A NOTE ON THE DIMENSION THEORY OF RINGS

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1. Introduction. Let \( O \) be an integral domain. If in \( O \) there is a proper chain

\[
(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset (1)
\]

of prime ideals, but no such chain

\[
(0) \subset P'_1 \subset \cdots \subset P'_{n+1} \subset (1),
\]

then \( O \) will be said to be \( n \)-dimensional. Let \( O \) be of dimension \( n \): the question is whether the polynomial ring \( O[x] \) is necessarily \((n + 1)\)-dimensional. Here, as throughout, \( x \) is an indeterminate.

By an \( F \)-ring we shall mean a 1-dimensional ring \( O \) such that \( O[x] \) is not 2-dimensional (i.e., the proposed assertion that \( O[x] \) is necessarily 2-dimensional fails). Given an \( F \)-ring, we try by definite constructions to pass to a larger \( F \)-ring having the same quotient field: this restricts the class of rings in which to look for an \( F \)-ring—a priori we do not know they exist. In this way we also come (in Theorem 8 below) to a complete characterization of \( F \)-rings: if \( O \) is 1-dimensional, then \( O[x] \) is 2-dimensional if and only if every quotient ring of \( O \), the integral closure of \( O \), is a valuation ring. The rings \( O \) thus coincide (for dimension 1) with Krull's Multiplikationsringe \([5; p. 554]\).

2. Preliminary results. The first five theorems are of a preparatory character, and the proofs offer no difficulties.

**Theorem 1.** Let \( O \) be an arbitrary commutative ring with 1, \( P_1, P_2, P_3 \) distinct ideals in \( O[x] \). If \( P_1 \subset P_2 \subset P_3 \), and \( P_2 \) and \( P_3 \) are prime ideals, then \( P_1 \), \( P_2, P_3 \) cannot have the same contraction to \( O \).

**Proof.** Let

\[
P_1 \cap O = P_2 \cap O = p,
\]

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and consider

\[ O[x]/P_2 = \overline{O}[x], \]

where \( \overline{x} \) is the residue of \( x \) and \( \overline{O} \cong O/p. \) Since

\[ \overline{O}[x] \cdot p \subseteq P_1 \subset P_2, \]

\( \overline{x} \) is algebraic over the integral domain \( \overline{O}. \) Let \( \overline{P}_3 \) be the image of \( P_3 \); then \( \overline{P}_3 \neq (0); \) but also \( \overline{P}_3 \cap \overline{O} \neq (0). \) In fact, let \( \gamma \in \overline{P}_3, \gamma \neq 0. \) Then

\[ c_0 \gamma^n + c_1 \gamma^{n-1} + \cdots + c_n = 0 \]

for some \( c_i \in \overline{O}, \ c_n \neq 0; \) and \( c_n \in \overline{P}_3 \cap \overline{O}. \) Hence also \( P_3 \cap O \neq p, \)

**Corollary.** If \( O \) is 1-dimensional, and \( P_1, P_2, P_3 \) are distinct prime ideals in \( O[x] \) different from \( (0) \) with \( P_1 \subset P_2 \subset P_3, \) then \( P_1 \cap O = (0), P_2 \) is the extension of its contraction to \( O, \) and \( P_3 \) is maximal.

**Proof.** If \( P_1 \cap O \neq (0), \) then \( P_1, P_2, P_3 \) would all have to contract to the same maximal ideal in \( O. \) So

\[ P_1 \cap O = (0) \text{ and } P_2 \cap O = p \neq (0). \]

Were \( O[x] \cdot p \subset P_2 \) properly, then, since \( O[x] \cdot p \) is prime,

\[ O[x] \cdot p \cap O = (0), \]

whereas

\[ O[x] \cdot p \cap O = p. \]

So \( O[x] \cdot p = P_2. \) Were \( P_3 \) not maximal, we would have \( P_2 \cap O = (0). \)

For the foregoing theorem, see also [4; Th. 10, p. 375].

**Theorem 2.** If \( O \) is \( n \)-dimensional, then \( O[x] \) is at least \( (n + 1) \)-dimensional and at most \( (2n + 1) \)-dimensional.

**Proof.** Let

\[ (0) \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset (1) \]

be a proper chain of prime ideals in \( O. \) Then

\[ (0) \subset O[x] \cdot P_1 \subset O[x] \cdot P_2 \subset \cdots \subset O[x] \cdot P_n \subset (1) \]
is also a proper chain of prime ideals in \( O[x] \); and \( O[x] \cdot P_n \) is not maximal, since, for example,

\[
O[x] \cdot P_n \subset (O[x] \cdot P_n, x) \subset (1).
\]

(Here, as throughout, we use the symbol \( \subset \) for proper inclusion.) Hence \( O[x] \) is at least \((n + 1)\)-dimensional. Let now \( O \) be \( n \)-dimensional, and consider a chain

\[
(0) \subset P_1' \subset \cdots \subset P_m' \subset (1)
\]

of prime ideals in \( O[x] \). Let there be \( s \) distinct ideals among the contractions

\[
(0) \cap O, P_1' \cap O, \ldots, P_m' \cap O.
\]

Then

\[
m + 1 < 2s \leq 2(n + 1), \text{ so } m \leq 2n + 1.
\]

**Theorem 3.** If \( O \) is \( n \)-dimensional but \( O[x] \) is not \((n + 1)\)-dimensional, then for at least one minimal prime ideal \( p \) of \( O \) either the quotient ring \( O_p \) is an \( F \)-ring or \( O/p \) is \( m \)-dimensional and \( O/p[x] \) is not \((m + 1)\)-dimensional, and \( m < n \).

**Proof.** Suppose that for some minimal prime ideal \( p \) of \( O \), \( O[x] \cdot p \) is not minimal in \( O[x] \); that is, there exists a prime ideal \( P \) such that

\[
(0) \subset P \subset O[x] \cdot p.
\]

Then

\[
(0) \subset O_p[x] \cdot P \subset O_p[x] \cdot p
\]

is also a chain of prime ideals in \( O_p[x] \), as one easily verifies. Since \( O_p[x] \cdot p \) is not maximal, this shows that \( O_p \) is an \( F \)-ring. We pass then to the case that \( O[x] \cdot p \) is minimal for every minimal prime ideal \( p \) of \( O \). Let

\[
(0) \subset P_1' \subset \cdots \subset P_{n+2}' \subset (1)
\]

be a chain of prime ideals in \( O[x] \). If

\[ P_1' \cap O = p \neq (0), \]

then \( O/p \) is at most \((n - 1)\)-dimensional, and \( O[x]/O[x] \cdot p \) is a polynomial ring in one variable over \( O/p \) and is at least \((n + 1)\)-dimensional. So we must suppose
but then

\[ P' \cap O = (0); \]

let \( p \) be a minimal prime ideal contained in \( p_2 \)—such exists since \( O \) is finite dimensional; then \( O[x] \cdot p \subset P'_2 \), properly, since \( O[x] \cdot p \) is minimal but \( P'_2 \) is not. Replacing \( P'_1 \) by \( O[x] \cdot p \), we come back to a previous case, and the proof is complete.

**Corollary.** If \( O \) is an \( F \)-ring, then so is some quotient ring of \( O \).

The foregoing theorem shows that if for some \( n \) there exists a ring \( O \) which is \( n \)-dimensional, while \( O[x] \) is not \((n + 1)\)-dimensional, then there exist \( F \)-rings. Thus we may provisionally confine our attention to 1-dimensional rings \( O \).

**Theorem 4.** If \( O \) is 1-dimensional, and \( O \) is a valuation ring, then \( O[x] \) is 2-dimensional.

**Proof.** Let \( p \) be a proper prime ideal of \( O \), and let

\[ (0) \subset P \subset O[x] \cdot p, \]

where \( P \) is prime. Let

\[ f(x) \in P, \quad f(x) \neq 0. \]

Then one can factor out from \( f(x) \) a coefficient of least value, that is, write

\[ f(x) = c \cdot g(x), \]

where \( c \in p \), and \( g(x) \) has at least one coefficient equal to 1; in particular, then \( g(x) \notin O[x] \cdot p \); hence \( c \in P \). So \( P \cap O \neq (0) \), whence

\[ P \cap O = p \quad \text{and} \quad P = O[x] \cdot p. \]

This proves that \( O[x] \) is 2-dimensional (see Corollary to Theorem 1).

Theorem 4 restricts the size of an \( F \)-ring, since a maximal ring is a valuation ring. The following theorem reduces the considerations to integrally closed rings.

**Theorem 5.** Let \( \overline{O} \) be the integral closure of the integral domain \( O \). Then \( O \) is an \( F \)-ring if and only if \( \overline{O} \) is an \( F \)-ring.
Proof. Let \( R \) be an integral domain integrally dependent on \( O \); a basic theorem of Krull (see, for example, [2; Th. 4, p. 254]) says that if \( P_1 \subset P_2 \) are prime ideals in \( R \), then they contract to distinct prime ideals in \( O \); hence \( \text{dim } R \leq \text{dim } O \). Another theorem (loc. cit., p. 254) says that if \( p_1 \subset p_2 \) are prime ideals in \( O \), and \( p_1 \) is a prime ideal in \( R \) contracting to \( p_1 \), then there exists a prime ideal \( P_2 \supset P_1 \), contracting to \( p_2 \). Hence \( \text{dim } R \geq \text{dim } O \), and so \( \text{dim } R = \text{dim } O \) hence \( O \) is 1-dimensional if and only if \( O \) is 1-dimensional, and \( O[x] \) is 2-dimensional if and only if \( O[x] \) is 2-dimensional.

Thus if there exist \( F \)-rings, then there exist integrally closed \( F \)-rings, and, taking an appropriate quotient ring, we see that there would exist an integrally closed \( F \)-ring \( O \) having just one proper prime ideal. In view of Theorem 4 (and the close association of integrally closed rings with valuation rings) one may ask whether an integrally closed ring with only one proper prime ideal is necessarily a valuation ring. Were it so, there would be no \( F \)-rings, but it is not so: Krull has an example [6; p. 670f]. For convenience, we may mention the example: let \( K \) be an algebraically closed field, \( x \) and \( y \) indeterminates; \( O \) consists of the rational functions \( r(x, y) \) which, when written in lowest terms, have denominators not divisible by \( x \), and which are such that \( r(0, y) \in K \).

3. Principal results. We now establish:

Theorem 6. If \( O \) is integrally closed with only one maximal ideal \( p \), \( \alpha \) an element of the quotient field of \( O \), and \( 1/\alpha \notin O \), then \( O[\alpha] \cdot p \) is prime. If also \( \alpha \notin O \), then \( O[\alpha] \cdot p \) is not maximal.

Proof. We first observe that

\[
(O[\alpha] \cdot p, \alpha) \neq (1),
\]
as an equation

\[
1 = c_0 + c_1 \alpha + \cdots + c_s \alpha^s \quad (c_0 \in p, c_i \in O),
\]
leads to an equation of integral dependence for \( 1/\alpha \) over \( O \). Let now \( g(x) \in O[x] \) be a monic polynomial of positive degree. We may assume, trivially, that \( \alpha \notin O \); then \( g(\alpha) = c \in O \) is impossible, as \( g(\alpha) \cdot c = 0 \) would be an equation of integral dependence for \( \alpha \) over \( O \); in particular, \( g(\alpha) \neq 0 \). Also \( 1/g(\alpha) \notin O \), for if it were in \( O \), it would be a nonunit in \( O \), and hence would be in \( p \), so that

\[
1 \in g(\alpha) \cdot p \subseteq O[\alpha] \cdot p,
\]
and this is not so. By the result on \( \alpha \),
(O[g(\alpha)] \cdot p, g(\alpha)) \neq (1).

Since \alpha satisfies \( g(x) - g(\alpha) = 0 \), \( O[\alpha] \) is integral over \( O[g(\alpha)] \); over any prime ideal in \( O[g(\alpha)] \) containing \((O[g(\alpha)] \cdot p, g(\alpha))\), there lies a prime ideal in \( O[\alpha] \), hence

\[(O[\alpha] \cdot p, g(\alpha)) \neq (1).\]

Since \( 1 + g(x) \) is monic of positive degree, also

\[(O[\alpha] \cdot p, 1 + g(\alpha)) \neq (1).\]

This shows that \( g(\alpha) \notin O[\alpha] \cdot p \), a conclusion that also holds if \( g(x) \) is of degree zero; that is, \( g(x) = 1 \).

We now prove that under the homomorphism \( g(x) \to g(\alpha) \) of \( O[x] \) onto \( O[\alpha] \), the inverse image of \( O[\alpha] \cdot p \) is \( O[x] \cdot p \); this will complete the proof, as \( O[x] \cdot p \) is prime but not maximal. Let, then,

\[g(x) \in O[x],\ g(x) \notin O[x] \cdot p.\]

We write

\[g(x) = g_1(x) + g_2(x),\]

where \( g_2(x) \in O[x] \cdot p \) and no coefficient of \( g_1(x) \) is in \( p \); in particular, this is so for the leading coefficient \( c \). Then \( g_1(\alpha)/c \notin O[\alpha] \cdot p \), since \( g_1(\alpha)/c \) is monic. A fortiori, \( g_1(\alpha) \notin O[\alpha] \cdot p \), whence also \( g(\alpha) \notin O[\alpha] \cdot p \).

**Corollary.** In the case \( \alpha \notin O \), if \( g(x) \in O[x] \) and \( g(\alpha) \in O[\alpha] \cdot p \), then \( g(x) \in O[x] \cdot p \).

**Theorem 7.** Let \( O \) be an integrally closed integral domain, \( p \) a proper ideal therein, \( a \) an element in the quotient-field of \( O \), but \( a \notin O_p, 1/a \notin O_p \). Then \( O[a] \cdot p \) is prime but not maximal; in fact,

\[O[\alpha] \cdot p \cap O = p \quad \text{and} \quad O[\alpha]/O[\alpha] \cdot p \cong O/p[x].\]

**Proof.** We know that \( O_p[\alpha] \cdot p \) is prime, and

\[O_p[\alpha] \cdot p \cap O[\alpha] = O[\alpha] = O[\alpha] \cdot p\]

by the last corollary (and the fact that \( O_p \cdot p \cap O = p \)). Hence \( O[\alpha] \cdot p \) is prime.

Also here, as in the corollary, we have that if \( g(x) \in O[x] \) and \( g(\alpha) \in O[\alpha] \cdot p \), then \( g(x) \in O[x] \cdot p \); the required isomorphism follows at once.
Theorem 7 is known in the case that $O$ is a finite discrete principal order [3, §49, p.134-136]. The class of rings dealt with in the theorem includes this class properly; for example, the ring $O$ of the example of Krull is not a finite discrete principal order, as $xy^\rho \in O$ for all $\rho$, but $y \notin O$.

**Theorem 8.** If $O$ is 1-dimensional, then $O[x]$ is 2-dimensional if and only if every quotient ring of the integral closure of $O$ is a valuation ring.

**Proof.** By Theorem 5, we may assume $O$ to be integrally closed. If $O$ is an $F$-ring, then so is one of its quotient rings (Theorem 3, Corollary). This quotient ring is not a valuation ring (Theorem 4). Conversely, suppose some quotient ring $O_1 = O_p$ is not a valuation ring. Let $\alpha$ be an element of the quotient field of $O_1$ such that $\alpha \notin O_1$ and $\alpha^{-1} \notin O_1$. Then $O_1[\alpha]$ is at least 2-dimensional, by Theorem 6, and $O_1[x]$ is at least 3-dimensional, as one sees by considering the homomorphism of $O_1[x]$ onto $O_1[\alpha]$ determined by mapping $x$ into $\alpha$. So $O_1$ is an $F$-ring. Thus $O_p[x] \cdot p$ is not minimal in $O_p[x]$, and it follows at once that $O[x] \cdot p$ is not minimal in $O[x]$, whence $O$ is an $F$-ring.

Let $O$ be the ring of Krull's example above, and let $X$ be an indeterminate. The single prime ideal $p$ in $O$ is constituted by the rational fractions $r(x, y)$ which, when written in lowest terms, have numerator divisible by $x$, i.e., are of the form $x \cdot g(x, y)$, where $g(x, y) \in K[x, y]$. The polynomials in $O[X]$ which vanish for $X = y$ form a prime ideal, different from $(0)$ since $xX - xy$ is in it, properly contained in $O[X] \cdot p$.

The following theorem is well known [4, Th. 13, p.376].

**Theorem 9.** If $O$ is a Noetherian ring of dimension $n$, then $O[x]$ is $(n + 1)$-dimensional.

**Proof.** Taking a quotient ring or residue class does not destroy the Noetherian character of $O$, so by Theorem 3 we may suppose $O$ is 1-dimensional. Let then $p$ be a proper prime ideal in $O$. Then $O[x] \cdot p$ is minimal for every principal ideal $O[x] \cdot (a)$, where $a \in p$, $a \neq 0$, so by the Principal Ideal Theorem [3, p.37], $O[x] \cdot p$ is minimal in $O[x]$, and $O[x]$ is 2-dimensional by Theorem 1, Corollary. Instead of the Principal Ideal Theorem, one could use instead that the integral closure $\overline{O}$ is also Noetherian (see, for example, [1, Th. 3, p.29]; see also [3, §39, p.108]). Neither proof makes use of the full force of the quoted theorems, so it might be of some interest to find a direct proof using less technical means.

**Note.** In a forthcoming paper we will show that if $O$ is a 1-dimensional ring
such that $O[x]$ is 2-dimensional, then $O[x_1, \ldots, x_n]$ is $(n + 1)$-dimensional. Theorem 2, above, will also be completed by examples showing that for any $m, n$ with $n + 1 \leq m \leq 2n + 1$, there exist $n$-dimensional rings such that $O[x]$ is $m$-dimensional.

References


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