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THE NORM FUNCTION OF AN ALGEBRAIC FIELD EXTENSION

HARLEY M. FLANDERS

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1. Introduction. Let k be an algebraic field, K a finite extension field of degree n over k , and $\omega_1, \dots, \omega_n$ a linear basis of K over k . (For the standard results of field theory which we have used in this paper, the reader is referred to the texts [2; 4; 5].) If $X = (X_1, \dots, X_m)$ is a set of indeterminates over K , then $[K(X) : k(X)] = n$, and in fact $\omega_1, \dots, \omega_n$ is a basis of $K(X)$ over $k(X)$. We set $m = n$ and form the so-called *general element*

$$\Xi = \omega_1 X_1 + \dots + \omega_n X_n$$

of K over k . We may, without confusion, use the symbol $N_{K/k}$ both for the norm function of K/k and for that of $K(X)/k(X)$. The *general norm of K over k* is the polynomial

$$N(X) = N(X_1, \dots, X_n) = N_{K/k}(\Xi) \in k[X].$$

We propose here to discuss the factorization of this polynomial and the possibility of characterizing the norm function $N_{K/k}$ of K/k intrinsically. We are indebted to Professor E. Artin for a helpful suggestion communicated orally.

2. Factorization of the general norm. If we take a new basis η_1, \dots, η_n , we simply effect a nonsingular linear transformation on the n variables X_i ; hence nothing essential is changed. The possibility of selecting a convenient basis will be used to advantage in the proofs below. Our first result, while not complete, admits a simple proof; consequently we give it before giving a more general result.

THEOREM 1. *Let $K = k(\theta)$ be a simple extension of k . Then the general norm $N(X)$ is irreducible in $k[X]$.*

Proof. Let $f(X) = (X - \theta_1) \dots (X - \theta_n)$ be the minimum function of $\theta = \theta_1$ over k , and take $1, \theta, \dots, \theta^{n-1}$ as a basis of K over k . Then

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$$N(X) = \prod_{i=1}^n (X_1 + \theta_i X_2 + \cdots + \theta_i^{n-1} X_n).$$

Since this is a complete factorization of $N(X)$ into linear factors, it follows that any factor of $N(X)$ must be the product of a constant and certain of the linear factors displayed. Consequently, if $G(X)$ is an irreducible factor of $N(X)$ in $k[X]$ with

$$\deg G(X) = r \quad (1 \leq r \leq n),$$

then, by properly renumbering and adjusting the coefficient of X_1^r , we have

$$G(X) = \prod_{i=1}^r (X_1 + \theta_i X_2 + \cdots + \theta_i^{n-1} X_n).$$

It follows that $G(X, -1, 0, \dots, 0) \in k[X]$. But this means that

$$\prod_{i=1}^r (X - \theta_i) \in k[X].$$

Since $f(X)$ is irreducible over k , we must have $r = n$.

We can generalize this theorem as follows.

THEOREM 2. *Let $[K : k] = n$, and let $m = \max \{[k(\theta) : k] \text{ for } \theta \in K\}$. Then m divides n , and the complete factorization in $k[X]$ of the general norm $N(X)$ of K over k is given by*

$$N(X) = [F(X)]^{n/m},$$

where $F(X)$ is an irreducible polynomial in $k[X]$.

Proof. If K/k is a separable extension, then it is a simple one and Theorem 1 applies. Consequently, we may assume that k has finite characteristic p , and that K/k is inseparable. Let S be the maximal separable subfield of K over k , and let $s = [S : k]$, so that $n = sp^u$. We let e denote the least whole number such that $K^{p^e} \subset S$. Then $1 \leq e \leq u$ and it is known [2; 4; 5] that $m = sp^e$. Finally, we let α be a generator of S/k , thus $S = k(\alpha)$; and let

$$\Omega_1 = 1, \Omega_2, \dots, \Omega_{p^u}$$

be a linear basis of K/S with

$$(\Omega_j)^{p^e} = \beta_j \in S.$$

The general element Ξ of K/k is given by

$$\Xi = \sum \alpha^i \Omega_j X_{ij} \quad (i = 0, \dots, s - 1; j = 1, \dots, p^u),$$

and the general norm by

$$N(X) = [F(X)]^{p^{u-e}}$$

with

$$F(X) = N_{S/k} \left(\sum \alpha^{ip^e} \beta_j X_{ij}^{p^e} \right).$$

This is the case because

$$N_{K/k} = N_{S/k} \circ N_{K/S}$$

and

$$N_{K/S} A = A^{p^u} = (A^{p^e})^{p^{u-e}} \text{ for } A \in K.$$

We next assert that the polynomial

$$\Pi(X) = \Xi^{p^e} = \sum \alpha^{ip^e} \beta_j X_{ij}^{p^e}$$

is irreducible in the ring $S[X]$. Suppose this is not the case and let $\Gamma(X)$ be an irreducible factor. We normalize the coefficient of the highest power of X_{01} in $\Gamma(X)$; we may thus write

$$\Gamma(X) = \Xi^{p^f v},$$

where $0 \leq f < e$ and $(v, p) = 1$. We clearly have

$$(p^f v, p^e) = p^f,$$

and so there exist rational integers a, b such that

$$p^f v a + p^e b = p^f.$$

This implies that

$$\Xi^{p^f} = (\Xi^{p^f v})^a (\Xi^{p^e})^b \in S[X];$$

hence

$$\Xi^{p^f} \in S[X], \sum \alpha^{i p^f} (\Omega_j)^{p^f} X_{ij}^{p^f} \in S[X].$$

Thus, for each i and j ($i = 0, \dots, s-1; j = 1, \dots, p^u$), we have

$$\alpha^{i p^f} (\Omega_j)^{p^f} \in S.$$

In particular, setting $i = 0$, we obtain

$$(\Omega_j)^{p^f} \in S \text{ for } j = 1, \dots, p^u.$$

Hence $K^{p^f} \subset S$, a contradiction of the definition of e .

It will be convenient in the remainder of the proof to have a "sufficiently large" field at our disposal. We form the splitting field U over k of any polynomial $f(X)$ in $k[X]$ which has amongst its roots the quantities $\alpha, \Omega_1, \dots, \Omega_{p^u}$. Then we may assume $k \subset S \subset K \subset U$, and any relative isomorphism on K over k into any field containing K is already into U .

Now let σ be any relative isomorphism of S over k into U . The fact that $\Pi(X)$ is irreducible over $S[X]$ clearly implies that $\Pi^\sigma(X)$ is irreducible over $S^\sigma[X]$. We also assert that if $\sigma \neq \iota$, the identity isomorphism, then $\Pi(X)$ and $\Pi^\sigma(X)$ are relatively prime in $U[X]$. To prove this, we first note that, since K is a pure inseparable extension of S , σ has a unique prolongation to an isomorphism (also denoted by σ) of K/k . Thus

$$\Pi(X) = \Xi^{p^e}, \quad \Pi^\sigma(X) = (\Xi^\sigma)^{p^e}.$$

These can have a proper common factor if and only if

$$\lambda \Xi = \Xi^\sigma \text{ for } \lambda \text{ in } K.$$

If this is the case, then we compare the coefficients on either side of X_{01} and X_{11} , obtaining $\lambda = 1$ and $\alpha = \alpha^\sigma$, an impossibility if $\sigma \neq \iota$.

To complete the proof, we let $\sigma_1, \dots, \sigma_s$ be all of the relative isomorphisms of S over k into U . We have

$$F(X) = N_{S/k} [\Pi(X)] = \prod_{h=1}^s [\Pi^{\sigma_h}(X)].$$

Let $G(X)$ be any irreducible factor of $F(X)$ in $k[X]$. It follows from the facts (a) each $\Pi^{\sigma_h}(X)$ is irreducible in $S^{\sigma_h}[X]$ and (b) the s polynomials $\Pi^{\sigma_h}(X)$ of $U[X]$ are pairwise relatively prime—an immediate consequence of the result of the last paragraph—that $G(X)$, after a trivial modification of leading coefficient, is necessarily of the form

$$G(X) = \prod_{h=1}^r [\Pi^{\sigma_h}(X)] \qquad (1 \leq r \leq s),$$

where, of course, we have rearranged the indices h as needed. Since $G(X) \in k[X]$, it follows that the polynomial

$$g(X) = \prod_{h=1}^r (X^{p^e} - \alpha^{\sigma_h}),$$

which results from the specialization

$$[X_{01} = X, X_{11} = -1, X_{ij} = 0 \text{ for all other } i, j],$$

is in $k[X]$. This implies $r = s$, $G(X) = F(X)$, as desired.

3. Characterization of the norm function.¹ In this section, let k, K be fields such that $[K : k] = n$. The norm function $N_{K/k}$ has the following properties:

$$(N_1) \qquad N_{K/k}(AB) = (N_{K/k}A) (N_{K/k}B) \text{ for all } A, B \in K,$$

$$(N_2) \qquad N_{K/k}(a) = a^n \text{ for all } a \in k.$$

These properties mean that $N_{K/k}0 = 0$ and that $N_{K/k}$ is a homomorphism on the multiplicative group K^* of nonzero elements of K into k^* such that

$$N_{K/k}a = a^n \text{ on } k^*.$$

¹A somewhat different characterization is given in [1].

DEFINITION 1. A function f on K into k is a *norm-like function* if

$$(N_1) \quad f(AB) = f(A)f(B) \text{ for all } A, B \in K,$$

$$(N_2) \quad f(a) = a^n \text{ for all } a \in k.$$

It is evident from group-theoretic considerations that in general there are many norm-like functions. We wish here to impose further restrictions which will distinguish the norm function $N_{K/k}$ from amongst all norm-like functions. The considerations of §1 suggest a "continuity" condition which we proceed to formulate.

DEFINITION 2. Let L be an n -dimensional linear space over a field k . A function f on L into k will be called a *polynomial function* if there is a basis x_1, \dots, x_n of L and a polynomial

$$F(X_1, \dots, X_n) \in k[X]$$

such that whenever

$$x = \sum a_i x_i \in L,$$

then

$$f(x) = F(a_1, \dots, a_n).$$

It is clear that there is no real dependence on a particular basis in this definition. Similarly we may define a *homogeneous polynomial function of degree m on L to k* by insisting that $F(X)$ be homogeneous of degree m . The norm function $N_{K/k}$ is a homogeneous norm-like function of degree n on K into k .

THEOREM 3. Let k be an infinite field, $[K:k] = n$, and let f be a polynomial norm-like function on K into k . Then $f = N_{K/k}$.

Proof. Let $\omega_1 = 1, \omega_2, \dots, \omega_n$ be a basis of K/k . $F(X_1, \dots, X_n)$ a polynomial such that

$$f(\sum a_i \omega_i) = F(a_1, \dots, a_n).$$

Since k is infinite, F is necessarily unique. It is known that there exist polynomials $g_1(X), \dots, g_n(X) \in k[X]$ such that if

$$A = \sum a_i \omega_i \in K$$

and we set

$$\bar{B} = \sum g_i(a_1, \dots, a_n) \omega_i,$$

then $AB = N_{K/k}A$. Thus

$$f(A\bar{B}) = f(A)f(\bar{B}) = (N_{K/k}A)^n,$$

and so we have

$$F(a_1, \dots, a_n) F(g_1(a_1, \dots, a_n), \dots) = [N(a_1, \dots, a_n)]^n,$$

where $N(X)$ is the general norm of K/k . Since k is infinite, this is an identity; that is,

$$F(X) F(g_1(X), \dots, g_n(X)) = N(X)^n.$$

By Theorem 2, we have

$$N(X) = M(X)^h,$$

where $M(X)$ is irreducible in $k[X]$. It follows that

$$F(X) = cM(X)^r$$

for some power r and $c \in k$. We specialize:

$$X \rightarrow (a, 0, \dots, 0),$$

obtaining

$$a^n = F(a, 0, \dots, 0) = cM(a, 0, \dots, 0)^r.$$

We raise to the h -power, noting that

$$N(a, 0, \dots, 0) = a^n; \quad a^{nh} = c^h a^{nr}.$$

This is true for all $a \in k$; hence

$$nh = nr, \quad h = r, \quad c^h = 1, \quad F(X) = cM(X)^h = cN(X).$$

It is immediate that $c = 1$, and hence $f = N_{K/k}$.

In the case that k is a finite field we get a somewhat different result unless we strengthen the hypotheses. We first have the following result.

THEOREM 4. *Let k be a finite field of q elements and let $[K : k] = n$. Suppose that f is a norm-like function on K into k . Then either $f = (N_{K/k})^r$, where $0 < r < q - 1$ and $nr \equiv n \pmod{q - 1}$, or $n \equiv 0 \pmod{q - 1}$ and f is given by $f(0) = 0$ and $f(A) = 1$ for all $A \neq 0$. Conversely, each such function is norm-like.*

Proof. Let A be a generator of the (cyclic) group K^* . Then

$$a = N_{K/k}A = A^u$$

is a generator of k^* . Here we have set $u = (q^n - 1)/(q - 1)$ for convenience. The norm-like function f , being a homomorphism on K^* , is completely determined by its effect on A . Thus we have $f(A) = a^r$ for some rational integer r . Since $a^{q-1} = 1$, we may assume that $0 \leq r < q - 1$. If $B \in K^*$, then $B = A^c$ and so

$$f(B) = f(A)^c = (N_{K/k}A)^{rc} = (N_{K/k}A^c)^r = (N_{K/k}B)^r.$$

Thus our function f is given by

$$f(B) = (N_{K/k}B)^r \text{ for } B \neq 0, f(0) = 0.$$

So far we have used only the property (N_1) . Property (N_2) asserts that $f(a) = a^n$. But in our case we have

$$f(a) = (N_{K/k}a)^r = a^{nr};$$

hence $a^n = a^{nr}$ is a necessary and sufficient condition that f be norm-like. This is equivalent to

$$nr \equiv n \pmod{q - 1},$$

since $k^* = \langle a \rangle$ is a cyclic group of $q - 1$ elements.

In our next proof we shall use the following results of Chevalley [3]. Let k be a finite field of q elements, and let L denote the linear space of all n -tuples $\mathbf{a} = (a_1, \dots, a_n)$ of elements of k . Let I denote the ideal in $k[X_1, \dots, X_n]$ of all polynomials $F(X)$ such that $F(\mathbf{a}) = 0$ identically on L . Then

$$I = (X_1^q - X_1, \dots, X_n^q - X_n).$$

If $F(X) \in k[X]$, then there is a unique polynomial $F^*(X)$ such that (a) $F \equiv F^* \pmod{I}$ and (b) $\deg_{X_i} F^* \leq q - 1$ for each $i = 1, \dots, n$. The polynomial F^* is called the *reduced form* of F , and has degree at most that of F . Finally, if $F(\mathbf{a}) = 1$ for all $\mathbf{a} \neq 0$ and $F(0) = 0$, then

$$F^* = (-1)^{n-1} (X_1^{q-1} - 1) \dots (X_n^{q-1} - 1) + 1.$$

THEOREM 5. *Let k be a finite field and let $[K : k] = n$. Suppose that f is a norm-like function on K into k , and that f is also a polynomial function of degree at most n . Then $f = N_{K/k}$.*

Proof. As before, we let q be the number of elements of k , and we may apply Theorem 4. If $q = 2$, we clearly have $f = N_{K/k}$ since

$$f(0) = 0 = N_{K/k} 0;$$

whilst if $A \neq 0$, then $f(A) \neq 0$, and hence

$$f(A) = 1 = N_{K/k} A.$$

We may henceforth assume that $q > 2$.

Next, let $\omega_1, \dots, \omega_n$ be a basis of K/k , and let $N(X)$ be the general norm of K/k with respect to this basis. By hypothesis, there exists a polynomial $F(X)$ of degree at most n such that

$$f(A) = F(a_1, \dots, a_n) \text{ for all } A = \sum a_i \omega_i.$$

Suppose that the second alternative of Theorem 4 is the case. Then

$$f(0) = 0 \text{ and } f(A) = 1 \text{ for all } A \neq 0.$$

This implies that

$$F^* = (-1)^{n-1} (X_1^{q-1} - 1) \dots (X_n^{q-1} - 1) + 1,$$

and so

$$(q - 1)n = \deg F^* \leq \deg F = n.$$

Hence $q - 1 \leq 1$, $q = 2$. We have already ruled out this possibility.

Finally suppose that $f = (N_{K/k})^r$, where $1 \leq r < q - 1$. We set

$$G(X) = F(X) [N(X)]^{q-1-r},$$

and have $G(0) = 0$. If $\mathbf{a} \neq 0$, then

$$A = \sum a_i \omega_i \neq 0,$$

and

$$G(\mathbf{a}) = f(A) (N_{K/k} A)^{q-1-r} = (N_{K/k} A)^{q-1} = 1.$$

This implies that

$$G^* = (-1)^{n-1} (X_1^{q-1} - 1) \cdots (X_n^{q-1} - 1) + 1;$$

hence

$$(q-1)n = \deg G^* \leq \deg G \leq n + (q-1-r)n = (q-r)n,$$

so that

$$q-1 \leq q-r, \quad r \leq 1, \quad r=1.$$

We are left with the single possibility $f = N_{K/k}$, as desired.

It is worth noting that the proof can still be pushed through under the weaker assumption that f is a polynomial function of degree at most $2n-1$. However, the most interesting case is that in which f is a homogeneous polynomial function of degree n .

4. Conjecture. It would be interesting to prove Theorem 3 under weakened conditions. We make the following definition.

DEFINITION 3. Let L be an n -dimensional linear space over a field k . A function f on L to k will be called an *algebraic function* if there is a basis x_1, \dots, x_n of L and a polynomial

$$F(X_0, X_1, \dots, X_n) \in k[X],$$

such that $F(X) \neq 0$, and such that whenever $x = \sum a_i x_i$ then

$$F(f(x), a_1, \dots, a_n) = 0.$$

Our conjecture is the following.

If k is an infinite field, $[K : k] = n$, and f is an algebraic norm-like function on K into k , then $f = N_{K/k}$.

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