expressions

$$|M| = \frac{1}{2} \int_0^{2\pi} r^2(\omega) d\omega = \frac{1}{2} \int_0^{2\pi} \left(\int_{-r(\omega-\pi/2)}^{r(\omega+\pi/2)} |\rho| d\rho \right) d\omega$$

for the area $|M|$ of a plane domain M , given in polar coordinates ρ, ω by the inequalities $0 \leq \rho \leq r(\omega)$, $0 \leq \omega \leq 2\pi$, the first has the well-known extension

$$(1) \quad |M| = \frac{1}{n} \int_{\Omega_n} r^n(u) d\omega_u^n$$

to n dimensions. Here Ω_n is the surface of the unit sphere in the n -dimensional Euclidean space, $d\omega_u^n$ is its area element at the point u , and M is given by $0 \leq \rho \leq r(u)$, $u \in \Omega_n$.

In the second expression, $|\rho|$ may be interpreted as (1-dimensional) volume of the simplex with one vertex at the origin z and the other at a variable point $p = (\rho, \omega \pm \pi/2)$ in the cross-section of M with the line normal to ω . The purpose of the present note is *the proof and the application of the following extension of this second expression to $n - 1$ sets M_1, \dots, M_{n-1} in E_n :*

$$(2) \quad |M_1| \cdots |M_{n-1}| \\ = \frac{(n-1)!}{2} \int_{\Omega_n} \left(\int_{M_1(u)} \cdots \int_{M_{n-1}(u)} T(p_1, \dots, p_{n-1}, z) dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1} \right) d\omega_u^n.$$

Here $M_j(u)$ is the cross-section of M_j with the hyperplane $H(u)$ through z normal to the unit vector u , the point p_j varies in $M_j(u)$, the differential $dV_{p_j}^{n-1}$ is the $((n-1)$ -dimensional) volume element of $M_j(u)$ at p_j , and $T(p_1, \dots, p_{n-1}, z)$ is the volume of the simplex with vertices p_1, \dots, p_{n-1}, z .

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Replacing the sets $M_{n-r+1}, \dots, M_{n-1}$ by the unit sphere U_n with center z yields expressions for $|M_1| \dots |M_{n-r}|$ in terms of the volume $T(p_1, \dots, p_{n-r}, z)$, in particular (1) for $r = n - 1$.

With the notation

$$\kappa_n = |U_n| = \pi^{n/2} / \Gamma\left(\frac{n}{2} + 1\right),$$

Steiner's symmetrization leads from (2) to the following result:

If M_1, \dots, M_{n-1} are convex bodies in E_n ($n \geq 3$) with interior points, z is a given point in E_n , and $M_j(u)$ the cross-section of M_j with the plane normal to u and through z , then

$$(3) \quad |M_1| \dots |M_{n-1}| \geq \frac{1}{n} \frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{\Omega_n} |M_1(u)|^{n/(n-1)} \dots |M_{n-1}(u)|^{n/(n-1)} d\omega_u^n,$$

and the equality sign holds only when the M_j are homothetic ellipsoids with center z .

It follows in particular for a convex body M that, for $n \geq 3$,

$$(4) \quad |M|^{n-1} \geq \frac{1}{n} \frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{\Omega_n} |M(u)|^n d\omega_u^n,$$

with the equality (if $|M| > 0$) only for ellipsoids with center z . The efforts to prove this inequality, which has applications in Finsler spaces, led to the present investigation. The—because of Jensen's inequality—weaker estimate

$$(5) \quad |M| \geq \frac{1}{n} \frac{\kappa_n^{-n/(n-1)}}{\kappa_{n-1}^n} \int_{\Omega_n} |M(u)|^{n/(n-1)} d\omega_u^n,$$

with equality sign (if $|M| > 0$) only for the spheres with center z , was found previously by L. A. Santaló who communicated it to the author. It is also the special case $M_{n-1} = M$, $M_j = U_n$ for $j < n - 1$, of (3).

2. Let M_1, \dots, M_{n-1} be bounded Jordan measurable sets in E_n , $n > 3$, such that the intersection of M_j with any ν -dimensional linear subspace (which in the future will be indicated by L_ν) through a fixed given point z possesses a

ν -dimensional Jordan measure. Since the subscripts run sometimes from 1 to n and other times from 1 to $n - 1$, we agree to use α, β for the former type, and j, k for the latter, and may then omit mentioning the range.

Let x_α be rectangular coordinates in E_n with z as origin. Take $n - 1$ copies E_n^j of E_n with coordinates x_α^j , and let M_j' be the image of M_j in E_n^j ; that is, $x^j \in M_j'$ if and only if the point x with $x_\alpha = x_\alpha^j$ lies in M_j . Then x_α^j may be considered as rectangular coordinates in the product space

$$E = E_n' \times \dots \times E_n^{n-1} = \prod E_n^j ;$$

hence

$$(6) \quad \prod |M_j| = \prod |M_j'| = \int_{\prod M_j'} dx_1^1 \dots dx_n^1 \dots dx_1^{n-1} \dots dx_n^{n-1}.$$

In E we introduce new coordinates

$$\bar{x}_1^1, \dots, \bar{x}_{n-1}^1, v_1, \dots, \bar{x}_1^{n-1}, \dots, \bar{x}_{n-1}^{n-1}, v_{n-1}$$

through the relations

$$(7) \quad x_k^j = \bar{x}_k^j, x_n^j = v_1 \bar{x}_1^j + \dots + v_{n-1} \bar{x}_{n-1}^j.$$

These equations fail to define v_j if $|x_k^j| = |\bar{x}_k^j| = 0$, i.e. if the points x^j in E_n are contained in an L_ν with $\nu < n - 2$, or if the L_{n-2} spanned by the x^j is parallel to the x_n -axis. The geometric meaning of the right side of (2) shows that a special discussion of this case is superfluous.

To evaluate $\prod |(M_j')|$ in the new coordinates, observe that the first n rows in the $n(n - 1)$ -rowed Jacobian J of the transformation (7) are in blocks of $n \times n$ matrices:

$$\begin{matrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ v_1 & v_2 & \dots & v_{n-1} & \bar{x}_1^1, & v_1 & v_2 & \dots & v_{n-1} & \bar{x}_2^1, & v_1 & v_2 & \dots & v_{n-1} & \bar{x}_{n-1}^1; \end{matrix}$$

hence

$$(8) \quad J = |\overline{x_k^j}|.$$

The unit normal u in E_n to the plane

$$x_n = x_1 v_1 + \dots + x_{n-1} v_{n-1}$$

is, with $w = (1 + v_1^2 + \dots + v_{n-1}^2)^{1/2}$, either

$$u_j = v_j w^{-1/2}, \quad u_n = -w^{-1/2} \quad \text{or} \quad u_j = -v_j w^{-1/2}, \quad u_n = w^{-1/2}.$$

Then $w^{-1} = |\cos \theta|$, where θ is the angle between u and the x_n -axis, so that $d\omega_u^n = w du_1 \dots du_{n-1}$ is the area element of Ω_n . Here we disregard again planes parallel to the x_n -axis. Now

$$\left| \frac{\partial u_j}{\partial v_k} \right| = w^{-3(n-1)} \begin{vmatrix} w^2 - v_1^2 & -v_1 v_2 & \dots & -v_1 v_{n-1} \\ -v_2 v_1 & w^2 - v_2^2 & \dots & -v_2 v_{n-1} \\ \cdot & \cdot & \dots & \cdot \\ -v_{n-1} v_1 & -v_{n-1} v_2 & \dots & w^2 - v_{n-1}^2 \end{vmatrix}$$

Since all principal minors of the determinant $|-v_j v_k|$ of order greater than 1 vanish, it follows (compare [4, pp. 125, 126]) that

$$\left| \frac{\partial u_j}{\partial v_k} \right| = w^{-3(n-1)} (w^{2(n-1)} - w^{2(n-2)} \sum v_j^2) = w^{-n-1}$$

and

$$(9) \quad d\omega_u^n = w^{1-n-1} dv_1 \dots dv_{n-1} = |\cos^n \theta| dv_1 \dots dv_{n-1}.$$

The volume element $dV_{x_i}^{n-1}$ of the hyperplane $x_n^j = \sum x_k^j v_k$ is

$$(10) \quad dV_{x_i}^{n-1} = dx_1^j \dots dx_{n-1}^j |\sec \theta|.$$

If we now interpret the points x^1, \dots, x^{n-1} as lying in the same E_n , then (8) shows that $|J|/(n-1)!$ is the volume of the projection of the simplex with vertices x^1, \dots, x^{n-1}, z on the plane $x_n = 0$. Since these points determine a hyperplane $H(u)$ with normal u through z ,

$$(11) \quad (n - 1)! T(x^1, \dots, x^{n-1}, z) = |J \sec \theta|.$$

Replacing x^j by p_j , we briefly summarize the results (9), (10), (11) as

$$(12) \quad dV_{p_1}^n \dots dV_{p_{n-1}}^n = (n - 1)! T(p_1, \dots, p_{n-1}, z) dV_{p_1}^{n-1} \dots dV_{p_{n-1}}^{n-1} d\omega_u^n.$$

After observing that in (2) by integrating over Ω_n every $M^j(u)$ is counted twice (once for u and once for $-u$), we see that *the relation (2) follows from (12)*.

For brevity we introduce, for sets M_1, \dots, M_r in E_s with $r \leq s$, the notation

$$\tau_s(M_1, \dots, M_r, z) = \int_{M_1} \dots \int_{M_r} T(p_1, \dots, p_r, z) dV_{p_1}^s \dots dV_{p_s}^s,$$

and may then write (2) in the form

$$(13) \quad |M_1| \dots |M_{n-1}| = \frac{(n - 1)!}{2} \int_{\Omega_n} \tau_{n-1}(M_1(u), \dots, M_{n-1}(u), z) d\omega_u^n.$$

3. In order to obtain expressions for $|M_1| \dots |M_r|$ with $r < n - 1$, we replace successively $M_{n-1}, \dots, M_{n-r+1}$ by the unit sphere U_n . The contribution of the latter sets to the right side of (13) can then be integrated out by using the following fact:

Let an L_μ , $0 < \mu < n - 1$, through the center z of the unit sphere U_{n-1} in E_{n-1} , intersect U_{n-1} in U_μ . For any point q in U_{n-1} , denote by r the distance qz , and by ϕ the angle between the ray qz and the L_μ . Then

$$(14) \quad \int_{U_{n-1}} r |\sin \phi| dV_q^{n-1} = \frac{\omega_{\nu-1}}{\omega_\nu} \cdot \kappa_\nu, \quad \nu = n - \mu,$$

where $\omega_\nu = \nu \cdot \kappa_\nu = 2\pi^{\nu/2} \Gamma^{-1}(\nu/2)$ is the area of the surface Ω_ν of U_ν , in particular $\omega_1 = 2$.

To prove (14), let the $L_{\nu-1}$ normal to the L_μ through q intersect U_μ in p , and U_{n-1} in the sphere $S_{\nu-1}$. If $\rho = pz$ then $S_{\nu-1}$ has radius $\sigma = (1 - \rho^2)^{1/2}$. Then $s = pq = r |\sin \phi|$; hence

$$\int_{U_{n-1}} r |\sin \phi| dV_q^{n-1} = \int_{U_\mu} \left(\int_{S_{\nu-1}} s dV_q^{\nu-1} \right) dV_p^\mu,$$

If $d\omega_{\frac{\nu-1}{q}}$ denotes the area element of the $\Omega_{\nu-1}$ with center p in the $L_{\nu-1}$ at the

point \bar{q} of the ray pq , then

$$s_{\nu-1} \int s dV_q^{\nu-1} = \Omega_{\nu-1} \int_0^\sigma s s^{\nu-2} ds d\omega_{\bar{q}}^{\nu-1} = \frac{\sigma^\nu}{\nu} \omega_{\nu-1} = (1-\rho^2)^{\nu/2} \frac{\omega_{\nu-1}}{\nu}.$$

Therefore, with a similar notation,

$$\begin{aligned} U_{n-1} \int r |\sin \phi| dV_q^{\nu-1} &= \frac{\omega_{\nu-1}}{\nu} \int_{\Omega_\mu} \int_0^1 (1-\rho^2)^{\nu/2} \rho^{\mu-1} d\rho d\omega_{\bar{p}}^\mu \\ &= \frac{\omega_{\nu-1} \omega_\mu}{\nu} \frac{\Gamma(\mu/2) \Gamma(\nu/2+1)}{2\Gamma(\mu/2+\nu/2+1)} = \frac{\omega_{\nu-1} \omega_\mu}{\nu} \frac{\Gamma(\mu/2)\nu/2 \Gamma(\nu/2)}{2\Gamma(n/2+1)} = \frac{\omega_{\nu-1}}{\omega_\nu} \kappa_n. \end{aligned}$$

Returning to (13), we replace M_{n-1} by U_n . Then $M_{n-1}(u)$ becomes the $(n-1)$ -dimensional unit sphere $U_n(u)$ in the hyperplane with normal u . If ϕ is the angle between the ray zp_{n-1} and the L_{n-2} spanned by p_1, \dots, p_{n-2}, z , then, with $r = zp_{n-1}$,

$$T(p_1, \dots, p_{n-1}, z) = (n-1)^{-1} r |\sin \phi| T(p_1, \dots, p_{n-2}, z).$$

Hence, carrying out the integration over $U_n(u)$, by (14) we obtain

$$\begin{aligned} |M_1| \cdots |M_{n-2}| \cdot \kappa_n \\ = \frac{1}{2} (n-2)! \frac{\omega_1}{\omega_2} \kappa_n \int_{\Omega_n} \tau_{n-1}(M_1(u), \dots, M_{n-2}(u), z) d\omega_u^n \end{aligned}$$

or

$$\begin{aligned} (15) \quad |M_1| \cdots |M_{n-2}| \\ = \frac{(n-2)!}{2\pi} \int_{\Omega_n} \tau_{n-1}(M_1(u), \dots, M_{n-2}(u), z) d\omega_u^n. \end{aligned}$$

If now M_{n-2} is replaced by U_n , then because of (14) the factor

$$\frac{1}{n-2} \frac{\omega_2}{\omega_3} \kappa_n$$

is introduced on the right. Continuing in this manner leads to the *general relation*

$$(16) \quad |M_1| \cdots |M_{n-r}| = \frac{(n-r)!}{\omega_r} \int_{\Omega_n} \tau_{n-1}(M_1(u), \dots, M_{n-r}(u), z) d\omega_u^n.$$

The integrand occurs in many $H(u)$, and it would be more natural to replace the integration over Ω_n by an integration over all L_{n-r} . The results of integral geometry [5] lead to such a reduction for general r ; however, we restrict our attention here to the two simplest cases, where no new formulas of the type (12) are required.

It is clear that the last formula in the sequence (16),

$$|M_1| = \frac{1}{\omega_n} \int_{\Omega_n} \tau_{n-1}(M_1(u), z) d\omega_u^n,$$

must be essentially identical with (1). Indeed, if M_1 can be represented in the form $0 \leq \rho \leq r(u)$, $u \in \Omega_n$, and we write the induced representation of $M_1(u)$ in the form

$$0 \leq \rho \leq r(v), \quad v \in \Omega_{n-2}(u) = H(u) \cap \Omega_n,$$

then, with $pz = \rho$, we have

$$\begin{aligned} \tau(M_1(u), z) &= \int_{M_1(u)} \rho dV_p^{n-1} = \int_{\Omega_{n-1}(u)} \int_0^{r(v)} \rho \rho^{n-2} d\omega_v^{n-1} \\ &= \frac{1}{n} \int_{\Omega_{n-1}(u)} r^n(v) d\omega_v^{n-1} \end{aligned}$$

and

$$|M_1| = \frac{1}{n \cdot \omega_{n-2}} \int_{\Omega_n} \left(\int_{\Omega_{n-1}(u)} r^n(v) d\omega_v^{n-2} \right) d\omega_u^n.$$

Now according to the results on cinematic measure on the sphere (see [5], for $n = 3$ already [3]), integrating over the v -normal to u first, and then over u , leads to the same result as integrating over the $H(w)$ that contain v , that is, those for which w is normal to v , and then over v . The first of the latter two integrations yields $\omega_{n-2} r^n(v)$, and (1) follows.

As second example, we indicate briefly the reduction of (15). Denoting by L_{n-2}^P the L_{n-2} spanned by p_1, \dots, p_{n-2}, z , and by M_i^P the intersection of M_i with L_{n-2}^P , we obtain from (12) that if L_{n-2}^P lies in $H(u)$ and has there the normal v , then

$$\begin{aligned} & \tau_{n-1}(M_1(u), \dots, M_{n-2}(u), z) \\ &= \int_{M_1(u)} \dots \int_{M_{n-2}(u)} T(p_1, \dots, p_{n-2}, z) dV_{P_1}^{n-1} \dots dV_{P_{n-2}}^{n-1} \\ &= \frac{(n-2)!}{2} \int_{\Omega_{n-1}(u)} \int_{M_1^P} \dots \int_{M_{n-2}^P} T^2(p_1, \dots, p_{n-2}, z) dV_{P_1}^{n-2} \dots dV_{P_{n-2}}^{n-2} d\omega_v^{n-1}. \end{aligned}$$

Substituting this in (15) leads besides the integrations over the M_i^P to integrations over $\Omega_{n-1}(u)$ and Ω_n . Similarly as in the preceding case, these latter two may be reduced to one integration by using the cinematic measure dL_{n-2}^P on Ω_n of the Ω_{n-2} in which L_{n-2}^P intersects Ω_n (compare [5]). The result (given without verification because it will not be used) is

$$(17) \quad |M_1| \dots |M_{n-2}| \\ = \frac{[(n-2)!]^2}{2} \int_{\Omega_n} \int_{M_1^P} \dots \int_{M_{n-2}^P} T^2(p_1, \dots, p_{n-2}, z) dV_{P_1}^{n-2} \dots dV_{P_{n-2}}^{n-2} dL_{n-2}^P.$$

4. To obtain the estimate (3) we use Steiner's symmetrization in the form suggested by Blaschke's treatment of Sylvester's Problem (compare [1, § 24]). In the following the subscripts i, h run from 1 to m .

Let M_1, \dots, M_m be convex bodies with interior points in E_m . In an arbitrary system of rectangular coordinates with origin z , symmetrize each M_i with respect to the (x_1, \dots, x_{m-1}) -plane P ; that is, slide a segment in which a line L_1 parallel to the x_m -axis intersects M_i along L_1 such that its center falls on P . Call \bar{M}_i the image of M_i under this transformation, and \bar{p}_i the image in \bar{M}_i of a given point p_i in M_i . The mapping preserves volume, $dV_{\bar{P}_i}^m = dV_{P_i}^m$. We are going to show that $\tau_m(M_1, \dots, M_m, z)$ does not increase.

If $p_i \in M_i$, denote by p_i' the point symmetric to p_i with respect to the center of that chord of M_i parallel to the x_m -axis which goes through p_i . If p_i^1, \dots, p_i^m are the coordinates of p_i , then with $\eta = 1/m!$

$$\pm T(p_1, \dots, p_m, z) = \eta |p_i^h|, \quad \pm T(p_1', \dots, p_m', z) = \eta |p_i'^h|.$$

The images \bar{p}_i of p_i and \bar{p}'_i of p'_i satisfy the relation

$$|\bar{p}_i^h| = -|\bar{p}'_i{}^h|;$$

hence

$$2T(\bar{p}_1, \dots, \bar{p}_m, z) = 2T(\bar{p}'_1, \dots, \bar{p}'_m, z) = \eta \left| |\bar{p}_i^h| - |\bar{p}'_i{}^h| \right|.$$

But

$$p_i^h = p_i{}^{\prime h} = \bar{p}_i^h = \bar{p}'_i{}^h \quad \text{for } 1 \leq h \leq m-1, \quad \text{and } p_i^m - p_i{}^{\prime m} = \bar{p}_i^m - \bar{p}'_i{}^m.$$

so that

$$|\bar{p}_i^h| - |\bar{p}'_i{}^h| = |p_i^h| - |p_i{}^{\prime h}|;$$

hence

$$(18) \quad T(p_1, \dots, p_m, z) + T(p'_1, \dots, p'_m, z) \geq 2T(\bar{p}_1, \dots, \bar{p}_m, z).$$

Since

$$\begin{aligned} \tau_m(M_1, \dots, M_m, z) &= \int_{M_1} \dots \int_{M_m} T(p_1, \dots, p_m, z) dV_{P_1}^m \dots dV_{P_m}^m \\ &= \int_{M_1} \dots \int_{M_m} T(p'_1, \dots, p'_m, z) dV_{P'_1}^m \dots dV_{P'_m}^m, \end{aligned}$$

we conclude from (18) and

$$dV_{\bar{P}_i}^m = dV_{P_i}^m, \quad dV_{\bar{P}'_i}^m = dV_{P'_i}^m$$

that

$$(19) \quad \tau_m(M_1, \dots, M_m, z) \geq \tau_m(\bar{M}_1, \dots, \bar{M}_m, z).$$

To discuss the equality sign, consider points p_i in M_i which are centers of chords parallel to the x_n -axis. Then $p_i = p'_i$, and the points $\bar{p}_i = \bar{p}'_i$ lie in P , so that the right side of (18) vanishes. Therefore the equality sign can hold in (18) only when the points p_1, \dots, p_m, z are coplanar. Choosing p_1, \dots, p_{i-1} ,

p_{i+1}, \dots, p_m such that they and z do not lie in an L_{m-2} (the M_i have interior points!) we see that all centers of chords of M_i parallel to the x_m -axis must lie in the L_{m-1} spanned by $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_m, z$. Moreover, this same L_{m-1} must contain the centers of the chords parallel to the x_m -axis of all the other M_n . Thus we have proved:

(20) If M_1, \dots, M_m are convex bodies in E_m with interior points, then simultaneous symmetrization of the M_i with respect to any plane P through z decreases $\tau_m(M_1, \dots, M_m, z)$ unless z and the centers of the chords perpendicular to P of all M_i are coplanar.

For given positive values $|M_1|, \dots, |M_m|$, the expression $\tau_m(M_1, \dots, M_m, z)$ can therefore be minimal only if the centers of every family of parallel chords of the different M_j lie on the same plane through z . This implies, first, that *each M_j is an ellipsoid with center z ,¹ and then that all these ellipsoids are homothetic.*

That *the minimum is actually reached* in this case is proved by the following standard argument (see [1, § 24]). Using a suitable sequence P_ν of planes through z , and symmetrizing M_1, \dots, M_m successively in P_1, P_2, \dots , yields a sequence M_1^ν, \dots, M_m^ν of convex bodies which tend to spheres S_1, \dots, S_m with center z and, of course, $|S_i| = |M_i^\nu| = |M_i|$ (compare [2, § 41]).

The functional $\tau_m(M_1, \dots, M_m, z)$ is monotone [that is, $M_i' \subset M_i$ implies $\tau_m(M_1', \dots, M_m', z) \leq \tau_m(M_1, \dots, M_m, z)$] and positive homogeneous:

$$\tau_m(\lambda M_1, \dots, \lambda M_m, z) = \lambda^{m(m+1)} \tau_m(M_1, \dots, M_m, z) \text{ for } \lambda > 0.$$

For a given $\epsilon > 0$, choose $N(\epsilon) > 0$ such that $S_i \subset (1 + \epsilon)M_i^\nu$ for $\nu > N(\epsilon)$ and all i . Then for $\nu > N(\epsilon)$, because of (20) and the two mentioned properties, we have

$$\begin{aligned} \tau_m(S_1, \dots, S_m, z) &\leq (1 + \epsilon)^{m(m+1)} \tau_m(M_1^\nu, \dots, M_m^\nu, z) \\ &\leq (1 + \epsilon)^{m(m+1)} \tau_m(M_1, \dots, M_m, z), \end{aligned}$$

which proves $\tau_m(S_1, \dots, S_m, z) \leq \tau_m(M_1, \dots, M_m, z)$ and hence the mini-

¹The proofs found in the literature all refer to the cases $m = 2, 3$; for references [2, § 70]. However, the extension to arbitrary m is immediate. A particularly simple proof, which works for all m and is not found in the literature, is obtained by using Loewner's result, that there is exactly one ellipsoid which has a given center, contains a given convex body, and has minimal volume.

imum property for homothetic ellipsoids with center z .

To evaluate $\tau_m(S_1, \dots, S_m, z)$, denote the radius of S_i by r_i . Then the results of section 3 show that

$$\begin{aligned} \tau_m(S_1, \dots, S_m, z) &= \frac{1}{m} r_m^{m+1} \frac{\omega_1}{\omega_2} \kappa_{m+1} \tau_m(S_1, \dots, S_{m-1}, z) \\ &= \frac{1}{m(m-1)} \frac{\omega_1}{\omega_3} r_m^{m+1} r_{m+1}^{m+1} \kappa_{m+1}^2 \tau_m(S_1, \dots, S_{m-2}, z) = \dots \\ &= \frac{1}{m!} \prod r_i^{m+1} \frac{\omega_1}{\omega_{m+1}} \kappa_{m+1}^m = \frac{1}{m!} \frac{2}{\omega_{m+1}} \frac{\kappa_{m+1}^m}{\kappa_m^{m+1}} \prod |S_i|^{(m+1)/m}. \end{aligned}$$

Therefore we have:

(21) *If M_1, \dots, M_m are convex bodies in E_m with interior points, then*

$$\tau_m(M_1, \dots, M_m, z) \geq \frac{2}{(m+1)!} \frac{\kappa_{m+1}^{m-1}}{\kappa_m^{m+1}} \prod |M_i|^{(m+1)/m},$$

and the equality sign holds only for homothetic ellipsoids with center z .

Applying this result to (13) yields the inequalities (3) and (4) with the conditions for the equality sign. The latter result may also be formulated as follows:

Among all convex bodies M with a given volume, the ellipsoids with center z (and only these) maximize $\int_{\Omega_n} |M(u)|^n d\omega_u^n$.

To ask for the minimum is senseless since for any convex body M the integral $\int_{\Omega_n} |M(u)|^n d\omega_u^n$ will tend to zero when M moves to infinity. However, it is a meaningful, but unsolved, problem to find the minimum of this integral for all convex bodies with a given volume and center z . This is equivalent to the problem of finding the smallest constant K such that for any convex M with center z the inequality

$$K \int_{\Omega_n} |M(u)|^n d\omega_u^n \geq |M|^{n-1}$$

holds. The existence of K follows readily from (2).

Finally (5) shows:

Among all convex bodies with center z the spheres (and only these) yield the maximum of

$$\min_u |M(u)|^n |M|^{1-n}.$$

The corresponding minimum maximum problem seems quite difficult.

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