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1. Introduction. Given the sequence $\{a_m\}$ and $p \neq 0$, Schur [5] defined the derivative a'_m by

$$(1.1) \qquad a'_m = \Delta a_m = (a_{m+1} - a_m)/p^{m+1};$$

higher derivatives are defined by means of

$$a_m^{(r)} = \Delta^r a_m = \Delta(a_m^{(r-1)}), \quad a_m^{(0)} = a_m.$$

In particular if p is a prime, a an integer and $a_m = a^{p^m}$, then by Fermat's theorem

$$a'_m = (a^{p^{m+1}} - a^{p^m})/p^{m+1}$$

is integral. Schur proved that if $p \nmid a$, then also the derivatives

$$\Delta^2 a^{p^m}, \ \Delta^3 a^{p^m}, \ \cdots, \ \Delta^{p-1} a^{p^m}$$

are all integral. Moreover if $a'_0 \equiv 0 \pmod p$ then all the derivatives $\Delta^r a^{p^m}$ are integral, while if $a'_0 \not\equiv 0 \pmod p$ then every number of $\Delta^p a^{p^m}$ has the denominator p.

A. Brauer [1] gave another proof of Schur's results. About the same time Zorn [6] proved these results by p-adic methods and indeed proved the following stronger theorem. For $x \equiv 1 \pmod p$, define

$$X_m = (x^{p^m} - 1)/p^{m+1},$$

and as above let $\Delta^r X_m$ denote the r-th derivative of X_m; then

$$(1.2) \qquad \Delta^r X_m \equiv \frac{(p-1)(p^2-1)\cdots(p^r-1)}{(r+1)!} X_m^{r+1} \pmod{p^m}$$

provided $r < p$; for $r < p-2$, the congruence (1.2) holds $\pmod{p^{m+1}}$. It is also shown that Schur's theorem is an easy consequence of Zorn's results.

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In the present paper we shall give a simple elementary proof of Zorn's congruences. In addition we prove, for example, that for $r \le p$,

$$(1.3) \qquad \Delta^r a^{p^m} \equiv \frac{1}{r!} a^{p^m} q_m^r \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m},$$

where

$$a^{(p-1)p^m} = 1 + p^{m+1} q_m;$$

for $r < p - 1$, (1.3) holds $\pmod{p^{m+1}}$.

We next (§4) extend Schur's and Zorn's theorems to algebraic numbers. In §5 we consider a generalization of another kind suggested by the arithmetic function (see for example [2, p. 84-86])

$$(1.4) \qquad F(a, m) = \sum_{de=m} \mu(d)\, a^e.$$

Finally (§6), we give some applications of Schur's theorem to the Euler and Bernoulli polynomials and numbers; the results are analogous to Kummer's congruences [3, Ch. 12]. In particular $\Delta^r E_{k+p^m}$ is integral $\pmod p$ for $p > 2, r < p$, $r \le m$; also $\Delta^r (B_{k+p^m}/(k + p^m))$ is integral $\pmod p$ for $p - 1 \nmid k + 1$, $r < p$, $r \le m$. Here E_k and B_k denote the Euler and Bernoulli numbers in the notation of Nörlund [3].

2. Formulas for $\Delta^r a_m$. We shall require some preliminary results.

LEMMA 1. *The following identity holds*:

$$(2.1) \qquad \prod_{i=0}^{r=1} (x - p^i) = \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} x^{r-i},$$

where

$$(2.2) \qquad \left[{r \atop i}\right] = \frac{(p^r - 1)(p^{r-1} - 1) \cdots (p^{r-i+1} - 1)}{(p-1)(p^2-1) \cdots (p^i - 1)} = \left[{r \atop r-i}\right], \left[{r \atop 0}\right] = 1.$$

LEMMA 2. *Put*

$$W_{k,r} = \sum_{i=0}^{k} (-1)^i \left[{k \atop i}\right] \binom{p^{k-i}}{r} p^{i(i-1)/2},$$

where $\binom{m}{r}$ *denotes a binomial coefficient. Then*

$$(2.3) \qquad W_{k,r} = \begin{cases} 0 & (r < k) \\ \dfrac{1}{r!} \prod_{i=0}^{r-1} (p^r - p^i) & (r = k) \\ \dfrac{1}{r!}\, p^{k(k-1)/2}\, U_{k,r} & (r > k), \end{cases}$$

where $U_{k,r}$ *is an integer.*

Lemma 1 is will known. To prove Lemma 2, we note first that the binomial coefficient $\binom{x}{r}$ is a polynomial in x of degree r. Since by (2.1)

$$\sum_{i=0}^{k} (-1)^i \left[{k \atop i}\right] p^{i(i-1)/2}\, p^{r(k-i)} = \prod_{i=0}^{k-1} (p^r - p^i),$$

the several parts of (2.3) follow without much difficulty.

LEMMA 3. *For an arbitrary sequence* $\{a_m\}$,

$$(2.4) \qquad \Delta^r a_m = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2}\, a_{m+r-i}\,.$$

This formula, which is given by Schur, is easily proved. In view of (2.1) it can be put in the following symbolic form:

$$(2.5) \qquad \Delta^r a_m = p^{-rm - r(r+1)/2}\, a^m \prod_{i=0}^{r-1} (a - p^i),$$

where it is understood that after expansion of the right member a^k is to be replaced by a_k.

Suppose now that $p \nmid a$ and put

$$(2.6) \qquad a^{(p-1)p^m} = 1 + p^{m+1} q_m,$$

so that q_m is integral. Then by the binomial theorem we have

$$a^{(p-1)p^{m+s}} = \sum_{i=0}^{p^r} \binom{p^s}{i} p^{(m+1)i} q_m^i \qquad (r \geq s),$$

and by (2.4) this implies

$$p^{rm+r(r+1)/2}\,\Delta^r a^{(p-1)p^m}$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{p^r} \binom{p^s}{i} p^{(m+1)i} q_m^i$$

$$= \sum_{i=0}^{p^r} p^{(m+1)i} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2}$$

$$= \sum_{i=0}^{p^r} p^{(m+1)i} q_m^i W_{r,i}$$

$$= \frac{1}{r!} p^{rm+r(r+1)/2} q_m^r \prod_{i=1}^{r} (p^i - 1)$$

$$+ \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)i+r(r-1)/2} q_m^i U_{r,i},$$

by (2.3); $W_{r,i}$ and $U_{r,i}$ have the same meaning as in Lemma 2. We thus get

$$(2.7)\qquad \Delta^r a^{(p-1)p^m} = \frac{1}{r!} q_m^r \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)(i-r)} q_m^i U_{r,i}.$$

We next set up a similar formula for $\Delta^r q_m$, where q_m is defined by (2.6). Indeed substitution in (2.4) gives

$$p^{rm+r(r+1)/2}\,\Delta^r q_m = \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2-(m+s+1)} \left(a^{(p-1)p^{m+s}} - 1\right)$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2-(m+s+1)} \sum_{i=1}^{p^r} \binom{p^s}{i} p^{(m+1)i} q_m^i$$

$$= \sum_{i=1}^{p^r} p^{(m+1)(i-1)} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2-s}$$

$$= \frac{1}{(r+1)!} p^{rm+r(r+1)/2} q_m^{r+1} \prod_{i=1}^{r} (p^i - 1)$$

$$+ \sum_{i=r+2}^{p^r} \frac{1}{i!} p^{(m+1)(i-1)+r(r-1)/2} q_m^i U'_{r,i},$$

by a slight modification of Lemma 2; the coefficient $U'_{r,i}$ is integral and is defined by

$$\frac{1}{i!}\, p^{r(r-1)/2}\, U'_{r,i} = \sum_{s=0}^{r} (-1)^s \left[{r \atop s}\right] (p_i^{r-s})\, p^{s(s-1)/2-(r-s)}.$$

Hence

$$(2.8)\quad \Delta^r q_m = \frac{1}{(r+1)!}\, q_m^{r+1} \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+2}^{p^r} \frac{1}{i!}\, p^{(m+1)(i-r-1)}\, q_m^i\, U'_{r,i}.$$

Using the same method we can also evaluate $\Delta^r a^{p^m}$. It follows from (2.6) that

$$(2.9)\qquad a^{p^{m+s}} = a^{p^m} (1 + p^{m+1} q_m)^{e_s} \qquad \left(e_s = \frac{p^s - 1}{p-1}\right),$$

and thus substitution in (2.4) yields

$$p^{rm+r(r+1)/2}\, \Delta^r a^{p^m} = a^{p^m} \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{e_r} \binom{e_s}{i} p^{(m+1)i} q_m^i$$

$$= a^{p^m} \sum_{i=0}^{e_r} p^{(m+1)i} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{e_s}{i} p^{(r-s)(r-s-1)/2}.$$

Since $\binom{e_s}{i}$ is a polynomial in p^s of degree i, the same reasoning as before applies and we get after a little manipulation

$$(2.10)\qquad \Delta^r a^{p^m} = \frac{1}{r!}\, a^{p^m}\, q_m^r\, \frac{\prod_{i=1}^{r} (p^i - 1)}{(p-1)^r}$$

$$+ a^{p^m} \sum_{i=r+1}^{e_r} \frac{1}{i!}\, p^{(m+1)(i-r)}\, q_m^i\, U''_{r,i},$$

where $U''_{r,i}$ is integral.

Comparison of (2.7) and (2.10) shows that (2.7) is included in (2.10). Indeed it is easy to set up the following formula which includes both (2.7) and (2.10):

$$(2.11)\qquad \Delta^r a^{kp^m} = \frac{1}{r!}\, a^{kp^m}\, q_m^r\, k^r\, \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} + a^{kp^m} \sum_{i=r+1}^{e_r} \frac{1}{i!}\, p^{(m+1)(i-r)}\, q_m^i\, V_{r,i},$$

where $V_{r,i} = V_{r,i}^{(k)}$ is integral and $k \geq 1$. The proof of (2.11) is exactly like the proof of (2.10); the first step is to raise both members of (2.9) to the k-th power.

3. The main results. In order to make use of (2.7) and (2.10) it is evidently necessary to examine $p^{(m+1)(i-r)}/i!$. We suppose $i > r$, $r \leq p$. Then in the first place it is easily seen [6, p. 462] that $p^{i-r}/i!$ is integral (mod p), and a simple discussion shows that $p^{i-r}/i!$ is divisible by p unless (i) $i = p$, $r = p-1$, or (ii) $i = p+1$, $r = p$. We now state:

THEOREM 1. *The derivative* $\Delta^r a^{(p-1)p^m}$ *is integral for* $1 \leq r \leq p-1$, *while* $\Delta^p a^{(p-1)p^m}$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r a^{(p-1)p^m}$ *are integral.*

THEOREM 2. *For* $1 \leq r \leq p$, $m \geq 0$,

$$(3.1)\qquad \Delta^r a^{(p-1)p^m} \equiv \frac{1}{r!}\, q_m^r \prod_{i=1}^r (p^i - 1) \pmod{p^m};$$

if $r < p-1$, *the congruence is valid* $\pmod{p^{m+1}}$.

THEOREM 3. *The derivative* $\Delta^r a^{p^m}$ *is integral for* $1 \leq r \leq p-1$, *while* $\Delta^p a^{p^m}$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r a^{(p-1)p^m}$ *are integral.*

THEOREM 4. *For* $1 \leq r \leq p$, $m \geq 0$,

$$(3.2)\qquad \Delta^r a^{p^m} \equiv \frac{1}{r!}\, a^{p^m}\, q_m^r\, \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m};$$

if $r < p-1$, *the congruence is valid* $\pmod{p^{m+1}}$.

If we make use of (2.11) rather than (2.7) or (2.10) we get the following more general result.

THEOREM 4′. *For* $1 \leq r \leq p$, $m \geq 0$

$$\Delta^r a^{kp^m} \equiv \frac{1}{r!} a^{kp^m} q_m^r k^r \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m};$$

if $r < p - 1$, *the congruence is valid* $\pmod{p^{m+1}}$.

To apply (2.8) we first examine $p^{i-r-1}/i!$ for $i > r + 1$, $r + 1 \leq p$. We have:

THEOREM 5. *The derivative* $\Delta^r q_m$ *is integral for* $1 \leq r \leq p - 2$, *while* $\Delta^{p-1} q_m$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r q_m$ *are integral.*

THEOREM 6. *For* $1 \leq r \leq p - 1$, $m \geq 0$,

$$\Delta^r q_m \equiv \frac{1}{(r+1)!} q_m^{r+1} \prod_{i=1}^{r} (p^i - 1) \pmod{p^m}; \tag{3.3}$$

if $r < p - 2$, *the congruence is valid* $\pmod{p^{m+1}}$.

Theorem 3 is of course Schur's theorem; Theorems 5 and 6 are due to Zorn. The remaining theorems are presumably new.

4. Generalization for algebraic numbers. Let k be an algebraic number field of degree n and let $\mathfrak{p}$ denote a prime ideal of k; also let

$$N\mathfrak{p} = p^f; \qquad \mathfrak{p}^e \mid p, \qquad \mathfrak{p}^{e+1} \nmid p; \tag{4.1}$$

for simplicity we assume $p > n$. If α k is integral (mod $\mathfrak{p}$) and $\mathfrak{p} \nmid \alpha$, then by Fermat's Theorem

$$\alpha^{p^f - 1} = 1 + \beta, \qquad \beta \equiv 0 \pmod{\mathfrak{p}}. \tag{4.2}$$

It follows from (4.2) that

$$\alpha^{(p^f-1)p^m} = 1 + \beta_m, \qquad \beta_m \equiv 0 \pmod{\mathfrak{p}^{me+1}}, \tag{4.3}$$

while (4,3) implies

$$\alpha^{(p^f-1)p^{m+s}} = \sum_{i=0}^{p^r} \binom{p^s}{i} \beta_m^i \qquad (r \geq s). \tag{4.4}$$

Then, exactly as in § 2,

$$p^{rm+r(r+1)/2}\ \Delta^r\ \alpha^{(p^f-1)p^m} = \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{p^r} \binom{p^s}{i} \beta_m^i$$

$$= \sum_{i=0}^{p^r} \beta_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2};$$

application of Lemma 2 now leads to

$$(4.5)\ \Delta^r\ \alpha^{(p^f-1)p^m} = \frac{1}{r!}\ p^{-r(m+1)}\ \beta_m^r \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+1}^{p^r} \frac{1}{i!}\ p^{-r(m+1)}\ \beta_m^i \omega_{r,i},$$

where $\omega_{r,i}$ is integral. Note that for $e > 1$ the right member of (4.5) need not be integral. Accordingly we assume $e = 1$; the assumption $p > n$ is then no longer needed.

We now have:

THEOREM 7. *Let* $N\mathfrak{p} = p^f$, $\mathfrak{p}^2 \nmid p$, $\mathfrak{p} \nmid \alpha$; *then* $\Delta^r\ \alpha^{(p^f-1)p^m}$ *is integral for* $1 \leq r \leq p - 1$, *while* $\Delta^p\ \alpha^{(p^f-1)p^m}$ *has the denominator* p *provided* $\alpha^{p^f-1} \not\equiv 1$ $(\text{mod}\ \mathfrak{p}^2)$; *if* $\alpha^{p^f-1} \equiv 1\ (\text{mod}\ \mathfrak{p}^2)$ *then all* $\Delta^r\ \alpha^{(p^f-1)p^m}$ *are integral.*

THEOREM 8. *With the hypotheses of Theorem* 7,

$$(4.6) \qquad \Delta^r\ \alpha^{(p^f-1)p^m} \equiv \frac{1}{r!} \left(\frac{\beta_m}{p^{m+1}}\right)^r \prod_{i=1}^{r} (p^i - 1) \pmod{\mathfrak{p}^m}$$

for $r \leq p$; *if* $r < p - 1$ *the congruence is valid* $(\text{mod}\ \mathfrak{p}^{m+1})$.

In order to extend Theorems 3 and 4′ it is convenient to suppose that $\mathfrak{p}$ is a prime ideal of the first degree. The following two theorems may be proved.

THEOREM 9. *Let* $N\mathfrak{p} = p$, $\mathfrak{p}^2 \nmid p$, $\mathfrak{p} \nmid \alpha$; *then* $\Delta^r\ \alpha^{p^m}$ *is integral for* $1 \leq r \leq p - 1$, *while* $\Delta^p\ \alpha^{p^m}$ *has the denominator* p *provided* $\alpha^{p-1} \not\equiv 1\ (\text{mod}\ \mathfrak{p}^2)$; *if* $\alpha^{p-1} \equiv 1\ (\text{mod}\ \mathfrak{p}^2)$ *then all* $\Delta^r\ \alpha^{p^m}$ *are integral.*

THEOREM 10. *With the hypotheses of Theorem* 9,

$$(4.7) \qquad \Delta^r\ \alpha^{kp^m} \equiv \frac{1}{r!} \left(\frac{k\,\beta_m}{p^{m+1}}\right)^r \frac{\prod_{i=1}^{r} (p^i - 1)}{(p-1)^r} \pmod{\mathfrak{p}^m}$$

for $r \leq p$; *if* $r < p - 1$ *the congruence is valid* $(\text{mod}\ \mathfrak{p}^{m+1})$.

For brevity we omit the extension of Theorems 5 and 6 for algebraic numbers.

5. Another generalization. Changing slightly the notation (1.1) we put

$$\Delta_p a_{mp^i} = (a_{mp^{i+1}} - a_{mp^i})/p^{i+1}, \tag{5.1}$$

and

$$\Delta_p^r a_{mp^i} = (\Delta_p^{r-1} a_{mp^{i+1}} - \Delta_p^{r-1} a_{mp^i})/p^{i+1}.$$

Then clearly $\Delta_p \Delta_q = \Delta_q \Delta_p$. If a and k are arbitrary integers then if follows from a well-known theorem concerning (1.4) that

$$\delta_k a^k = \Delta_{p_1} \cdots \Delta_{p_s} a^k \qquad (k = p_1^{e_1} \cdots p_s^{e_s}) \tag{5.2}$$

is integral. In view of Schur's theorem we can state the following generalization.

THEOREM 11. *Let* $(a,k) = 1$ *and let* $r <$ *the smallest prime dividing* k; *define*

$$\delta_k^r a^k = \delta_k \delta_k^{r-1} a^k. \tag{5.3}$$

Then $\delta_k^r a_k$ *is integral for* $k > 1$.

Indeed because of the commutativity of the operators Δ_{p_i} we need only observe that (5.2) and (5.3) imply

$$\delta_k^r a^k = \Delta_{p_1}^r \cdots \Delta_{p_s}^r a^k \tag{5.4}$$

and the theorem follows immediately.

The restriction $(a,k) = 1$ can be removed by taking k sufficiently large as we shall see below.

A slight extension of Theorem 11 is contained in:

THEOREM 12. *Let*

$$(a,k) = 1, \quad k = p_1^{e_1} \cdots p_s^{e_s},$$

and let $r_i < p_i$, $j = 1, \cdots, s$; *then*

$$\Delta_{p_1}^{r_1} \cdots \Delta_{p_s}^{r_s} a^k \tag{5.5}$$

is integral for all $k > 1$.

We remark that the function defined in (5.2) can also be expressed in the form

$$\delta_k a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d)\, a^{dk},$$

where $\mu(d)$ is the Möbius function and

$$k_1 = p_1^{e_1+1} \cdots p_s^{e_s+1};$$

similarly (5.3) becomes

$$\delta_k^r a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d)\, \delta_k^{r-1} a^{dk}.$$

Formulas of a different kind can be obtained by applying (2.4) to (5.4) and (5.5); for example, (2.5) suggests the following symbolic formula:

$$\delta_k^r a^k = k^{-r} \prod_{j=1}^{s} p_j^{r(r+1)/2} \cdot \prod_{j=1}^{s} a_j^{e_j} \prod_{i=0}^{r=1} (a_j - p_j^i),$$

where after expansion $a_1^{f_1} \cdots a_s^{f_s}$ is to be replaced by a^m,

$$m = p_1^{f_1} \cdots p_s^{f_s}.$$

A similar but slightly more complicated formula can be stated for (5.5). We shall omit the generalization of Theorems 11 and 12 to algebraic numbers.

6. Applications. In the theorems of § 2 it is assumed that $p \nmid a$. However Theorem 3, for example, is easily extended to the case $p \mid a$. We can state that $\Delta^r a^{p^m}$ is integral for $r \le p-1$ and arbitrary a provided $m \ge r$. For let $p \mid a$; then, in view of (2.4), it is only necessary to verify that

$$p^{m+r-i} + \frac{1}{2}\, i(i-1) \ge rm + \frac{1}{2}\, r(r+1)$$

for $0 \le i \le r \le p-1$, $r \ge m$. This can be proved by induction with respect to m. In the next place since Theorem 11 is a direct consequence of Theorem 3 we infer that it also holds for all a provided $r \le \min(e_1, \cdots, e_s)$ in the notation of Theorem 11.

Now consider the number

$$C_k = \sum_{a=1}^{n} A_a a^k, \tag{6.1}$$

where A_a denote integers (mod p) and $n \geq 1$ is arbitrary. Then

$$\Delta^r C_{k+p^m} = \sum_{a=1}^{n} A_a \Delta^r a^{k+p^m} \qquad (k \geq 0), \tag{6.2}$$

so that by the remark in the previous paragraph $\Delta^r C_{p^m}$ is certainly integral (mod p) provided $r \leq p-1$ and $r \leq m$. In the second place we may apply the operator δ_k^r defined in (5.2) and (5.3) and get

$$\delta_k^r C_{h+k} = \sum_{a=1}^{n} A_a \delta_k^r a^{h+k}; \tag{6.3}$$

we infer that $\delta_k^r C_k$ is integral provided $r <$ the smallest prime dividing k and $r \leq \min(i_1, \cdots, i_s)$, the notation being that of (5.2). Indeed a somewhat more general result can be obtained by applying Theorem 15, namely,

$$\Delta_{p_1}^{r_1} \cdots \Delta_{p_s}^{r_s} C_{h+k} \qquad (h \geq 0) \tag{6.4}$$

is integral provided $r_t < p_t$, $r_t \leq e_t$, $t = 1, \cdots, s$.

As an instance of (6.1) we take the well-known formula for the Euler polynomial

$$E_m(x) = \sum_{s=0}^{m} \frac{1}{2^s} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (x+i)^m. \tag{6.5}$$

(We use the notation of Nörlund [4] for the Euler and Bernoulli polynomials.) If $p > 2$ and x is integral (mod p) the preceding discussion applies. In particular using (2.4) we have:

THEOREM 13. *Let $p > 2$ and x be integral* (mod p). *Then*

$$\Delta^r E_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i} \right] p^{i(i-1)/2} E_{k+p^{m-i}}(x)$$

is integral (mod p) *provided* $r < p$, $r \leq m$.

For brevity we omit the generalizations corresponding to (6.3) and (6.4). The special case

$$\sum_{de=m} \mu(d) E_{k+e}(x) \equiv 0 \qquad (\text{mod } m) \tag{6.6}$$

may be noted

As for the Bernoulli polynomials, it can be shown that if $p \nmid a$ and x is integral (mod p) then a formula of the type (6.1) holds for

$$\beta_k(x) = \frac{a^{k+1} - 1}{k+1} B_{k+1}(x). \tag{6.7}$$

(See for example Nielsen [3, Ch. 14].) Thus it follows that

$$\Delta^r \beta_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} \beta_{k+p^{m-i}}(x)$$

is integral for $r < p$, $r \le m$. If now we assume $p - 1 \nmid k$ and take a a primitive root (mod p) such that $a^{p-1} \equiv 1 \pmod{p^r}$ we get:

THEOREM 14. *Let* $p > 2$ *and* x *be integral* (mod p); *put* $H_k(x) = B_k(x)/k$. *Then if* $p - 1 \nmid k + 1$,

$$\Delta^r H_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} H_{k+p^{m-i}}(x)$$

is integral for $r < p$, $r \le m$.

Finally corresponding to (6.6) we state

$$\sum_{de=m} \mu(d)\, \beta_{k+e}(x) \equiv 0 \pmod{m},$$

for $\beta_k(x)$ as defined in (6.7).

REFERENCES

1. A. Brauer, *Über eine Erweiterung des kleinen Fermatschen Satzes*, Math. Z. 42 (1937), 255-262.

2. L. E. Dickson, *History of the theory of numbers*, vol. 1, Washington, 1919.

3. Niels Nielsen, *Traité élémentaire des nombres de Bernoulli*, Paris, 1923.

4. N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924.

5. I. Schur, *Ein Beitrag zur elementaren Zahlentheorie*, Sitzungsberichte der Preussischen Akademie der Wissenschaften (1933), 145-151.

6. Max Zorn, *p-adic analysis and elementary number theory*, Ann. of Math. (2) 38(1937) 451-464.

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