Pacific Journal of Mathematics

ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE HEAT EQUATION

WATSON BRYAN FULKS

Vol. 3, No. 2 April 1953

ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE HEAT EQUATION

W. FULKS

1. Introduction. Recently it has been shown independently by Hartman and Wintner [5] and by the present author [4] that if $u(x, t)$ has continuous derivatives u_{xx} and u_t, and is a nonnegative solution of the heat equation

$$(1) \qquad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle R: $\{0 < x < 1;\ 0 < t < k \le \infty\}$, then $u(x, t)$ can be represented in the form

$$(2) \quad u(x, t) = \int_{0+}^{1-0} G(x, t; y, 0)\, dA(y) + \int_0^t G_y(x, t; 0, s)\, dB(s) - \int_0^t G_y(x, t; 1, s)\, dC(s),$$

where

$$(3) \qquad G(x, t; y, s) = \frac{1}{2}\left[\vartheta_3\left(\frac{x-y}{2}, t-s\right) - \vartheta_3\left(\frac{x+y}{2}, t-s\right)\right],$$

and where ϑ_3 is the Jacobi theta function. The integrals are Riemann-Stieltjes integrals with nondecreasing integrator functions, A, B, and C. The first integral may be improper but is absolutely convergent. It was further shown (see [5] and [3]) that

$$(4) \qquad u(x, 0+) = A'(x)$$

and

$$(5) \qquad u(0+, t) = B'(t-0); \quad u(1-0, t) = C'(t-0)$$

at every point where the derivatives in question exist.

Received January 30, 1952. The preparation of this paper was sponsored, in part, by the Office of Naval Research, Contract Nonr-386(00).

Pacific J. Math. 3 (1953), 387-391

2. Theorem. As to the question of the extent to which (4) and (5) uniquely determine $u(x, t)$, it is clear that they do not do so completely, for the singular solution $G_y(x, t; 0, 0)$, called a heat explosion by Doetsch [2], has normal boundary values identically zero on the three boundaries $x = 0$, $x = 1$, and $t = 0$ of R. Yet A, B, C, through formula (2), do uniquely determine u; hence one might expect that by proper choice of the path of approach to the boundary, zero boundary values would assure the vanishing of u. In particular, because of the central role played by G and G_y in the representation (2), one might expect those paths to be the curves along which these functions become unbounded. This leads us to the following:

THEOREM. *Suppose*

(a) $u(x, t)$ *is a nonnegative solution of* (1) *in* R;

(b) u_{xx} *and* u_t *are continuous in* R;

(c) $u(x, 0+) = 0$ $\qquad (0 < x < 1)$;

(d) *for every* s $(0 \leq s < k)$, $\lim u(x, t) = 0$ *as* (x, t) *tends to* $(0, s)$ *along some parabolic arc of the form* $t - s = ax^2$, $a > 0$, *and* $\lim u(x, t) = 0$ *as* (x, t) *tends to* $(1, s)$ *along some parabolic arc of the form* $t - s = a(x - 1)^2$, $a > 0$.

Then $u(x, t) \equiv 0$ *in* R.

3. Proof. As we remarked in the first sentence, conditions (a) and (b) permit representation of u in the form (2). From the formula

$$\vartheta_3(x/2, t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x + 2n)^2}{4t}\right], \tag{6}$$

which can be found in [2], it is easily seen that for $0 < x < 1$ the two latter integrals in formula (2) $\longrightarrow 0$ as $t \longrightarrow 0+$. Furthermore,

$$\int_{0+}^{1-0} G(x, t; y, 0)\, dA(y) = \int_{0+}^{\delta} G(x, t; y, 0)\, dA(y)$$

$$+ \int_{\delta}^{1-\delta} G(x, t; y, 0)\, dA(y) + \int_{1-\delta}^{1-0} G(x, t; y, 0)\, dA(y),$$

where $\delta < (1/2) \min [x, 1 - x]$ and is taken so small that, given $\epsilon > 0$,

$$\left|\int_{0+}^{\delta} G(x, t; y, 0)\, dA(y)\right| < \epsilon \text{ and } \left|\int_{1-\delta}^{1-0} G(x, t; y, 0)\, dA(y)\right| < \epsilon$$

uniformly in t, for $0 < t \leq t_0$ for some t_0. Possibility to do this is ensured by [5, Lemma 2, p. 385]. Now

$$\int_{\delta}^{1-\delta} G(x, t; y, 0)\, dA(y) = \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y)$$

$$+ \int_{\delta}^{1-\delta} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y+2n)^2}{4t}\right] dA(y)$$

$$- \int_{\delta}^{1-\delta} \sum_{n=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x+y+2n)^2}{4t}\right] dA(y).$$

The two latter integrals are easily seen to vanish with t. Since also the left side of (2) $\longrightarrow 0$ as $t \longrightarrow 0$, it follows that, if $\delta' < \delta$,

$$\overline{\lim_{t\to 0+}} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y)$$

$$\leq \overline{\lim_{t\to 0+}} \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \leq 2\epsilon.$$

Let $\epsilon \longrightarrow 0$ and obtain

$$\lim_{t\to 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(y-x)^2}{4t}\right] dA(y) = 0.$$

By [6, Th. 7], we see that $A(y)$ is constant between δ', and $1-\delta'$. Let $\delta' \longrightarrow 0$. This ensures the vanishing of the first integral of (2).

Now let us turn to the boundary $x = 0$. Suppose that for some t_0 the boundary function $B(s)$ is not continuous. If σ is the jump (positive since $B(s)$ is increasing) in $B(s)$ at $s = t_0$, then for $t > t_0$, since $G_y(x, t; 0, s) \geq 0$ (see [5, p. 370]).

$$u(x, t) \geq \int_0^t G_y(x, t; 0, s)\, dB(s) \geq \sigma\, G_y(x, t; 0, t_0)$$

$$= \frac{1}{2}\, \sigma x \pi^{-1/2} (t-t_0)^{-3/2} \exp\left[\frac{-x^2}{4(t-t_0)}\right]$$

$$+ \frac{1}{2}\, \sigma \pi^{-1/2} (t-t_0)^{-3/2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (2n+x) \exp\left[\frac{-(2n+x)^2}{4(t-t_0)}\right].$$

Since $u(x, t) \longrightarrow 0$ as $(x, t) \longrightarrow (0, t_0)$ along $t - t_0 = ax^2$ for some $a > 0$, we have

$$u(x, t) \geq \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp\left[\frac{-1}{4a}\right]$$

$$+ \frac{1}{2} \sigma \pi^{-1/2} a^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2n+x}{x^3} \exp\left[\frac{-(2n+x)^2}{4ax^2}\right],$$

As $x \longrightarrow 0+$, the sum clearly $\longrightarrow 0$; but

$$\lim_{(x,t) \to (0,t_0)} u(x, t) = 0 \geq \underline{\lim}_{x \to 0} \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp\left[\frac{-1}{4a}\right] = \infty .$$

This is a contradiction. Hence $\sigma = 0$, and $B(s)$ is continuous for $0 \leq s < k$.

Now let $t = t_0 + ax^2$. Then

$$u(x, t) \geq \int_{t_0}^{t_0 + ax^2/2} G_y(x, t; 0, s)\, dB(s)$$

$$= \int_{t_0}^{t_0 + ax^2/2} \frac{1}{2} x \pi^{-1/2} (t-s)^{-3/2} \exp\left[\frac{-x^2}{4(t-s)}\right] dB(s)$$

$$+ \int_{t_0}^{t_0 + ax^2/2} \frac{1}{2} \pi^{-1/2} (t-s)^{-3/2} Q(x, t; s)\, dB(s),$$

where

$$Q(x, t; s) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n+x) \exp\left[\frac{-(2n+x)^2}{4(t-s)}\right]$$

Clearly the latter integral vanishes with x, Since in the interval of integration we have

$$\exp\left[\frac{-x^2}{4(t-s)}\right] \geq \exp\left[\frac{-x^2}{4(ax^2/2)}\right] = \exp\left[\frac{-1}{2a}\right]$$

and

$$t - s \leq ax^2,$$

it follows that

$$u(x,t) \geq \frac{1}{2}\,\pi^{-1/2}\,a^{-3/2}\,x^{-2}\,\exp\left[\frac{-1}{2a}\right]\left[B\left(t_0 + \frac{ax^2}{2}\right) - B(t_0)\right] + o(1)$$

$$\geq K\,\frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} + o(1),$$

where K is a positive constant. Letting $x \longrightarrow 0$, we obtain

$$0 \geq \overline{\lim_{x \to 0}}\,\frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} = D^{+}[B(t_0)].$$

Hence, by [1, p.580], $B(s)$ is a monotone decreasing function. Since it is non-decreasing, it must be constant. Similarly it can be shown that $C(s)$ is constant. This completes the proof.

It seems probable that conditions (b), (c) and (d) would ensure the vanishing of $u(x,t)$ if it were represented by (2) with A, B, C of bounded variation, but the proof eludes the author.

References

1. C. Caratheodory, *Vorlesungen Über Reelle Funktionen*, Berlin, 1927.

2. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, New York, 1943.

3. W. Fulks, *On the boundary values of solutions of the heat equation*, Pacific J. Math. 2 (1952), 141-145.

4. ———, *On integral representations and uniqueness of solutions of the heat equation*, University of Minnesota Thesis, June, 1949.

5. P. Hartman and A. Wintner, *On the solutions of the equation of heat conduction*, American J. Math. 72 (1950), 367-395.

6. D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. 55 (1944), 85-95.

THE UNIVERSITY OF MINNESOTA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors except Robinson, whose term expires with the completion of the present volume; they might also be sent to M. M. Schiffer, Stanford University, Stanford, California, who is succeeding Robinson. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $8.00; single issues, $2.50. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics
Vol. 3, No. 2 April, 1953