

# Pacific Journal of Mathematics

**ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE  
HEAT EQUATION**

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# ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE HEAT EQUATION

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**1. Introduction.** Recently it has been shown independently by Hartman and Wintner [5] and by the present author [4] that if  $u(x, t)$  has continuous derivatives  $u_{xx}$  and  $u_t$ , and is a nonnegative solution of the heat equation

$$(1) \quad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle  $R: \{0 < x < 1; 0 < t < k \leq \infty\}$ , then  $u(x, t)$  can be represented in the form

$$(2) \quad u(x, t) = \int_{0+}^{1-0} G(x, t; y, 0) dA(y) \\ + \int_0^t G_y(x, t; 0, s) dB(s) - \int_0^t G_y(x, t; 1, s) dC(s),$$

where

$$(3) \quad G(x, t; y, s) = \frac{1}{2} \left[ \vartheta_3 \left( \frac{x-y}{2}, t-s \right) - \vartheta_3 \left( \frac{x+y}{2}, t-s \right) \right],$$

and where  $\vartheta_3$  is the Jacobi theta function. The integrals are Riemann-Stieltjes integrals with nondecreasing integrator functions,  $A$ ,  $B$ , and  $C$ . The first integral may be improper but is absolutely convergent. It was further shown (see [5] and [3]) that

$$(4) \quad u(x, 0+) = A'(x)$$

and

$$(5) \quad u(0+, t) = B'(t-0); \quad u(1-0, t) = C'(t-0)$$

at every point where the derivatives in question exist.

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**2. Theorem.** As to the question of the extent to which (4) and (5) uniquely determine  $u(x, t)$ , it is clear that they do not do so completely, for the singular solution  $G_\gamma(x, t; 0, 0)$ , called a heat explosion by Doetsch [2], has normal boundary values identically zero on the three boundaries  $x = 0$ ,  $x = 1$ , and  $t = 0$  of  $R$ . Yet  $A, B, C$ , through formula (2), do uniquely determine  $u$ ; hence one might expect that by proper choice of the path of approach to the boundary, zero boundary values would assure the vanishing of  $u$ . In particular, because of the central role played by  $G$  and  $G_\gamma$  in the representation (2), one might expect those paths to be the curves along which these functions become unbounded. This leads us to the following:

THEOREM. Suppose

- (a)  $u(x, t)$  is a nonnegative solution of (1) in  $R$ ;
- (b)  $u_{xx}$  and  $u_t$  are continuous in  $R$ ;
- (c)  $u(x, 0+) = 0$  ( $0 < x < 1$ );
- (d) for every  $s$  ( $0 \leq s < k$ ),  $\lim u(x, t) = 0$  as  $(x, t)$  tends to  $(0, s)$  along some parabolic arc of the form  $t - s = ax^2$ ,  $a > 0$ , and  $\lim u(x, t) = 0$  as  $(x, t)$  tends to  $(1, s)$  along some parabolic arc of the form  $t - s = a(x - 1)^2$ ,  $a > 0$ .

Then  $u(x, t) \equiv 0$  in  $R$ .

**3. Proof.** As we remarked in the first sentence, conditions (a) and (b) permit representation of  $u$  in the form (2). From the formula

$$(6) \quad \mathfrak{J}_3(x/2, t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp \left[ \frac{-(x + 2n)^2}{4t} \right],$$

which can be found in [2], it is easily seen that for  $0 < x < 1$  the two latter integrals in formula (2)  $\rightarrow 0$  as  $t \rightarrow 0+$ . Furthermore,

$$\begin{aligned} \int_{0+}^{1-0} G(x, t; y, 0) dA(y) &= \int_{0+}^{\delta} G(x, t; y, 0) dA(y) \\ &+ \int_{\delta}^{1-\delta} G(x, t; y, 0) dA(y) + \int_{1-\delta}^{1-0} G(x, t; y, 0) dA(y), \end{aligned}$$

where  $\delta < (1/2) \min [x, 1 - x]$  and is taken so small that, given  $\epsilon > 0$ ,

$$\left| \int_{0+}^{\delta} G(x, t; y, 0) dA(y) \right| < \epsilon \quad \text{and} \quad \left| \int_{1-\delta}^{1-0} G(x, t; y, 0) dA(y) \right| < \epsilon$$

uniformly in  $t$ , for  $0 < t \leq t_0$  for some  $t_0$ . Possibility to do this is ensured by [5,

Lemma 2, p. 385]. Now

$$\begin{aligned} \int_{\delta}^{1-\delta} G(x, t; y, 0) dA(y) &= \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \\ &+ \int_{\delta}^{1-\delta} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y+2n)^2}{4t}\right] dA(y) \\ &- \int_{\delta}^{1-\delta} \sum_{n=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x+y+2n)^2}{4t}\right] dA(y). \end{aligned}$$

The two latter integrals are easily seen to vanish with  $t$ . Since also the left side of (2)  $\rightarrow 0$  as  $t \rightarrow 0$ , it follows that, if  $\delta' < \delta$ ,

$$\begin{aligned} \overline{\lim}_{t \rightarrow 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \\ \leq \overline{\lim}_{t \rightarrow 0+} \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \leq 2\epsilon. \end{aligned}$$

Let  $\epsilon \rightarrow 0$  and obtain

$$\lim_{t \rightarrow 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(y-x)^2}{4t}\right] dA(y) = 0.$$

By [6, Th. 7], we see that  $A(y)$  is constant between  $\delta'$ , and  $1 - \delta'$ . Let  $\delta' \rightarrow 0$ . This ensures the vanishing of the first integral of (2).

Now let us turn to the boundary  $x = 0$ . Suppose that for some  $t_0$  the boundary function  $B(s)$  is not continuous. If  $\sigma$  is the jump (positive since  $B(s)$  is increasing) in  $B(s)$  at  $s = t_0$ , then for  $t > t_0$ , since  $G_y(x, t; 0, s) \geq 0$  (see [5, p. 370]).

$$\begin{aligned} u(x, t) &\geq \int_0^t G_y(x, t; 0, s) dB(s) \geq \sigma G_y(x, t; 0, t_0) \\ &= \frac{1}{2} \sigma x \pi^{-1/2} (t - t_0)^{-3/2} \exp\left[\frac{-x^2}{4(t - t_0)}\right] \\ &+ \frac{1}{2} \sigma \pi^{-1/2} (t - t_0)^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n + x) \exp\left[\frac{-(2n + x)^2}{4(t - t_0)}\right]. \end{aligned}$$

Since  $u(x, t) \rightarrow 0$  as  $(x, t) \rightarrow (0, t_0)$  along  $t - t_0 = ax^2$  for some  $a > 0$ , we have

$$u(x, t) \geq \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp \left[ \frac{-1}{4a} \right] \\ + \frac{1}{2} \sigma \pi^{-1/2} a^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2n+x}{x^3} \exp \left[ \frac{-(2n+x)^2}{4ax^2} \right],$$

As  $x \rightarrow 0+$ , the sum clearly  $\rightarrow 0$ ; but

$$\lim_{(x,t) \rightarrow (0,t_0)} u(x, t) = 0 \geq \lim_{x \rightarrow 0} \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp \left[ \frac{-1}{4a} \right] = \infty.$$

This is a contradiction. Hence  $\sigma = 0$ , and  $B(s)$  is continuous for  $0 \leq s < k$ .

Now let  $t = t_0 + ax^2$ . Then

$$u(x, t) \geq \int_{t_0}^{t_0+ax^2/2} G_y(x, t; 0, s) dB(s) \\ = \int_{t_0}^{t_0+ax^2/2} \frac{1}{2} x \pi^{-1/2} (t-s)^{-3/2} \exp \left[ \frac{-x^2}{4(t-s)} \right] dB(s) \\ + \int_{t_0}^{t_0+ax^2/2} \frac{1}{2} \pi^{-1/2} (t-s)^{-3/2} Q(x, t; s) dB(s),$$

where

$$Q(x, t; s) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n+x) \exp \left[ \frac{-(2n+x)^2}{4(t-s)} \right]$$

Clearly the latter integral vanishes with  $x$ , Since in the interval of integration we have

$$\exp \left[ \frac{-x^2}{4(t-s)} \right] \geq \exp \left[ \frac{-x^2}{4(ax^2/2)} \right] = \exp \left[ \frac{-1}{2a} \right]$$

and

$$t - s \leq ax^2,$$

it follows that

$$\begin{aligned} u(x, t) &\geq \frac{1}{2} \pi^{-1/2} a^{-3/2} x^{-2} \exp \left[ \frac{-1}{2a} \right] \left[ B \left( t_0 + \frac{ax^2}{2} \right) - B(t_0) \right] + o(1) \\ &\geq K \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} + o(1), \end{aligned}$$

where  $K$  is a positive constant. Letting  $x \rightarrow 0$ , we obtain

$$0 \geq \lim_{x \rightarrow 0} \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} = D^+[B(t_0)].$$

Hence, by [1, p.580],  $B(s)$  is a monotone decreasing function. Since it is non-decreasing, it must be constant. Similarly it can be shown that  $C(s)$  is constant. This completes the proof.

It seems probable that conditions (b), (c) and (d) would ensure the vanishing of  $u(x, t)$  if it were represented by (2) with  $A, B, C$  of bounded variation, but the proof eludes the author.

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# Pacific Journal of Mathematics

Vol. 3, No. 2

April, 1953

William George Bade, <i>An operational calculus for operators with spectrum in a strip</i> .....	257
E. F. Beckenbach and Lloyd Kenneth Jackson, <i>Subfunctions of several variables</i> .....	291
David Blackwell, <i>Extension of a renewal theorem</i> .....	315
L. Carlitz, <i>Some theorems on the Schur derivative</i> .....	321
Paul Arnold Clement, <i>Generalized convexity and surfaces of negative curvature</i> .....	333
Merrill M. Flood, <i>On the Hitchcock distribution problem</i> .....	369
Watson Bryan Fulks, <i>On the unique determination of solutions of the heat equation</i> .....	387
John W. Green, <i>Length and area of a convex curve under affine transformation</i> .....	393
William Gustin, <i>An isoperimetric minimax</i> .....	403
Arthur Eugene Livingston, <i>Some Hausdorff means which exhibit the Gibbs' phenomenon</i> .....	407
Charles Loewner, <i>On generation of solutions of the biharmonic equation in the plane by conformal mappings</i> .....	417
Gábor Szegő, <i>Remark on the preceding paper of Charles Loewner</i> .....	437
Immanuel Marx and G. Piranian, <i>Lipschitz functions of continuous functions</i> .....	447
Ting-Kwan Pan, <i>The spherical curvature of a hypersurface in Euclidean space</i> .....	461
Ruth Lind Potter, <i>On self-adjoint differential equations of second order</i> ....	467
E. H. Rothe, <i>A note on the Banach spaces of Calkin and Morrey</i> .....	493
Eugene Schenkman, <i>A generalization of the central elements of a group</i> ....	501
A. Seidenberg, <i>A note on the dimension theory of rings</i> .....	505