

# Pacific Journal of Mathematics

**THE SPHERICAL CURVATURE OF A HYPERSURFACE IN  
EUCLIDEAN SPACE**

TING-KWAN PAN

# THE SPHERICAL CURVATURE OF A HYPERSURFACE IN EUCLIDEAN SPACE

T. K. Pan

**1. Introduction.** Let  $V_n$  be a hypersurface immersed in a Euclidean space  $S_{n+1}$ . Let  $P$  be a point of  $V_n$  corresponding to the point  $P'$  of the hyperspherical representation  $G_n$  of  $V_n$ . Let  $V$  denote the extension of a region  $\phi$  of  $V_n$ , and  $V'$  the extension of the corresponding hyperspherical region  $\phi'$  of  $G_n$ . If the region around  $P$  tends to zero, the ratio  $V'/V$  tends to a limit  $\Gamma$ , which is called the *spherical curvature* of  $V_n$  at  $P$  [1, pp. 258-261]. It is found that  $\Gamma = |\Omega/g|$ , where  $g = |g_{ij}|$  and  $\Omega = |\Omega_{ij}|$  are respectively the determinants of the coefficients of the first and the second fundamental forms of  $V_n$ . In this note, some properties of the spherical curvature are studied, and new interpretations of the Gaussian curvature are derived.

The notation of Eisenhart [2] will be used for the most part.

**2. Some properties.** Let a real and analytic hypersurface  $V_n$  be defined by

$$y^\alpha = y^\alpha(x^1, \dots, x^n) \quad (\alpha = 1, \dots, n+1),$$

referred to a Cartesian coordinate system  $y^\alpha$  in a Euclidean space  $S_{n+1}$ . Let a vector-field  $v$  in  $V_n$  be defined by

$$v^\alpha = p^i \partial y^\alpha / \partial x^i \quad (i = 1, \dots, n),$$

where the  $v^\alpha$  are real and analytic functions of the  $x^i$ . Let  $C$  be a curve of  $V_n$ . The normal curvature vector of  $v$  with respect to  $C$  at  $P$  is defined as the normal component of the derived vector of the vector-field  $v$  along  $C$  at  $P$  [3]. Let  $\kappa$  denote a nonzero extreme value of the magnitudes of the normal curvature vectors of  $v$  with respect to all curves of  $V_n$  at  $P$ . Then  $\kappa$ , which is called a principal curvature of  $v$  at  $P$ , is defined by

$$(2.1) \quad |\Psi_{ij} - \kappa^2 g_{ij}| = 0,$$

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where

$$\Psi_{ij} = \Omega_{ik} \Omega_{jl} p^k p^l / g_{kl} p^k p^l.$$

Since  $\|\Psi_{ij}\|$  is of rank 1, there is one such extreme corresponding to a vector-field  $v$ . Its value is evidently equal to

$$(2.2) \quad \kappa = (\Psi_{ij} g^{ij})^{1/2} = (H_{ij} p^i p^j / g_{ij} p^i p^j)^{1/2},$$

where  $H_{ij}$  is the fundamental tensor of the hyperspherical representation  $G_n$ .

The extreme of the principal curvature of a vector-field  $v$  at  $P$ , as the field varies, is defined by

$$(2.3) \quad |H_{ij} - \kappa^2 g_{ij}| = 0.$$

There are  $n$  such extremes  $\bar{\kappa}_i$  corresponding to the principal directions for the tensor  $H_{ij}$ . Their product is found to be

$$\prod_{i=1}^n \bar{\kappa}_i = |H/g|^{1/2} = |\Omega/g|,$$

since  $H = |H_{ij}| = \Omega^2/g$ , [1, p. 260]. The principal directions for the tensor  $H_{ij}$  and those determined by the tensor  $\Omega_{ij}$  are identical, since the principal curvature of a principal vector-field can easily be shown equal to the normal curvature of the corresponding line of curvature. Hence we have:

**THEOREM 2.1.** *The spherical curvature of a  $V_n$  at  $P$  is equal to the product of the extreme principal curvatures of vector-fields in  $V_n$  at  $P$ , which is the same as the product of principal curvatures of  $V_n$  at  $P$ .*

Since  $S_{n+1}$  is Euclidean, the equations of Gauss are

$$(2.4) \quad R_{ijkl} = \Omega_{ik} \Omega_{jl} - \Omega_{il} \Omega_{jk}.$$

Multiplying (2.4) by  $g^{ik}$  and summing with respect to  $i$  and  $k$ , we obtain

$$(2.5) \quad H_{jl} = M \Omega_{jl} + R_{jl},$$

where  $M$  is the mean curvature of  $V_n$ , and where  $R_{jl}$  is the Ricci tensor. When  $V_n$  is a minimal hypersurface, we have  $M = 0$ , and the Ricci tensor is identical

with the fundamental tensor of  $G_n$ . If  $M \neq 0$ , we have

$$(2.6) \quad H_{ij} p^i p^j = R_{ij} p^i p^j$$

if and only if  $v$  is an asymptotic vector-field. If  $v$  is a unit asymptotic vector-field, we notice, from (2.2), (2.6), and the equality

$$R_{ij} \lambda_h^i \lambda_h^j = - \sum_{k=1}^n \gamma_{hk},$$

that the square of the principal curvature of  $v$  at  $P$  is numerically equal to the sum of the Riemannian curvatures determined by  $v$  and  $n - 1$  other mutually orthogonal unit vectors orthogonal to  $v$  at  $P$ . Hence we have established the following result:

**THEOREM 2.2.** *The square of the principal curvature of an asymptotic vector-field at  $P$  in  $V_n$  is numerically equal to the mean curvature of  $V_n$  at  $P$  for the corresponding asymptotic direction.*

The extreme of the principal curvatures  $\kappa$  of asymptotic vector-fields at  $P$  in  $V_n$  is defined by

$$|R_{ij} - \kappa^2 g_{ij}| = 0.$$

There are  $n$  such extreme values corresponding to the principal directions for the Ricci tensor  $R_{ij}$ . Their product is evidently equal to  $|\Omega/g|$ , if  $V_n$  is minimal. Hence we have:

**THEOREM 2.3.** *The principal curvatures of asymptotic vector-fields at  $P$  in  $V_n$  attain their extreme values in the principal directions for the Ricci tensor.*

**THEOREM 2.4.** *The spherical curvature of a minimal  $V_n$  at  $P$  is the product of the principal curvatures of the  $n$  vector-fields at  $P$  corresponding to the principal directions for the Ricci tensor.*

**3. The Gaussian curvature.** When  $n = 2$ ,  $\Gamma$  is called the spherical curvature of a surface  $S$  in an ordinary space. It coincides in absolute value with the Gaussian curvature  $K$  of  $S$ . The principal curvature of a vector-field  $v$  in  $V_n$  for  $n = 2$  coincides in absolute value with the principal curvature of  $v$  in  $S$ , [3]. The extreme principal curvatures of vector-fields in  $V_n$  for  $n = 2$  coincide in absolute value with the principal curvatures of  $S$ . The mean curvature of  $V_n$  for

$n = 2$  is identical with the Gaussian curvature of  $S$ . Hence Theorems 2.1 and 2.2 lead directly to the following new interpretations of the Gaussian curvature:

**THEOREM 3.1.** *The Gaussian curvature of  $S$  at  $P$  is the product of the extreme principal curvatures of vector fields of  $S$  at  $P$ , and is the negative of the square of the magnitude of the Gaussian representation of a unit arc along an asymptotic line from  $P$  in  $S$ .*

Let  $p^\alpha$  and  $q^\alpha$  be two distinct conjugate vector fields in  $S$ . Then we have

$$q^\beta = e^{\beta\mu} d_{\alpha\mu} p^\alpha \quad (\alpha, \beta, \mu = 1, 2),$$

where  $d_{\alpha\mu}$  is the second fundamental tensor of  $S$ . The principal curvatures of the vector-fields  $p^\alpha$  and  $q^\alpha$  are respectively equal to

$$e\rho_p = (h_{\alpha\beta} p^\alpha p^\beta / g_{\alpha\beta} p^\alpha p^\beta)^{1/2},$$

$$e\rho_q = (hg_{\alpha\beta} p^\alpha p^\beta / gh_{\alpha\beta} p^\alpha p^\beta)^{1/2},$$

where  $h_{\alpha\beta}$  is the third fundamental tensor of  $S$ . Hence their product is

$$(3.1) \quad (e\rho_p)(e\rho_q) = (h/g)^{1/2}.$$

The expression on the right side of (3.1) is equal to  $eK$ , where  $e$  is +1 or -1 according as  $K$  is positive or negative at the point under consideration. At an elliptic point, the principal curvatures of all vector-fields are of the same sign. At a hyperbolic point, the principal curvatures of two vector-fields are different in sign if they lie in different sections separated by the asymptotic lines of  $S$ . Consequently, the principal curvatures of two conjugate vector-fields have opposite signs, since conjugate directions are separated by the asymptotic directions of the surface. Hence at an elliptic point of  $S$ , the product of the principal curvatures of two conjugate vector-fields is positive; while at a hyperbolic point of  $S$ , it is negative. At a parabolic point the normal curvature of any vector-field with respect to any curve is zero. We may consider that every direction in  $S$  at a parabolic point is both an asymptotic direction and a principal direction of a vector-field which is to be considered. Hence at a parabolic point the principal curvature of any vector-field is zero; consequently, the product of the principal curvatures of two conjugate vector-fields is zero. Thus the following theorem is proved:

**THEOREM 3.2.** *The Gaussian curvature of  $S$  at  $P$  is the product of the principal curvatures of any two distinct conjugate vector-fields in  $S$  at  $P$ .*

The sum of the squares of the principal curvatures of the two conjugate vector-fields is found to be

$$(e\rho_p)^2 + (e\rho_q)^2 = M(\kappa_p + \kappa_q) - 2K,$$

where  $\kappa_p$  and  $\kappa_q$  are the normal curvatures of the curves of the two fields, and where  $M$  is the mean curvature of  $S$ . By Theorem 3.2 the above equation can be written as

$$(3.2) \quad (e\rho_p + e\rho_q)^2 = M(\kappa_p + \kappa_q).$$

Since the product of the normal radii at a point in conjugate directions is a maximum for characteristic lines, and a minimum for lines of curvature, and since the sum of normal radii in conjugate directions is constant, we obtain from (3.2) the following result:

**THEOREM 3.3.** *The sum of the principal curvatures of two conjugate vector-fields at  $P$  is the mean proportional between the mean curvature at  $P$  of  $S$  and the sum of the normal curvatures in the two conjugate directions at  $P$ . The square of the sum of the principal curvatures of two conjugate vector-fields at  $P$  is a maximum for the principal vector-fields of  $S$ , and a minimum for the characteristic vector-fields of  $S$ .*

Let  $m$  ( $m > 2$ ) directions be such that the angle of two adjoining directions is  $2\pi/m$ . Let the principal curvatures of the vector-fields in such directions be denoted by  $e\rho_1, e\rho_2, \dots, e\rho_m$ . Then

$$\frac{1}{m} \sum_{i=1}^{m>2} (e\rho_i)^2 = \frac{1}{2} M^2 - K,$$

since

$$\frac{1}{m} \left( \sum_{i=1}^{m>2} \kappa_{p_i} \right) = \frac{1}{2} M,$$

where  $\kappa_{p_i}$  are the normal curvatures of the curves of the corresponding vector-fields.

**THEOREM 3.4.** *One  $m$ th of the sum of the squares of the principal curvatures of  $m$  ( $> 2$ ) vector-fields at  $P$ , such that the angle of two adjoining vectors of these fields at  $P$  is  $2\pi/m$ , is constant and is the same for any  $m$  greater than two. The constant is half of the square of the mean curvature of  $S$  minus the Gaussian curvature of  $S$  at  $P$ .*

It is easy to prove that the principal direction of a vector-field in  $S$  is orthogonal to the curve of the field if and only if the vector-field is an asymptotic field. Let  $p^\alpha$  be an asymptotic vector-field in  $S$ . Then its orthogonal trajectories are defined by

$$du^\beta = e^{\beta\mu} g_{\alpha\mu} p^\alpha.$$

The principal curvature of the asymptotic vector-field  $p^\alpha$  is given by

$$(e\rho_p) = d_{\alpha\beta} p^\alpha e^{\beta\mu} g_{\gamma\mu} p^\gamma / [(g_{\alpha\beta} p^\alpha p^\beta) (g_{\alpha\beta} e^{\alpha\mu} g_{\gamma\mu} p^\gamma e^{\beta\lambda} g_{\sigma\lambda} p^\sigma)]^{1/2},$$

which after simplification becomes

$$(e\rho_p) = \epsilon^{\beta\mu} d_{\alpha\beta} g_{\gamma\mu} p^\alpha p^\gamma / g_{\alpha\beta} p^\alpha p^\beta = \tau_g,$$

where  $\tau_g$  is the geodesic torsion of the curve of the asymptotic vector-field.

**THEOREM 3.5.** *The principal curvature of an asymptotic vector-field at  $P$  in  $S$  is equal to the geodesic torsion at  $P$  of the curve of the field, or simply the torsion at  $P$  of the corresponding asymptotic line.*

From Theorem 3.1 and Theorem 3.5 we immediately obtain the first part of the theorem of Enneper, that the square of the torsion of a real asymptotic line at a point is equal to the absolute value of the total curvature of the surface at the point. By the second part of the same theorem we notice that *the principal curvatures of the asymptotic vector-fields in  $S$  are different in sign.*

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