SOME THEOREMS ON THE SCHUR DERIVATIVE

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1. Introduction. Given the sequence \( \{a_m\} \) and \( p \neq 0 \), Schur [5] defined the derivative \( a'_m \) by

\[
a'_m = \Delta a_m = (a_{m+1} - a_m)/p^{m+1};
\]

higher derivatives are defined by means of

\[
a^{(r)}_m = \Delta^r a_m = \Delta (a^{(r-1)}_m), \quad a^{(0)}_m = a_m.
\]

In particular if \( p \) is a prime, \( a \) an integer and \( a_m = a^{p^m} \), then by Fermat's theorem

\[
a'_m = (a^{p^m+1} - a^{p^m})/p^{m+1}
\]

is integral. Schur proved that if \( p \nmid a \), then also the derivatives

\[
\Delta^2 a^{p^m}, \Delta^3 a^{p^m}, \ldots, \Delta^{p-1} a^{p^m}
\]

are all integral. Moreover if \( a'_0 \equiv 0 \pmod{p} \) then all the derivatives \( \Delta^r a^{p^m} \) are integral, while if \( a'_0 \not\equiv 0 \pmod{p} \) then every number of \( \Delta^p a^{p^m} \) has the denominator \( p \).

A. Brauer [1] gave another proof of Schur's results. About the same time Zorn [6] proved these results by \( p \)-adic methods and indeed proved the following stronger theorem. For \( x \equiv 1 \pmod{p} \), define

\[
X_m = (x^{p^m} - 1)/p^{m+1},
\]

and as above let \( \Delta^r X_m \) denote the \( r \)-th derivative of \( X_m \); then

\[
\Delta^r X_m = \frac{(p-1) (p^2 - 1) \cdots (p^r - 1)}{(r+1)!} X_m^{r+1} \pmod{p^m}
\]

provided \( r < p \); for \( r < p - 2 \), the congruence (1.2) holds \( (\mod{p^{m+1}}) \). It is also shown that Schur's theorem is an easy consequence of Zorn's results.
In the present paper we shall give a simple elementary proof of Zorn's congruences. In addition we prove, for example, that for $r \leq p$,

$$
\Delta^r a^p m \equiv \frac{1}{r!} a^p m - \frac{\prod_{i=1}^r (p^i - 1)}{(p - 1)^r} \pmod{p^m},
$$

where

$$a^{(p-1)p^m} = 1 + p^{m+1} q_m,$$

for $r < p - 1$, (1.3) holds $\pmod{p^{m+1}}$.

We next (§4) extend Schur's and Zorn's theorems to algebraic numbers. In §5 we consider a generalization of another kind suggested by the arithmetic function (see for example [2, p. 84-86])

$$F(a, m) = \sum_{d e = m} \mu(d) a^e.$$

Finally (§6), we give some applications of Schur's theorem to the Euler and Bernoulli polynomials and numbers; the results are analogous to Kummer's congruences [3, Ch. 12]. In particular $\Delta^r E_{k+p^m}$ is integral $\pmod{p}$ for $p > 2, r < p, r \leq m$; also $\Delta^r (B_{k+p^m}/(k + p^m))$ is integral $\pmod{p}$ for $p - 1 \nmid k + 1, r < p, r \leq m$. Here $E_k$ and $B_k$ denote the Euler and Bernoulli numbers in the notation of Nörlund [3].

2. Formulas for $\Delta^r a_m$. We shall require some preliminary results.

**Lemma 1.** The following identity holds:

$$\prod_{i=0}^{r-1} (x - p^i) = \sum_{i=0}^r (-1)^i \left[ \begin{array}{c} r \\ i \end{array} \right] p^{i(i-1)/2} x^{r-i},$$

where

$$\left[ \begin{array}{c} r \\ i \end{array} \right] = \frac{(p^r - 1)(p^{r-1} - 1) \cdots (p^{r-i+1} - 1)}{(p - 1)(p^2 - 1) \cdots (p^i - 1)} = \left[ \begin{array}{c} r \\ r-i \end{array} \right], \left[ \begin{array}{c} r \\ 0 \end{array} \right] = 1.$$

**Lemma 2.** Put

$$w_{k,r} = \sum_{i=0}^{k} (-1)^i \left[ \begin{array}{c} k \\ i \end{array} \right] (p^{k-i}) p^{i(i-1)/2},$$
where \( \binom{m}{r} \) denotes a binomial coefficient. Then

\[
\mathbb{W}_{k,r} = \begin{cases} 
0 & (r < k) \\
\frac{1}{r!} \prod_{i=0}^{r-1} (p^r - p^i) & (r = k) \\
\frac{1}{r!} p^{k(k-1)/2} U_{k,r} & (r > k),
\end{cases}
\]

where \( U_{k,r} \) is an integer.

Lemma 1 is well known. To prove Lemma 2, we note first that the binomial coefficient \( \binom{x}{r} \) is a polynomial in \( x \) of degree \( r \). Since by (2.1)

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} p^{i(i-1)/2} p^{r(k-i)} = \prod_{i=0}^{k-1} (p^r - p^i),
\]

the several parts of (2.3) follow without much difficulty.

**Lemma 3.** For an arbitrary sequence \( \{ a_m \} \),

\[
\Delta' a_m = p^{-rm-r(r+1)/2} \sum_{i=0}^{r} (-1)^i \binom{r}{i} p^{i(i-1)/2} a_{m+r-i}.
\]

This formula, which is given by Schur, is easily proved. In view of (2.1) it can be put in the following symbolic form:

\[
\Delta' a_m = p^{-rm-r(r+1)/2} a^m \prod_{i=0}^{r-1} (a - p^i),
\]

where it is understood that after expansion of the right member \( a^k \) is to be replaced by \( a_k \).

Suppose now that \( p \nmid a \) and put

\[
a^{(p-1)p^m} = 1 + p^m q_m,
\]

so that \( q_m \) is integral. Then by the binomial theorem we have

\[
a^{(p-1)p^{m+s}} = \sum_{i=0}^{p^s} (p^s_i) p^{(m+1)i} q_m^i, \quad (r \geq s),
\]
and by (2.4) this implies

\[ p^{m+r(r+1)/2} \Delta^r a^{(p-1)p^m} \]

\[ = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} p^{(r-s)(r-s-1)/2} \sum_{i=0}^{p^r} (p^s_i) p^{(m+1)i} q^i_m \]

\[ = \sum_{i=0}^{p^r} p^{(m+1)i} q^i_m \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (p^s_i) p^{(r-s)(r-s-1)/2} \]

\[ = \sum_{i=0}^{p^r} p^{(m+1)i} q^i_m \left[ \prod_{i=1}^{r} (p^i - 1) \right] + \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)i+(r-1)/2} q^i_m U_{r,i}, \]

by (2.3); \( U_{r,i} \) and \( U_{r,i} \) have the same meaning as in Lemma 2. We thus get

(2.7) \( \Delta^r a^{(p-1)p^m} = \frac{1}{r!} q^r_m \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)(i-r)/2} q^i_m U_{r,i} \).

We next set up a similar formula for \( \Delta^r q_m \), where \( q_m \) is defined by (2.6). Indeed substitution in (2.1) gives

\[ p^{m+r(r+1)/2} \Delta^r q_m = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} p^{(r-s)(r-s-1)/2} \left(- (m+s+1) \right) (a^{(p-1)p^{m+s}} - 1) \]

\[ = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} p^{(r-s)(r-s-1)/2} \left(- (m+s+1) \right) \sum_{i=1}^{p^s} (p^s_i) p^{(m+1)i} q^i_m \]

\[ = \sum_{i=1}^{p^r} p^{(m+1)(i-1)} q^i_m \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (p^s_i) p^{(r-s)(r-s-1)/2} - s \]

\[ = \frac{1}{(r+1)!} \right) \sum_{i=1}^{r} (p^i - 1) + \sum_{i=r+2}^{p^r} \frac{1}{i!} p^{(m+1)(i-1)+(r-1)/2} q^i_m U_{r,i}, \]
by a slight modification of Lemma 2; the coefficient $U_{r,i}$ is integral and is defined by

$$\frac{1}{i!} p^{(r-1)/2} U_{r,i} = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left( p_i^{r-s} \right) p^{s(s-1)/2} - (r-s).$$

Hence

$$\Delta^r q_m = \frac{1}{(r+1)!} q_m^{r+1} \prod_{i=1}^{r} (p_i - 1) + \sum_{i=r+2}^{r} \frac{1}{i!} p^{(m+1)(i-r-1)} q_m^i U_{r,i}.$$ 

Using the same method we can also evaluate $\Delta^r a^{m+1}$. It follows from (2.6) that

$$a^{m+s} = a^{m+1} \left( 1 + p^{m+1} q_m \right)^{e_s} \quad \left( e_s = \frac{p^s - 1}{p - 1} \right),$$

and thus substitution in (2.4) yields

$$p^{r+m+(r+1)/2} \Delta^r a^{m+1} = a^{m+1} \sum_{s=0}^{e_r} (-1)^{r-s} \binom{r}{s} p^{(r-s)(r-s-1)/2} \sum_{i=0}^{e_r} \binom{e_s}{i} p^{(m+1)i} q_m^i$$

$$= a^{m+1} \sum_{i=0}^{e_r} p^{(m+1)i} q_m^i \sum_{s=0}^{e_r} (-1)^{r-s} \binom{r}{s} \binom{e_s}{i} p^{(r-s)(r-s-1)/2}.$$ 

Since $\binom{e_s}{i}$ is a polynomial in $p^s$ of degree $i$, the same reasoning as before applies and we get after a little manipulation

$$\Delta^r a^{m+1} = \frac{1}{r!} a^{m+1} q_m^r \prod_{i=1}^{r} (p_i - 1) \left( p - 1 \right)^r$$

$$+ a^{m+1} \sum_{i=r+1}^{e_r} \frac{1}{i!} p^{(m+1)(i-r)} q_m^i U_{r,i},$$

where $U_{r,i}$ is integral.

Comparison of (2.7) and (2.10) shows that (2.7) is included in (2.10). Indeed it is easy to set up the following formula which includes both (2.7) and (2.10):
\[(2.11) \quad \Delta' a^{kp^m} = \frac{1}{r!} a^{kp^m} q_m^r k^r \prod_{i=1}^r (p^i - 1) (p - 1)^r \]

\[+ \quad a^{kp^m} \sum_{i=r+1}^{e_r} \frac{1}{i!} p(m+1)(i-r) q_m^i V_{r,i}, \]

where \(V_{r,i} = V^{(k)}_{r,i}\) is integral and \(k \geq 1\). The proof of (2.11) is exactly like the proof of (2.10); the first step is to raise both members of (2.9) to the \(k\)-th power.

3. The main results. In order to make use of (2.7) and (2.10) it is evidently necessary to examine \(p^{(m+1)(i-r)/i!}\). We suppose \(i > r, r \leq p\). Then in the first place it is easily seen [6, p. 462] that \(p^{i-r}/i!\) is integral \((\text{mod } p)\), and a simple discussion shows that \(p^{i-r}/i!\) is divisible by \(p\) unless (i) \(i = p, r = p - 1\), or (ii) \(i = p + 1, r = p\). We now state:

**Theorem 1.** The derivative \(\Delta' a^{(p-1)p^m}\) is integral for \(1 \leq r \leq p - 1\), while \(\Delta a^{(p-1)p^m}\) has the denominator \(p\) provided \(a^{p-1} \not\equiv 1 \pmod{p^2}\); if \(a^{p-1} \equiv 1 \pmod{p^2}\) then all \(\Delta' a^{(p-1)p^m}\) are integral.

**Theorem 2.** For \(1 \leq r \leq p, m \geq 0\),

\[(3.1) \quad \Delta' a^{(p-1)p^m} \equiv \frac{1}{r!} q_m^r \prod_{i=1}^r (p^i - 1) \pmod{p^m}; \]

if \(r < p - 1\), the congruence is valid \((\text{mod } p^{m+1})\).

**Theorem 3.** The derivative \(\Delta' a^{p^m}\) is integral for \(1 \leq r \leq p - 1\), while \(\Delta a^{p^m}\) has the denominator \(p\) provided \(a^{p-1} \not\equiv 1 \pmod{p^2}\); if \(a^{p-1} \equiv 1 \pmod{p^2}\) then all \(\Delta' a^{(p-1)p^m}\) are integral.

**Theorem 4.** For \(1 \leq r \leq p, m \geq 0\),

\[(3.2) \quad \Delta' a^{p^m} \equiv \frac{1}{r!} a^{p^m} q_m^r \prod_{i=1}^r (p^i - 1) (p - 1)^r \pmod{p^m}; \]

if \(r < p - 1\), the congruence is valid \((\text{mod } p^{m+1})\).

If we make use of (2.11) rather than (2.7) or (2.10) we get the following more general result.

**Theorem 4'.** For \(1 \leq r \leq p, m \geq 0\)
if \( r < p - 1 \), the congruence is valid \((\text{mod } p^{m+1})\).

To apply (2.8) we first examine \( p^{i-r-1}/i! \) for \( i > r + 1 \), \( r + 1 \leq p \). We have:

**Theorem 5.** The derivative \( \Delta^r q_m \) is integral for \( 1 \leq r \leq p - 2 \), while \( \Delta^{p-1} q_m \) has the denominator \( p \) provided \( a^{p-1} \not\equiv 1 \pmod{p^2} \); if \( a^{p-1} \equiv 1 \pmod{p^2} \) then all \( \Delta^r q_m \) are integral.

**Theorem 6.** For \( 1 \leq r \leq p - 1 \), \( m \geq 0 \),

\[
\Delta^r q_m = \frac{1}{(r+1)!} q_m^{r+1} \prod_{i=1}^{r} (p^i - 1) \pmod{p^m};
\]

if \( r < p - 2 \), the congruence is valid \((\text{mod } p^{m+1})\).

Theorem 3 is of course Schur's theorem; Theorems 5 and 6 are due to Zorn. The remaining theorems are presumably new.

**4. Generalization for algebraic numbers.** Let \( k \) be an algebraic number field of degree \( n \) and let \( \mathfrak{p} \) denote a prime ideal of \( k \); also let

\[
N\mathfrak{p} = p^f; \quad \mathfrak{p}^e \mid p, \quad \mathfrak{p}^{e+1} \not\mid p;
\]

for simplicity we assume \( p > n \). If \( \alpha \in k \) is integral \((\text{mod } \mathfrak{p})\) and \( \mathfrak{p} \not\mid \alpha \), then by Fermat's Theorem

\[
\alpha^{p^f-1} = 1 + \beta, \quad \beta \equiv 0 \pmod{\mathfrak{p}}.
\]

It follows from (4.2) that

\[
\alpha^{(p^f-1)p^m} = 1 + \beta_m, \quad \beta_m \equiv 0 \pmod{\mathfrak{p}^{me+1}},
\]

while (4.3) implies

\[
\alpha^{(p^f-1)p^{m+s}} = \sum_{i=0}^{p^r} \left( \begin{array}{c} p^s \end{array} \right) \beta_m^i
\]

\((r \geq s)\).

Then, exactly as in \( \S 2 \),
\[ p^{rm+r(r+1)/2} \Delta^r \alpha^{(p^f-1)p^m} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} p^{(r-s)(r-s-1)/2} \sum_{i=0}^{r} \binom{p^s}{i} \beta_m^{i} \]

application of Lemma 2 now leads to

\[ (4.5) \Delta^r \alpha^{(p^f-1)p^m} = \frac{1}{r!} p^{-r(m+1)} \beta_m^r \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{-r(m+1)} \beta_m^{i} \omega_{r,i}, \]

where \( \omega_{r,i} \) is integral. Note that for \( e > 1 \) the right member of (4.5) need not be integral. Accordingly we assume \( e = 1 \); the assumption \( p > n \) is then no longer needed.

We now have:

**Theorem 7.** Let \( N \not\mid p, \ not \ p, \ not \ \alpha \); then \( \Delta^r \alpha^{(p^f-1)p^m} \) is integral for \( 1 \leq r \leq p - 1 \), while \( \Delta^P \alpha^{(p^f-1)p^m} \) has the denominator \( p \) provided \( \alpha^{p^f-1} \not\equiv 1 \) (mod \( p^2 \)); if \( \alpha^{p^f-1} \equiv 1 \) (mod \( p^2 \)) then all \( \Delta^r \alpha^{(p^f-1)p^m} \) are integral.

**Theorem 8.** With the hypotheses of Theorem 7,

\[ (4.6) \Delta^r \alpha^{(p^f-1)p^m} \equiv \frac{1}{r!} p^{-r(m+1)} \beta_m^r \prod_{i=1}^{r} (p^i - 1) \quad (\text{mod } p^m) \]

for \( r \leq p \); if \( r < p - 1 \) the congruence is valid (mod \( p^{m+1} \)).

In order to extend Theorems 3 and 4′ it is convenient to suppose that \( \varphi \) is a prime ideal of the first degree. The following two theorems may be proved.

**Theorem 9.** Let \( N \not\mid p, \ not \ p, \ not \ \alpha \); then \( \Delta^r \alpha^{p^m} \) is integral for \( 1 \leq r \leq p - 1 \), while \( \Delta^P \alpha^{p^m} \) has the denominator \( p \) provided \( \alpha^{p^f-1} \not\equiv 1 \) (mod \( p^2 \)); if \( \alpha^{p^f-1} \equiv 1 \) (mod \( p^2 \)) then all \( \Delta^r \alpha^{p^m} \) are integral.

**Theorem 10.** With the hypotheses of Theorem 9,

\[ (4.7) \Delta^r \alpha^{kp^m} \equiv \frac{1}{r!} \left( \frac{k \beta_m}{p^{m+1}} \right)^r \prod_{i=1}^{r} \frac{(p^i - 1)}{(p-1)^r} \quad (\text{mod } p^m) \]

for \( r \leq p \); if \( r < p - 1 \) the congruence is valid (mod \( p^{m+1} \)).
For brevity we omit the extension of Theorems 5 and 6 for algebraic numbers.

5. Another generalization. Changing slightly the notation (1.1) we put

\[(5.1) \quad \Delta_p a_{mp^i} = (a_{mp^{i+1}} - a_{mp^i})/p^{i+1},\]

and

\[(5.1) \quad \Delta_p a_{mp^i} = (\Delta_p a_{mp^{i+1}} - \Delta_p a_{mp^i})/p^{i+1}.\]

Then clearly \(\Delta_p \Delta_q = \Delta_q \Delta_p\). If \(a\) and \(k\) are arbitrary integers then if follows from a well-known theorem concerning (1.4) that

\[(5.2) \quad \delta_k a^k = \Delta_{p_1} \cdots \Delta_{p_s} a^k \quad (k = p_1^{e_1} \cdots p_s^{e_s})\]

is integral. In view of Schur's theorem we can state the following generalization.

**Theorem 11.** Let \((a, k) = 1\) and let \(r < \) the smallest prime dividing \(k\); define

\[(5.3) \quad \delta^r_k a^k = \delta_k \delta^{r-1}_k a^k.\]

Then \(\delta^r_k a_k\) is integral for \(k > 1\).

Indeed because of the commutativity of the operators \(\Delta_{p_i}\) we need only observe that (5.2) and (5.3) imply

\[(5.4) \quad \delta^r_k a^k = \Delta^r_{p_1} \cdots \Delta^r_{p_s} a^k\]

and the theorem follows immediately.

The restriction \((a, k) = 1\) can be removed by taking \(k\) sufficiently large as we shall see below.

A slight extension of Theorem 11 is contained in:

**Theorem 12.** Let

\[(a, k) = 1, \quad k = p_1^{e_1} \cdots p_s^{e_s},\]

and let \(r_j < p_j, \quad j = 1, \cdots, s\); then

\[(5.5) \quad \Delta^r_{p_1} \cdots \Delta^r_{p_s} a^k\]

is integral for all \(k > 1\).

We remark that the function defined in (5.2) can also be expressed in the form
\[ \delta_k a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d) a^{dk}, \]

where \( \mu(d) \) is the Möbius function and

\[ k_1 = p_1^{e_1+1} \cdots p_s^{e_s+1}; \]

similarly (5.3) becomes

\[ \delta_k a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d) \delta_{k}^{-1} a^{dk}. \]

Formulas of a different kind can be obtained by applying (2.4) to (5.4) and (5.5); for example, (2.5) suggests the following symbolic formula:

\[ \delta_k a^k = k^{-r} \prod_{j=1}^s p_j^{r(r+1)/2} \cdot \prod_{j=1}^s a_j^{e_j} \prod_{i=0}^{r-1} (a_j - p_j^i), \]

where after expansion \( a_1^{e_1} \cdots a_s^{e_s} \) is to be replaced by \( a^m \),

\[ m = p_1^{f_1} \cdots p_s^{f_s}. \]

A similar but slightly more complicated formula can be stated for (5.5). We shall omit the generalization of Theorems 11 and 12 to algebraic numbers.

6. Applications. In the theorems of § 2 it is assumed that \( p \nmid a \). However Theorem 3, for example, is easily extended to the case \( p \mid a \). We can state that \( \Delta a^m \) is integral for \( r \leq p - 1 \) and arbitrary \( a \) provided \( m \geq r \). For let \( p \mid a \); then, in view of (2.4), it is only necessary to verify that

\[ p^{m+r-i} + \frac{1}{2} i(i - 1) \geq rm + \frac{1}{2} r(r + 1) \]

for \( 0 \leq i \leq r \leq p - 1, r \geq m \). This can be proved by induction with respect to \( m \). In the next place since Theorem 11 is a direct consequence of Theorem 3 we infer that it also holds for all \( a \) provided \( r \leq \min (e_1, \cdots, e_s) \) in the notation of Theorem 11.

Now consider the number

\[ (6.1) \quad C_k = \sum_{a=1}^n \Delta a_a a^k, \]
where \( A_a \) denote integers \((\text{mod } p)\) and \( n \geq 1 \) is arbitrary. Then

\[
\Delta^r C_{k+p^m} = \sum_{a=1}^{n} A_a \Delta^r a^{k+p^m} \quad (k \geq 0),
\]

so that by the remark in the previous paragraph \( \Delta^r C_{pm} \) is certainly integral \((\text{mod } p)\) provided \( r \leq p - 1 \) and \( r \leq m \). In the second place we may apply the operator \( \delta_k^r \) defined in (5.2) and (5.3) and get

\[
\delta_k^r C_{h+k} = \sum_{a=1}^{n} A_a \delta_k^r a^{h+k};
\]

we infer that \( \delta_k^r C_k \) is integral provided \( r < \) the smallest prime dividing \( k \) and \( r \leq \min (i_1, \ldots, i_s) \), the notation being that of (5.2). Indeed a somewhat more general result can be obtained by applying Theorem 15, namely,

\[
\Delta_{p_1}^{r_1} \cdots \Delta_{p_s}^{r_s} C_{h+k} \quad (h \geq 0)
\]

is integral provided \( r_t < p_t, r_t \leq e_t \), \( t = 1, \ldots, s \).

As an instance of (6.1) we take the well-known formula for the Euler polynomial

\[
E_m(x) = \sum_{s=0}^{m} \frac{1}{2^s} \sum_{i=0}^{s} (-1)^i (\begin{array}{c} s \\ i \end{array}) (x+i)^m.
\]

(We use the notation of Nörlund [4] for the Euler and Bernoulli polynomials.) If \( p > 2 \) and \( x \) is integral \((\text{mod } p)\) the preceding discussion applies. In particular using (2.4) we have:

**Theorem 13.** Let \( p > 2 \) and \( x \) be integral \((\text{mod } p)\). Then

\[
\Delta^r E_{k+p^m}(x) = p^{-rm} - r(r+1)/2 \sum_{i=0}^{r} (-1)^i \left[ r \atop i \right] p^{i(i-1)/2} E_{k+p^m-i}(x)
\]

is integral \((\text{mod } p)\) provided \( r < p, r \leq m \).

For brevity we omit the generalizations corresponding to (6.3) and (6.4). The special case

\[
\sum_{de = m} \mu(d) E_{k+e}(x) \equiv 0 \quad (\text{mod } m)
\]
may be noted

As for the Bernoulli polynomials, it can be shown that if \( p \not| a \) and \( x \) is integral (mod \( p \)) then a formula of the type (6.1) holds for

\[
\beta_k(x) = \frac{a^{k+1} - 1}{k+1} B_{k+1}(x).
\]

(See for example Nielsen [3, Ch. 14].) Thus it follows that

\[
\Delta^r \beta_{k+p^m}(x) = p^{-rm} r(r+1)/2 \sum_{i=0}^{r} (-1)^i \binom{r}{i} p^{i(i-1)/2} \beta_{k+p^m-i}(x)
\]

is integral for \( r < p, r < m \). If now we assume \( p - 1 \not| k \) and take \( a \) a primitive root (mod \( p \)) such that \( a^{p-1} \equiv 1 \) (mod \( p^r \)) we get:

**Theorem 14.** Let \( p > 2 \) and \( x \) be integral (mod \( p \)); put \( H_k(x) = B_k(x)/k \). Then if \( p - 1 \not| k + 1 \),

\[
\Delta^r H_{k+p^m}(x) = p^{-rm} r(r+1)/2 \sum_{i=0}^{r} (-1)^i \binom{r}{i} p^{i(i-1)/2} H_{k+p^m-i}(x)
\]

is integral for \( r < p, r \leq m \).

Finally corresponding to (6.6) we state

\[
\sum_{d|e=m} \mu(d) \beta_{k+e}(x) \equiv 0 \pmod{m},
\]

for \( \beta_k(x) \) as defined in (6.7).

**References**


**Duke University**
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