

Pacific Journal of Mathematics

EXTENSION OF A RENEWAL THEOREM

DAVID BLACKWELL

EXTENSION OF A RENEWAL THEOREM

DAVID BLACKWELL

1. Introduction. A chance variable x will be called a d -lattice variable if

$$(1) \quad \sum_{n=-\infty}^{\infty} \Pr\{x = nd\} = 1$$

and

$$(2) \quad d \text{ is the largest number for which (1) holds.}$$

If x is not a d -lattice variable for any d , x will be called a *nonlattice variable*. The main purpose of this paper is to give a proof of:

THEOREM 1. *Let x_1, x_2, \dots be independent identically distributed chance variables with $E(x_1) = m > 0$ (the case $m = +\infty$ is not excluded); let $S_n = x_1 + \dots + x_n$; and, for any $h > 0$, let $U(a, h)$ be the expected number of integers $n \geq 0$ for which $a \leq S_n < a + h$. If the x_n are nonlattice variables, then*

$$U(a, h) \rightarrow \frac{h}{m}, 0 \quad \text{as } a \rightarrow +\infty, -\infty.$$

If the x_n are d -lattice variables, then

$$U(a, d) \rightarrow \frac{d}{m}, 0 \quad \text{as } a \rightarrow +\infty, -\infty.$$

(If $m = +\infty$, h/m and d/m are interpreted as zero.)

This theorem has been proved (A) for nonnegative d -lattice variables by Kolmogorov [5] and by Erdős, Feller, and Pollard [4]; (B) for nonnegative nonlattice variables by the writer [1], using the methods of [4]; (C) for d -lattice variables by Chung and Wolfowitz [3]; (D) for nonlattice variables such that the distribution of some S_n has an absolutely continuous part and $m < \infty$ by Chung

Received June 28, 1952. This paper was written under an Office of Naval Research contract.

and Pollard [2], using a purely analytical method; and (E) in the form given here by Harris (unpublished). Harris' proof does not essentially use the results of the special cases (A), (B), (C), (D); the proof given here obtains Theorem 1 almost directly from the special cases (A) and (B) by way of an integral identity and an equation of Wald.

2. An integral identity. Let N_1 be the smallest n for which $S_n > 0$, and write $z_1 = S_{N_1}$; let N_2 be the smallest $n > 0$, for which $S_{N_1+n} - S_{N_1} > 0$, and write $z_2 = S_{N_1+N_2} - S_{N_1}$, and so on. Continuing in this way, we obtain sequences $N_1, N_2, \dots; z_1, z_2, \dots$ of independent, positive, identically distributed chance variables such that

$$S_{N_1+\dots+N_K} = z_1 + \dots + z_K.$$

Let $V(t), R(t)$ denote the expected number of integers $n \geq 0$ for which

$$T_n = z_1 + \dots + z_n \leq t \text{ and } -t \leq S_n \leq 0,$$

$n < N_1$, respectively. That $V(t) < \infty$ follows from a theorem of Stein [6], and that $R(t) < \infty$ follows from $E(N_1) < \infty$, which we show in the next section. The integral identity is:

THEOREM 2. $U(a, h) = \int_0^\infty [R(t-a) - R(t-a-h)] dV(t).$

Proof. If n_K is the number of integers n with

$$N_1 + \dots + N_K \leq n < N_1 + \dots + N_{K+1} \text{ and } a \leq S_n < a + h,$$

we have

$$E(n_K | T_K = t) = R(t-a) - R(t-a-h),$$

so that

$$E(n_K) = \int_0^\infty [R(t-a) - R(t-a-h)] dF_K(t),$$

where $F_K(t) = \Pr\{T_K \leq t\}$. Summing over $K = 0, 1, 2, \dots$, and using the fact that

$$V(t) = \sum_{K=0}^\infty F_K(t),$$

we obtain the theorem.

3. Wald's equation. The main purpose of this section is to note that $E(N_1)$ is finite, so that an equation of Wald [7, p. 142] holds.

THEOREM 3. $E(N_1) < \infty$ and $mE(N_1) = E(z_1)$, so that $m, E(z_1)$ are both finite or both infinite.

Proof. In showing $E(N_1)$ finite, we may suppose $\{x_n\}$ bounded above; for defining $x_n^* = \min\{s_n, M\}$ yields an $N_1^* \geq N_1$; choosing M sufficiently large makes $E(x_n^*) > 0$, and $E(N_1^*) < \infty$ implies $E(N_1) < \infty$. Since

$$\frac{T_K}{K} = \frac{S_{N_1 + \dots + N_K}}{N_1 + \dots + N_K} \cdot \frac{N_1 + \dots + N_K}{K},$$

we obtain, letting $K \rightarrow \infty$ and using the strong law of large numbers, first that $E(z_1) = mE(N_1)$ and next since if $\{x_n\}$ is bounded above and $\{z_n\}$ is bounded, that $E(N_1)$ is finite in this case and consequently in general.

4. The d-lattice case. For d -lattice variables, Theorem 2 yields

$$(3) \quad U(nd, d) = \sum_{s=0}^{\infty} r(s-n) v(s) = \sum_{s=0}^{\infty} r(s) v(s+n),$$

where $r(s) = R(sd) - R([s-1]d)$ and $v(s) = V(sd) - V([s-1]d)$. Now

$$\sum_{s=0}^{\infty} r(s) = \lim_{t \rightarrow \infty} R(t) = E(N_1) < \infty.$$

Theorem (A) asserts that

$$v(n) \rightarrow \frac{d}{E(z_1)}, 0 \quad \text{as } n \rightarrow \infty, -\infty;$$

applying this to (1) yields

$$U(nd, d) \rightarrow \frac{dE(N_1)}{E(z_1)}, 0 \quad \text{as } n \rightarrow \infty, -\infty,$$

and Wald's equation yields Theorem 1 for d -lattice variables.

5. The nonlattice case. For nonlattice variables we have, rewriting Theorem

2 with a change of variable,

$$U(a, h) = \int_M^\infty [R(t) - R(t-h)] dV(t+a).$$

For any $M > 0$, write

$$U(a, h) = I_1(M, a, h) + I_2(M, a, h),$$

where

$$I_1 = \int_0^M [R(t) - R(t-h)] dV(t+a)$$

and

$$I_2 = \int_0^\infty [R(t) - R(t-h)] dV(t+a).$$

Theorem B applied to $\{z_n\}$ yields

$$V(t+h) - V(t) \rightarrow \frac{h}{E(z_1)}$$

for all $h > 0$ as $t \rightarrow \infty$, so that, since $R(t)$ is monotone,

$$\begin{aligned} I_1 &= \int_0^M R(t) dV(t+a) - \int_0^{M-h} R(t) dV(t+a+h) \\ &\rightarrow \frac{1}{E(z_1)} \cdot \int_{M-h}^M R(t) dt, \quad 0 \quad \text{as } a \rightarrow \infty, -\infty \end{aligned}$$

for fixed M, h . We now show that, for fixed h , $I_2(M, a, h) \rightarrow 0$ as $M \rightarrow \infty$ uniformly in a . We have

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty} \int_{M+nh}^{M+(n+1)h} [R(t) - R(t-h)] dV(t+a) \\ &\leq \sum_{n=0}^{\infty} R_1(M, n) [V(a+M+(n+1)h) - V(a+M+nh)], \end{aligned}$$

where

$$R_1(M, n) = \sup [R(t) - R(t-h)]$$

as t varies over the interval $(M + nh, M + (n + 1)h)$. Since, by Theorem (B),

$$V(b+h) - V(b) \rightarrow \frac{h}{E(z_1)} \quad \text{as } b \rightarrow \infty,$$

there is a constant c (for the given h) such that

$$I_2(M, a, h) \leq c \sum_{n=0}^{\infty} R_1(M, n) \quad \text{for all } M \text{ and } a.$$

Now

$$\sum_{n=0}^{\infty} R_1(M, 2n) \leq E(N_1) - R(M) \quad \text{and} \quad \sum_{n=0}^{\infty} R_1(M, 2n+1) \leq E(N_1) - R(M),$$

and $R(M) \rightarrow E(N_1)$ as $M \rightarrow \infty$. Thus

$$|U(a, h) - I_1(M, a, h)| < \epsilon(M, h)$$

for all a , where $\epsilon(M, h) \rightarrow 0$ as $M \rightarrow \infty$ for fixed h . Then

$$\begin{aligned} \left| U(a, h - \frac{hE(N_1)}{E(z_1)}) \right| &\leq \epsilon(M, h) + \left| I_1(M, a, h) - \frac{1}{E(z_1)} \int_{M-h}^M R(t) dt \right| \\ &\quad + \left| \frac{1}{E(z_1)} \int_{M-h}^M R(t) dt - hE(N_1) \right|, \end{aligned}$$

so that

$$\begin{aligned} \limsup_{a \rightarrow \infty} \left| U(a, h) - \frac{hE(N_1)}{E(z_1)} \right| \\ \leq \epsilon(M, h) + \frac{1}{E(z_1)} \left| \int_{M-h}^M R(t) dt - hE(N_1) \right|. \end{aligned}$$

Letting $M \rightarrow \infty$ yields

$$U(a, h) \rightarrow \frac{hE(N_1)}{E(z_1)} \quad \text{as } a \rightarrow \infty,$$

and Wald's equation yields Theorem 1 for $a \rightarrow \infty$. Similarly,

$$U(a, h) \leq \epsilon(M, h) + |I_1(M, a, h)|$$

for all a , so that

$$\limsup_{a \rightarrow -\infty} U(a, h) \leq \epsilon(M, h)$$

and $U(a, h) \rightarrow 0$ as $a \rightarrow -\infty$. This completes the proof.

REFERENCES

1. D. Blackwell, *A renewal theorem*, Duke Math. J. **15** (1948), 145-151.
2. K. L. Chung and Harry Pollard, *An extension of renewal theory*, Proc. Amer. Math. Soc. **3** (1952), 303-309.
3. K. L. Chung and J. Wolfowitz, *On a limit theorem in renewal theory*, Ann. of Math. **55** (1952), 1-6.
4. P. Erdős, W. Feller, and H. Pollard, *A theorem on power series*, Bull. Amer. Math. Soc. **55** (1949), 201-204.
5. A. Kolmogorov, *Anfangsgründe der Theorie der Markoffschen Ketten mit unendlich vielen möglichen Zuständen*, Mat. Sbornik N.S. **1** (1936), 607-610.
6. C. Stein, *A note on cumulative sums*, Ann. Math. Statistics **17** (1946), 498-499.
7. A. Wald, *Sequential tests of statistical hypotheses*, Ann. Math. Statistics **16** (1945) 117-186.

STANFORD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

R. M. ROBINSON
University of California
Berkeley 4, California

E. HEWITT
University of Washington
Seattle 5, Washington

R. F. DILWORTH
California Institute of Technology
Pasadena 4, California

E. F. BECKENBACH
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

H. BUSEMANN	P. R. HALMOS	BØRGE JESSEN	J. J. STOKER
HERBERT FEDERER	HEINZ HOPF	PAUL LÉVY	E. G. STRAUS
MARSHALL HALL	R. D. JAMES	GEORGE PÓLYA	KÔSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA	UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY	STANFORD RESEARCH INSTITUTE
UNIVERSITY OF CALIFORNIA, BERKELEY	STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA, DAVIS	WASHINGTON STATE COLLEGE
UNIVERSITY OF CALIFORNIA, LOS ANGELES	UNIVERSITY OF WASHINGTON
UNIVERSITY OF CALIFORNIA, SANTA BARBARA	* * *
UNIVERSITY OF NEVADA	AMERICAN MATHEMATICAL SOCIETY
OREGON STATE COLLEGE	NATIONAL BUREAU OF STANDARDS,
UNIVERSITY OF OREGON	INSTITUTE FOR NUMERICAL ANALYSIS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors except Robinson, whose term expires with the completion of the present volume; they might also be sent to M. M. Schiffer, Stanford University, Stanford, California, who is succeeding Robinson. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$8.00; single issues, \$2.50. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS · BERKELEY AND LOS ANGELES

COPYRIGHT 1953 BY PACIFIC JOURNAL OF MATHEMATICS

William George Bade, <i>An operational calculus for operators with spectrum in a strip</i>	257
E. F. Beckenbach and Lloyd Kenneth Jackson, <i>Subfunctions of several variables</i>	291
David Blackwell, <i>Extension of a renewal theorem</i>	315
L. Carlitz, <i>Some theorems on the Schur derivative</i>	321
Paul Arnold Clement, <i>Generalized convexity and surfaces of negative curvature</i>	333
Merrill M. Flood, <i>On the Hitchcock distribution problem</i>	369
Watson Bryan Fulks, <i>On the unique determination of solutions of the heat equation</i>	387
John W. Green, <i>Length and area of a convex curve under affine transformation</i>	393
William Gustin, <i>An isoperimetric minimax</i>	403
Arthur Eugene Livingston, <i>Some Hausdorff means which exhibit the Gibbs' phenomenon</i>	407
Charles Loewner, <i>On generation of solutions of the biharmonic equation in the plane by conformal mappings</i>	417
Gábor Szegő, <i>Remark on the preceding paper of Charles Loewner</i>	437
Immanuel Marx and G. Piranian, <i>Lipschitz functions of continuous functions</i>	447
Ting-Kwan Pan, <i>The spherical curvature of a hypersurface in Euclidean space</i>	461
Ruth Lind Potter, <i>On self-adjoint differential equations of second order</i>	467
E. H. Rothe, <i>A note on the Banach spaces of Calkin and Morrey</i>	493
Eugene Schenkman, <i>A generalization of the central elements of a group</i>	501
A. Seidenberg, <i>A note on the dimension theory of rings</i>	505