

# Pacific Journal of Mathematics

**THE RECIPROCITY THEOREM FOR DEDEKIND SUMS**

L. CARLITZ

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**1. Introduction.** Let  $((x)) = x - [x] - 1/2$ , where  $[x]$  denotes the greatest integer  $\leq x$ , and put

$$(1.1) \quad \bar{s}(h, k) = \sum_{r(\bmod k)} \left( \left( \frac{r}{k} \right) \right) \left( \left( \frac{hr}{k} \right) \right),$$

the summation extending over a complete residue system (mod  $k$ ). Then if  $(h, k) = 1$ , the sum  $\bar{s}(h, k)$  satisfies (see for example [4])

$$(1.2) \quad 12hk \{ \bar{s}(h, k) + \bar{s}(k, h) \} = h^2 + 3hk + k^2 + 1.$$

Note that  $\bar{s}(h, k) = s(h, k) + 1/4$ , where  $s(h, k)$  is the sum defined in [4].

In this note we shall give a simple proof of (1.2) which was suggested by Redei's proof [5]. The method also applies to Apostol's extension [1]; [2].

**2. A formula for  $\bar{s}(h, k)$ .** We start with the easily proved formula

$$(2.1) \quad \left( \left( \frac{r}{k} \right) \right) = -\frac{1}{2k} + \frac{1}{k} \sum_{s=1}^{k-1} \frac{\rho^{-rs}}{\rho^s - 1} \quad (\rho = e^{2\pi i/k}),$$

which is equivalent to a formula of Eisenstein. (Perhaps the quickest way to prove (2.1) is to observe that

$$\sum_{r=0}^{k-1} \left( \left( \frac{r}{k} \right) \right) \rho^{rs} = \begin{cases} 1/(\rho^s - 1) & (k \nmid s) \\ -1/2 & (k \mid s); \end{cases}$$

inverting leads at once to (2.1)).

Now substituting from (2.1) in (1.1) we get

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$$\begin{aligned} \bar{s}(h, k) &= \sum_r \left\{ -\frac{1}{2k} + \frac{1}{k} \sum_{t=1}^{k-1} \frac{\rho^{-ts}}{\rho^t - 1} \right\} \left\{ -\frac{1}{2k} + \frac{1}{k} \sum_{s=1}^{k-1} \frac{\rho^{-hrs}}{\rho^s - 1} \right\} \\ &= \frac{1}{4k} + \frac{1}{k^2} \sum_{s, t=1}^{k-1} \frac{1}{(\rho^s - 1)(\rho^t - 1)} \sum_{r=0}^{k-1} \rho^{-r(sh+t)}. \end{aligned}$$

Since the inner sum vanishes unless  $s + ht \equiv 0 \pmod{k}$ , we get

$$\bar{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{k=1}^{k-1} \frac{1}{(\rho^{-s} - 1)(\rho^{hs} - 1)},$$

or, what is the same thing,

$$(2.2) \quad \bar{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{\zeta \neq 1} \frac{1}{(\zeta^{-1} - 1)(\zeta^h - 1)},$$

where  $\zeta$  runs through the  $k$ th roots of unity distinct from 1.

**3. Proof of (1.2)** In the next place consider the equation

$$(3.1) \quad (x^h - 1)f(x) + (x^k - 1)g(x) = x - 1,$$

where  $f(x), g(x)$  are polynomials,  $\deg f(x) < k - 1, \deg g(x) < h - 1$ . Then if  $\zeta$  has the same meaning as in (2.2), it is clear from (3.1) that

$$(\zeta^h - 1)f(\zeta) = \zeta - 1.$$

Thus by the Lagrange interpolation formula

$$(3.2) \quad f(x) = (x^k - 1) \left\{ \frac{f(1)}{k(x - 1)} + \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{x - \zeta} \frac{\zeta - 1}{\zeta^h - 1} \right\}.$$

Similarly, if  $\eta$  runs through the  $h$ th roots of unity,

$$(3.3) \quad g(x) = \left\{ \frac{g(1)}{h(x - 1)} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{x - \eta} \frac{\eta - 1}{\eta^k - 1} \right\}.$$

Now it follows from (3.1) that  $hf(1) + kg(1) = 1$ ; hence substituting from (3.2) and (3.3) in (3.1) we get the identity

$$(3.4) \quad \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{x - \zeta} \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{x - \eta} \frac{\eta - 1}{\eta^k - 1} \\ = \frac{x - 1}{(x^k - 1)(x^h - 1)} - \frac{1}{hk(x - 1)}.$$

Next put  $x = 1 + t$  in (3.4) and expand both members in ascending powers of  $t$ . We find without difficulty that the right member of (3.4) becomes

$$(3.5) \quad -\frac{h + k - 2}{2hk} + \frac{h^2 + 3hk + k^2 - 3h - 3k + 1}{12hk} t + \dots$$

Comparison of coefficients of  $t$  in both sides of (3.4) leads at once to

$$-\frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{\zeta - 1} \frac{1}{\zeta^h - 1} - \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{\eta - 1} \frac{1}{\eta^k - 1} \\ = \frac{h^2 + 3hk + k^2 - 3h - 3k + 1}{12hk}.$$

Therefore by (2.2) and the corresponding formula for  $s(k, h)$ , we have

$$\bar{s}(h, k) + \bar{s}(k, h) = \frac{h^2 + 3hk + k^2 + 1}{12hk},$$

which is the same as (1.2).

**4. The generalized reciprocity formula.** The identity (3.4) implies a good deal more than (1.2). For example, for  $x = 0$ , we get

$$(4.1) \quad \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta - 1}{\eta^k - 1} = 1 - \frac{1}{hk},$$

while if we use the constant term in (3.5), we find that

$$(4.2) \quad \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{\eta^k - 1} = \frac{h + k - 2}{2hk}.$$

Again if we multiply by  $x$  and let  $x \rightarrow \infty$ , we get

$$(4.3) \quad \frac{1}{k} \sum_{\zeta \neq 1} \zeta \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \eta \frac{\eta - 1}{\eta^k - 1} = -\frac{1}{hk}.$$

More generally, expanding (3.4) in descending powers of  $x$ , we have

$$(4.4) \quad \frac{1}{k} \sum_{\zeta \neq 1} \zeta^r \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \eta^r \frac{\eta - 1}{\eta^k - 1} = \begin{cases} -\frac{1}{hk} & (1 \leq r < h + k - 1) \\ 1 - \frac{1}{hk} & (r = h + k - 1). \end{cases}$$

By continuing the expansion of (3.5) we can also show that

$$h \sum_{\zeta \neq 1} \frac{\zeta}{(\zeta - 1)^r (\zeta^h - 1)} + k \sum_{\eta \neq 1} \frac{\eta}{(\eta - 1)^r (\eta^k - 1)} \quad (r \geq 1)$$

is a polynomial in  $h, k$ , but the explicit expression seems complicated. A more interesting result can be obtained as follows. First we divide both sides of (3.4) by  $x - 1$  so that the left member becomes

$$\begin{aligned} & \frac{1}{k} \sum_{\zeta} \frac{\zeta}{\zeta^h - 1} \left( \frac{1}{x - \zeta} - \frac{1}{x - 1} \right) + \frac{1}{h} \sum_{\eta} \frac{\eta}{\eta^k - 1} \left( \frac{1}{x - \eta} - \frac{1}{x - 1} \right) \\ &= \frac{1}{k} \sum_{\zeta} \frac{\zeta}{\zeta^h - 1} \frac{1}{x - \zeta} + \frac{1}{h} \sum_{\eta} \frac{\eta}{\eta^k - 1} \frac{1}{x - \eta} - \frac{h + k - 2}{2hk(x - 1)} \end{aligned}$$

by (4.2). We now put  $x = e^t$ . Transposing the last term above to the right we find that the right member has the expansion

$$(4.5) \quad \frac{1}{hk} \sum_{m=0}^{\infty} \frac{(Bh + Bk)^m t^{m-2}}{m!} + \frac{h + k}{2hk} \sum_{m=0}^{\infty} \frac{B_m t^{m-1}}{m!} + \frac{1}{hk} \sum_{m=0}^{\infty} \frac{(m-1) B_m t^{m-2}}{m!},$$

where the  $B_m$  are the Bernoulli numbers. In the left member we put

$$\frac{1 - \zeta}{e^t - \zeta} = \sum_{m=0}^{\infty} H_m(\zeta) \frac{t^m}{m!},$$

where the  $H_m(\zeta)$  are the so-called ‘‘Eulerian numbers’’; we thus get

$$(4.6) \quad \frac{1}{k} \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{\zeta} \frac{H_m(\zeta^{-1})}{(\zeta-1)(\zeta^{-h}-1)} + \frac{1}{h} \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{\eta} \frac{H_m(\eta^{-1})}{(\eta-1)(\eta^{-k}-1)}.$$

But by [3, formula (6.6)], for  $p$  odd  $> 1$ ,

$$\frac{p}{k^p} \sum_{\zeta} \frac{H_{p-1}(\zeta)}{(\zeta-1)(\zeta^{-h}-1)} = s_p(h, k)$$

where [1]

$$s_p(h, k) = \sum_{r \pmod{k}} \bar{B}_1\left(\frac{r}{k}\right) \bar{B}_p\left(\frac{hr}{k}\right),$$

and  $\bar{B}_r(x)$  is the Bernoulli function. Thus the coefficient of  $t^{p-1}/(p-1)!$  in (4.6) is

$$(4.7) \quad \frac{1}{p} \left\{ k^{p-1} s_p(h, k) + h^{p-1} s_p(k, h) \right\},$$

while the corresponding coefficient in (4.5) is

$$(4.8) \quad \frac{1}{p(p+1)hk} (Bh + Bk)^{p+1} + \frac{1}{(p+1)hk} B_{p+1}.$$

Hence equating (4.7) and (4.8) we get Apostol's formula [1, Theorem 1]:

$$(p+1) \{ hk^p s_p(h, k) + kh^p s_p(k, h) \} = (Bh + Bk)^{p+1} + pB_{p+1}$$

for  $p$  odd  $> 1$ . Note that  $s_1(h, k) = \bar{s}(h, k)$ .

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