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**FORCES ON THE BOUNDARY OF A DIELECTRIC**

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**1. Introduction.** It has been shown [1, ch. VII] that the component parallel to the axis of  $x$  of the resultant force on the matter inside any closed surface  $S_1$  drawn in a medium of specific inductive capacity  $K$  is given by

$$X = - \iint_{S_1} (lP_{xx} + mP_{xy} + nP_{xz}) dS,$$

where  $(l, m, n)$  are the direction-cosines of the normal to the surface,

$$P_{xx} = \frac{K}{8\pi} (\bar{X}^2 - \bar{Y}^2 - \bar{Z}^2),$$

$$P_{xy} = \frac{K}{4\pi} \bar{X}\bar{Y},$$

$$P_{xz} = \frac{K}{4\pi} \bar{X}\bar{Z},$$

and  $\bar{X}, \bar{Y}, \bar{Z}$  are given in terms of the potential by  $-\partial\phi/\partial x, -\partial\phi/\partial y, -\partial\phi/\partial z$ , respectively, provided the effect of electrostriction is neglected.

If any other surface  $S_2$  is taken, surrounding  $S_1$ , and if

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial z} = 0$$

at all points between the surfaces, that is to say provided  $\nabla^2\phi = 0$  at all such points, then, by Green's theorem,

$$X = - \iint_{S_2} (lP_{xx} + mP_{xy} + nP_{xz}) dS,$$

and similarly for the other components of the resultant force on the matter inside  $S_1$ .

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This method can be used to find the resultant forces caused by the refraction of the lines of force at a surface of discontinuity separating one medium of specific inductive capacity  $K_i$  from a second of specific inductive capacity  $K_0$ .

**2. Two-dimensional fields.** Instead of applying the foregoing method to two-dimensional fields, we can best obtain the results by using the complex potential. Let

$$\Omega_0 \equiv \phi_0 + i\psi_0$$

be the complex potential of the field in the dielectric  $K_0$ . The components of the resultant force on the boundary  $C$  are then given by

$$Y_0 + iX_0 = \frac{K_0}{8\pi} \int_C \left( \frac{d\Omega_0}{dz} \right)^2 dz,$$

and the couple  $\Gamma_0$  is the real part of

$$- \frac{K_0}{8\pi} \int_C \left( \frac{d\Omega_0}{dz} \right)^2 z dz.$$

These results follow from the equations of the Introduction with  $Z = 0$ . The details are omitted since the proof is identical with that of the well-known theorem of Blasius [3, p. 163; 2, p. 91] in fluid flow. The substitution of

$$\Omega_0 = \sum_{n=1}^p \frac{c_n - ic'_n}{n} z^n + \sum_{n=1}^{\infty} \frac{d_n + id'_n}{n} z^{-n}$$

and

$$\Omega_i = \sum_{n=1}^{p'} \frac{a_n - ia'_n}{n} z^n + \sum_{n=1}^{\infty} \frac{b_n + ib'_n}{n} z^{-n},$$

and separation into real and imaginary parts, yields the explicit forms

$$(1a) \quad X = \frac{K_0}{2} \sum_{n=1}^{p-1} (d_n c_{n+1} + d'_n c'_{n+1}) - \frac{K_i}{2} \sum_{n=1}^{p'-1} (b_n a_{n+1} + b'_n a'_{n+1}),$$

$$(1b) \quad Y = \frac{K_0}{2} \sum_{n=1}^{p-1} (d_n c'_{n+1} - d'_n c_{n+1}) - \frac{K_i}{2} \sum_{n=1}^{p'-1} (b_n a'_{n+1} - b'_n a_{n+1}),$$

$$(1c) \quad \Gamma = \frac{K_0}{2} \sum_{n=1}^P (c_n d'_n - c'_n d_n) - \frac{K_i}{2} \sum_{n=1}^{P'} (a_n b'_n - a'_n b_n).$$

CIRCULAR CYLINDER IN A GENERAL FIELD. If a circular cylinder of radius  $a$ , filled with homogeneous dielectric of specific inductive capacity  $K$ , is placed at the origin of coordinates in a two-dimensional field whose complex potential is  $f(z)$  in air, having no singularities inside or on  $r = a$ , then the complex potentials inside and outside the cylinder are respectively

$$(2) \quad \begin{cases} \Omega_i = \frac{2}{(1+K)} f(z), \\ \Omega_o = f(z) + \frac{(1-K)}{(1+K)} \bar{f}\left(\frac{a^2}{z}\right). \end{cases}$$

It is assumed that there are no other boundaries present, and that the field is caused by isolated singularities (charges, dipoles, etc.). The result can easily be obtained by considering the boundary conditions. Note that by putting  $K = 0$  in  $\Omega_o$  above we obtain the Circle Theorem [4, p. 84].

If the original real potential is taken to be

$$\phi(r, \theta) = \sum_{n=1}^P \left( \frac{E_n r^n \cos n\theta}{n} + \frac{E'_n r^n \sin n\theta}{n} \right),$$

then the potentials inside and outside the dielectric are

$$\phi_i = \frac{2}{(1+K)} \phi(r, \theta)$$

and

$$\phi_o = \phi(r, \theta) + \frac{(1-K)}{(1+K)} \phi\left(\frac{a^2}{r}, \theta\right).$$

Thus with the above notation we have

$$a_n = \frac{2E_n}{(1+K)}, \quad a'_n = \frac{2E'_n}{(1+K)}, \quad b_n = b'_n = 0,$$

$$c_n = E_n, \quad c'_n = E'_n, \quad d_n = \frac{(1-K)}{(1+K)} a^{2n} E_n, \quad d'_n = \frac{(1-K)}{(1+K)} a^{2n} E'_n.$$

Hence the resultant forces on the boundary are given by

$$(3) \quad \left\{ \begin{array}{l} X = \frac{(1-K)}{2(1+K)} \sum_{n=1}^{p-1} a^{2n} (E_n E_{n+1} + E'_n E'_{n+1}), \\ Y = \frac{(1-K)}{2(1+K)} \sum_{n=1}^{p-1} a^{2n} (E_n E'_{n+1} - E'_n E_{n+1}), \\ \Gamma = 0. \end{array} \right.$$

Equations (2) can be extended in the form of infinite series to the case where there is also present a conducting surface  $r = b$  ( $b < a$ ). Infinite series are also obtained when  $r = b$  is a line of flow. These two cases can then be used to obtain results for a dielectric elliptic cylinder.

**3. Three-dimensional fields.** In spherical polar coordinates  $(r, \theta, \omega)$ , the components of force are

$$(4) \quad \left\{ \begin{array}{l} Z = \iint (F \cos \theta - G \sin \theta) dS, \\ Y = \iint [(F \sin \theta + G \cos \theta) \sin \omega + H \cos \omega] dS, \\ X = \iint [(F \sin \theta + G \cos \theta) \cos \omega - H \sin \omega] dS, \end{array} \right.$$

where

$$\begin{aligned} F &= F_i - F_0, \quad G = G_i - G_0, \quad H = H_i - H_0, \\ F_0 &= \frac{K_0}{8\pi} \left\{ \left( \frac{\partial \phi_0}{\partial r} \right)^2 - \left( \frac{\partial \phi_0}{r \partial \theta} \right)^2 - \left( \frac{\partial \phi_0}{r \sin \theta \partial \omega} \right)^2 \right\}, \\ G_0 &= \frac{K_0}{4\pi} \left( \frac{\partial \phi_0}{\partial r} \right) \left( \frac{\partial \phi_0}{r \partial \theta} \right), \\ H_0 &= \frac{K_0}{4\pi} \left( \frac{\partial \phi_0}{\partial r} \right) \left( \frac{\partial \phi_0}{r \sin \theta \partial \omega} \right), \end{aligned}$$

with similar expressions for  $F_i, G_i, H_i$ . As before,  $\phi_0$  is the potential inside the dielectric  $K_0$ . The integration is performed over the sphere of radius  $r$ .

The couple components are

$$(5) \quad \left\{ \begin{array}{l} N = \iint H r \sin \theta dS, \\ M = \iint [G r \cos \omega - H r \sin \omega \cos \theta] dS, \\ L = \iint [H r \cos \omega \cos \theta - G r \sin \omega] dS, \end{array} \right.$$

Considering components due to  $F_0, G_0, H_0$  only, and making the change of variable

$$\mu = -\cos \theta, \quad dS = r^2 \, d\omega \, d\mu,$$

we obtain

$$(6) \quad \frac{8\pi Z_0}{K_0} \int_0^{2\pi} \int_{-1}^1 \left\{ (1-\mu^2) \left( \frac{\partial \phi_0}{\partial \mu} \right)^2 + \frac{1}{(1-\mu^2)} \left( \frac{\partial \phi_0}{\partial \omega} \right)^2 - r^2 \left( \frac{\partial \phi_0}{\partial r} \right)^2 \right\} \mu \, d\omega \, d\mu \\ - 2 \int_0^{2\pi} \int_{-1}^1 \left( \frac{\partial \phi_0}{\partial r} \right) \left( \frac{\partial \phi_0}{\partial \mu} \right) (1-\mu^2) \, r \, d\omega \, d\mu,$$

and proceed similarly for  $X_0, Y_0, N_0, M_0, L_0$ .

These integrals can be evaluated if the potential  $\phi_0$  is expanded with the usual notation [1, ch. VII, p. 239, and elsewhere], in the form

$$(7) \quad \phi_0 = \sum_{n=1}^p r^n S_n + \sum_{n=1}^{\infty} \frac{W_n}{r^{n+1}},$$

where

$$(8) \quad S_n = A_{0,n} P_n + \sum_{s=1}^n (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega) P_n^s,$$

$$(9) \quad W_n = a_{0,n} P_n + \sum_{s=1}^n (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega) P_n^s,$$

and  $P_n^s$  satisfies the differential equation

$$\frac{d}{d\mu} \left\{ (1-\mu^2) \frac{dP_n^s}{d\mu} \right\} = - \left\{ n(n+1) - \frac{s^2}{(1-\mu^2)} \right\} P_n^s.$$

With the usual notation for associated Legendre functions of the first kind, we have

$$P_n^s = (1 - \mu^2)^{s/2} \frac{d^s P_n}{d\mu^s} = \frac{1}{2^n n!} (1 - \mu^2)^{s/2} \frac{d^{n+s}}{d\mu^{n+s}} (\mu^2 - 1)^n.$$

The potential  $\phi_i$  has a similar expansion.

The recurrence and integral formulae used are

- (a) 
$$\sqrt{(1 - \mu^2)} P_n^s = \frac{1}{(2n + 1)} (P_{n+1}^{s+1} - P_{n-1}^{s+1}),$$
- (b) 
$$(2n + 3) \mu P_{n+1}^s = (n + s + 1) P_n^s + (n - s + 2) P_{n+2}^s,$$
- (c) 
$$2s\mu P_{n+1}^s = \sqrt{(1 - \mu^2)} P_{n+1}^{s+1} + (n + s + 1)(n - s + 2) \sqrt{(1 - \mu^2)} P_{n+1}^{s-1},$$
- (d) 
$$(n + s + 1) P_n^s = (n - s + 1) \mu P_{n+1}^s + \sqrt{(1 - \mu^2)} P_{n+1}^{s+1},$$
- (e) 
$$\begin{aligned} (1 - \mu^2) \frac{dP_{n+1}^s}{d\mu} &= (n + 2) \mu P_{n+1}^s - (n - s + 2) P_{n+2}^s \\ &= (n + s + 1) P_n^s - \mu(n + 1) P_{n+1}^s \\ &= \sqrt{(1 - \mu^2)} P_{n+1}^{s+1} - s\mu P_{n+1}^s, \end{aligned}$$
- (f) 
$$\int_{-1}^1 P_n^s P_{n'}^s d\mu = \begin{cases} 0 & \text{if } n \neq n', \\ \frac{(n + s)!}{(n - s)!} \cdot \frac{2}{(2n + 1)} & \text{if } n = n', \end{cases}$$
- (g) 
$$\int_{-1}^1 P_n^{s+1} P_{n'}^{s-1} d\mu = \begin{cases} 0 & \text{if } n' > n \text{ or } n - n' \text{ odd,} \\ -\frac{(n + s - 1)!}{(n - s - 1)!} \cdot \frac{2}{(2n + 1)} & \text{if } n = n', \\ \frac{4s(n + s - 3)!}{(n - s - 1)!} & \text{if } n = n' + 2, \end{cases}$$
- (h) 
$$\int_{-1}^1 \mu P_n^s P_{n'}^s d\mu = \begin{cases} 0 & \text{if } n' \neq n \pm 1, \\ \frac{(n + s + 1)!}{(n - s)!} \cdot \frac{2}{(2n + 1)(2n + 3)} & \text{if } n' = n + 1, \end{cases}$$

$$(i) \quad \int_{-1}^1 \frac{P_n^s P_{n+1}^{s+1}}{\sqrt{1-\mu^2}} d\mu = \frac{2(n+s)!}{(n-s)!},$$

$$(j) \quad \int_{-1}^1 \frac{P_n^s P_{n+1}^{s-1}}{\sqrt{1-\mu^2}} d\mu = 0,$$

$$(k) \quad \int_{-1}^1 \sqrt{1-\mu^2} P_n^s P_{n'}^{s+1} d\mu = \begin{cases} 0 & \text{if } n' \neq n \pm 1, \\ \frac{2}{(2n+1)(2n+3)} \cdot \frac{(n+s+2)!}{(n-s)!} & \text{if } n' = n+1, \end{cases}$$

$$(l) \quad \int_{-1}^1 \frac{P_{n+1}^s P_{n'+1}^s}{(1-\mu^2)} d\mu = \begin{cases} \frac{(n+s-1)!}{s(n-s-1)!} & \text{if } n = n'+2, \\ \frac{(n+s+1)!}{s(n-s+1)!} & \text{if } n = n'. \end{cases}$$

Some of these formulae may be found in textbooks [ 1 ].

The Z force is given by

$$Z = Z_i - Z_0,$$

where  $Z_0$  is given by (6) above with a similar expression for  $Z_i$ . Consider, first, the integral

$$\begin{aligned} I_1 &= \int_0^{2\pi} \int_{-1}^1 \mu(1-\mu^2) \left( \frac{\partial \phi_0}{\partial \mu} \right)^2 d\omega d\mu \\ &= \int_0^{2\pi} \int_{-1}^1 \mu(1-\mu^2) \left[ \sum_{n=1}^P r^n S'_n + \sum_{n=1}^{\infty} \frac{W'_n}{r^{n+1}} \right]^2 d\omega d\mu, \end{aligned}$$

where, from (8) and (9),

$$S'_n = A_{0,n} \frac{\partial P_n}{\partial \mu} + \sum_{s=1}^n (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega) \frac{\partial P_n^s}{\partial \mu}$$

and



$$\dot{W}'_n = a_{0,n} \frac{\partial P_n}{\partial \mu} + \sum_{s=1}^n (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega) \frac{\partial P_n^s}{\partial \mu}.$$

Now the integral has to be independent of  $r$ , so that

$$\begin{aligned} I_1 &= \int_0^{2\pi} \int_{-1}^1 2\mu(1-\mu^2) \sum_{n=1}^{p-1} S'_{n+1} \dot{W}'_n d\omega d\mu \\ &= 2\pi \sum_{n=1}^{p-1} \int_{-1}^1 \left[ 2A_{0,n+1} a_{0,n+1} \frac{\partial P_{n+1}}{\partial \mu} \frac{\partial P_n}{\partial \mu} \right. \\ &\quad \left. + \sum_{s=1}^n (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) \frac{\partial P_{n+1}^s}{\partial \mu} \frac{\partial P_n^s}{\partial \mu} \right] \mu(1-\mu^2) d\mu. \end{aligned}$$

But

$$\begin{aligned} (10) \quad & \int_{-1}^1 \mu(1-\mu^2) \frac{\partial P_{n+1}^s}{\partial \mu} \frac{\partial P_n^s}{\partial \mu} d\mu \\ &= \int_{-1}^1 (1-\mu^2) \frac{\partial P_n^s}{\partial \mu} \left[ \left( \frac{n+s+1}{2n+3} \right) \frac{\partial P_n^s}{\partial \mu} + \left( \frac{n-s+2}{2n+3} \right) \frac{\partial P_{n+2}^s}{\partial \mu} \right] d\mu \\ &\quad + (n-s+1) \int_{-1}^1 P_{n+1}^s P_{n+1}^s d\mu - (n+1) \int_{-1}^1 \mu P_n^s P_{n+1}^s d\mu \text{ (by (b), (e))} \\ &= \left( \frac{n+s+1}{2n+3} \right) \int_{-1}^1 P_n^s \left[ n(n+1) - \frac{s^2}{(1-\mu^2)} \right] P_n^s d\mu \\ &\quad + \left( \frac{n-s+2}{2n+3} \right) \int_{-1}^1 P_n^s \left[ (n+2)(n+3) - \frac{s^2}{(1-\mu^2)} \right] P_{n+2}^s d\mu \\ &\quad + (n-s+1) \frac{(n+s+1)!}{(n-s+1)!} \cdot \frac{2}{(2n+3)} \\ &\quad - (n+1) \frac{(n+s+1)!}{(n-s)!} \cdot \frac{2}{(2n+1)(2n+3)} \quad \text{(by (f), (h))} \\ &= \frac{(n+s+1)n(n+1)}{(2n+3)} \int_{-1}^1 P_n^s P_n^s d\mu - s^2 \int_{-1}^1 \frac{\mu P_n^s P_{n+1}^s}{(1-\mu^2)} d\mu \\ &\quad + \frac{(n+s+1)!}{(n-s)!} \frac{2n}{(2n+3)(2n+1)}, \end{aligned}$$

which gives an expression for  $I_1$ .

In a similar manner,

$$(11) \quad I_2 = \int_0^{2\pi} \int_{-1}^1 r^2 \left( \frac{\partial \phi_0}{\partial r} \right)^2 \mu d\omega d\mu$$

$$= -2\pi \sum_{n=1}^{p-1} (n+1)^2 \int_{-1}^1 \left[ 2A_{0,n+1} a_{0,n} P_{n+1} P_n \right. \\ \left. + \sum_{s=1}^n (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) P_{n+1}^s P_n^s \right] \mu d\mu,$$

$$(12) \quad I_3 = \int_0^{2\pi} \int_{-1}^1 \left( \frac{\partial \phi_0}{\partial \omega} \right)^2 \frac{\mu}{(1-\mu^2)} d\omega d\mu$$

$$= 2\pi \sum_{n=1}^{p-1} \int_{-1}^1 \sum_{s=1}^n s^2 (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) P_{n+1}^s P_n^s \frac{\mu}{(1-\mu^2)} d\mu,$$

$$(13) \quad I_4 = \int_0^{2\pi} \int_{-1}^1 r \left( \frac{\partial \phi_0}{\partial \mu} \right) \left( \frac{\partial \phi_0}{\partial r} \right) (1-\mu^2) d\omega d\mu$$

$$= \pi \sum_{n=1}^{p-1} (n+1) \int_{-1}^1 \left[ 2A_{0,n+1} a_{0,n} \frac{\partial P_n}{\partial \mu} P_{n+1} \right. \\ \left. + \sum_{s=1}^n (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) \frac{\partial P_n^s}{\partial \mu} P_{n+1}^s \right] (1-\mu^2) d\mu$$

$$- \pi \sum_{n=1}^{p-1} (n+1) \int_{-1}^1 \left[ 2A_{0,n+1} a_{0,n} \frac{\partial P_{n+1}}{\partial \mu} P_n \right. \\ \left. + \sum_{s=1}^n (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) \frac{\partial P_{n+1}^s}{\partial \mu} P_n^s \right] (1-\mu^2) d\mu.$$

We see that from (6),

$$\frac{8\pi Z_0}{K_0} = I_1 - I_2 + I_3 - 2I_4,$$

from which, by using (10), (11), (12), (13), we get

$$(14) \quad Z_0 = \frac{K_0}{2} \sum_{n=1}^{p-1} \left[ 2(n+1) A_{0,n+1} a_{0,n} + \sum_{s=1}^n (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) \frac{(n+s+1)!}{(n-s)!} \right].$$

This determines  $Z$ , for the component  $Z_i$  can be written down in terms of the coefficients of the expansion of  $\phi_i$ . In the same way, when the components  $Y_0, X_0, N_0, M_0, L_0$  are determined then the components  $Y, X, N, M, L$  may be written down.

By using the same method as for  $Z_0$  above, the following results are obtained:

$$(15) \quad Y_0 = \frac{K_0}{4} \sum_{n=1}^{p-1} \left[ 2(n+1)(n+2) B_{1,n+1} a_{0,n} - 2n(n+1) A_{0,n+1} b_{1,n} - \sum_{s=1}^n \frac{(n+s+2)!}{(n-s)!} (A_{s+1,n+1} b_{s,n} - B_{s+1,n+1} a_{s,n}) - \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s,n+1} b_{s+1,n} - B_{s,n+1} a_{s+1,n}) \right],$$

$$(16) \quad X_0 = \frac{K_0}{4} \sum_{n=1}^{p-1} \left[ 2(n+1)(n+2) A_{1,n+1} a_{0,n} - 2n(n+1) A_{0,n+1} a_{1,n} + \sum_{s=1}^n \frac{(n+s+2)!}{(n-s)!} (A_{s+1,n+1} a_{s,n} + B_{s+1,n+1} b_{s,n}) - \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s,n+1} a_{s+1,n} + B_{s,n+1} b_{s+1,n}) \right],$$

$$(17) \quad N_0 = \frac{K_0}{2} \sum_{n=1}^p \sum_{s=1}^n \frac{s(n+s)!}{(n-s)!} (B_{s,n} a_{s,n} - A_{s,n} b_{s,n}),$$

$$(18) \quad M_0 = \frac{K_0}{4} \sum_{n=1}^p \left[ 2n(n+1) (A_{1,n} a_{0,n} - A_{0,n} a_{1,n}) \right]$$

$$+ \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s+1,n} a_{s,n} + B_{s+1,n} b_{s,n} - A_{s,n} a_{s+1,n} - B_{s,n} b_{s+1,n}) \Bigg].$$

$$(19) \quad L_0 = \frac{K_0}{4} \sum_{n=1}^p \left[ 2n(n+1) (A_{0,n} b_{1,n} - B_{1,n} a_{0,n}) + \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s+1,n} b_{s,n} + A_{s,n} b_{s+1,n} - B_{s+1,n} a_{s,n} - B_{s,n} a_{s+1,n}) \right].$$

It can be shown that the field becomes two-dimensional if

$$(20) \quad \begin{cases} s \text{ even, } b_{s,n} = B_{s,n} = 0, a_{s,n} = \frac{2(n-s)!}{n!} a_{0,n}, A_{s,n} = \frac{2n!}{(n+s)!} A_{0,n}, \\ s \text{ odd, } a_{s,n} = A_{s,n} = 0, b_{s,n} = \frac{(n-s)!}{(n-1)!} b_{1,n}, B_{s,n} = \frac{(n+1)!}{(n+s)!} B_{1,n}. \end{cases}$$

SPHERE IN A GENERAL FIELD. If a sphere of radius  $a$ , filled with homogeneous dielectric of specific inductive capacity  $K$ , is placed in air, with its center at the origin of coordinates in any electrostatic field whose potential function is  $\phi(x, y, z)$  having no singularities inside or on  $r = a$ , then the potential inside and outside the sphere are respectively

$$\phi_i = \frac{2}{(1+K)} \phi(x, y, z) + \frac{(K-1)}{(K+1)^2} \int_0^1 t^{-K/(K+1)} \phi(xt, yt, zt) dt$$

and

$$\begin{aligned} \phi_o &= \phi(x, y, z) - \frac{(K-1)}{(K+1)} \frac{a}{r} \phi(x_1, y_1, z_1) \\ &+ \frac{(K-1)}{(K+1)^2} \frac{a}{r} \int_0^1 t^{-K/(K+1)} \phi(x_1 t, y_1 t, z_1 t) dt, \end{aligned}$$

where

$$x_1 = \frac{a^2 x}{r^2}, \quad y_1 = \frac{a^2 y}{r^2}, \quad z_1 = \frac{a^2 z}{r^2} \quad \text{and} \quad r^2 = x^2 + y^2 + z^2.$$

It is assumed that there are no other boundaries present.

These results can be obtained either by the method of P. Weiss [6], or by expanding  $\phi(x, y, z)$  in harmonics and using the boundary conditions

$$\phi_i = \phi_0, \quad K \frac{\partial \phi_i}{\partial r} = \frac{\partial \phi_0}{\partial r} \quad \text{on } r = a.$$

If the original potential is given by

$$\phi(x, y, z) = \sum_{n=1}^p r^n S_n,$$

where

$$S_n = A_{0,n} P_n + \sum_{s=1}^n (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega) P_n^s,$$

and if we assume the interference potential to be given by

$$\phi_1 = \sum_{n=1}^{\infty} \frac{\ddot{w}_n}{r^{n+1}},$$

where

$$\ddot{w}_n = a_{0,n} P_n + \sum_{s=1}^n (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega) P_n^s,$$

then, by using the above result, we get

$$a_{0,n} = \begin{cases} \frac{n(1-K)}{n(1+K)+1} a^{2n+1} A_{0,n} & \text{if } n \leq p, \\ 0 & \text{if } n > p, \end{cases}$$

with similar expressions for  $a_{s,n}$ ,  $b_{s,n}$ . The forces are thus

$$Z = \frac{1}{2} \sum_{n=1}^{p-1} \frac{n(K-1) a^{2n+1}}{n(K+1)+1} \left[ 2(n+1) A_{0,n} A_{0,n+1} + \sum_{s=1}^n (A_{s,n} A_{s,n+1} + B_{s,n} B_{s,n+1}) \frac{(n+s+1)!}{(n-s)!} \right],$$

$$\begin{aligned}
 X = \frac{1}{4} \sum_{n=1}^{p-1} \frac{n(K-1) a^{2n+1}}{n(K+1)+1} & \left[ 2(n+1)(n+2) A_{0,n} A_{1,n+1} - 2n(n+1) A_{1,n} A_{0,n+1} \right. \\
 & + \sum_{s=1}^n (A_{s,n} A_{s+1,n+1} + B_{s,n} B_{s+1,n+1}) \frac{(n+s+2)!}{(n-s)!} \\
 & \left. - \sum_{s=1}^{n-1} (A_{s+1,n} A_{s,n+1} + B_{s+1,n} B_{s,n+1}) \frac{(n+s+1)!}{(n-s-1)!} \right],
 \end{aligned}$$

$$\begin{aligned}
 Y = \frac{1}{4} \sum_{n=1}^{p-1} \frac{n(K-1) a^{2n+1}}{n(K+1)+1} & \left[ 2(n+1)(n+2) A_{0,n} B_{1,n+1} - 2n(n+1) A_{0,n+1} B_{1,n} \right. \\
 & - \sum_{s=1}^n (B_{s,n} A_{s+1,n+1} - A_{s,n} B_{s+1,n+1}) \frac{(n+s+2)!}{(n-s)!} \\
 & \left. - \sum_{s=1}^{n-1} (B_{s+1,n} A_{s,n+1} - B_{s,n+1} A_{s+1,n}) \frac{(n+s+1)!}{(n-s-1)!} \right],
 \end{aligned}$$

$N = M = L = 0.$

The potential inside, being of the form

$$\phi_i = \sum_{n=1}^p r^n S_n,$$

contributes nothing to the forces.

The forces on bodies with surfaces  $r = a + \epsilon P_n$  ( $\epsilon$  small) can also be easily evaluated.

As an example [1, p. 290, ex. 31], take a positive point-charge  $e$  at the point  $(0, 0, c)$ ,  $c > a$ . We have

$$A_{0,n} = \frac{e}{c^{n+1}}, \quad A_{s,n} = 0,$$

and so

$$Z = e^2 \sum_{n=1}^{\infty} \frac{n(n+1)(K-1)}{n(K+1)+1} \cdot \frac{a^{2n+1}}{c^{2n+3}}, \quad Y = 0, \quad X = 0.$$

The resultant attraction between the sphere and point-charge may thus be written

$$\frac{e^2 a^3 \alpha}{2 c^3} \left\{ \frac{2 c^2}{(c^2 - a^2)^2} + \frac{(1 + \alpha)}{(c^2 - a^2)} - \frac{c(1 - \alpha^2)}{a^3} \left(\frac{a}{c}\right)^2 \int_0^{a/c} \frac{x^{2-\alpha}}{(1-x^2)} dx \right\},$$

where

$$\alpha = \frac{K-1}{K+1}.$$

Equations (14) - (19) can be used for the forces on a body in a liquid moving irrotationally and extending to infinity, simply by putting  $K_0 = 4\pi\rho$ . Elementary cases have been considered [5].

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