SOME THEOREMS ON GENERALIZED DEDEKIND SUMS

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1. Introduction. Using a method developed by Rademacher [5], Apostol [1] has proved a transformation formula for the function

\[
G_p(x) = \sum_{m,n=1}^{\infty} n^{-p} x^{mn} \quad (|x| < 1),
\]

where \( p \) is a fixed odd integer \( > 1 \). The formula involves the coefficients

\[
c_r(h,k) = \sum_{\mu \pmod{k+1-r}} P_{p+1-r} \left( \frac{\mu}{k} \right) P_r \left( \frac{h\mu}{k} \right) \quad (0 \leq r \leq p + 1),
\]

where \((h,k) = 1\), the summation is over a complete residue system \( \pmod{k} \), and \( P_r(x) = \overline{B}_r(x) \), the Bernoulli function.

We shall show in this note that the transformation formula for (1.1) implies a reciprocity relation involving \( c_r(h,k) \), which for \( r = p \) reduces to Apostol's reciprocity theorem [1, Th. 1; 2, Th. 2] for the generalized Dedekind sum \( c_p(h,k) \). In addition, we prove some formulas for \( c_r(h,k) \) which generalize certain results proved by Rademacher and Whiteman [6]. Finally we derive a representation of \( c_r(h,k) \) in terms of so-called "Eulerian numbers".

2. Some preliminaries. It will be convenient to recall some properties of the Bernoulli function \( P_r(x) \); by definition, \( P_r(x) = B_r(x) \) for \( 0 \leq x < 1 \), and \( P_r(x+1) = P_r(x) \). Also we have the formulas

\[
\sum_{r=0}^{k-1} P_r \left( t + \frac{r}{k} \right) = k^{1-m} P_r(kt), \quad P_r(-x) = (-1)^r P_r(x).
\]

It follows from the second of (2.1) that \( c_r(h,k) = 0 \) for \( p \) even and \( 0 \leq r \leq p + 1 \). We have also

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\( (2.2) \quad c_0(h, k) = c_{p+1}(h, k) = k^{-p} B_{p+1} \)

provided \((h, k) = 1\). Further, it is clear from the second of \((2.1)\) that

\( (2.3) \quad c_r(-h, k) = (-1)^r c_r(h, k). \)

Now as in [5, 321] put \( x = e^{2\pi i \tau} \),

\[ \tau = \frac{iz + h}{k}, \quad \tau' = \frac{iz^{-1} + h'}{k}, \]

so that, on eliminating \( z \), we get

\( (2.4) \quad \tau' = \frac{h' \tau + k'}{k \tau - h} \quad (hh' + kk' + 1 = 0); \)

thus \((2.4)\) is a unimodular transformation. Now Apostol's transformation formula [1, Th. 2] reads (in our notation)

\[
G_p(e^{2\pi i \tau}) = (iz)^{p-1} G_p(e^{2\pi i \tau'}) - \frac{1}{2} \left( \frac{2\pi i}{k} \right)^p \frac{B_{p+1}}{(p+1)!} \\
+ \frac{i^{p-1}}{2z} \left( \frac{2\pi}{k} \right)^p \frac{B_{p+1}}{(p+1)!} + \frac{(2\pi i)^p}{2 \cdot p!} c_p(h, k) \\
+ \frac{(2\pi)^p z^{p-1}}{2(p+1)!} \sum_{r=0}^{p-2} \left( \frac{p+1}{r+1} \right) e^{\pi i (r-1)/2} z^{-r} \sum_{\mu=1}^{k} \sum_{\mu=1}^{p} \left( \frac{h' \mu}{k} \right) P_{p-r} \left( \frac{\mu}{k} \right). 
\]

Making use of \((1.2)\), \((2.2)\), and \((2.3)\), we easily verify that this result can be put in the form

\( (2.5) \quad G_p(e^{2\pi i \tau}) = (k\tau - h)^{p-1} G_p(e^{2\pi i \tau'}) + \frac{(2\pi i)^p}{2(p+1)!} f(h, k; \tau), \)

where

\( (2.6) \quad f(h, k; \tau) = \sum_{r=0}^{p+1} \binom{p+1}{r} (k\tau - h)^{p-r} c_r(h, k). \)

We remark that \((2.6)\) can be written in the symbolic form
(2.7) \[(k\tau - h)f(h, k; \tau) = (k\tau - h + c(h, k))^{p+1},\]

where it is understood that after expanding the right member of (2.7) by the binomial theorem, \(c'(h, k)\) is replaced by \(c_r(h, k)\).

We shall require an explicit formula for \(f(0, 1; \tau)\). Since, by (1.2),

\[c_r(0, 1) = P_{p+1} - r(0) P_r(0) = B_{p+1} \tau r,\]

it is clear that (2.6) implies

(2.8) \[f(0, 1; \tau) = \frac{1}{\tau} \sum_{r=0}^{p+1} \binom{p+1}{r} B_{p+1} \tau r \tau^{p+1-r} = \frac{1}{\tau} (B + \tau B)^{p+1}.\]

If in (2.4) we replace \(\tau\) by \(-1/\tau\), then \(\tau'\) becomes

(2.9) \[\tau' = \frac{-k'\tau + h'}{h\tau + k},\]

and (2.5) becomes

(2.10) \[G_p(e^{-2\pi i/\tau}) = \left(\frac{h\tau + k}{\tau}\right)^{p-1} G_p(e^{2\pi i \tau'}) + \frac{(2\pi i)^p}{2(p+1)!} f(h, k; -\frac{1}{\tau}).\]

By (2.5) and (2.8) we have

(2.11) \[G_p(e^{2\pi i \tau}) = \tau^{p-1} G_p(e^{-2\pi i/\tau}) + \frac{(2\pi i)^p}{2\tau(p+1)!} (B + \tau B)^{p+1},\]

and by (2.5) and (2.9),

(2.12) \[G_p(e^{2\pi i \tau}) = (h\tau + k)^{p-1} G_p(e^{2\pi i \tau'}) + \frac{2\pi i}{2(p+1)!} f(-k, h; \tau).\]

Comparison of (2.10), (2.11), (2.12) yields

\[f(-k, h; \tau) = \tau^{p-1} f(h, k; -\frac{1}{\tau}) + \frac{1}{\tau} (B + \tau B)^{p+1},\]

or with \(\tau\) replaced by \(-1/\tau\),
(2.13) \[ f(h, k; \tau) = \tau^{p-1} f\left(-k, h; -\frac{1}{\tau}\right) + \frac{1}{\tau} (B + \tau B)^{p+1}. \]

(For the above, compare [3, pp. 162-163]).

3. The main results. In (2.7) replace \(h, k, \tau\) by \(-h, -k, -1/\tau\) respectively; we get

\[
\frac{k\tau - h}{\tau} f\left(-k, h; -\frac{1}{\tau}\right) = \left(\frac{k\tau - h}{\tau} + c(-k, h)\right)^{p+1}.
\]

By (2.3), it is clear that (2.13) becomes

\[
\tau(k\tau - h + c(h, k))^{p+1}
\]

\[= (\tau c(k, h) - \tau k + h)^{p+1} + (k\tau - h)(B + \tau B)^{p+1}.\]

Comparison of the coefficients of \(\tau^{r+1}\) in both members of (3.1) leads immediately to:

**Theorem 1.** For \(p\) odd \(> 1\), \(0 < r < p\),

\[
\left(\begin{array}{c} p+1 \\ r \end{array}\right) h^{r}(c(h, k) - h)^{p+1-r} = \left(\begin{array}{c} p+1 \\ r+1 \end{array}\right) h^{p-r}(c(k, h) - k)^{r+1}
\]

\[+ kB_{p+1-r}B_r - hB_{p-r}B_{r+1}.\]

In the next place, if for brevity we put \(w = k\tau - h\), then (3.1) becomes

\[
k^p(w + h)(w + c(h, k))^{p+1}
\]

\[= ((w + h)c(k, h) - wk)^{p+1} + w(Bk + (w + h)B)^{p+1}.\]

We now compare coefficients of \(w^{r+1}\) in both members of (3.3); a little care is required in connection with the extreme right member. We state the result as:

**Theorem 2.** For \(p\) odd \(> 1\), \(0 \leq r \leq p\),

\[
\left(\begin{array}{c} p+1 \\ r+1 \end{array}\right)hk^p c_{p-r}(h, k) + \left(\begin{array}{c} p+1 \\ r \end{array}\right)k^p c_{p+1-r}(h, k)
\]
\[
\left(\frac{p + 1}{r + 1}\right) h^{p-r}(c(k, h) - k)^{r+1} c^{p-r}(k, h) + \left(\frac{p + 1}{r}\right) (Bk + B'h)^{p+1-r} B'^r,
\]

where

\[
(Bk + B'h)^{p+1-r} B'^r = \sum_{s=0}^{p+1-r} \binom{p + 1 - r}{s} B_{p+1-r-s} B_{r+s} k^{p+1-r-s} h^s.
\]

For \( r = 0 \), (3.4) becomes

\[
(p + 1)hkPc_p(h, k) + kPc_{p+1}(h, k)
\]

\[
= (p + 1)hP\{c_{p+1}(k, h) - kfc_p(k, h)\} + (p + 1)(Bk + Bh)^{p+1},
\]

which reduces to

\[
(3.5) \quad (p + 1)\{hkPc_p(h, k) + kPc_p(k, h)\} = (p + 1)(Bk + Bh)^{p+1} + pB_{p+1}.
\]

This is Apostol's reciprocity theorem.

If we take \( r = 1 \) in (3.4), we get

\[
\begin{align*}
& p \{ h^2 kPc_{p-1}(h, k) - k^2 h^{p} c_{p-1}(k, h) \} \\
& = -2\{ hk^p c_p(h, k) + pkh^p c_p(h, k) \} + pB_{p+1} + 2(Bk + B'h)^p B'^h.
\end{align*}
\]

If in this formula we interchange \( h \) and \( k \) and add we again get (3.5), while if we subtract we get

\[
(3.6) \quad p \{ h^2 kPc_{p-1}(h, k) - k^2 h^{p} c_{p-1}(k, h) \}
\]

\[
= (p - 1)\{hkPc_p(h, k) - kh^p c_p(k, h)\} - (Bk + Bh)^p(Bk - Bh).
\]

In view of (3.6), it does not seem likely that Theorem 2 will yield a simple expression for

\[
h^{r+1} k^p c_{p-r}(h, k) + (-1)^r k^{r+1} h^p c_{p-r}(k, h) \quad (r > 0).
\]

We remark that Theorems 1 and 2 are equivalent. Indeed it is evident that
(3.2) is equivalent to (3.1), and (3.4) is equivalent to (3.3); also it is clear that (3.1) and (3.3) are equivalent.

4. Some additional results. We next prove (compare [6, Th. 1]):

**Theorem 3.** For \( p, q \geq 1, 0 < r \leq p + 1 \), we have

\[
(4.1) \quad c_r(qh, qk) = q^{r-p} c_r(h, k).
\]

Note that we now do not assume \( p \) odd, \((h, k) = 1\).

To prove (4.1), we have, using (1.2),

\[
c_r(qh, qk) = \sum_{\mu \pmod{qk}} P_{p+1-r} \left( \frac{\mu}{qk} \right) P_r \left( \frac{h \mu}{k} \right).
\]

\[
= \sum_{\nu \pmod{q}} P_{p+1-r} \left( \frac{\nu k + \rho}{qk} \right) P_r \left( \frac{h (\nu k + \rho)}{k} \right)
\]

\[
= \sum_{\rho} P_r \left( \frac{h \rho}{k} \right) \sum_{\nu} P_{p+1-r} \left( \frac{\nu}{q} + \frac{\rho}{qk} \right)
\]

\[
= q^{r-p} \sum_{\rho} P_{p+1-r} \left( \frac{\rho}{k} \right) P_r \left( \frac{h \rho}{k} \right)
\]

\[
= q^{r-p} c_r(h, k).
\]

For brevity we define

\[
(4.2) \quad b_r(h, k) = (c(h, k) - h)^r = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} c_s(h, k),
\]

which occurs in Theorem 1. Clearly

\[
c_r(h, k) = (b(h, k) + h)^r.
\]

**Theorem 4.** For \( p, q \geq 1, 0 \leq r \leq p + 1 \), we have

\[
(4.3) \quad b_r(qh, qk) = q^{r-p} b_r(h, k).
\]
By (4.1) and (4.2) we have

\[ b_r(qh, qk) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (qh)^{r-s} c_s(qh, qk) \]

\[ = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} q^{r-p} c_s(h, k) \]

\[ = q^{r-p} b_r(h, k). \]

If we define

\[(4.4) \quad a_r(h, k) = (c(h, k) - h) r^{p+1-r}(h, k),\]

which is suggested by Theorem 2, we get:

**Theorem 5.** For \( p, q \geq 1, 0 \leq r \leq p + 1, \)

\[(4.5) \quad a_r(qh, qk) = qa_r(h, k). \]

The proof, which is exactly like the proof of (4.3), will be omitted.

We note that (4.4) implies

\[(4.6) \quad h^{r} c^{p+1-r}(h, k) = \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} a_s(h, k) = (1 - a(h, k))^r. \]

Also using (4.2) and (4.6), we get

\[(4.7) \quad h^{p+1-r} b_r(h, k) = (1 - a(h, k))^{p+1-r} a^r(h, k), \]

and reciprocally from (4.4),

\[(4.8) \quad a_r(h, k) = (b(h, k) + h) ^{p+1-r} b^r(h, k). \]

Using \( a_r(h, k) \) and \( b_r(h, k) \), we can state Theorems 1 and 2 somewhat more compactly.

**5. Another property of** \( c_r(h, k) \). For the next theorem compare [6, Th. 2].

**Theorem 6.** For \( p \geq 1, 0 \leq r \leq p, \) and \( q \) prime, we have
(5.1) \[ \sum_{m=0}^{q-1} c_r(h + mk, qk) = (q + q^{1-r})c_r(h, k) - q^{1-r}c_r(ph, k). \]

By (1.2), the left member of (5.1) is equal to

\[
\sum_{m=0}^{q-1} \sum_{\mu=1}^{qk} P_{p+1-r} \left( \frac{\mu}{qk} \right) P_r \left( \frac{(h + mk)\mu}{qk} \right) = \sum_{\mu=1}^{qk} P_{p+1-r} \left( \frac{\mu}{qk} \right) \sum_{m=0}^{\frac{qk}{\mu}} P_r \left( \frac{\mu}{qk} + \frac{m\mu}{q} \right)
\]

\[
= \sum_{\mu=1}^{qk} P_{p+1-r} \left( \frac{\mu}{qk} \right) \left( \frac{h\mu}{k} \right) q^{1-r}
\]

\[
+ \sum_{\nu=1}^{k} P_{p+1-r} \left( \frac{\nu}{k} \right) \left\{ qP_r \left( \frac{h\nu}{k} \right) - P_r \left( \frac{qh\nu}{k} \right) q^{1-r} \right\}
\]

\[
= q^{1-r}c_r(qh, qk) + qc_r(h, k) - q^{1-r}c_r(qh, k)
\]

\[
= (q^{1-p} + q)c_r(h, k) - q^{1-r}c_r(qh, k),
\]

by (4.1).

It does not seem possible to frame a result like (5.1) for the expressions \( b_r(h, k) \) or \( a_r(h, k) \) defined by (4.2) and (4.3).

6. Representation by Eulerian numbers. If \( k > 1, \rho^k = 1, \rho \neq 1 \), we define the "Eulerian number" \( H_m(\rho) \) by means of [4, p.825]

\[
(6.1) \quad \frac{1 - \rho}{e^t - \rho} = \sum_{m=0}^{\infty} H_m(\rho) \frac{t^m}{m!}.
\]

Then it is easily verified that [4, p.825]

\[
k^{m-1} \sum_{r=0}^{k-1} \rho^r B_m \left( \frac{r}{k} \right) = \frac{m}{\rho - 1} H_{m-1}(\rho^{-1}),
\]

which may be put in the more convenient form
Now consider the representation (finite Fourier series)

\begin{equation}
P_m\left(\frac{r}{k}\right) = \sum_{s=0}^{k-1} A_s \zeta^{-rs} \quad (\zeta = e^{2\pi i/k}).
\end{equation}

If we multiply both members of (6.3) by \(\zeta^rt\) and sum, we get

\begin{equation}
kA_t = \sum_r \zeta^rt \sum_{s=0}^{k-1} A_s \zeta^{-rs} = \begin{cases}
\frac{mk^{1-m}}{\zeta^t - 1} H_{m-1}(\zeta^{-t}) & (t \neq 0) \\
k^{1-m} B_m & (t = 0),
\end{cases}
\end{equation}

by (6.2) and (2.1). Thus (6.3) becomes

\begin{equation}
P_m\left(\frac{\mu}{k}\right) = k^m B_m + mk^{1-m} \sum_{s=1}^{k-1} \frac{H_{m-1}(\zeta^{-s})}{\zeta^s - 1} \zeta^{-\mu s}.
\end{equation}

Thus substituting from (6.4) in (1.2), we get after a little reduction

\begin{equation}
c_r(h, k) = \frac{B_{p+1-r} B_r}{k^p} + \frac{r(p + 1 - r)}{k^p} \sum_{t=1}^{k-1} \frac{H_{p-r}(\zeta^{-ht}) H_{r-1}(\zeta^{-t})}{(\zeta^{-ht} - 1)(\zeta^{-t} - 1)}.
\end{equation}

Thus \(c_r(h, k)\) has been explicitly evaluated in terms of the Eulerian numbers. One or two special cases of (6.5) may be mentioned. For \(r = p\) we have

\begin{equation}
c_p(h, k) = \frac{p}{k^p} \sum_{t=1}^{k-1} \frac{H_{p-1}(\zeta^{-t})}{(\zeta^{-ht} - 1)(\zeta^{-t} - 1)} \quad (p > 1),
\end{equation}

while for \(r = p = 1\) we have

\begin{equation}
\overline{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{t=1}^{k-1} \frac{1}{(\zeta^{-ht} - 1)(\zeta^{-t} - 1)},
\end{equation}

where \(\overline{s}(h, k) = c_1(h, k)\). Note that \(\overline{s}(h, k) = s(h, k) + 1/4\), where \(s(h, k)\) is the ordinary Dedekind sum \([6]\). We also note that (6.4) becomes, for \(m = 1\),
which is equivalent to a formula of Eisenstein.

Possibly (6.5) can be used to give a direct proof of Theorem 1 or Theorem 2.

REFERENCES


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