THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

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1. Introduction. Let \( ((x)) = x - [x] - 1/2 \), where \([x]\) denotes the greatest integer \( \leq x \), and put

\[
(1.1) \quad \overline{s}(h, k) = \sum_{r (\mod k)} \left( \left( \frac{r}{k} \right) \left( \frac{hr}{k} \right) \right),
\]

the summation extending over a complete residue system \((\mod k)\). Then if \((h, k) = 1\), the sum \( \overline{s}(h, k) \) satisfies (see for example [4])

\[
(1.2) \quad 12hk \{ \overline{s}(h, k) + \overline{s}(k, h) \} = h^2 + 3hk + k^2 + 1.
\]

Note that \( \overline{s}(h, k) = s(h, k) + 1/4 \), where \( s(h, k) \) is the sum defined in [4].

In this note we shall give a simple proof of (1.2) which was suggested by Redei's proof [5]. The method also applies to Apostol's extension [1]; [2].

2. A formula for \( \overline{s}(h, k) \). We start with the easily proved formula

\[
(2.1) \quad \left( \left( \frac{r}{k} \right) \right) = -\frac{1}{2k} + \frac{1}{k} \sum_{s=1}^{k-1} \frac{\rho^{-rs}}{\rho^s - 1} \quad (\rho = e^{2\pi i/k}),
\]

which is equivalent to a formula of Eisenstein. (Perhaps the quickest way to prove (2.1) is to observe that

\[
\sum_{r=0}^{k-1} \left( \left( \frac{r}{k} \right) \right) \rho^{rs} = \begin{cases} 1/(\rho^s - 1) & (k \nmid s) \\ -1/2 & (k \mid s) \end{cases}
\]

inverting leads at once to (2.1)).

Now substituting from (2.1) in (1.1) we get
\[
\overline{s}(h, k) = \sum_r \left\{-\frac{1}{2k} + \frac{1}{k} \sum_{t=1}^{k-1} \frac{\rho^{-ts}}{\rho^t - 1}\right\}\left\{-\frac{1}{2k} + \frac{1}{k} \sum_{s=1}^{k-1} \frac{\rho^{-hrs}}{\rho^s - 1}\right\}
\]

\[
= \frac{1}{4k} + \frac{1}{k^2} \sum_{s, t=1}^{k-1} \frac{1}{(\rho^s - 1)(\rho^t - 1)} \sum_{r=0}^{k-1} \rho^{-r(sh+t)}.
\]

Since the inner sum vanishes unless \(s + ht \equiv 0 \pmod{k}\), we get

\[
\overline{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{k=1}^{k-1} \frac{1}{(\rho^{-s} - 1)(\rho^{hs} - 1)}
\]

or, what is the same thing,

(2.2) \[
\overline{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{\zeta \neq 1} \frac{1}{(\zeta^{-1} - 1)(\zeta^h - 1)},
\]

where \(\zeta\) runs through the \(k\)th roots of unity distinct from 1.

3. Proof of (1.2) In the next place consider the equation

(3.1) \[
(x^h - 1)f(x) + (x^k - 1)g(x) = x - 1,
\]

where \(f(x), g(x)\) are polynomials, \(\deg f(x) < k - 1, \deg g(x) < h - 1\). Then if \(\zeta\) has the same meaning as in (2.2), it is clear from (3.1) that

\[(\zeta^h - 1)f(\zeta) = \zeta - 1.\]

Thus by the Lagrange interpolation formula

(3.2) \[
f(x) = (x^k - 1)\left\{\frac{f(1)}{k(x - 1)} + \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{x - \zeta} \frac{\zeta - 1}{\zeta^h - 1}\right\}.
\]

Similarly, if \(\eta\) runs through the \(h\)th roots of unity,

(3.3) \[
g(x) = \left\{\frac{g(1)}{h(x - 1)} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{x - \eta} \frac{\eta - 1}{\eta^k - 1}\right\}.
\]

Now it follows from (3.1) that \(hf(1) + kg(1) = 1\); hence substituting from (3.2) and (3.3) in (3.1) we get the identity
Next put \( x = 1 + t \) in (3.4) and expand both members in ascending powers of \( t \).

We find without difficulty that the right member of (3.4) becomes

\[
(3.5) \quad \frac{h + k - 2}{2hk} + \frac{h^2 + 3hk + k^2 - 3h - 3k + 1}{12hk} t + \cdots.
\]

Comparison of coefficients of \( t \) in both sides of (3.4) leads at once to

\[
- \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{\zeta - 1} \frac{1}{\zeta^h - 1} - \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{\eta - 1} \frac{1}{\eta^k - 1} = \frac{h^2 + 3hk + k^2 - 3h - 3k + 1}{12hk}.
\]

Therefore by (2.2) and the corresponding formula for \( s(k, h) \), we have

\[
\overline{s}(h, k) + \overline{s}(k, h) = \frac{h^2 + 3hk + k^2 + 1}{12hk},
\]

which is the same as (1.2).

4. The generalized reciprocity formula. The identity (3.4) implies a good deal more than (1.2). For example, for \( x = 0 \), we get

\[
(4.1) \quad \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta - 1}{\eta^k - 1} = 1 - \frac{1}{hk},
\]

while if we use the constant term in (3.5), we find that

\[
(4.2) \quad \frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{\eta^k - 1} = \frac{h + k - 2}{2hk}.
\]

Again if we multiply by \( x \) and let \( x \to \infty \), we get
More generally, expanding (3.4) in descending powers of $x$, we have

\[
\frac{1}{k} \sum_{\zeta \neq 1} \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta - 1}{\eta^k - 1} = -\frac{1}{hk}.
\]

By continuing the expansion of (3.5) we can also show that

\[
\frac{1}{k} \sum_{\zeta \neq 1} \zeta^r \frac{\zeta - 1}{\zeta^h - 1} + \frac{1}{h} \sum_{\eta \neq 1} \eta^r \frac{\eta - 1}{\eta^k - 1} = \begin{cases} 
-\frac{1}{hk} & (1 \leq r < h + k - 1) \\
1 - \frac{1}{hk} & (r = h + k - 1).
\end{cases}
\]

is a polynomial in $h$, $k$, but the explicit expression seems complicated. A more interesting result can be obtained as follows. First we divide both sides of (3.4) by $x - 1$ so that the left member becomes

\[
\frac{1}{k} \zeta \sum_{\zeta \neq 1} \left( \frac{1}{x - \zeta} - \frac{1}{x - 1} \right) + \frac{1}{h} \sum_{\eta \neq 1} \frac{\eta}{ \eta^k - 1} \left( \frac{1}{x - \eta} - \frac{1}{x - 1} \right)
\]

\[
= \frac{1}{k} \sum_{\zeta} \frac{\zeta}{\zeta^h - 1} \frac{1}{x - \zeta} + \frac{1}{h} \sum_{\eta} \frac{\eta}{\eta^k - 1} \frac{1}{x - \eta} - \frac{h + k - 2}{2hk(x - 1)}
\]

by (4.2). We now put $x = e^t$. Transposing the last term above to the right we find that the right member has the expansion

\[
\frac{1}{h} \sum_{m=0}^{\infty} \frac{(Bh + Bk)^m t^{m-2}}{m!} + \frac{h + k}{2hk} \sum_{m=0}^{\infty} \frac{B_m t^{m-1}}{m!} + \frac{1}{hk} \sum_{m=0}^{\infty} \frac{(m-1)B_m t^{m-2}}{m!},
\]

where the $B_m$ are the Bernoulli numbers. In the left member we put

\[
\frac{1 - \zeta}{e^t - \zeta} = \sum_{m=0}^{\infty} H_m(\zeta) \frac{t^m}{m!},
\]

where the $H_m(\zeta)$ are the so-called "Eulerian numbers"; we thus get
(4.6) \[ \frac{1}{k} \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{\zeta} \frac{H_m(\zeta^{-1})}{(\zeta - 1)(\zeta^{-h} - 1)} + \frac{1}{h} \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{\eta} \frac{H_m(\eta^{-1})}{(\eta - 1)(\eta^{-k} - 1)}. \]

But by [3, formula (6.6)], for \( p \) odd \( > 1 \),
\[ \frac{p}{k^p} \sum_{\zeta} \frac{H_{p-1}(\zeta)}{(\zeta - 1)(\zeta^{-h} - 1)} = s_p(h, k) \]
where [1]
\[ s_p(h, k) = \sum_{r \pmod{k}} \bar{B}_1 \left( \frac{r}{k} \right) \bar{B}_p \left( \frac{hr}{k} \right), \]
and \( \bar{B}_r(x) \) is the Bernoulli function. Thus the coefficient of \( t^{p-1}/(p - 1)! \) in (4.6) is
\[ (4.7) \quad \frac{1}{p} \left\{ k^{p-1} s_p(h, k) + h^{p-1} s_p(k, h) \right\}, \]
while the corresponding coefficient in (4.5) is
\[ (4.8) \quad \frac{1}{p(p + 1)hk} (Bh + Bk)^{p+1} + \frac{1}{(p + 1)hk} B_{p+1}. \]

Hence equating (4.7) and (4.8) we get Apostol's formula [1, Theorem 1]:
\[ (p + 1) \left\{ hk^p s_p(h, k) + kh^p s_p(k, h) \right\} = (Bh + Bk)^{p+1} + pB_{p+1} \]
for \( p \) odd \( > 1 \). Note that \( s_1(h, k) = \bar{s}(h, k) \).

REFERENCES


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