STRUCTURED THEOREMS FOR RELATIVELY COMPLEMENTED LATTICES

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Introduction. In a previous paper [3] a study was made of the projectivities between the points of a simple relatively complemented lattice of finite dimension. It was shown that for a given dimension there is an upper bound for the number of transposes required to establish the projectivities between the points. The examples given in which this upper bound is attained have a particularly simple structure—they are closely related to a direct union. We shall prove here some general structure theorems for relatively complemented lattices and then apply these to the case of maximal projectivities.

The notation will be that of [3]. The lattice $L$ to which we refer is always relatively complemented.

1. Structure Theorems. Our arguments depend heavily upon the simplicity or indecomposability [2] of $L$, and it is convenient to have the following characterization of a direct union:

**Theorem 1.1.** If $L$ has dimension $n$, and $a$, $b$ are two elements of $L$, then $L \cong a/z \lor b/z$ if and only if

1. $\rho(a) + \rho(b) \leq n$, and
2. $p \subseteq a$ if and only if $p \nsubseteq b$ for all points $p \in L$.

**Proof.** Certainly if $L \cong a/z \lor b/z$, conditions (1) and (2) will hold. Suppose (1) and (2) hold in $L$. We shall proceed by induction on $n$. The theorem is true when $n = 1, 2$. Suppose it is true for all lattices of dimension less than $n$, but $L \ncong a/z \lor b/z$.

It is clear that

$$x = (a \cap x) \lor (b \cap x)$$

for all $x \in L$. Consider the mapping

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\[
x \rightarrow \sigma x = (a \cap x, b \cap x) \subseteq a/z \lor b/z.
\]

Now \(x \supseteq y\) if and only if
\[
a \cap x \supseteq a \cap y \text{ and } b \cap x \supseteq b \cap y;
\]
and the latter occurs if and only if \(\sigma x \supseteq \sigma y\). Hence \(L\) is isomorphic, as a partially ordered set, to a subset of \(a/z \lor b/z\), where
\[
\sigma u = (a, b), \ \sigma z = (z, z).
\]

These remarks show that if any two elements \(a, b\) of \(L\) satisfy (2), we must have
\[
\rho(a) + \rho(b) \geq n.
\]

If \(L \not\subseteq a/z \lor b/z\), there are points \(p \subseteq a\) and \(q \subseteq b\) such that \(p/z P_2 q/z\). Hence there is a maximal element \(m\) such that \(m \not\subseteq p, m \not\subseteq q\). Then \(s_1\) and \(s_2\) exist with
\[
a > s_1 \supseteq m \cap a, \ b > s_2 \supseteq m \cap b.
\]

Furthermore,
\[
a \cup s_2 = b \cup s_1 = u.
\]

Let \(u = x_0 > x_1 > \cdots > x_{n-1} > x_n = z\) be a complete chain in \(L\). This chain maps onto
\[
\sigma u = (a, b) > \sigma x_1 > \cdots > \sigma x_{n-1} > \sigma x_n = (z, z).
\]

Either (i) \(\sigma x_1 = (a, t_2)\), where \(b > t_2\), or (ii) \(\sigma x_1 = (t_1, b)\), where \(a > t_1\). Suppose the former is true. The points of \(x_1\) are in either \(a\) or \(t_2\), but not both. Then \(a\) and \(t_2\) satisfy (2) in \(x_1/z\), and since
\[
\rho(x_1) = n - 1,
\]
we have
\[
\rho(a) + \rho(t_2) \geq n - 1.
\]

But
\[
\rho(a) + \rho(b) \leq n, \text{ so } \rho(t_2) = \rho(b) - 1.
\]

Then by the induction hypothesis, \(x_1/z \cong a/z \lor t_2/z\). This gives the exist-
ence of a chain from $s_1$ through $a$ to $u$ of length $1 + \rho(b) - 1 + 1$, or $\rho(b) + 1$. By Lemma 3.6 of [3], there is a chain from $b$ to $z$ of length at least $\rho(b) + 1$, which is a contradiction. A similar contradiction arises if $\sigma x_1 = (t_1, b)$. Therefore $L \cong a/z \lor b/z$, and thus the theorem is proved.

The following theorem gives more information about the quotient lattices $a_p^k/z$ introduced in Lemma 3.5 of [3].

**Theorem 1.2.** Let $L$ be simple of dimension $n > 1$. If $p$ is any point and $k$ is a nonnegative integer such that $k < [(n + 1)/2]$, then $a_p^k/z$ has dimension at least $2k + 1$.

**Proof.** The theorem is true when $k = 0$. Suppose it is true for all $k$ less than the one in which we are interested. Then $a_p^{k-1}$ has dimension at least $2k - 1$, and $a_p^k \supseteq a_p^{k-1}$. If $a_p^k = u$, we are through, so assume $u \nsubseteq a_p^k$. Then there is a point $s \in L$ with $s \nsubseteq a_p^k$, but $s/z P_2 t/z$ for some $t \in C_p$. Hence there is a maximal element $m$ such that $m \nsubseteq s$, $m \nsubseteq a_p^k$. Since $s \in C_p$, we have $m \supseteq a_p^{k-1}$. Therefore $a_p^k \supseteq a_p^{k-1}$, and the dimension of $a_p^k/z$ is at least $2k$. Suppose $\dim(a_p^k/z) = 2k$.

Let $b$ be the join of all points of $L$ which are not in $a_p^k$. All of these points are in $x_p^k = \bigcap M_p^k$, where

$$M_p^k = \{ m \in L \mid u > m \nsubseteq a_p^{k-1} \}.$$  

(See proof of Lemma 3.5 of [3].) Hence $x_p^k \supseteq b$ and $b \cap a_p^k = z$. The latter follows from the assumption $\dim(a_p^k/z) = 2k$, since, by Theorem 3.1 of [3] for any point $q$ we would have $q \subseteq a_p^k$ if and only if $q \subseteq C_p$. On the other hand, it is shown, in the proof of Lemma 3.5 of [3], that $q \subseteq C_p$ if and only if $q \nsubseteq x_p^k$.

Since $L$ is simple, there exists an $x$ such that

$$u > x, \quad x \nsubseteq a_p^k, \quad x \nsubseteq b.$$  

But $x \nsubseteq b$ implies $x \supseteq a_p^{k-1}$. Then

$$x = a_p^{k-1} \cup (b \cap x), \quad u = b \cup x = a_p^{k-1} \cup b.$$  

Hence if $u > m$ we have $m \supseteq a_p^{k-1}$, if and only if $m \nsubseteq b$. Therefore $a_p^k$, $a_p^{k-1}$, and $b$ satisfy the conditions of Lemma 3.6 of [3], and there exists a chain of length at least $2k$ from $u$ to $b$. Then

$$\rho(b) \leq n - 2k, \quad \text{so} \quad \rho(a_p^k) + \rho(b) \leq n.$$  

But by Theorem 1.1 we would have $L = a_p^k/z \lor b/z$, contrary to the simplicity of $L$. Therefore $\rho(a_p^k) \geq 2k + 1$ for all $k < [(n + 1)/2]$.  

Let \( \mathfrak{P} \) denote the partially ordered subset of \( L \) consisting of \( u \), the maximal elements, the points, and \( z \). Let \( \mathfrak{P}_\nu \) be the normal completion of \( \mathfrak{P} \). Consider the mapping \( A \rightarrow U_A \) from \( \mathfrak{P}_\nu \) into \( L \). (\( A \) is a normally closed subset of \( L \).) If \( A \supseteq B \), then \( U_A \supseteq U_B \). Suppose \( U_A \supseteq U_B \); then \( x \in A^* \) implies \( x \supseteq a \), all \( a \in A \) implies \( x \supseteq U_A \), so \( x \supseteq U_B \), and hence \( x \supseteq b \) all \( b \in B \) and \( x \in B^* \); therefore \( A^* \subseteq B^* \), so \((A^*)_* \supseteq (B^*)_*\), or \( A \supseteq B \). Thus the mapping is order preserving both ways.

Suppose \( a \in L \), \( a \neq u \), \( a \neq z \). Set

\[
P(a) = \{ p \in \mathfrak{P} \mid a \supseteq p > z \},
\]

\[
M(a) = \{ m \in \mathfrak{P} \mid u > m \supseteq a \}.
\]

Now \( x \supseteq p \), all \( p \in P(a) \), if and only if \( x \supseteq a \), so \( M(a) \subseteq (P(a))^* \). Also \( P(a) \subseteq (P(a)^*)_* \). Suppose \( y \in (P(a)^*)_* \); then \( y \subseteq x \), all \( x \in P(a)^* \) implies \( y \subseteq m \), all \( m \in M(a) \) implies \( y \subseteq a \). Suppose \( a' \supseteq y \), all \( y \in (P(a)^*)_* \); then \( a' \supseteq p \), all \( p \in P(a) \) implies \( a' \supseteq a \), so \( a = U(P(a)^*)_* \). If \( a = u \), then \( a = U(u) \); if \( a = z \) then \( a = U(z) \). (Here \( (x) \) denotes the principal ideal generated by \( x \).)

Hence each \( a \in L \) has an inverse image under the above mapping, and \( \mathfrak{P}_\nu \cong L \); see [2]. This proves the following:

**Theorem 1.3.** The structure of \( L \) is completely determined by the structure of \( \mathfrak{P} \).

**Remark.** From the nature of the proof it is seen that the above theorem will be true for any lattice each of whose elements is a join of points and the meet of maximal elements.

2. **Lattices with maximal projectivities.** In this section we shall study simple lattices of odd dimension in which there occurs a maximal projectivity. We shall show that these lattices are quite close to a direct union in the sense that their structure can be completely described in terms of sublattices. Throughout this section \( L \) will be a simple lattice of dimension \( 2n + 1 \), and \( p, q \) are two points in \( L \) such that \( p/z P q/z \) requires \( 2n + 2 \) transposes. Then we have:

**Theorem 2.1.** If \( k \leq n \), the following statements are true:

1. \( \rho(a^k_p) = 2k + 1 \), \( \rho(a^{n-k}_q) = 2n - 2k + 1 \);
2. \( x^k_p = a^{n-k}_q \), \( x^{n-k} = a^k_p \);
3. \( a^k_p/z \) has a maximal projectivity if and only if \( a^{n-k}_q/z \) has a maximal projectivity;
(4) if \( a^k_p/z \) has a maximal projectivity then \( a^k_p \cap a^{n-k}_q = r > z \), otherwise \( a^k_p \cap a^{n-k}_q = z \).

**Proof.** Note that \( s \subseteq C^n_q \) implies \( s \subseteq C^k_p \) implies \( s \subseteq x^k_p \) implies \( a^{n-k}_q \subseteq x^k_p \). Suppose there is a maximal element \( m \) such that \( m \supseteq a^{k-1}_p \), \( m \supseteq x^k_p \). If \( m/z \) is simple, we contradict the assumption of a maximal projectivity between \( p/z \) and \( q/z \), since \( \rho(m) \leq 2n \).

Write

\[ m/z = L_1 \vee L_2 \vee \cdots \vee L_\nu, \]

where the \( L_i \) are simple nontrivial quotient lattices, and \( \nu > 1 \). Now \( a^{n-k}_q/z \) and \( a^{k-1}_p/z \) are both simple; if they are in the same \( L_i \), we again contradict our maximal projectivity assumption. Hence they are in different components and we must have

\[ \rho(a^{k-1}_p) + \rho(a^{n-k}_q) \leq 2n. \]

By Theorem 1.2,

\[ \rho(a^{k-1}_p) \geq 2k - 1; \quad \rho(a^{n-k}_q) \geq 2n - 2k + 1. \]

Therefore

\[ \rho(a^{n-k}_q) = 2n - 2k + 1. \]

The elements \( a^{k-1}_p \) and \( a^{n-k}_q \) are in different \( L_i \), so

\[ \rho(a^{k-1}_p \cup a^{n-k}_q) = \rho(a^{k-1}_p) + \rho(a^{n-k}_q) \geq 2n, \]

and hence

\[ m = a^{k-1}_p \cup a^{n-k}_q \text{ or } m/z = a^{k-1}_p/z \cup a^{n-k}_q/z. \]

Now let \( s > z \), \( s \subseteq x^k_p \). Then \( s \subseteq C^k_p \), so \( s \nsubseteq a^{k-1}_p \). But \( m \supseteq x^k_p \), \( m \supseteq s \), and therefore \( s \subseteq a^{n-k}_q \). This shows that \( x^k_p \subseteq a^{n-k}_q \), and hence \( x^k_p = a^{n-k}_q \). Thus we have shown that if \( a^{k-1}_p \cup x^k_p \neq u \), then \( x^k_p = a^{n-k}_q \) and \( \rho(a^{n-k}_q) = 2n - 2k + 1 \).

Suppose \( a^{k-1}_p \cup x^k_p = u \). Then for each maximal element \( m \), \( m \supseteq a^{k-1}_p \) if and only if \( m \nsubseteq x^k_p \). We have \( \rho(a^{k-1}_p) \geq 2k - 1 \), so \( \dim(u/a^{k-1}_p) \leq 2n + 2 - 2k \).

Since \( L \) is simple, \( \dim(u/x^k_p) \geq 2k \), by Theorem 1.1. Hence \( \rho(x^k_p) \leq 2n - 2k + 1 \). But \( x^k_p \geq a^{n-k}_q \), and \( \rho(a^{n-k}_q) \geq 2n - 2k + 1 \). Hence, in all cases, \( x^k_p = a^{n-k}_q \) and \( \rho(a^{n-k}_q) = 2n - 2k + 1 \). By a similar argument, \( x^{n-k}_q = a^{k}_p \) and \( \rho(a^{k}_p) = 2k + 1 \). This demonstrates (1) and (2).
Suppose \( r > z \), \( r \subseteq a_p^k \) such that \( r/z \) \( p/z \) requires \( 2k + 2 \) transposes. Now \( r \notin C_p \) implies \( r \subseteq x_p^k = a_q^{n-k} \). Furthermore, \( r \subseteq a_p^k = x_q^{n-k} \) implies \( r \notin C_q \) implies that \( r/z \) \( p/z \) requires \( 2n - 2k + 2 \) transposes. The argument is symmetric in \( p \) and \( q \), and this proves (3).

Suppose \( s > z \) and \( s/z \) \( p/z \) requires \( 2n + 2 \) transposes. Then \( x_p^n = a_s^0 = s = a_q^0 = q \), so there is at most one point \( q \) such that \( p/z \) \( q/z \) requires \( 2n + 2 \) transposes. This shows that the \( r \) in the preceding paragraph, if it exists, is unique, and we have (4).

We are now in a position to characterize the maximal elements of \( L \) in terms of the structure of \( a_p^k/z \) and \( a_q^{n-k}/z \). When we know these maximal elements, we will know the structure of \( L \), by Theorem 1.3. First we prove two useful lemmas.

**Lemma 2.1.** There is a chain of length \( 2n + 1 \) through \( a_p^k \).

Suppose \( a_p^k \cup a_q^{n-k-1} = u \). Then the maximal elements of \( L \) are in two disjoint classes—those containing \( a_p^k \) and those containing \( a_q^{n-k-1} \); and by Theorem 1.1,

\[
\dim \left( \frac{u}{a_p^k} \right) + \dim \left( \frac{u}{a_q^{n-k-1}} \right) > 2n + 1.
\]

But

\[
\dim \left( \frac{u}{a_p^k} \right) \leq 2n + 1 - (2k + 1);
\]

\[
\dim \left( \frac{u}{a_q^{n-k-1}} \right) \leq 2n + 1 - (2n - 2k - 1).
\]

Hence \( \dim \left( \frac{u}{a_p^k} \right) = 2n - 2k \).

Suppose \( u > m \geq a_p^k \cup a_q^{n-k-1} \). Now \( m/z \) is not simple, since \( \rho(m) \leq 2n \) and \( m \geq p, m \geq q \). Suppose

\[
m/z = L_1 \lor L_2 \lor \cdots \lor L_\nu,
\]

where \( \nu > 1 \). Then \( a_p^k/z \) and \( a_q^{n-k-1}/z \) are in different components and again there is a chain from \( a_p^k \) to \( u \) of length at least \( 2n - 2k \) since \( \rho(a_q^{n-k-1}) = 2n - 2k - 1 \). This proves the lemma.

**Lemma 2.2.** If \( s > z, a \vdash s, b \vdash s \), but \( a \cup b \geq s \), then there are points \( s_1 \subseteq a \), \( s_2 \subseteq b \) such that \( s_1/z \) \( P_2 \) \( s/z \) and \( s_2/z \) \( P_2 \) \( s/z \).

Let \( a \cup b > x \geq b \), and let \( x' \) be a relative complement of \( a \cup b \) in \( a \cup b/x \) such that \( a \cup b > x' \). Then \( x' \vdash a \), \( x' \vdash s \); hence \( x' \vdash s_1 \), for some point \( s_1 \subseteq a \). Therefore \( s/z \) \( T \) \( a \cup b/x' \) \( T \) \( s_1/z \) \( T \). Similarly we can show the existence of \( s_2 \).
proving the lemma.

**Lemma 2.3.** The following relation holds: \( \dim \left( \frac{a^k}{a^k_{p-1}} \right) = 2. \)

For since \( L \) is simple there is a maximal \( m_0 \) such that \( m_0 \nmid a^k_p, m_0 \nmid a^{n-k}_q \). Then \( m_0 \geq a^{k-1}_p, m_0 \geq a^{n-k-1}_q \). Assume \( a^k_p > a^{k-1}_p \). Then \( m_0 \cap a^k_p = a^{k-1}_p \). Set \( w = a^{n-k}_q \cap m_0 \). Then \( y \) exists such that \( a^{n-k}_q \cap y \geq w \). Since \( m_0 = a^{k-1}_p \cup w \), we have \( u = a^{k-1}_p \cup y = w \cup a^k_p \). Since there is a chain of length \( 2k \) from \( a^{n-k}_q \) to \( u \), there exists a maximal \( m \) such that \( m \nmid a^{n-k}_q \) and such that there exists a chain of length at least \( 2k \) from \( m \) to \( y \). Now \( m \nmid a^k_p \) since \( a^k_p \cup y = u \). But \( m \geq a^{k-1}_p \) and \( m/z = a^{k-1}_p/z \lor y/z \) in contradiction with the length of the chain from \( y \) to \( m \). Hence \( a^k > a^{k-1}_p \), and we must have \( \dim \left( \frac{a^{n-k}_q}{a^{n-k-1}_q} \right) = 2. \)

**Corollary.** The following relation holds:

\[ \dim \left( \frac{a^{n-k}_q}{a^{n-k-1}_q} \right) = 2. \]

This follows by symmetry.

3. **Maximal elements when** \( a^k_p \cap a^{n-k}_q \neq z \). The following theorem gives the possibilities for maximal elements when \( a^k_p/z \) and \( a^{n-k}_q/z \) each have a maximal projectivity. We assume throughout that \( 1 \leq k \leq n - 1 \).

**Theorem 3.1.** Let \( a^k_p \cap a^{n-k}_q = r > z \), and let \( u > m \). If \( m \geq r \), either

1. \( m \geq a^k_p \) and \( a^{n-k}_q > m \cap a^{n-k}_q \),

or

2. \( a^k_p > m \cap a^k_p \) and \( m \geq a^{n-k}_q \).

If \( m \nmid r \), then \( a^k_p > a^k_p \cap m \) and \( a^{n-k}_q > a^{n-k}_q \cap m \).

**Proof.** Let \( u > m \geq r \), and suppose \( m \nmid a^k_p, m \nmid a^{n-k}_q \). Then \( m \geq a^{k-1}_p \), and \( m \geq a^{n-k-1}_q \), for otherwise we would not have a maximal projectivity in \( L \). For the same reason, we have \( r \nmid a^{k-1}_p, r \nmid a^{n-k-1}_q \). Then since

\[
\rho(a^{k-1}_p) = 2k - 1, \quad \rho(a^k_p) = 2k + 1,
\]

\[
\rho(a^{n-k}_q) = 2n - 2k - 1, \quad \rho(a^{n-k}_q) = 2n - 2k + 1,
\]

we must have

\[
a^k_p > m \cap a^k_p = r \cup a^{k-1}_p \quad \text{and} \quad a^{n-k}_q > m \cap a^{n-k}_q = r \cup a^{n-k-1}_q.
\]

Hence
\[ m = (r \cup a_p^{k-1}) \cup (r \cup a_q^{n-k-1}) \quad \text{and} \quad u = a_p^k \cup m = a_p^k \cup a_q^{n-k-1}. \]

Similarly, \( u = a_q^{n-k} \cup a_p^{k-1} \).

By Lemma 2.1, there is a chain from \( a_p^k \) to \( u \) of length \( 2n - 2k \). Since \( L \) is relatively complemented, it is easy to see that \( v \) exists such that \( u > v, v \cap a_p^k = r \cup a_p^{k-1} \), and there is a chain from \( r \cup a_p^{k-1} \) to \( v \) of length at least \( 2n - 2k \). There is an \( s \subseteq c_p^{-1} \) such that \( s \nsubseteq a_p^k \cup r \). Hence \( s \nsubseteq v \), and this implies \( v \geq a_q^{n-k-1} \). Therefore \( v \geq m \) and \( v = m \). Then by Theorem 1.1,

\[ m/z = a_p^{k-1} \cup r/z \cup a_q^{n-k-1}/z; \]

but this contradicts the existence of a chain from \( a_p^k \cup r \) to \( m \) of length \( 2n - 2k \). Hence we must have either \( m \geq a_p^k \), or \( m \geq a_q^{n-k} \).

Suppose \( m \geq a_q^{n-k} \), but \( a_p^k > x \cap m \cap a_p^k \). Let \( y \) be a relative complement of \( x \) in \( a_p^k/a_p^k \cap m \). Then \( y \geq a_p^k \cap m \), since \( a_p^k > x \). Hence \( m \nsubseteq x, m \nsubseteq y \), so

\[ x \cup a_p^{n-k} = y \cup a_q^{n-k} = u. \]

Since

\[ \rho(a_p^k) = \rho(a_p^{k-1}) + 2 \quad \text{and} \quad r \cup a_p^{k-1} \supset a_p^{k-1}, \]

it follows that \( m \nsubseteq a_p^{k-1} \). Hence either \( x \nsubseteq a_p^{k-1} \) or \( y \nsubseteq a_p^{k-1} \). Suppose the latter is the case. Then there is an \( s \subseteq c_p^{-1} \) such that \( s \nsubseteq y \). But \( s \subseteq u = y \cup a_q^{n-k} \).

Hence, by Lemma 2.2, \( s/z \cap L_{2n-2k+2} q/z \cap p/z \cap L_{2n} q/z \) contrary to our assumption of a maximal projectivity between \( p/z \) and \( q/z \). A similar contradiction arises if \( x \nsubseteq a_p^{k-1} \). Hence \( a_p^k > a_p^k \cap m \). The roles of \( p \) and \( q \) are symmetric, so if \( m \geq a_p^k \), then \( a_q^{n-k} > m \cap a_q^{n-k} \).

Now let \( u > m \nsubseteq r \). Since \( m \nsubseteq r \), we have \( m \geq a_p^{k-1} \) and \( m \geq a_q^{n-k-1} \). Suppose

\[ a_p^k > x > a_p^{k-1} = a_p^k \cap m \quad \text{and} \quad a_q^{n-k} > y > a_q^{n-k-1} = a_q^{n-k} \cap m. \]

Let \( x' \) be a relative complement of \( x \) in \( a_p^k/a_p^{k-1} \). Suppose \( x' \nsubseteq r \), and let \( x'' \) be a relative complement of \( a_p^k \) in \( u/x' \). Since \( a_p^k > x' \), we can assume \( u > x'' \). Now \( x'' \nsubseteq r \), so \( x'' \geq a_q^{n-k-1} \). Hence \( x'' = m \), contrary to \( a_p^{k-1} = m \cap a_p^k \). A similar contradiction arises if \( x \nsubseteq r \); and since \( a_p^{k-1} \nsubseteq r \), we must have \( a_p^k > m \cap a_p^k \).

Therefore either

\[ a_p^k > m \cap a_p^k \quad \text{or} \quad a_q^{n-k} > m \cap a_q^{n-k}. \]

Suppose
$a_p^k > m \cap a_p^k > a_p^{k-1}$ but $a_q^{n-k} > y > a_q^{n-k-1} = m \cap a_q^{n-k}$.

As before, $v$ exists with $u > v$, $v \cap a_p^k = m \cap a_p^k$, and there is a chain from $m \cap a_p^k$ to $v$ of length at least $2n - 2k$. There is a point $s \subset C_p^k$ such that $s \notin m \cap a_p^k$, and hence $s \notin v$. Therefore $v \supseteq a_q^{n-k-1}$, so $v = m$. But $m/z$, by Theorem 1.1, is equal to $m \cap a_p^k/\mathbb{Z} \vee a_q^{n-k-1}/\mathbb{Z}$ in contradiction with the length of the chain from $m \cap a_p^k$ to $v = m$. Hence $a_q^{n-k} > m \cap a_q^{n-k}$, and whenever $u > m \notin r$, we have

$$a_p^k > m \cap a_p^k, a_q^{n-k} > m \cap a_q^{n-k}.$$

The converse of this theorem is not true; however we do have the following result:

**Theorem 3.2.** If $a_p^k > x \supseteq r$, then $u > x \cup a_q^{n-k}$, while if $a_q^{n-k} > y \supseteq r$, then $u > a_p^k \cup y$. If $a_p^k > x \notin r$ and $a_q^{n-k} > y \notin r$, then $u > x \cup y$ if and only if for any points $t \subseteq x$, $s \subseteq y$, we have $t \cup s \notin r$.

**Proof.** Let $a_p^k > x \supseteq r$, and let $x'$ be a relative complement of $a_p^k$ in $u/x$ such that $u > x'$. Then by Theorem 3.1 we have $x' \supseteq a_q^{n-k}$ and $x' = x \cup a_q^{n-k}$. A similar argument shows that if $a_q^{n-k} > y \supseteq r$, then $u > a_p^k \cup y$.

Suppose

$$a_p^k > x \notin r, a_q^{n-k} \supseteq y \notin r.$$

If

$$x \supseteq t \supseteq z \text{ and } y \supseteq s \supseteq z,$$

such that $s \cup t \supseteq r$, then

$$x \cup y \supseteq r \text{ and } x \cup y = (x \cup r) \cup (y \cup r) = u.$$

Suppose $x \cup y = u$. Since

$$x \notin r, y \notin r,$$

it follows that

$$x \supseteq a_p^{k-1}, y \supseteq a_q^{n-k-1}.$$

If $x = a_p^{k-1}$ or $y = a_q^{n-k-1}$, Lemma 2.2 tells us that $r \subset C_p^k$ or $r \subset C_q^{n-k}$. Hence

$$x > a_p^{k-1} \text{ and } y > a_q^{n-k-1}.$$

So points $s$ and $t$ exist such that $x = t \cup a_p^{k-1}$ and $y = s \cup a_q^{n-k-1}$. Therefore
and applying Lemma 2.2 twice we get \( t \cup s \supseteq r \). All that is required to finish the proof of the theorem is to show that if \( u > m \supseteq x \cup y \), then \( m = x \cup y \). Suppose \( m \supseteq r \); then

\[
m \supseteq (x \cup r) \cup (y \cup r) = u.
\]

So \( m \nsubseteq r \). Hence, by Theorem 3.1, \( m = x_1 \cup y_1 \), where

\[
a_p^k > x_1 \nsubseteq r \quad \text{and} \quad a_q^{n-k} > y_1 \nsubseteq r.
\]

But this implies \( x = x_1 \), \( y = y_1 \), and \( m = x \cup y \).

### 4. Maximal elements when \( a_p^k \cap a_q^{n-k} = z \).

Here as before we assume that \( 1 \leq k \leq n - 1 \).

**Theorem 4.1.** If \( u > m \) then \( m \) is one of the following three types:

1. \( m \supseteq a_p^k, a_q^{n-k} \supseteq a_q^{n-k} \cap m \nsubseteq a_q^{n-k-1} \), or dually;
2. \( m \supseteq a_p^k, a_q^{n-k} \cap m = a_q^{n-k-1} \), or dually;
3. \( a_p^k > m \cap a_p^k \supseteq a_p^k \), and \( a_q^{n-k} > m \cap a_q^{n-k} \supseteq a_q^{n-k-1} \).

**Proof.** Suppose

\[
u > m \supseteq a_p^k, m \nsubseteq a_q^{n-k-1}, \text{ but } a_q^{n-k} > x \supseteq a_q^{n-k} \cap m.
\]

Then not all elements of \( a_q^{n-k} / z \) covering \( m \cap a_q^{n-k} \) will contain \( a_q^{n-k-1} \). On the other hand,

\[
m = (m \cap a_q^{n-k}) \cup a_p^k,
\]

so for any point

\[
s \subseteq a_q^{n-k}, s \nsubseteq m \cap a_q^{n-k},
\]

we have

\[
s \cup (m \cap a_q^{n-k}) \cup a_p^k \supseteq a_q^{n-k-1}.
\]

Then by Lemma 2.2 we must have

\[
s \cup (m \cap a_q^{n-k}) \supseteq a_q^{n-k-1},
\]

contrary to the above assertion. Therefore if
\[ u > m \geq a^k_p \quad \text{and} \quad m \not\leq a^{n-k-1}_q, \]

then \( a^{n-k}_q > m \cap a^{n-k}_q \).

Now suppose \( u > m \geq a^k_p \) and \( m \geq a^{n-k-1}_q \). If \( m/z \) is simple, we contradict our maximal projectivity assumption; but arguing as before on the direct split of \( m/z \), we see that

\[ m/z = a^k_p/z \lor a^{n-k-1}_q/z, \]

and hence \( m \cap a^{n-k}_q = a^{n-k-1}_q \).

Finally suppose \( u > m \), but \( m \not\subseteq a^k_p, m \not\subseteq a^{n-k}_q \). Then \( m \supseteq a^{n-k-1}_p \) and \( m \supseteq a^{n-k-1}_q \).

Assume \( m \cap a^k_p = a^{k-1}_p \), and let \( a^k_p > x > a^{k-1}_p \), by Lemma 2.3. Let \( v \) be a relative complement of \( a^k_p \) in \( u/x \) such that \( u > v \). Since \( v \not\subseteq a^k_p \), we have \( v \supseteq a^{n-k-1}_q \). Now \( v \neq m \), so \( a^{n-k}_q > m \cap a^{n-k}_q \). Then \( m' \) exists such that \( u > m', m' \not\subseteq a^{n-k}_q \), and there is a chain from \( m \cap a^{n-k}_q \) to \( m' \) of length at least \( 2k \). Since \( m' \not\subseteq a^{n-k}_q \), it follows that \( m' \supseteq a^{k-1}_p \), and hence \( m' = m \). But \( m/z \) is not simple; \( a^{k-1}_p \) and \( a^{n-k}_q \cap m \) are in different components. This is contrary to the length of the above chain, since \( \rho(a^{k-1}_p) = 2k - 1 \). Hence we must have \( a^k_p > m \cap a^k_p \), and dually \( a^{n-k}_q > m \cap a^{n-k}_q \).

Examples show that it is impossible from the structures of \( a^k_p/z \) and \( a^{n-k}_q/z \) to tell whether \( u > a^k_p \cup a^{n-k-1}_q \) or \( u = a^k_p \cup a^{n-k-1}_q \), and dually. However, for the other maximal elements we have:

**Theorem 4.2.** If \( a^{n-k}_q > y \not\subseteq a^{n-k-1}_q \), then \( u > y \cup a^k_p \), and dually. If

\[ a^k_p > x \supseteq a^{k-1}_p \quad \text{and} \quad a^{n-k}_q > y \supseteq a^{n-k-1}_q, \]

then \( u > x \cup y \) if and only if for every pair of points \( s \subseteq x, t \subseteq y \) the lattice \( s \cup t/z \) is a Boolean algebra.

**Proof.** Suppose \( a^{n-k}_q > y \supseteq a^{n-k-1}_q \) and \( u = a^k_p \cup y \).

Then there is a point \( t \subseteq a^{n-k-1}_q \) such that \( t \not\subseteq y, t \not\subseteq a^k_p \), but \( t \subseteq a^k_p \cup y \); and using Lemma 2.2 we obtain a contradiction of our maximal projectivity hypothesis. On the other hand, if \( u > m \supseteq a^k_p \cup y \), then by Theorem 4.1 we get \( m = a^k_p \cup y \).

Let \( a^k_p > x \supseteq a^{k-1}_p \) and \( a^{n-k}_q > y \supseteq a^{n-k-1}_q \). By Theorem 4.1, either \( u = x \cup y \).
or \( u > x \cup y \). If \( u = x \cup y \), there are points \( s \subseteq x \), \( t \subseteq y \) such that

\[
t \cup a_p^{k-1} \cup s \cup a_q^{n-k-1} \geq a_q^{n-k}.
\]

Then by Lemma 2.2, we have

\[
t \cup s \cup a_q^{n-k-1} \geq a_q^{n-k};
\]

thus \( t \cup a_q^{n-k}/z \) is not a direct union, so there is another point \( r \subseteq a_p^k \) such that \( t \cup s \cup a_q^{n-k-1} \supseteq r \), and hence \( t \cup s \supseteq r \). But this tells us that \( t \cup s/z \) is not a Boolean algebra.

If \( u > x \cup y \), we must have \( x \cup y/z = x/z \lor y/z \), and the condition is satisfied.

Here again, then, save for the one exception, the structure of \( L \) is determined by the structure of sublattices and the relations between points.

**References**


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