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SOME THEOREMS ON GENERALIZED DEDEKIND SUMS

L. CARLITZ

1. Introduction. Using a method developed by Rademacher [5], Apostol [1] has proved a transformation formula for the function

$$G_p(x) = \sum_{m,\, n=1}^{\infty} n^{-p} x^{mn} \qquad (|x| < 1), \tag{1.1}$$

where p is a fixed odd integer > 1. The formula involves the coefficients

$$c_r(h, k) = \sum_{\mu \,(\mathrm{mod}\, k)} P_{p+1-r}\left(\frac{\mu}{k}\right) P_r\left(\frac{h\mu}{k}\right) \qquad (0 \le r \le p + 1), \tag{1.2}$$

where $(h, k) = 1$, the summation is over a complete residue system (mod k), and $P_r(x) = \overline{B}_r(x)$, the Bernoulli function.

We shall show in this note that the transformation formula for (1.1) implies a reciprocity relation involving $c_r(h, k)$, which for $r = p$ reduces to Apostol's reciprocity theorem [1, Th. 1; 2, Th. 2] for the generalized Dedekind sum $c_p(h, k)$. In addition, we prove some formulas for $c_r(h, k)$ which generalize certain results proved by Rademacher and Whiteman [6]. Finally we derive a representation of $c_r(h, k)$ in terms of so-called "Eulerian numbers".

2. Some preliminaries. It will be convenient to recall some properties of the Bernoulli function $P_r(x)$; by definition, $P_r(x) = B_r(x)$ for $0 \le x < 1$, and $P_r(x + 1) = P_r(x)$. Also we have the formulas

$$\sum_{r=0}^{k-1} P_r\left(t + \frac{r}{k}\right) = k^{1-m} P_r(kt), \quad P_r(-x) = (-1)^r P_r(x). \tag{2.1}$$

It follows from the second of (2.1) that $c_r(h, k) = 0$ for p even and $0 \le r \le p + 1$. We have also

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$$(2.2)\qquad c_0(h,k) = c_{p+1}(h,k) = k^{-p}B_{p+1}$$

provided $(h, k) = 1$. Further, it is clear from the second of (2.1) that

$$(2.3)\qquad c_r(-h,k) = (-1)^r c_r(h,k).$$

Now as in [5, 321] put $x = e^{2\pi i\tau}$,

$$\tau = \frac{iz+h}{k}, \quad \tau' = \frac{iz^{-1}+h'}{k},$$

so that, on eliminating z, we get

$$(2.4)\qquad \tau' = \frac{h'\tau + k'}{k\tau - h} \qquad (hh' + kk' + 1 = 0);$$

thus (2.4) is a unimodular transformation. Now Apostol's transformation formula [1, Th. 2] reads (in our notation)

$$G_p(e^{2\pi i\tau}) = (iz)^{p-1}G_p(e^{2\pi i\tau'}) - \frac{1}{2}\left(\frac{2\pi z}{k}\right)^p \frac{B_{p+1}}{(p+1)!}$$

$$+ \frac{i^{p-1}}{2z}\left(\frac{2\pi}{k}\right)^p \frac{B_{p+1}}{(p+1)!} + \frac{(2\pi i)^p}{2\cdot p!}c_p(h,k)$$

$$+ \frac{(2\pi)^p z^{p-1}}{2(p+1)!}\sum_{r=0}^{p-2}\binom{p+1}{r+1} e^{\pi i(r-1)/2} z^{-r}\sum_{\mu=1}^{k} P_{p-r}\left(\frac{h'\mu}{k}\right)P_{r+1}\left(\frac{\mu}{k}\right).$$

Making use of (1.2), (2.2), and (2.3), we easily verify that this result can be put in the form

$$(2.5)\qquad G_p(e^{2\pi i\tau}) = (k\tau - h)^{p-1}G_p(e^{2\pi i\tau'}) + \frac{(2\pi i)^p}{2(p+1)!}f(h,k;\tau),$$

where

$$(2.6)\qquad f(h,k;\tau) = \sum_{r=0}^{p+1}\binom{p+1}{r}(k\tau - h)^{p-r}c_r(h,k).$$

We remark that (2.6) can be written in the symbolic form

$$(2.7) \qquad (k\tau - h)\, f(h, k; \tau) = (k\tau - h + c(h, k))^{p+1},$$

where it is understood that after expanding the right member of (2.7) by the binomial theorem, $c^r(h, k)$ is replaced by $c_r(h, k)$.

We shall require an explicit formula for $f(0, 1; \tau)$. Since, by (1.2),

$$c_r(0, 1) = P_{p+1-r}(0)\, P_r(0) = B_{p+1-r} B_r,$$

it is clear that (2.6) implies

$$(2.8) \qquad f(0, 1; \tau) = \frac{1}{\tau} \sum_{r=0}^{p+1} \binom{p+1}{r} B_{p+1-r} B_r \tau^{p+1-r} = \frac{1}{\tau}(B + \tau B)^{p+1}.$$

If in (2.4) we replace τ by $-1/\tau$, then τ' becomes

$$(2.9) \qquad \tau^* = \frac{-k'\tau + h'}{h\tau + k},$$

and (2.5) becomes

$$(2.10) \qquad G_p(e^{-2\pi i/\tau}) = \left(\frac{h\tau + k}{\tau}\right)^{p-1} G_p(e^{2\pi i\tau^*}) + \frac{(2\pi i)^p}{2(p+1)!}\, f\left(h, k; -\frac{1}{\tau}\right).$$

By (2.5) and (2.8) we have

$$(2.11) \qquad G_p(e^{2\pi i\tau}) = \tau^{p-1} G_p(e^{-2\pi i/\tau}) + \frac{(2\pi i)^p}{2\tau(p+1)!}(B + \tau B)^{p+1},$$

and by (2.5) and (2.9),

$$(2.12) \qquad G_p(e^{2\pi i\tau}) = (h\tau + k)^{p-1} G_p(e^{2\pi i\tau^*}) + \frac{2\pi i}{2(p+1)!}\, f(-k, h; \tau).$$

Comparison of (2.10), (2.11), (2.12) yields

$$f(-k, h; \tau) = \tau^{p-1} f\left(h, k; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^{p+1},$$

or with τ replaced by $-1/\tau$,

$$(2.13) \qquad f(h, k; \tau) = \tau^{p-1} f\left(-k, h; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^{p+1}.$$

(For the above, compare [3, pp. 162-163]).

3. The main results. In (2.7) replace h, k, τ by $-k, h, -1/\tau$ respectively; we get

$$\frac{k\tau - h}{\tau} f\left(-k, h; -\frac{1}{\tau}\right) = \left(\frac{k\tau - h}{\tau} + c(-k, h)\right)^{p+1}.$$

By (2.3), it is clear that (2.13) becomes

$$(3.1) \qquad \tau(k\tau - h + c(h, k))^{p+1}$$
$$= (\tau c(k, h) - \tau k + h)^{p+1} + (k\tau - h)(B + \tau B)^{p+1}.$$

Comparison of the coefficients of τ^{r+1} in both members of (3.1) leads immediately to:

THEOREM 1. *For p odd* > 1, $0 \leq r \leq p$,

$$(3.2) \qquad \binom{p+1}{r} k^r(c(h, k) - h)^{p+1-r} = \binom{p+1}{r+1} h^{p-r}(c(k, h) - k)^{r+1}$$
$$+ kB_{p+1-r}B_r - hB_{p-r}B_{r+1}.$$

In the next place, if for brevity we put $w = k\tau - h$, then (3.1) becomes

$$(3.3) \qquad k^p(w + h)(w + c(h, k))^{p+1}$$
$$= ((w + h)c(k, h) - wk)^{p+1} + w(Bk + (w + h)B)^{p+1}.$$

We now compare coefficients of w^{r+1} in both members of (3.3); a little care is required in connection with the extreme right member. We state the result as:

THEOREM 2. *For p odd* > 1, $0 \leq r \leq p$,

$$(3.4) \qquad \binom{p+1}{r+1} hk^p c_{p-r}(h, k) + \binom{p+1}{r} k^p c_{p+1-r}(h, k)$$

$$= \binom{p+1}{r+1} h^{p-r}(c(k, h) - k)^{r+1} c^{p-r}(k, h) + \binom{p+1}{r} (Bk + B'h)^{p+1-r} B'^{r},$$

where

$$(Bk + B'h)^{p+1-r} B'^{r} = \sum_{s=0}^{p+1-r} \binom{p+1-r}{s} B_{p+1-r-s} B_{r+s} k^{p+1-r-s} h^{s}.$$

For $r = 0$, (3.4) becomes

$$(p+1) hk^p c_p(h, k) + k^p c_{p+1}(h, k)$$

$$= (p+1) h^p \{ c_{p+1}(k, h) - kc_p(k, h) \} + (p+1)(Bk + Bh)^{p+1},$$

which reduces to

$$(3.5) \qquad (p+1)\{ hk^p c_p(h, k) + k^p hc_p(k, h) \} = (p+1)(Bk + Bh)^{p+1} + pB_{p+1}.$$

This is Apostol's reciprocity theorem.

If we take $r = 1$ in (3.4), we get

$$p\{ h^2 k^p c_{p-1}(h, k) - k^2 h^p c_{p-1}(k, h) \}$$

$$= -2\{ hk^p c_p(h, k) + pkh^p c_p(h, k) \} + pB_{p+1} + 2(Bk + B'h)^p B'h.$$

If in this formula we interchange h and k and add we again get (3.5), while if we subtract we get

$$(3.6) \qquad p\{ h^2 k^p c_{p-1}(h, k) - k^2 h^p c_{p-1}(k, h) \}$$

$$= (p-1)\{ hk^p c_p(h, k) - kh^p c_p(k, h) \} - (Bk + Bh)^p (Bk - Bh).$$

In view of (3.6), it does not seem likely that Theorem 2 will yield a simple expression for

$$h^{r+1} k^p c_{p-r}(h, k) + (-1)^r k^{r+1} h^p c_{p-r}(k, h) \qquad (r > 0).$$

We remark that Theorems 1 and 2 are equivalent. Indeed it is evident that

(3.2) is equivalent to (3.1), and (3.4) is equivalent to (3.3); also it is clear that (3.1) and (3.3) are equivalent.

4. Some additional results. We next prove (compare [6, Th. 1]):

THEOREM 3. *For p, $q \geq 1$, $0 \leq r \leq p+1$, we have*

$$c_r(qh, qk) = q^{r-p} c_r(h, k). \tag{4.1}$$

Note that we now do not assume p odd, $(h, k) = 1$.

To prove (4.1), we have, using (1.2),

$$\begin{aligned} c_r(qh, qk) &= \sum_{\mu \,(\mathrm{mod}\ qk)} P_{p+1-r}\left(\frac{\mu}{qk}\right) P_r\left(\frac{h\mu}{k}\right). \\ &= \sum_{\substack{\nu \,(\mathrm{mod}\ q) \\ \rho \,(\mathrm{mod}\ k)}} P_{p+1-r}\left(\frac{\nu k + \rho}{qk}\right) P_r\left(\frac{h(\nu k + \rho)}{k}\right) \\ &= \sum_{\rho} P_r\left(\frac{h\rho}{k}\right) \sum_{\nu} P_{p+1-r}\left(\frac{\nu}{q} + \frac{\rho}{qk}\right) \\ &= q^{r-p} \sum_{\rho} P_{p+1-r}\left(\frac{\rho}{k}\right) P_r\left(\frac{h\rho}{k}\right) \\ &= q^{r-p} c_r(h, k). \end{aligned}$$

For brevity we define

$$b_r(h, k) = (c(h, k) - h)^r = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} c_s(h, k), \tag{4.2}$$

which occurs in Theorem 1. Clearly

$$c_r(h, k) = (b(h, k) + h)^r.$$

THEOREM 4. *For p, $q \geq 1$, $0 \leq r \leq p+1$, we have*

$$b_r(qh, qk) = q^{r-p} b_r(h, k). \tag{4.3}$$

By (4.1) and (4.2) we have

$$b_r(qh, qk) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (qh)^{r-s} c_s(qh, qk)$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} q^{r-p} c_s(h, k)$$

$$= q^{r-p} b_r(h, k).$$

If we define

$$a_r(h, k) = (c(h, k) - h)^r c^{p+1-r}(h, k), \tag{4.4}$$

which is suggested by Theorem 2, we get:

THEOREM 5. *For p, $q \geq 1$, $0 \leq r \leq p+1$,*

$$a_r(qh, qk) = qa_r(h, k). \tag{4.5}$$

The proof, which is exactly like the proof of (4.3), will be omitted.

We note that (4.4) implies

$$h^r c^{p+1-r}(h, k) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} a_s(h, k) = (1 - a(h, k))^r. \tag{4.6}$$

Also using (4.2) and (4.6), we get

$$h^{p+1-r} b_r(h, k) = (1 - a(h, k))^{p+1-r} a^r(h, k), \tag{4.7}$$

and reciprocally from (4.4),

$$a_r(h, k) = (b(h, k) + h)^{p+1-r} b^r(h, k). \tag{4.8}$$

Using $a_r(h, k)$ and $b_r(h, k)$, we can state Theorems 1 and 2 somewhat more compactly.

5. Another property of $c_r(h, k)$. For the next theorem compare [**6**, Th. 2].

THEOREM 6. *For $p \geq 1$, $0 \leq r \leq p$, and q prime, we have*

$$\sum_{m=0}^{q-1} c_r(h+mk,\, qk) = (q+q^{1-p})\,c_r(h,\,k) - q^{1-r}c_r(ph,\,k). \tag{5.1}$$

By (1.2), the left member of (5.1) is equal to

$$\sum_{m=0}^{q-1}\sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)P_r\left(\frac{(h+mk)\mu}{qk}\right)$$

$$= \sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)\sum_{m=0}^{q-1} P_r\left(\frac{h\mu}{qk}+\frac{m\mu}{q}\right)$$

$$= \sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)P_r\left(\frac{h\mu}{k}\right)q^{1-r}$$

$$+ \sum_{\nu=1}^{k} P_{p+1-r}\left(\frac{\nu}{k}\right)\left\{qP_r\left(\frac{h\nu}{k}\right) - P_r\left(\frac{qh\nu}{k}\right)q^{1-r}\right\}$$

$$= q^{1-r}c_r(qh,\,qk) + qc_r(h,\,k) - q^{1-r}c_r(qh,\,k)$$

$$= (q^{1-p}+q)\,c_r(h,\,k) - q^{1-r}c_r(qh,\,k),$$

by (4.1).

It does not seem possible to frame a result like (5.1) for the expressions $b_r(h,\,k)$ or $a_r(h,\,k)$ defined by (4.2) and (4.3).

6. Representation by Eulerian numbers. If $k > 1$, $\rho^k = 1$, $\rho \neq 1$, we define the "Eulerian number" $H_m(\rho)$ by means of [4, p. 825]

$$\frac{1-\rho}{e^t-\rho} = \sum_{m=0}^{\infty} H_m(\rho)\frac{t^m}{m!}. \tag{6.1}$$

Then it is easily verified that [4, p. 825]

$$k^{m-1}\sum_{r=0}^{k-1}\rho^r B_m\left(\frac{r}{k}\right) = \frac{m}{\rho-1}H_{m-1}(\rho^{-1}),$$

which may be put in the more convenient form

$$(6.2) \qquad k^{m-1} \sum_{r \,(\mathrm{mod}\ k)} \rho^r P_m\left(\frac{r}{k}\right) = \frac{m}{\rho - 1} H_{m-1}(\rho^{-1}).$$

Now consider the representation (finite Fourier series)

$$(6.3) \qquad P_m\left(\frac{r}{k}\right) = \sum_{s=0}^{k-1} A_s \zeta^{-rs} \qquad (\zeta = e^{2\pi i/k}).$$

If we multiply both members of (6.3) by ζ^{rt} and sum, we get

$$kA_t = \sum_r \zeta^{rt} P_m\left(\frac{r}{k}\right) = \begin{cases} \dfrac{mk^{1-m}}{\zeta^t - 1} H_{m-1}(\zeta^{-t}) & (t \neq 0) \\ k^{1-m} B_m & (t = 0), \end{cases}$$

by (6.2) and (2.1). Thus (6.3) becomes

$$(6.4) \qquad P_m\left(\frac{\mu}{k}\right) = k^{-m} B_m + mk^{-m} \sum_{s=1}^{k-1} \frac{H_{m-1}(\zeta^{-s})}{\zeta^s - 1} \zeta^{-\mu s}.$$

Thus substituting from (6.4) in (1.2), we get after a little reduction

$$(6.5) \qquad c_r(h, k) = \frac{B_{p+1-r} B_r}{k^p} + \frac{r(p+1-r)}{k^p} \sum_{t=1}^{k-1} \frac{H_{p-r}(\zeta^{ht}) H_{r-1}(\zeta^{-t})}{(\zeta^{-ht} - 1)(\zeta^t - 1)}.$$

Thus $c_r(h, k)$ has been explicitly evaluated in terms of the Eulerian numbers. One or two special cases of (6.5) may be mentioned. For $r = p$ we have

$$(6.6) \qquad c_p(h, k) = \frac{p}{k^p} \sum_{t=1}^{k-1} \frac{H_{p-1}(\zeta^{-t})}{(\zeta^{-ht} - 1)(\zeta^t - 1)} \qquad (p > 1),$$

while for $r = p = 1$ we have

$$\overline{s}(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{t=1}^{k-1} \frac{1}{(\zeta^{-ht} - 1)(\zeta^t - 1)},$$

where $\overline{s}(h, k) = c_1(h, k)$. Note that $\overline{s}(h, k) = s(h, k) + 1/4$, where $s(h, k)$ is the ordinary Dedekind sum [6]. We also note that (6.4) becomes, for $m = 1$,

$$P_1\left(\frac{\mu}{k}\right) = -\frac{1}{2k} + \frac{1}{k}\sum_{s=1}^{k-1}\frac{\zeta^{-\mu s}}{\zeta^s - 1},$$

which is equivalent to a formula of Eisenstein.

Possibly (6.5) can be used to give a direct proof of Theorem 1 or Theorem 2.

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DUKE UNIVERSITY

THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

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1. Introduction. Let $((x)) = x - [x] - 1/2$, where $[x]$ denotes the greatest integer $\leq x$, and put

$$\overline{s}(h, k) = \sum_{r(\mathrm{mod}\ k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{1.1}$$

the summation extending over a complete residue system $(\mathrm{mod}\ k)$. Then if $(h, k) = 1$, the sum $\overline{s}(h, k)$ satisfies (see for example [4])

$$12hk\{\overline{s}(h, k) + \overline{s}(k, h)\} = h^2 + 3hk + k^2 + 1. \tag{1.2}$$

Note that $\overline{s}(h, k) = s(h, k) + 1/4$, where $s(h, k)$ is the sum defined in [4].

In this note we shall give a simple proof of (1.2) which was suggested by Redei's proof [5]. The method also applies to Apostol's extension [1]; [2].

2. A formula for $\overline{s}(h, k)$. We start with the easily proved formula

$$\left(\left(\frac{r}{k}\right)\right) = -\frac{1}{2k} + \frac{1}{k}\sum_{s=1}^{k-1} \frac{\rho^{-rs}}{\rho^s - 1} \qquad (\rho = e^{2\pi i/k}), \tag{2.1}$$

which is equivalent to a formula of Eisenstein. (Perhaps the quickest way to prove (2.1) is to observe that

$$\sum_{r=0}^{k-1} \left(\left(\frac{r}{k}\right)\right) \rho^{rs} = \begin{cases} 1/(\rho^s - 1) & (k \nmid s) \\ -1/2 & (k \mid s); \end{cases}$$

inverting leads at once to (2.1)).

Now substituting from (2.1) in (1.1) we get

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$$\overline{s}(h,k)=\sum_r\left\{-\frac{1}{2k}+\frac{1}{k}\sum_{t=1}^{k-1}\frac{\rho^{-ts}}{\rho^t-1}\right\}\left\{-\frac{1}{2k}+\frac{1}{k}\sum_{s=1}^{k-1}\frac{\rho^{-hrs}}{\rho^s-1}\right\}$$

$$=\frac{1}{4k}+\frac{1}{k^2}\sum_{s,\,t=1}^{k-1}\frac{1}{(\rho^s-1)(\rho^t-1)}\sum_{r=0}^{k-1}\rho^{-r(sh+t)}.$$

Since the inner sum vanishes unless $s+ht\equiv 0 \pmod{k}$, we get

$$\overline{s}(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{k=1}^{k-1}\frac{1}{(\rho^{-s}-1)(\rho^{hs}-1)},$$

or, what is the same thing,

$$\overline{s}(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{\zeta\neq 1}\frac{1}{(\zeta^{-1}-1)(\zeta^h-1)}, \tag{2.2}$$

where ζ runs through the kth roots of unity distinct from 1.

3. Proof of (1.2) In the next place consider the equation

$$(x^h-1)f(x)+(x^k-1)g(x)=x-1, \tag{3.1}$$

where $f(x)$, $g(x)$ are polynomials, $\deg f(x)<k-1$, $\deg g(x)<h-1$. Then if ζ has the same meaning as in (2.2), it is clear from (3.1) that

$$(\zeta^h-1)f(\zeta)=\zeta-1.$$

Thus by the Lagrange interpolation formula

$$f(x)=(x^k-1)\left\{\frac{f(1)}{k(x-1)}+\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{x-\zeta}\,\frac{\zeta-1}{\zeta^h-1}\right\}. \tag{3.2}$$

Similarly, if η runs through the hth roots of unity,

$$g(x)=\left\{\frac{g(1)}{h(x-1)}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{x-\eta}\,\frac{\eta-1}{\eta^k-1}\right\}. \tag{3.3}$$

Now it follows from (3.1) that $hf(1)+kg(1)=1$; hence substituting from (3.2) and (3.3) in (3.1) we get the identity

$$\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{x-\zeta}\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{x-\eta}\,\frac{\eta-1}{\eta^k-1} \tag{3.4}$$

$$=\frac{x-1}{(x^k-1)(x^h-1)}-\frac{1}{hk(x-1)}.$$

Next put $x = 1 + t$ in (3.4) and expand both members in ascending powers of t. We find without difficulty that the right member of (3.4) becomes

$$-\frac{h+k-2}{2hk}+\frac{h^2+3hk+k^2-3h-3k+1}{12hk}\,t+\cdots. \tag{3.5}$$

Comparison of coefficients of t in both sides of (3.4) leads at once to

$$-\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{\zeta-1}\,\frac{1}{\zeta^h-1}-\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{\eta-1}\,\frac{1}{\eta^k-1}$$

$$=\frac{h^2+3hk+k^2-3h-3k+1}{12hk}.$$

Therefore by (2.2) and the corresponding formula for $s(k, h)$, we have

$$\bar{s}(h, k)+\bar{s}(k, h)=\frac{h^2+3hk+k^2+1}{12hk},$$

which is the same as (1.2).

4. The generalized reciprocity formula. The identity (3.4) implies a good deal more than (1.2). For example, for $x = 0$, we get

$$\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta-1}{\eta^k-1}=1-\frac{1}{hk}, \tag{4.1}$$

while if we use the constant term in (3.5), we find that

$$\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{\eta^k-1}=\frac{h+k-2}{2hk}. \tag{4.2}$$

Again if we multiply by x and let $x \longrightarrow \infty$, we get

$$(4.3)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\zeta\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\eta\,\frac{\eta-1}{\eta^k-1}=-\frac{1}{hk}\,.$$

More generally, expanding (3.4) in descending powers of x, we have

$$(4.4)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\zeta^r\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\eta^r\,\frac{\eta-1}{\eta^k-1}=\begin{cases}-\dfrac{1}{hk} & (1\leq r<h+k-1)\\[2ex] 1-\dfrac{1}{hk} & (r=h+k-1).\end{cases}$$

By continuing the expansion of (3.5) we can also show that

$$h\sum_{\zeta\neq 1}\frac{\zeta}{(\zeta-1)^r\,(\zeta^h-1)}+k\sum_{\eta\neq 1}\frac{\eta}{(\eta-1)^r(\eta^k-1)}\qquad (r\geq 1)$$

is a polynomial in h, k, but the explicit expression seems complicated. A more interesting result can be obtained as follows. First we divide both sides of (3.4) by $x-1$ so that the left member becomes

$$\frac{1}{k}\sum_{\zeta}\frac{\zeta}{\zeta^h-1}\left(\frac{1}{x-\zeta}-\frac{1}{x-1}\right)+\frac{1}{h}\sum_{\eta}\frac{\eta}{\eta^k-1}\left(\frac{1}{x-\eta}-\frac{1}{x-1}\right)$$

$$=\frac{1}{k}\sum_{\zeta}\frac{\zeta}{\zeta^h-1}\,\frac{1}{x-\zeta}+\frac{1}{h}\sum_{\eta}\frac{\eta}{\eta^k-1}\,\frac{1}{x-\eta}-\frac{h+k-2}{2hk(x-1)}$$

by (4.2). We now put $x=e^t$. Transposing the last term above to the right we find that the right member has the expansion

$$(4.5)\qquad \frac{1}{hk}\sum_{m=0}^{\infty}\frac{(Bh+Bk)^m t^{m-2}}{m!}+\frac{h+k}{2hk}\sum_{m=0}^{\infty}\frac{B_m t^{m-1}}{m!}+\frac{1}{hk}\sum_{m=0}^{\infty}\frac{(m-1)B_m t^{m-2}}{m!},$$

where the B_m are the Bernoulli numbers. In the left member we put

$$\frac{1-\zeta}{e^t-\zeta}=\sum_{m=0}^{\infty}H_m(\zeta)\,\frac{t^m}{m!},$$

where the $H_m(\zeta)$ are the so-called "Eulerian numbers"; we thus get

$$(4.6)\qquad \frac{1}{k}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\zeta}\frac{H_m(\zeta^{-1})}{(\zeta-1)(\zeta^{-h}-1)}+\frac{1}{h}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\eta}\frac{H_m(\eta^{-1})}{(\eta-1)(\eta^{-k}-1)}.$$

But by [3, formula (6.6)], for p odd > 1,

$$\frac{p}{k^p}\sum_{\zeta}\frac{H_{p-1}(\zeta)}{(\zeta-1)(\zeta^{-h}-1)}=s_p(h,k)$$

where [1]

$$s_p(h,k)=\sum_{r(\mathrm{mod}\ k)}\overline{B}_1\left(\frac{r}{k}\right)\overline{B}_p\left(\frac{hr}{k}\right),$$

and $\overline{B}_r(x)$ is the Bernoulli function. Thus the coefficient of $t^{p-1}/(p-1)!$ in (4.6) is

$$(4.7)\qquad \frac{1}{p}\left\{k^{p-1}s_p(h,k)+h^{p-1}s_p(k,h)\right\},$$

while the corresponding coefficient in (4.5) is

$$(4.8)\qquad \frac{1}{p(p+1)hk}(Bh+Bk)^{p+1}+\frac{1}{(p+1)hk}B_{p+1}.$$

Hence equating (4.7) and (4.8) we get Apostol's formula [1, Theorem 1]:

$$(p+1)\{hk^p s_p(h,k)+kh^p s_p(k,h)\}=(Bh+Bk)^{p+1}+pB_{p+1}$$

for p odd > 1. Note that $s_1(h,k)=\overline{s}(h,k)$.

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IDENTIFICATIONS IN SINGULAR HOMOLOGY THEORY

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INTRODUCTION

0.1. Given a Mayer complex M, a subcomplex M' is termed an *unessential identifier* for M if the natural projections from M onto the factor complex M/M' induce isomorphisms-onto on the homology level (see [1, § 1.2]). The present paper is a continuation and improvement of certain results obtained by Radó and Reichelderfer (see [1] and [3]) concerning unessential identifiers for the singular complex R of Radó (see [1, § 0.1]). We shall make use of the results, terminology, and notation in [1] and [3] with one exception. Because of a conflict in notation in [1] and [3], we shall use the notation η_p for the homomorphisms

$$\eta_p : C_p^S \longrightarrow C_p^R,$$

defined as the trivial homomorphism for $p < 0$, and for $p \geq 0$ as follows:

$$\eta_p(d_0, \cdots, d_p, T)^S = (d_0, \cdots, d_p, T)^R$$

(see [1, § 0.3]).

0.2. The principal results of the present paper may be described as follows. Let $N(\sigma_p \beta_p^R)$ denote the nucleus of the product homomorphism

$$\sigma_p \beta_p^R : C_p^R \longrightarrow C_p^S.$$

THEOREM. *The system* $\{N(\sigma_p \beta_p^R)\}$ *is an unessential identifier for* R.

Furthermore, for each p we have

$$N(\sigma_p \beta_p^R) \supset \hat{\Delta}_p^R \supset \hat{\Gamma}_p^R,$$

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where $\{\hat{\Delta}_p^R\}$ and $\{\hat{\Gamma}_p^R\}$ are the largest unessential identifiers for R obtained by Reichelderfer [3, § 3.6] and Radó [1, § 4.7], respectively. Thus $\{N(\sigma_p \beta_p^R)\}$ is the largest unessential identifier presently known for R and imposes all the classical identifications in R.

Let $N(\beta_p^S)$ denote the nucleus of the barycentric homomorphism

$$\beta_p^S : C_p^S \longrightarrow C_p^S.$$

THEOREM. *The system* $\{N(\beta_p^S)\}$ *is an unessential identifier for* S.

It is interesting to note that the foregoing theorem gives for the Eilenberg complex S the result corresponding to that of Reichelderfer for the Radó complex R (see [3, § 3.2]).

I. PRELIMINARIES

1.1. Let $v_0, \cdots, v_p$ denote $p+1$ points in Hilbert space E_∞. The barycenter $b = b(v_0, \cdots, v_p)$ of these points is given by

$$b = (v_0 + \cdots + v_p)/(p+1).$$

The following lemmas are easily verified.

1.2. LEMMA. *Let* v_j $(j = 0, \cdots, p)$ *denote* $p+1$ *points in* E_∞, *and*

$$x = \sum_{j=0}^{p} \mu_j \, b(v_0, \cdots, v_j),$$

where μ_j *is real for* $j = 0, \cdots, p$. *Then*

$$x = \sum_{j=0}^{p} \sum_{l=j}^{p} \frac{\mu_l}{l+1} v_j, \quad \text{with} \quad \sum_{j=0}^{p} \sum_{l=j}^{p} \frac{\mu_l}{l+1} = \sum_{j=0}^{p} \mu_j.$$

1.3. LEMMA. *Let* v_j $(j = 0, \cdots, p)$ *denote* $p+1$ *points in* E_∞, *and*

$$x = \sum_{j=0}^{p} \mu_j v_j,$$

with μ_j $(j = 0, \cdots, p)$ *real and satisfying*

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_p \geq 0.$$

Then

$$x = \sum_{j=0}^{p} \lambda_j b(v_0 \cdots v_j),$$

with

$$\lambda_j = (j+1)(\mu_j - \mu_{j+1}) \text{ for } j = 0, \cdots, p-1 \text{ (provided } p-1 \geq 0),$$

$$\lambda_p = (p+1)\mu_p,$$

and

$$\sum_{j=0}^{p} \lambda_j = \sum_{j=0}^{p} \mu_j .$$

1.4. As in [1], let $d_0, d_1, d_2, \cdots$ denote the sequence of points $(1, 0, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $(0, 0, 1, 0, \cdots)$, $\cdots$ in E_∞. For integers p, q such that $p \geq 0$, $0 \leq q \leq p+1$, the homomorphism

$$q_{*p} : C_p \longrightarrow C_{p+1}$$

in the formal complex K of E_∞ is defined by the relation

$$q_{*p}(v_0, \cdots, v_p) = \begin{cases} (d_{p+1}, v_0, \cdots, v_p) & \text{for } q = 0, \\ (-1)^q (v_0, \cdots, v_{q-1}, d_{p+1}, v_q, \cdots, v_p) & \text{for } 1 \leq q \leq p, \\ (-1)^{p+1}(v_0, \cdots, v_p, d_{p+1}) & \text{for } q = p+1. \end{cases}$$

1.5. For $p \geq 0$, let τ_p denote an element of $T_{p\,0}$ (see [**3**, § 1.9]), and let $(i_0, \cdots, i_p)$ denote the permutation of $0, \cdots, p$ which gives rise to τ_p. Then we let sgn τ_p denote the sign of the permutation $(i_0, \cdots, i_p)$: i.e., sgn τ_p is $+1$ or -1 according as an even or odd number of transpositions is required to obtain $(i_0, \cdots, i_p)$.

The following lemmas are then obvious.

1.6. LEMMA. *For $p \geq 0$ and $\tau_{p+1} \in T_{p+1\,0}$, there exists a unique $\pi_p \in T_{p0}$,* *and a unique q, $0 \le q \le p+1$, such that*

$$\tau_{p+1}(d_0, \cdots, d_{p+1}) = q_{*p}\, \pi_p\, (p+1)_{p+1}(d_0, \cdots, d_{p+1}).$$

1.7. LEMMA. *For $p \ge 0$, let E_{p+1} denote the set of ordered pairs (q, π_p), $0 \le q \le p+1$, $\pi_p \in T_{p0}$. There exists a biunique correspondence*

$$\xi : T_{p+1\,0} \longrightarrow E_{p+1}$$

with

$$\xi \tau_{p+1} = (q, \pi_p),$$

such that

$$\tau_{p+1}(d_0, \cdots, d_{p+1}) = q_{*p}\pi_p(p+1)_{p+1}(d_0, \cdots, d_{p+1})$$

and

$$\operatorname{sgn} \tau_{p+1} = (-1)^{p+q+1} \operatorname{sgn} \pi_p.$$

1.8. Let

$$h_p : C_p \longrightarrow C_q$$

denote a homomorphism in K such that

$$h_p(d_0 \cdots d_p) = \pm (w_0, \cdots, w_q).$$

Then $[h_p]$ will denote the usual affine mapping from the convex hull $|d_0, \cdots, d_q|$ of the points $d_0, \cdots, d_q$ onto the convex hull $|w_0, \cdots, w_q|$ of the points $w_0, \cdots, w_q$ such that $[h_p](d_i) = w_i$ for $i = 0, \cdots, q$.

1.9. Let β_p^R denote the barycentric homomorphism in R, and ρ_{*p}^R the barycentric homotopy operator in R of Reichelderfer (see [3, §2.1]). The barycentric homomorphism

$$\beta_p^S : C_p^S \longrightarrow C_p^S$$

in S may be given by

$$\beta_p^S = \sigma_p\, \beta_p^R\, \eta_p \qquad \text{(see [2, §3.7])}.$$

The corresponding homotopy operator

$$\rho^S_{*p} : C^S_p \longrightarrow C^S_{p+1}$$

is given by

$$\rho^S_{*p} = \sigma_{p+1}\, \rho^R_{*p}\, \eta_p \,,$$

1.10. Employing the structure theorems for β^R_p, ρ^R_{*p} (see [3, § 2.2]) we obtain the following:

LEMMA. *For* $p \geq 0$,

$$\beta^S_p(d_0, \cdots, d_p, T)^S = \sum_{\tau_p \in T_{p0}} \operatorname{sgn} \tau_p(d_0, \cdots, d_p, T[0_{p+1}\, b_{p0}\, \tau_p])^S,$$

$$\rho^S_{*p}(d_0, \cdots, d_p, T)^S = \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (-1)^k \operatorname{sgn} \tau_p(d_0, \cdots, d_{p+1}, T[b_{pk}\, \tau_p])^S.$$

Proof. We have

$$\begin{aligned} \beta^S_p(d_0, \cdots, d_p, T)^S &= \sigma_p\, \beta^R_p(d_0, \cdots, d_p, T)^R \\ &= \sigma_p \sum_{\tau_p \in T_{p0}} (0_{p+1}\, b_{p0}\, \tau_p(d_0, \cdots, d_p), T)^R \\ &= \sum_{\tau_p \in T_{p0}} \operatorname{sgn} \tau_p(d_0, \cdots, d_p, T[0_{p+1}\, b_{p0}\, \tau_p])^S, \end{aligned}$$

and

$$\begin{aligned} \rho^S_{*p}(d_0, \cdots, d_p, T)^S &= \sigma_{p+1}\, \rho^R_{*p}(d_0, \cdots, d_p, T)^R \\ &= \sigma_{p+1} \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (b_{pk}\, \tau_p(d_0, \cdots, d_p), T)^R \\ &= \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (-1)^k \operatorname{sgn} \tau_p(d_0, \cdots, d_{p+1}, T[b_{pk}\, \tau_p])^S. \end{aligned}$$

1.11. In [2], Radó makes use of the following identities which we state in terms of ρ_{*p}^{R} :

(1) $\sigma_{p+1}\, \rho_{*p}^{R}\, \eta_p\, \sigma_p = \sigma_{p+1}\, \rho_{*p}^{R}$, $\quad -\infty < p < \infty$,

(2) $\sigma_p\, \beta_p^{R}\, \eta_p\, \sigma_p = \sigma_p\, \beta_p^{R}$, $\quad -\infty < p < \infty$.

The proof of (1) may be modeled after the proof for the corresponding identity stated in terms of the classical homotopy operator ρ_p^R (see [2, § 3.5]). From identities (1) and (2), we have

(3) $\beta_p^{S}\, \sigma_p = \sigma_p\, \beta_p^{R}$,

(4) $\rho_{*p}^{S}\, \sigma_p = \sigma_{p+1}\, \rho_{*p}^{R}$,

(5) $\beta_{p+1}^{S}\, \rho_{*p}^{S}\, \sigma_p = \sigma_{p+1}\, \beta_{p+1}^{R}\, \rho_{*p}^{R}$

for all integers p.

1.12. Let P_1 and P_2 denote the following propositions:

P_1. Let c_p^S denote a p-chain of S such that

$$\beta_p^S\, c_p^S = 0.$$

Then

$$\beta_{p+1}^S\, \rho_{*p}^S\, c_p^S = 0.$$

P_2. Let c_p^R denote a p-chain of R such that

$$\sigma_p\, \beta_p^R\, c_p^R = 0.$$

Then

$$\sigma_{p+1}\, \beta_{p+1}^R\, \rho_{*p}^R\, c_p^R = 0.$$

THEOREM. $P_1 \equiv P_2$; *i.e.*, P_1 *is true if and only if* P_2 *is true.*

Proof. Assume P_1, and let c_p^R denote a p-chain of R such that

$$\sigma_p \ \beta_p^R \ c_p^R = 0.$$

Then via identity (3) we have

$$\beta_p^S \ \sigma_p \ c_p^R = 0.$$

Therefore

$$\beta_{p+1}^S \ \rho_{*p}^S \ \sigma_p \ c_p^R = 0.$$

But via identity (5), we have

$$\sigma_{p+1} \ \beta_{p+1}^R \ \rho_{*p}^R \ c_p^R = 0,$$

and P_2 follows.

Now assume P_2, and let c_p^S denote a p-chain of S such that

$$\beta_p^S \ c_p^S = 0.$$

Then since

$$\beta_p^S = \sigma_p \ \beta_p^R \ \eta_p,$$

we have

$$\sigma_p \ \beta_p^R \ \eta_p \ c_p^S = 0.$$

Therefore, via P_2, we have

$$\sigma_{p+1} \ \beta_{p+1}^R \ \rho_{*p}^R \ \eta_p \ c_p^S = 0.$$

But via (5) and the fact that $\sigma_p \ \eta_p = 1$, we have

$$\sigma_{p+1} \ \beta_{p+1}^R \ \rho_{*p}^R \ \eta_p \ c_p^S = \beta_{p+1}^S \ \rho_{*p}^S \ \sigma_p \ \eta_p \ c_p^S = \beta_{p+1}^S \ \rho_{*p}^S \ c_p^S = 0,$$

and P_1 follows.

II. The proof of P_1

2.1. We shall use throughout this section the notation T for the p-cell

$(d_0, \cdots, d_p, T)^S$ when there is little chance for ambiguity. Under this convention a chain c_p^S having the representation

$$c_p^S = \sum_{j=1}^{n} \lambda_j (d_0, \cdots, d_p, T_j)^S$$

may be written $\sum_{j=1}^{n} \lambda_j T_j$. Thus T represents both a transformation from the convex hull $|d_0, \cdots, d_p|$ into the topological space X and the p-cell $(d_0, \cdots, d_p, T)^S$.

2.2. For $p < 0$, the proposition P_1 is trivial. For $p = 0$, P_1 is also trivial. For since $\beta_0^R = 1$ and $\sigma_0 \eta_0 = 1$, we have

$$\beta_0^S c_0^S = 0$$

implying

$$\sigma_0 \beta_0^R \eta_0 c_0^S = \sigma_0 \eta_0 c_0^S = c_0^S = 0,$$

whence clearly

$$\beta_1^S \rho_{*0}^S c_0^S = 0.$$

Now, take a fixed $p \geq 1$. Let

$$c_p^S = \sum_{j=1}^{n} \lambda_j T_j \qquad (\lambda_j \neq 0)$$

denote a p-chain of S such that

$$\beta_p^S c_p^S = 0.$$

Via §1.10,

$$\beta_p^S c_p^S = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \lambda_j \operatorname{sgn} \tau_p T_j [0_{p+1} b_{p0} \tau_p]. \tag{1}$$

Let E denote the set of ordered pairs (j, τ_p), $1 \leq j \leq n$, $\tau_p \in T_{p0}$. Then

$$(2) \qquad \beta_p^S c_p^S = \sum_{(j,\tau_p)\in E} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1} b_{p0} \tau_p].$$

We now define a binary relation "$\equiv$" on E as follows:

$$(j, \tau_p) \equiv (j', \tau_p')$$

if and only if $T_j[0_{p+1} b_{p0} \tau_p]$, $T_{j'}[0_{p+1} b_{p0} \tau_p']$ are identical p-cells. Then "$\equiv$" as defined is obviously a true equivalence relation and induces a partitioning of E into nonempty, mutually disjoint sets E_s $(s = 1, \cdots, t)$ with

$$E = \bigcup_{s=1}^{t} E_s.$$

Therefore, via (2), we have

$$(3) \qquad \beta_p^S c_p^S = \sum_{s=1}^{t} \sum_{(j, \tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1} b_{p0} \tau_p].$$

Take $1 \leq s < s' \leq t$. Then for $(j, T_p) \in E_s$, $(j', T_p') \in E_{s'}$, the p-cells $T_j[0_{p+1} b_{p0} \tau_p]$, $T_{j'}[0_{p+1} b_{p0} \tau_p']$ are distinct. Therefore, since

$$\beta_p^S c_p^S = 0,$$

we must have for each s, $1 \leq s \leq t$,

$$(4) \qquad \sum_{(j, \tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1} b_{p0} \tau_p] = 0,$$

and hence

$$(5) \qquad \sum_{(j, \tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p = 0,$$

since all p-cells occuring in (4) are identical.

2.3. Again via §1.10,

$$(6) \qquad \beta^S_{p+1}\, \rho^S_{*p}\, c^S_p = \sum_{j=1}^{n} \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} \sum_{\tau_{p+1} \in T_{p+1\,0}} (-1)^k \operatorname{sgn} \tau_p \operatorname{sgn} \tau_{p+1}\, \lambda_j\, T_j[b_{pk}\, \tau_p]\, [0_{p+2}\, b_{p+1\,0}\, \tau_{p+1}].$$

Applying the lemma of § 1.7, we obtain

$$(7) \qquad \beta^S_{p+1}\, \rho^S_{*p}\, c^S_p = \sum_{k=0}^{p} \sum_{q=0}^{p+1} (-1)^{p+q+k+1} \left\{ \sum_{j=1}^{n} \sum_{\tau_p \in T_{pk}} \sum_{\pi_p \in T_{p0}} \lambda_j \operatorname{sgn} \tau_p \operatorname{sgn} \pi_p\, T_j[b_{pk}\, \tau_p]\, [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p(p+1)_{p+1}] \right\}.$$

Thus, to prove that

$$\beta^S_{p+1}\, \rho^S_{*p}\, c^S_p = 0,$$

we are led to consider for a fixed k and q, $0 \le k \le p$, $0 \le q \le p+1$, the expression

$$(8) \qquad Y_{kq} = \sum_{j=1}^{n} \sum_{\tau_p \in T_{pk}} \sum_{\pi_p \in T_{p0}} \lambda_j \operatorname{sgn} \tau_p \operatorname{sgn} \pi_p\, T_j[b_{pk}\, \tau_p]\, [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p(p+1)_{p+1}].$$

Now to prove P_1 we need only show that $Y_{kq} = 0$. Therefore k and q will remain fixed throughout the remainder of this section; and even though subsequent definitions will depend upon k and q, they will not be displayed in the notation.

2.4. For

$$\tau_p = \tau_p(i_0, \cdots, i_p) \in T_{p0}$$

(see [3, § 1.9]) there exists a unique permutation $(n_0, \cdots, n_k)$ of $0, \cdots, k$ such that $i_{n_0} < \cdots < i_{n_k}$. Let

$$\bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p),$$

where $j_l = i_{n_l}$ for $l = 0, \cdots, k$, and $j_l = i_l$ for $k+1 \le l \le p$. Then there exists a unique permutation $(m_0, \cdots, m_k)$ of $0, \cdots, k$, namely $(n_0, \cdots, n_k)^{-1}$, such that

$$\tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k}, j_{k+1}, \cdots, j_p).$$

Furthermore, let $A(\tau_p)$ denote the set of $\pi_p \in T_{p0}$ defined as follows. For

$$\pi_p = \pi_p(u_0, \cdots, u_p) \in T_{p0}$$

we have a unique set of integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$ such that $(u_{l_0}, \cdots, u_{l_k})$ is a permutation of $0, \cdots, k$. Set $\pi_p \in A(\tau_p)$ if and only if $m_0 = u_{l_0}, \cdots, m_k = u_{l_k}$.

2.5. Let B denote the set of ordered pairs (τ_p, π_p), $\tau_p \in T_{p0}$, $\pi_p \in A(\tau_p)$, and B' the set of ordered pairs (τ_p', π_p'), $\tau_p' \in T_{pk}$, $\pi_p' \in T_{p0}$. We define a mapping

$$\gamma : B \longrightarrow B'$$

as follows:

$$\gamma(\tau_p, \pi_p) = (\tau_p', \pi_p')$$

where $\tau_p' = \overline{\tau}_p$ and $\pi_p' = \pi_p$. One shows with little difficulty that γ is biunique. Therefore

$$(9) \qquad Y_{kq} = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \sum_{\pi_p \in A(\tau_p)} \lambda_j \operatorname{sgn} \overline{\tau}_p \operatorname{sgn} \pi_p \, T_j[b_{pk}\, \overline{\tau}_p]$$

$$[0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p(p+1)_{p+1}].$$

2.6. Let $A = A(\tau_p(0, \cdots, p))$. For $\tau_p \in T_{p0}$ we define

$$f_{\tau_p} : A \longrightarrow A(\tau_p)$$

as follows. For $\pi_p(u_0, \cdots, u_p) \in A$, there exist integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$, such that $u_{l_0} = 0, \cdots, u_{l_k} = k$. Define

$$f_{\tau_p} \pi_p = \pi_p'(u_0', \cdots, u_p')$$

as follows. Let

$$\bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p) \text{ and } \tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k}, j_{k+1}, \cdots, j_p),$$

where $(m_0, \cdots, m_k)$ is a permutation of $0, \cdots, k$. Set $u'_{l_0} = m_0, \cdots, u'_{l_k} = m_k$, and $u'_r = u_r$ for $r \neq l_0, \cdots, l_k$. Here again it is easy to show that f_{τ_p} is biunique. We have then

$$(10) \qquad Y_{kq} = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \sum_{\pi_p \in A} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p T_j [b_{pk} \bar{\tau}_p] [0_{p+2}\, b_{p+1\,0}\, q_{*p} f_{\tau_p} \pi_p (p+1)_{p+1}],$$

and hence

$$(11) \qquad Y_{kq} = \sum_{s=1}^{t} \sum_{\pi_p \in A} \sum_{(j,\tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p T_j [b_{pk} \bar{\tau}_p] [0_{p+2}\, b_{p+1\,0}\, q_{*p} f_{\tau_p} \pi_p (p+1)_{p+1}]$$

(see § 2.2).

2.7. LEMMA. *Take $\pi_p(u_0, \cdots, u_p) \in T_{p0}$ and let*

$$\alpha = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p (p+1)_{p+1}].$$

Let

$$x = \sum_{j=0}^{p+1} \mu_j d_j,$$

with

$$\mu_j \geq 0, \ j = 0, \cdots, p+1, \text{ and } \sum_{j=0}^{p+1} \mu_j = 1,$$

denote a point of $|d_0, \cdots, d_{p+1}|$. Then

$$\alpha(x) = \sum_{j=0}^{p+1} a_j d_j,$$

where

(i) $a_j \geq 0,\ j = 0, \cdots, p+1;$

(ii) $\sum_{j=0}^{p+1} a_j = 1;$

(iii) $a_{u_0} \geq a_{u_1} \geq \cdots \geq a_{u_p};$

(iv) $a_{u_0}, \cdots, a_{u_p}, a_{p+1}$ *are independent of* π_p; *i.e., if* $\pi_p' = \pi_p'(u_0', \cdots, u_p') \in T_{p0}$ *and*

$$\alpha' = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p'(p+1)_{p+1}],$$

then

$$\alpha'(x) = \sum_{j=0}^{p+1} a_j' d_j$$

with

$$a_{u_0} = a'_{u_0'}, \cdots, a_{u_p} = a'_{u_p'},\ a_{p+1} = a'_{p+1}.$$

Proof. We consider only the case $1 \leq q \leq p$ since the fringe cases $q = 0$, $p+1$ follow in a completely analogous manner. In case $1 \leq q \leq p$ we have

$$\alpha = [b(w_0)b(w_0, w_1) \cdots b(w_0, \cdots, w_{p+1})],$$

where

$$w_l = d_{u_l},\ l = 0, \cdots, q-1,\ w_q = d_{p+1},\ w_l = d_{u_{l-1}},\ l = q+1, \cdots, p+1.$$

Therefore,

$$\alpha(x) = \sum_{j=0}^{p+1} \mu_j\, b(w_0, \cdots, w_j) = \sum_{j=0}^{p+1} \left(\sum_{l=j}^{p+1} \frac{\mu_l}{l+1} \right) w_j$$

(see §1.2). Let

$$a_{p+1} = \sum_{l=q}^{p+1} \frac{\mu_l}{l+1}, \quad a_{u_r} = \sum_{l=r}^{p+1} \frac{\mu_l}{l+1} \quad \text{for } r = 0, \cdots, q-1$$

and

$$a_{u_r} = \sum_{l=r+1}^{p+1} \frac{\mu_l}{l+1} \quad \text{for } r = q, \cdots, p.$$

Clearly, $a_{u_0}, \cdots, a_{u_p}, a_{p+1}$ are independent of π_p in the sense of (iv), and $a_{u_0} \geq \cdots \geq a_{u_p}$. Furthermore, $a_j \geq 0$ $(j = 0, \cdots, p+1)$, and

$$\sum_{j=0}^{p+1} a_j = \sum_{j=0}^{p+1} \mu_j = 1.$$

Also,

$$\alpha(x) = \sum_{j=0}^{q-1} a_{u_j} d_{u_j} + a_{p+1} d_{p+1} + \sum_{j=q}^{p} a_{u_j} d_{u_j} = \sum_{j=0}^{p+1} a_j d_j,$$

and the lemma follows.

2.8. LEMMA. *Take* (j, τ_p) *and* $(j', \tau_p') \in E_s$ *(see* §2.2*),* $1 \leq s \leq t$*, and* $\pi_p^* \in A$*. Then*

$$T_j [b_{pk} \overline{\tau}_p] [0_{p+2} b_{p+1\,0} q_{*p} f_{\tau_p} \pi_p^* (p+1)_{p+1}]$$

$$= T_{j'} [b_{pk} \overline{\tau}_p'] [0_{p+2} b_{p+1\,0} q_{*p} f_{\tau_p'} \pi_p^* (p+1)_{p+1}].$$

Proof. Since (j, τ_p), (j', τ_p') lie in E_s, we have

$$T_j [0_{p+1} b_{p0} \tau_p] = T_{j'} [0_{p+1} b_{p0} \tau_p'],$$

Let

$$\pi_p = f_{\tau_p} \pi_p^* = \pi_p(u_0, \cdots, u_p), \quad \pi_p' = f_{\tau_p'} \pi_p^* = \pi_p'(u_0', \cdots, u_p'),$$

$$\alpha = [0_{p+2} b_{p+1\,0} q_{*p} \pi_p (p+1)_{p+1}], \quad \alpha' = [0_{p+2} b_{p+1\,0} q_{*p} \pi_p' (p+1)_{p+1}],$$

$$\gamma = [b_{pk} \overline{\tau}_p], \text{ and } \gamma' = [b_{pk} \overline{\tau}_p'].$$

Furthermore, let

$$\tau_p = \tau_p(i_0, \cdots, i_p), \ \bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p),$$

$$\tau_p' = \tau_p'(i_0', \cdots, i_p'), \ \bar{\tau}_p' = \bar{\tau}_p'(j_0', \cdots, j_p').$$

We have permutations $(m_0, \cdots, m_k)$, $(n_0, \cdots, n_k)$ of $0, \cdots, k$ such that

$$\tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k}, \ j_{k+1}, \cdots, j_p),$$

$$\tau_p' = \tau_p'(j_{n_0}', \cdots, j_{n_k}', \ j_{k+1}', \cdots, j_p')$$

Take an arbitrary point of $|d_0, \cdots, d_{p+1}|$, say

$$x = \sum_{j=0}^{p+1} \mu_j d_j \qquad\qquad \mu_j \geq 0, \ \sum_{j=0}^{p+1} \mu_j = 1.$$

Then via the lemma of § 2.7 we have

$$\alpha(x) = \sum_{j=0}^{p+1} a_j d_j \ \text{ with } \ a_j \geq 0, \ \sum_{j=0}^{p+1} a_j = 1, \ a_{u_0} \geq \cdots \geq a_{u_p},$$

and

$$\alpha'(x) = \sum_{j=0}^{p+1} a_j' d_j \ \text{ with } \ a_j' \geq 0, \ \sum_{j=0}^{p+1} a_j' = 1, \ a_{u_0'}' \geq \cdots \geq a_{u_p'}' ,$$

with

$$a_{u_0} = a_{u_0'}', \cdots, a_{u_p} = a_{u_p'}' \ \text{ and } \ a_{p+1} = a_{p+1}'.$$

Now

$$\gamma = [d_{j_0}, \cdots, d_{j_k}, \ b(d_{j_0}, \cdots, d_{j_k}), \cdots, b(d_{j_0}, \cdots, d_{j_p})].$$

Hence

$$\gamma\alpha(x) = a_0 d_{j_0} + \cdots + a_k d_{j_k} + a_{k+1} b(d_{j_0}, \cdots, d_{j_p}) + \cdots + a_{p+1} b(d_{j_0}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{j_{m_0}} + \cdots + a_{m_k} d_{j_{m_k}} + a_{k+1} b(d_{j_0}, \cdots, d_{j_k}) + \cdots + a_{p+1} b(d_{j_0}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{j_{m_0}} + \cdots + a_{m_k} d_{j_{m_k}} + a_{k+1} b(d_{j_{m_0}}, \cdots, d_{j_{m_k}}) + \cdots + a_{p+1} b(d_{j_{m_0}}, \cdots, d_{j_{m_k}}, d_{j_{k+1}}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{i_0} + \cdots + a_{m_k} d_{i_k} + a_{k+1} b(d_{i_0}, \cdots, d_{i_k}) + \cdots + a_{p+1} b(d_{i_0}, \cdots, d_{i_p}).$$

Now take integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$, such that $(u_{l_0}, \cdots, u_{l_k})$ is a permutation of $0, \cdots, k$. Since $\pi_p \in A(\tau_p)$, we have $m_0 = u_{l_0}, \cdots, m_k = u_{l_k}$. Hence $a_{m_0} \geq \cdots \geq a_{m_k}$.

In a similar fashion we obtain

$$\gamma'\alpha'(x) = a'_{n_0} d_{i'_0} + \cdots + a'_{n_k} d_{i'_k} + a'_{k+1} b(d_{i'_0}, \cdots, d_{i'_k}) + \cdots + a'_{p+1} b(d_{i'_0}, \cdots, d_{i'_p}),$$

with $a'_{n_0} \geq \cdots \geq a'_{n_k}$; and if $l'_0, \cdots, l'_k$, $0 \leq l'_0 < \cdots < l'_k \leq p$, are integers such that $(u'_{l'_0}, \cdots, u'_{l'_k})$ is a permutation of $0, \cdots, k$, we have

$$n_0 = u'_{l'_0}, \cdots, n_k = u'_{l'_k}.$$

Applying § 1.3, we get

$$a_{m_0} d_{i_0} + \cdots + a_{m_k} d_{i_k} = \sum_{l=0}^{k} \gamma_l b(d_{i_0}, \cdots, d_{i_l})$$

with

$$\gamma_l = (l+1)(a_{m_l} - a_{m_{l+1}}) \text{ for } l = 0, \cdots, k-1,$$

$$\gamma_k = (k+1)a_{m_k} ,$$

and

$$\sum_{l=0}^{k} \gamma_l = \sum_{l=0}^{k} a_{m_l} .$$

Similarly,

$$a'_{n_0} d_{i'_0} + \cdots + a'_{n_k} d_{i'_k} = \sum_{l=0}^{k} \gamma'_l b(d_{i'_0}, \cdots, d_{i'_l})$$

with

$$\gamma'_l = (l+1)(a'_{n_l} - a'_{n_{l+1}}) \text{ for } l = 0, \cdots, k-1,$$

$$\gamma'_k = (k+1)a'_{n_k}$$

and

$$\sum_{l=0}^{k} \gamma'_l = \sum_{l=0}^{k} a'_{n_l} .$$

However, since

$$\pi_p = f_{\tau_p} \pi_p^*, \ \pi'_p = f_{\tau'_p} \pi_p^*,$$

we have

$$l_0 = l'_0, \cdots, l_k = l'_k \text{ and } u_r = u'_r \text{ for } r \neq l_0, \cdots, l_k.$$

Therefore, $a_{u_{l_0}} = a'_{u'_{l'_0}}, \cdots, a_{u_{l_k}} = a'_{u'_{l'_k}}$, and hence

$$a_{m_0} = a'_{n_0}, \cdots, a_{m_k} = a'_{n_k} .$$

Thus

$$\gamma_r = \gamma'_r \text{ for } r = 0, \cdots, k.$$

Furthermore,

$$a_{u_r} = a'_{u'_r} \text{ for } r \neq l_0, \cdots, l_k, \text{ and } a_{p+1} = a'_{p+1}.$$

Therefore,

$$\gamma\alpha(x) = \sum_{l=0}^{k} \gamma_l\, b(d_{i_0}, \cdots, d_{i_l}) + \sum_{l=k}^{p} a_{l+1}\, b(d_{i_0}, \cdots, d_{i_l}),$$

$$\gamma'\alpha'(x) = \sum_{l=0}^{k} \gamma_l\, b(d_{i'_0}, \cdots, d_{i'_l}) + \sum_{l=k}^{p} a_{l+1}\, b(d_{i'_0}, \cdots, d_{i'_l}),$$

with

$$\sum_{l=0}^{k} \gamma_l + \sum_{l=k}^{p} a_{l+1} = \sum_{l=0}^{p+1} a_l = 1.$$

Let

$$y = \sum_{j=0}^{p} h_j\, d_j$$

with

$$h_j = \gamma_j \text{ for } j = 0, \cdots, k-1,$$

$$h_k = \gamma_k + a_{k+1},$$

$$h_j = a_{j+1} \text{ for } j = k+1, \cdots, p.$$

Clearly,

$$h_j \geq 0 \quad (j = 0, \cdots, p), \text{ and } \sum_{j=0}^{p} h_j = 1.$$

Then

$$\gamma\,\alpha(x) = \sum_{l=0}^{p} h_l\, b(d_{i_0}, \cdots, d_{i_l}) = [0_{p+1}\, b_{p0}\, \tau_p](y)$$

and

$$\gamma' \alpha'(x) = \sum_{l=0}^{p} h_l \, b(d_{i_0'}, \cdots, d_{i_l'}) = [0_{p+1} \, b_{p0} \, \tau_p'](y).$$

Therefore, since

$$T_j [0_{p+1} \, b_{p0} \, \tau_p](y) = T_{j'} [0_{p+1} \, b_{p0} \, \tau_p'](y),$$

we have

$$T_j \, \gamma \, \alpha(x) = T_{j'} \, \gamma' \, \alpha'(x).$$

Since x is arbitrary in $|d_0, \cdots, d_{p+1}|$, our lemma follows.

2.9. LEMMA. *For any s, $1 \leq s \leq t$, and $\pi_p^* \in A$,*

$$\sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = 0.$$

Proof. Since

$$\operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = \operatorname{sgn} \tau_p \operatorname{sgn} \pi_p^*,$$

we have

$$\sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = \operatorname{sgn} \pi_p^* \sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \tau_p = 0$$

via (5) of § 2.2.

2.10. Employing §§ 2.8, 2.9, and (11) of § 2.6, we see that $Y_{kq} = 0$, and hence P_1 follows. Let us note also that since $P_1 \equiv P_2$, P_2 also is valid.

III. RESULTS

3.1. In [1, § 4.2], Radó has established a lemma, which we state here for the barycentric homotopy operator ρ_{*p}^R.

LEMMA. *Let $\{G_p\}$ be an identifier for R, such that the following conditions hold:*

(i) $G_p \supset A_p^R$ *(see* [1, § 3.4]),

(ii) $c_p^R \in G_p$ *implies that* $\sigma_p \beta_p^R c_p^R = 0$,

(iii) $c_p^R \in G_p$ *implies that* $\rho_{*p}^R c_p^R \in G_{p+1}$.

Then $\{G_p\}$ *is an unessential identifier for* R.

The proof of this lemma is identical with the proof of the corresponding lemma as given by Radó with ρ_p^R (classical homotopy operator) replacing ρ_{*p}^R.

Since

$$\sigma_p \beta_p^R : C_p^R \longrightarrow C_p^S$$

is a chain mapping, the system $\{N(\sigma_p \beta_p^R)\}$ of nuclei of the homomorphisms $\sigma_p \beta_p^R$ is an identifier for R (see [**1**, §1.2]). Furthermore,

$$N(\sigma_p \beta_p^R) \supset A_p^R \text{ since } \sigma_p \beta_p^R = \beta_p^S \sigma_p$$

(see §**1.11**). Applying P_2 directly, we see that $N(\sigma_p \beta_p^R)$ satisfies (iii) of the foregoing lemma. Therefore, since $N(\sigma_p \beta_p^R)$ is the largest identifier, satisfying (ii), we have the following maximum result yielded by the same lemma:

THEOREM. *The system* $\{N(\sigma_p \beta_p^R)\}$ *is an unessential identifier for* R.

3.2. In order to compare our results with those of Radó [**1**] and Reichelderfer [**3**] let us first note that

$$\hat{N}(\sigma_p \beta_p^R) = N(\sigma_p \beta_p^R),$$

where $\hat{N}(\sigma_p \beta_p^R)$ is the division hull of $N(\sigma_p \beta_p^R)$, since C_p^R is a free Abelian group. Then since

$$N(\sigma_p \beta_p^R) \supset \Delta_p^R = N(\beta_p^R) + A_p^R$$

(see [**3**, §3.6]) we have

$$N(\sigma_p \beta_p^R) \supset \hat{\Delta}_p^R \supset \hat{\Gamma}_p^R$$

(see [**1**, §4.7]).

The writer has been unable to determine as yet whether $N(\sigma_p \beta_p^R)$ is effectively larger than either $\hat{\Delta}_p^R$ or $\hat{\Gamma}_p^R$.

3.3. The following lemma (see [**1**, §4.1]) is immediate from the fact that ρ_{*p}^S satisfies the well-known "homotopy identity,"

$$\partial_{p+1}^S \rho_{*p}^S + \rho_{*p-1}^S \partial_p^S = \beta_p^S - 1.$$

LEMMA. *Let* $\{G_p\}$ *be an identifier for* S *such that the following conditions hold*:

(i) $c_p^S \in G_p$ *implies that* $\beta_p^S c_p^S = 0$,

(ii) $c_p^S \in G_p$ *implies that* $\rho_{*p}^S c_p^S \in G_{p+1}$.

Then $\{G_p\}$ *is an unessential identifier for* S.

The system of nuclei $\{N(\beta_p^S)\}$ clearly is an identifier for S since β_p^S is a chain mapping. Therefore, applying P_1 we obtain the maximum result of the foregoing lemma.

THEOREM. *The system* $\{N(\beta_p^S)\}$ *is an unessential identifier for* S.

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A METHOD OF GENERAL LINEAR FRAMES IN RIEMANIAN GEOMETRY, I

HARLEY FLANDERS

1. Introduction. In this paper we shall derive the basic quantities of Riemannian geometry, such as parallelism, curvature tensors, and so on, from a consideration of all linear frames in the various tangent spaces. This procedure has the advantage of subsuming both the classical approach through local coordinate frames and the more modern approach through orthonormal frames. The exact connection between these methods is thus made quite explicit.

The principal machinery used here is the exterior differential calculus of É. Cartan. (See [**1**, p. 201-208; **2**, p. 33-44; **3**, p. 4-6; **4**, p. 146-152; **7**, p. 3-10].) We shall follow the notation of Chern [**3**] with exceptions that we shall note in the course of the paper. It is important to keep in mind the following specific points of this calculus.

On a differentiable manifold of dimension n one has associated with each $p = 0, 1, 2, \cdots$ the linear space of exterior differential forms of degree p (p-forms). The coefficients form the ring of differentiable functions on the manifold. The 0-forms are simply the functions themselves, and the only p-form with $p > n$ is the form 0. Locally, if $u^1, \cdots, u^n$ is a local coordinate system then a one-form ω may be written

$$(1.1) \qquad \omega = \sum f_i(u)\, du^i ;$$

and, more generally, a p-form ω may be written

$$(1.2) \qquad \omega = \sum_{(1 \le i_1 < \cdots < i_p \le n)} f_{i_1 \cdots i_p}(u)\, du^{i_1} \cdots du^{i_p}$$

$$= \frac{1}{p!} \sum f_{i_1 \cdots i_p}(u)\, du^{i_1} \cdots du^{i_p} \quad \text{with the } f_{(i)} \text{ skew-symmetric.}$$

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If ω is a p-form and η a q-form, then

$$\omega\eta = (-1)^{pq}\ \eta\omega \tag{1.3}$$

is the exterior product of ω and η, and is a $(p+q)$-form.

The operation d of exterior differentiation is intrinsically characterized by the following properties:

(A) d sends a p-form ω into a $(p+1)$-form $d\omega$;

(B) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$;

(C) $d(d\omega) = 0$;

(D) $df = \sum(\partial f/\partial u^i)du^i$, where f is a 0-form (function) and (u) is a local coordinate system;

(E) $d(\omega\eta) = d\omega\eta + (-1)^p\ \omega d\eta$, where ω is a p-form.

We shall also use matrices whose elements are differential forms. If A is such a matrix, ${}^tA$ will denote its transpose and dA will denote the matrix whose elements are obtained by applying d termwise to the elements of A. If A and B are square matrices of p-forms and q-forms respectively, then it follows from (1.3) that

$${}^t(AB) = (-1)^{pq}\ {}^tB\ {}^tA. \tag{1.4}$$

If A is a nonsingular matrix of functions (0-forms), then

$$d(A^{-1}) = -A^{-1}\ dA\ A^{-1}. \tag{1.5}$$

This is the case because $AA^{-1} = I$, the identity matrix, hence

$$dA\ A^{-1} + AdA^{-1} = 0.$$

2. Linear frames. We shall now define the objects of this investigation. We begin with a differentiable manifold $\mathfrak{R}$ of dimension n and class C^∞. (See [8, p. 20].) On such a manifold one may form the space $C(\mathfrak{R})$ of all infinitely differentiable real-valued functions on $\mathfrak{R}$. If P is a point of $\mathfrak{R}$, a *tangent vector* at P is an operator $\mathbf{v}$ on $C(\mathfrak{R})$ to the reals satisfying

(A) $\mathbf{v}(f+g) = \mathbf{v}(f) + \mathbf{v}(g)$,

(B) $\mathbf{v}(fg) = f(P)\ \mathbf{v}(g) + g(P)\ \mathbf{v}(f)$, for all f, g in $C(\mathfrak{R})$.

It is well known [5, p. 76-78; 6] that the set of all tangent vectors at P forms a linear space of dimension n under the usual operations of addition and scalar multiplication of operators. If $\mathfrak{U}$ is a coordinate neighborhood on $\mathfrak{R}$ with a local coordinate system $u^1, \cdots, u^n$, then the operators

$$\mathbf{e}_1 = \frac{\partial}{\partial u^1}, \cdots, \mathbf{e}_n = \frac{\partial}{\partial u^n} \tag{2.1}$$

may be considered as tangent vectors at each point P of $\mathfrak{U}$, where if f is in $C(\mathfrak{R})$ we have

$$\mathbf{e}_i(f) = (\partial f/\partial u^i)_P. \tag{2.2}$$

The vectors of (2.1) in fact form a basis for the tangent space of each P in $\mathfrak{U}$.

A *vector field* (*vector*, for short) is an assignment of a tangent vector $\mathbf{v}_P$ at P to each point P of $\mathfrak{R}$ [5, p. 82-83]. In terms of the basis (2.1), one may write a given vector field $\mathbf{v}$ on $\mathfrak{U}$ as follows:

$$\mathbf{v} = \lambda^i \mathbf{e}_i, \qquad \text{with } \lambda^i = \lambda^i(u^1, \cdots, u^n). \tag{2.3}$$

Here we use the Einstein summation convention, as we shall do in what follows. The vector field $\mathbf{v}$ is *infinitely differentiable* if each of the coordinate functions λ^i of the variables u^j is so. In the future we shall deal only with this kind of vector so that "vector field" or "vector" will always mean infinitely differentiable vector. It is important to note that that this definition is independent of the particular local coordinate system we have choosen, since a change in local coordinates merely effects a nonsingular linear transformation with C^∞ coefficients on the λ^i, in accordance with the usual tensor rules.

By a *linear frame* we shall mean a set $\mathbf{e}_1, \cdots, \mathbf{e}_n$ of vectors which form a basis for the tangent space at each point P of a given coordinate neighborhood $\mathfrak{U}$. One may visualize this as a choice of oblique coordinates in each of the tangent spaces at the various points of $\mathfrak{U}$ in such a way that the coordinate axes and units vary smoothly in moving from point to point. The vectors of (2.1) form an example of a linear frame, and we shall call such a frame a *coordinate frame* to indicate that it is derived from a local coordinate system.

The manifold $\mathfrak{R}$ is called a *Riemannian space* if it carries the following additional structure. For each P in $\mathfrak{R}$ one is given an inner product in the tangent space at P, making this space into a euclidean space. This assignment of inner products to the various tangent spaces must be infinitely differentiable in the following sense. If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors on $\mathfrak{R}$, then $\mathbf{v}\cdot\mathbf{w}$, the inner product of $\mathbf{v}$ and $\mathbf{w}$, which clearly is a point function on $\mathfrak{R}$, must be of class C^∞. This implies (and is equivalent to) the following. If $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is the coordinate frame of (2.1), then

$$\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik}(u^1, \cdots, u^n), \tag{2.4}$$

where the functions g_{ik} are C^∞ functions on $\mathfrak{U}$. In this case one custimarily writes

$$ds^2 = g_{ik}\{du^i du^k\}, \tag{2.5}$$

where $\{\,\}$ denotes the ordinary tensor product of the differentials, as distinguished from the exterior product.

An *orthonormal frame* $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a frame satisfying

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}, \quad \text{the Kronecker } \delta. \tag{2.6}$$

If $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a frame on the space $\mathfrak{R}$, then there is uniquelly determined a (dual) basis $\sigma^1, \cdots, \sigma^n$ of the space of differential forms of degree one. This is the case because the algebraic dual of the space of tangent vectors at a point is precisely the space of 1-forms at that point. (Cf. [**5**, p. 81].) As is customary, we shall formally write

$$dP = \sigma^i \mathbf{e}_i, \tag{2.7}$$

and think of this *displacement vector dP* as a tangent vector whose components are differentials. (See [**1**, p. 34, 52, 101; **3**, p. 10; **6**, Chapter 2].)

3. Existence of parallel displacement. We shall now generalize the development of [**3**, § 5]. We first of all select a linear frame $\mathbf{e}_1, \cdots, \mathbf{e}_n$, and have

$$dP = \sigma^i \mathbf{e}_i, \tag{3.1}$$

where P is the variable point of $\mathfrak{U}$ and the σ^i are one-forms on $\mathfrak{U}$. We set

$$\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik}, \tag{3.2}$$

which defines a positive definite symmetric matrix $\| g_{ik} \|$ of functions on $\mathfrak{U}$.

We next wish to define differential forms ω_i^j of degree 1 so that if we set

$$d\mathbf{e}_i = \omega_i^j \mathbf{e}_j, \tag{3.3}$$

then the equations

$$d(dP) = 0, \tag{3.4}$$

$$d\mathbf{e}_i \cdot \mathbf{e}_k + \mathbf{e}_i \cdot d\mathbf{e}_k = dg_{ik} \tag{3.5}$$

will be formally satisfied. The first yields

$$d(dP) = d(\sigma^i \mathbf{e}_i) = d\sigma^j \mathbf{e}_j - \sigma^i \omega_i^j \mathbf{e}_j = 0,$$

hence

$$d\sigma^j - \sigma^i \omega_i^j = 0. \tag{3.6}$$

The second equation becomes

$$\omega_i^j g_{jk} + \omega_k^l g_{il} = d g_{ik}. \tag{3.7}$$

THEOREM 3.1. *The equations* (3.4), (3.5) *define unique* 1-*forms* ω_i^j.

Proof. It is convenient to work with covariant components. We set

$$\omega_{ir} = \omega_i^j g_{jr}, \quad \eta_r = d\sigma^j g_{jr}, \tag{3.8}$$

and our equations become

$$\sigma^i \omega_{ir} = \eta_r, \tag{3.4'}$$

$$\omega_{ik} + \omega_{ki} = d g_{ik}. \tag{3.5'}$$

The one-forms $\sigma^1, \cdots, \sigma^n$ are linearly independent, and so we may write

$$\eta_r = \frac{1}{2} h_{rst} \sigma^s \sigma^t, \tag{3.9}$$

$$dg_{ik} = c_{ikl} \sigma^l, \tag{3.10}$$

where the h_{rst} and c_{ikl} are known functions on $\mathfrak{U}$ satisfying

$$h_{rst} + h_{rts} = 0, \quad c_{ikl} = c_{kil}. \tag{3.11}$$

We seek unknown functions Γ_{ik}^j such that

$$\omega_i^j = \Gamma_{ik}^j \sigma^k, \tag{3.12}$$

or

$$\omega_{ir} = \Gamma_{irk} \sigma^k \qquad \text{with } \Gamma_{irk} = \Gamma_{ik}^j g_{jr}. \tag{3.13}$$

We now have

$$\sigma^i \omega_{ir} = \sigma^i \Gamma_{irk} \sigma^k = \frac{1}{2} (\Gamma_{irk} - \Gamma_{kri}) \sigma^i \sigma^k,$$

and so our equations (3.4′), (3.5′) become

$$(3.4'') \qquad \Gamma_{irk} - \Gamma_{kri} = h_{rik},$$

$$(3.5'') \qquad \Gamma_{ikl} + \Gamma_{kil} = c_{ikl}.$$

These equations have a unique solution. To prove this, we derive as a consequence of our equations the expression

$$\begin{aligned}\Gamma_{irk} &= h_{rik} + \Gamma_{kri} = h_{rik} + c_{kri} - \Gamma_{rki} \\ &= h_{rik} + c_{kri} - h_{kri} - \Gamma_{ikr} = h_{rik} + c_{kri} - h_{kri} - c_{ikr} + \Gamma_{kir} \\ &= h_{rik} + c_{kri} - h_{kri} - c_{ikr} + h_{ikr} + \Gamma_{rik} \\ &= h_{rik} + c_{kri} - h_{kri} - c_{ikr} + h_{ikr} + c_{rik} - \Gamma_{irk}.\end{aligned}$$

This implies that the only possible solution is given by

$$(3.14) \qquad \Gamma_{irk} = \frac{1}{2}\,(h_{rik} + h_{ikr} - h_{kri}) + \frac{1}{2}\,(c_{rik} + c_{kri} - c_{ikr}).$$

Substitution of this expression into the original equations (3.4′), (3.5″) shows that this indeed is a solution.

The functions Γ_{irk} are the components of the Christoffel symbols of the first kind--with respect to a general frame rather than a coordinate frame as is usual. In case of a coordinate frame (2.1) we have

$$\sigma^i = du^i, \quad d\sigma^i = 0, \quad h_{rst} = 0;$$

only the terms in the c_{rik} appear in (3.14). Since in this case

$$dg_{ik} = c_{ikl}\, du^l,$$

we have

$$c_{ikl} = \frac{\partial g_{ik}}{\partial u^l},$$

and so (3.14) is precisely the formula of Cartan [**1**, p.37]. In case of an orthonormal frame, the g_{ik} are constant, hence the c_{rik} all vanish; only the terms in the h_{rik} appear in (3.14). Thus formula (3.1) of [**3**] results. In view of these special cases and the right side of (3.14), it would appear that somehow a general frame can be decomposed into a coordinate frame and an orthonormal frame. This possibility seems worthy of further investigation.

We now can express our result in a convenient matrix form. We set

$$(3.15)\quad G = \|g_{ik}\|, \quad \mathbf{e} = {}^t(\mathbf{e}_1, \cdots, \mathbf{e}_n), \quad \sigma = (\sigma^1, \cdots, \sigma^n), \quad \Omega = \|\omega_i^k\|.$$

We then have the vector equations

$$(3.16)\qquad dP = \sigma \mathbf{e}, \quad d\mathbf{e} = \Omega \mathbf{e}, \quad \mathbf{e} \cdot {}^t\mathbf{e} = G,$$

and the form equations

$$(3.17)\qquad d\sigma = \sigma\Omega, \quad dG = \Omega G + G\, {}^t\Omega.$$

It is perhaps well to keep in mind the relation in ordinary differentials

$$(3.18)\qquad ds^2 = dP \cdot dP = g_{ik}\{\sigma^i \sigma^k\} = \{\sigma G\, {}^t\sigma\}.$$

Suppose that $\mathbf{X} = \lambda \mathbf{e}$ is a (contravariant) vector field on $\mathfrak{U}$, where $\lambda = (\lambda^1, \cdots, \lambda^n)$. We have

$$(3.19)\qquad d\mathbf{X} = d\lambda \mathbf{e} + \lambda d\mathbf{e} = (d\lambda + \lambda\Omega)\mathbf{e}.$$

The vector field is said to be generated by parallel displacement along a subspace if the components of $d\mathbf{X}$ vanish on that subspace. Thus the condition is

$$(3.20)\qquad d\lambda + \lambda\Omega = 0.$$

If $\mathbf{Y} = \mu \mathbf{e}$ is a second vector field, also generated by parallel displacement, so that

$$d\mu + \mu\Omega = 0,$$

then we have

$$\mathbf{X} \cdot \mathbf{Y} = \lambda G\, {}^t\mu,$$

hence

$$\begin{aligned} d(\mathbf{X} \cdot \mathbf{Y}) &= d\lambda\, G\, {}^t\mu + \lambda\, dG\, {}^t\mu + \lambda\, G\, {}^t d\mu \\ &= -\lambda\Omega\, G\, {}^t\mu + \lambda(\Omega G + G\, {}^t\Omega)\, {}^t\mu - \lambda\, G\, {}^t\Omega\, {}^t\mu = 0. \end{aligned}$$

This shows that parallel displacement is a euclidean transformation.

The differential forms given in (3.19) are often called the components of the *absolute differential* of the given field $\mathbf{X}$. (See [**1**, p. 38].) These are given explicitly by

$$(3.21) \qquad D\lambda = d\lambda + \lambda\Omega, \quad D\lambda = (D\lambda^1, \cdots, D\lambda^n).$$

If we express these forms in terms of the basis σ, we obtain the coefficients of the *covariant derivative* of λ:

$$(3.22) \qquad D\lambda^i = \lambda^i{}_{,j}\, \sigma^j, \text{ or } D\lambda = \lambda, \sigma, \quad \text{where} \quad \lambda, = \|\lambda^i{}_{,j}\|.$$

One deals with covariant (form) fields and tensor fields similarly. Suppose for example that

$$\mathbf{T} = \lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j$$

is a contravariant tensor field of order two. Here

$$\mathbf{e}_i\, \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j$$

denotes the tensor product of the vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. We have

$$(3.23) \qquad d\mathbf{T} = d\lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j + \lambda^{ij}\, \omega_i^k\, \mathbf{e}_k\, \mathbf{e}_j + \lambda^{ij}\, \omega_j^l\, \mathbf{e}_i\, \mathbf{e}_l,$$

hence

$$(3.24) \qquad d\mathbf{T} = D\lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j, \quad D\lambda^{ij} = d\lambda^{ij} + \lambda^{kj}\, \omega_k^i + \lambda^{il}\, \omega_l^j.$$

This again defines the covariant derivative

$$D\lambda^{ij} = \lambda^{ij}{}_{,k}\, \sigma^k.$$

4. Consequences; the curvature forms and the Bianchi identities. We begin with the basic relations (3.17). By differentiating the first of these, $d\sigma = \sigma\Omega$, we obtain

$$0 = d\sigma\Omega - \sigma d\Omega = \sigma(\Omega^2 - d\Omega).$$

Thus if we set

$$(4.1) \qquad \Theta = d\Omega - \Omega^2,$$

we obtain the relation

$$(4.2) \qquad \sigma\Theta = 0.$$

The elements of the matrix

$$\Theta = \|\theta_i^k\|$$

are two-forms, usually called the *curvature forms*. We set

$$\theta_i^k = \frac{1}{2} R^k_{ilm} \sigma^l \sigma^m \tag{4.3}$$

with

$$R^k_{ilm} + R^k_{iml} = 0,$$

defining the Riemann symbols of the second kind. The relation (4.2) may now be written

$$R^k_{ilm} \sigma^i \sigma^l \sigma^m = 0.$$

By expressing this 3-form in skew-symmetric canonical form, we obtain

$$R^k_{ilm} + R^k_{lmi} + R^k_{mil} = 0. \tag{4.4}$$

We next differentiate the relation (4.1) to obtain

$$d\Theta = -d\Omega\Omega + \Omega d\Omega = -(\Theta + \Omega^2)\Omega + \Omega(\Theta + \Omega^2).$$

This gives the *Bianchi relations*:

$$d\Theta = \Omega\Theta - \Theta\Omega. \tag{4.5}$$

It is easily shown that further differentiation of this relation yields nothing new.

Now let us work on the second relation,

$$dG = \Omega G + G\,{}^t\Omega,$$

of (3.17). This implies

$$\begin{aligned} 0 &= d\Omega G - \Omega dG + dG\,{}^t\Omega + G\,{}^td\Omega \\ &= (\Theta + \Omega^2) G - \Omega(\Omega G + G\,{}^t\Omega) + (\Omega G + G\,{}^t\Omega)\,{}^t\Omega + G({}^t\Theta - ({}^t\Omega)^2); \end{aligned}$$

hence we have

$$\Theta G + G\,{}^t\Theta = 0. \tag{4.6}$$

One also verifies that differentiating this formula leads to nothing more. One now introduces the covariant components of Θ by setting

$$\theta_{ik} = \theta_i^j g_{jk}. \tag{4.7}$$

This implies

(4.8)
$$\theta_{ik} = \frac{1}{2} R_{iklm} \sigma^l \sigma^m$$

with

$$R_{iklm} = R^j_{ilm} g_{jk}.$$

These new symbols R are the Riemann symbols of the first kind (in the case of a coordinate frame) and are also called the components of the *covariant curvature tensor*. Their tensor nature will be verified in the next section. The relation (4.6) now has the simple expressions

(4.9)
$$\theta_{ik} + \theta_{ki} = 0, \quad R_{iklm} + R_{kilm} = 0.$$

We also have from the relations (4.3) and (4.4),

(4.10)
$$R_{iklm} + R_{ikml} = 0, \quad R_{iklm} + R_{lkmi} + R_{mkil} = 0.$$

On combining (4.9) with (4.10), one obtains in the usual way the symmetry relation

(4.11)
$$R_{iklm} = R_{lmik}.$$

5. Change of basis. Suppose that $\mathbf{e}^*$ is second frame on $\mathfrak{U}$. Then

(5.1)
$$\mathbf{e}^* = A\,\mathbf{e},$$

where A is a nonsingular matrix of functions. For convenience we set $B = A^{-1}$, so that

$$dB = -B\,dA\,B.$$

The relation (5.1) implies

(5.2)
$$\sigma = \sigma^* A, \quad \text{or} \quad \sigma^* = \sigma B.$$

From the relation (3.16) we have

$$\mathbf{e} \cdot {}^t\mathbf{e} = G.$$

This implies

$$G^* = A\,G\,{}^tA.$$

Next we obtain the main transformation law:

THEOREM 5.1. *Under the change of basis* (5.1) *we have*

$$\Omega^* = A\Omega A^{-1} + dA\, A^{-1}. \tag{5.4}$$

Proof. According to Theorem 3.1, the matrix Ω^* is uniquely determined by the formulas

$$d\sigma^* = \sigma^* \Omega^*, \quad \Omega^* G^* + G^* \, {}^t\Omega^* = dG^*.$$

By differentiating (5.2), we obtain

$$d\sigma^* = d\sigma B - \sigma dB = \sigma \Omega B + \sigma B dA B = \sigma^*(A\Omega B + dAB),$$

which shows that the expression given in (5.4) satisfies the first of these conditions. The verification of the second condition is this:

$$\begin{aligned}(A\Omega B + dA\, B)\, A G\, {}^tA &+ A G\, {}^tA\, ({}^tB\, {}^t\Omega\, {}^tA + {}^tB\, {}^tdA) \\ &= A\Omega G\, {}^tA + dA\, G\, {}^tA + AG\, {}^t\Omega\, {}^tA + A G\, {}^tdA \\ &= dA\, G\, {}^tA + A\, dG\, {}^tA + A G\, {}^td A = d(A G\, {}^tA) = dG^*.\end{aligned}$$

COROLLARY 1. *The curvature forms transform according to the law*

$$\Theta^* = A\, \Theta\, A^{-1} \tag{5.5}$$

Proof. We have

$$d\Omega^* = dA\Omega B + A\, d\Omega\, B + A\, d\Omega\, B + A\, \Omega B\, dA\, B + dA\, B\, dA\, B$$

and

$$\Omega^{*2} = A\, \Omega^2\, B + A\Omega B\, dA\, B + dA\Omega B + dA\, B\, dA\, B,$$

hence

$$\Theta^* = d\Omega^* - \Omega^{*2} = A\, d\Omega\, B - A\Omega^2\, B = A\Theta A^{-1}.$$

COROLLARY 2. *If* $\mathbf{X} = \lambda\, \mathbf{e} = \lambda^*\, \mathbf{e}^*$ *is a vector field on* $\mathfrak{U}$, *the following transformation laws hold*:

$$\lambda^* = \lambda A^{-1}, \quad D\lambda^* = D\lambda\, A^{-1}. \tag{5.6}$$

Proof. The first relation is simply the statement that λ satisfies the contravariant transformation law, and is obvious. The second relation is true because

$$D\lambda^* = d\lambda^* + \lambda^*\Omega^* = d\lambda\, B - \lambda B\, dA\, B + \lambda B(A\Omega\, B + dA\, B)$$
$$= d\lambda\, B + \lambda\Omega\, B = D\lambda\, B.$$

Corollary 1 asserts that the forms θ_i^j which compose the matrix Θ transform as a mixed tensor of order two. Theorem 5.1 gives the transformation law for the forms ω_i^j, and can easily be converted into a transformation law for the Christoffel symbols Γ_{ik}^j of §3. What is more important, however, is the assertion of Corollary 2, that the components $D\lambda^i$ of the absolute differential of X transform by the contravariant tensor rule. This proves incidentally that parallel displacement is intrinsic.

6. The volume element and Gaussian curvature. We set

$$\gamma = |G|^{1/2}\, \sigma^1, \cdots, \sigma^n. \tag{6.1}$$

Thus γ is a nonzero n-form on $\mathfrak{U}$. Here $|G|$ denotes the (positive) determinant of the positive-definite matrix G. It follows from equations (5.2) and (5.3) that under a change of frame we have

$$|G^*|^{1/2} = \epsilon_A\, |A| \cdot |G|^{1/2}, \qquad \sigma^1, \cdots, \sigma^n = |A|\, \sigma^{*1}, \cdots, \sigma^{*n}, \tag{6.2}$$

where

$$\epsilon_A = \operatorname{sgn} |A|.$$

Thus we have the transformation law satisfied by the *volume element* γ:

$$\gamma^* = \epsilon_A\, \gamma. \tag{6.3}$$

It is thus possible to define the volume of an orientable n-dimensional portion of $\mathfrak{R}$ by integrating γ over that portion.

We now borrow some information from the theory of skew-symmetric matrices. Let $S = \|x_{ij}\|$ be a generic skew-symmetric matrix of even dimension $n = 2m$. Then there is a unique homogeneous polynomial $P(x_{ij})$ of degree m with the following properties:

(a) $|S| = [P(x_{ij})]^2$;

(b) if $S^* = AS\,{}^tA$, *where A is nonsingular, then*

$$P(x_{ij}^*) = |A|\, P(x_{ij});$$

(c) *P has value* 1 *for the specialization*

$$S = \begin{bmatrix} O_m & I_m \\ -I_m & O_m \end{bmatrix}$$

Now assume that our space $\mathfrak{R}$ has even dimension $n = 2m$. The matrix $H = \Theta G$ is skew-symmetric, by equations (4.7) and (4.9). Also the elements of H are 2-forms, and hence lie in the commutative ring generated by all forms of even degree. We set

$$\xi = -P(H)/|G|^{1/2}. \tag{6.4}$$

This form ξ is of degree n and is called the *Gaussian curvature form* [5]. When we combine (b) above with equation (6.2), we obtain the transformation law

$$\xi^* = \epsilon_A \ \xi. \tag{6.5}$$

Since γ is a nonzero n-form, and there is only one linearly independent n-form, we have

$$\xi = K\gamma, \tag{6.6}$$

where K is a function called the *Gaussian curvature*. We may combine (6.3) with (6.5) to obtain the intrinsic character of this quantity:

$$K^* = K. \tag{6.7}$$

7. A property of $|G|$. In this section we shall set

$$g = |G|.$$

The equation (5.3) then implies that

$$g^* = a^2 g,$$

where

$$a = |A|.$$

The following result is known [1, p. 44] for the classical case of a local coordinate frame.

THEOREM 7.1. *If*

$$S(\Omega) = \sum \omega_i^i$$

denotes the trace of the matrix Ω, *then*

$$\frac{1}{2}\frac{dg}{g} = S(\Omega). \tag{7.1}$$

The proof of this theorem will rest on the following known lemma.

LEMMA 7.1. *If A is a nonsingular matrix of functions, and*

$$a = |A|,$$

then

$$\frac{da}{a} = S(dA \cdot A^{-1}). \tag{7.2}$$

We shall include a short proof of this result for completeness. We set

$$C = \operatorname{cof} A, \quad B = A^{-1} = a^{-1} C.$$

Then

$$da = \sum \eta_i,$$

where η_i is the determinant formed from $|A|$ be replacing the i^{th} row of $|A|$ by the row $(da_{i1}, \cdots, da_{in})$. Thus

$$\eta_i = \sum_{j=1}^{n} (da_{ij})\, c_{ji}.$$

It follows that

$$da = \sum (da_{ij})\, c_{ji}, \quad \text{summed on } i \text{ and } j.$$

On the other hand,

$$S(dA \cdot A^{-1}) = a^{-1} S(dA \cdot C) = a^{-1} \sum (da_{ij})\, c_{ji} = a^{-1}\, da,$$

as asserted.

Proof of Theorem 7.1. We shall first show that the formula (7.1) is valid, provided that it is valid for a single moving frame. We have, under the change of frame (5.1),

$$dg^* = 2ag\, da + a^2\, dg;$$

hence

$$\frac{1}{2}\frac{dg^*}{g^*} = \frac{1}{2}\frac{dg}{g} + \frac{da}{a}.$$

Next, from equation (5.4) we have

$$S(\Omega^*) = S(\Omega) + S(dA \cdot A^{-1}).$$

It now follows from Lemma 7.1 that

$$S(\Omega^*) - \frac{1}{2}\frac{dg^*}{g^*} = S(\Omega) - \frac{1}{2}\frac{dg}{g}.$$

Finally, we note that for an orthonormal frame, Ω is skew-symmetric, hence $S(\Omega) = 0$, while $G = I$, $g = 1$, and so $g^{-1}\, dg = 0$.

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THE NEUMANN PROBLEM FOR THE HEAT EQUATION

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1. Introduction. By the Neumann problem we mean the following boundary-value problem: to determine the solution $u(x, t)$ of the equation

$$u_{xx}(x, t) - u_t(x, t) = 0 \tag{1.1}$$

in the rectangle or semi-infinite strip $R^{(b,c)}$: $\{ b < x < c;\ a < t < T \leq \infty \}$, given $u(x, a)$ on $b < x < c$ and $u_x(b, t)$ and $u_x(c, t)$ on $a < t < T$. There is a formula in terms of the Green's function (essentially given by Doetsch in [2, p. 361]) which gives the answer to this problem if the closed rectangle is in the interior of a larger region in which $u(x, t)$ is a continuous solution of (1.1). This formula is as follows: let $d = c - b$, and let

$$F^{(b,c)}(x, t; y, s) = \frac{1}{2d}\left[\vartheta_3\left(\frac{x-y}{2d}, \frac{t-s}{d^2}\right) + \vartheta_3\left(\frac{x+y-2b}{2d}, \frac{t-s}{d^2}\right)\right]$$

where ϑ_3 is the Jacobi Theta function; then

$$u(x, t) = \int_b^c F^{(b,c)}(x, t; y, a)\, u(y, a)\, dy - \int_a^t F^{(b,c)}(x, t; b, s)\, u_x(b, s)\, ds$$
$$+ \int_a^t F^{(b,c)}(x, t; c, s)\, u_x(c, s)\, ds. \tag{1.2}$$

The purpose of this paper is to extend the use of formula (1.2) in the following manner: we will give conditions under which a solution of the heat equation can be written in the form (1.2) wherein $u(a, y)\, dy$, etc., are replaced by $dA(y)$ or by $a(y)\, dy$, where $A(y) \in$ BV (that is, of bounded variation) or $a(y) \in L$. And we will examine the senses in which these extensions of formula (1.2) solve the boundary-value problem; that is, the manner in which the solutions tend to the prescribed boundary data for approach to a boundary point. Furthermore, we will obtain criteria for the unique determination of the solutions of these generalized boundary-value problems.

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We will normalize our rectangle to be R: $\{0 < x < 1,\ 0 < t < T \leq \infty\}$, and for this region we will delete the superscripts from the Green's function and denote it simply by $F(x, t; y, s)$. And we will denote by H the class of solutions of (1.1) for which both u_{xx} and u_t are continuous.

It will be convenient to display here the formula (see [2, p. 307])

$$\vartheta_3\,(x/2, t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x+n)^2}{4t}\right], \tag{1.3}$$

from which it is clear that $F^{(b,c)}(x, t; y, s)$ is a uniformly continuous function of all six variables if d is bounded away from both zero and infinity, and if the point (x, t) is bounded away from the point (y, s).

It also follows easily from (1.3) that

$$F_x(x, t; 0, s) = -G_y(x, t; 0, s) \quad \text{and} \quad F_x(x, t; 1, s) = -F_x(1-x, t; 0, s),$$

where

$$G(x, t; y, s) = \frac{1}{2}\left[\vartheta_3\left(\frac{x-y}{2}, t-s\right) - \vartheta_3\left(\frac{x+y}{2}, t-s\right)\right]$$

is the Green's function for the corresponding Dirichlet problem. (See [3; 4; 5; 6; 7].)

2. The Stieltjes integral representation. Our first main theorem gives the solution to one of the generalized boundary-value problems.

THEOREM 1. *For $u(x, t)$ to be representable in R by*

$$u(x, t) = \int_0^1 F(x, t; y, 0)\, dA(y) - \int_0^t F(x, t; 0, s)\, dB(s) \tag{2.1}$$
$$+ \int_0^t F(x, t; 1, s)\, dC(s),$$

where $A(y) \in \mathrm{BV}\ (0 \leq y \leq 1)$ and $B(s)$, $C(s) \in \mathrm{BV}\ (0 \leq s \leq s_0)$ for every $s_0 < T$, it is necessary and sufficient that

(1) *$u(x, t) \in H$ in R,*

(2) *$\int_0^t |u_x(x, s)|\, ds < M_t$ uniformly for $0 < x \leq x_0$ and $x_1 \leq x < 1$ for some x_0, x_1, where M_t depends only on t,*

(3) $\int_0^1 |u(y, t)|\, dy \leq M$ *uniformly for* $0 < t \leq t_0$ *for some* t_0.

Proof. To prove the sufficiency, let $(x, t) \in R$. Then there exist $a, b, c > 0$ such that $u(x, t)$ is given by (1.2). But, by condition (3),

$$A_a(x) = \int_0^x u(y, a)\, dy \in \mathrm{BV}[0 \leq x \leq 1]$$

uniformly in a for $0 < a \leq t_0$. Hence the uniformity holds for any sequence of values of a tending to zero, and thus by the well-known convergence theorems of Helly and Bray (see, for example, [**9**, p. 29-31]) there exists a subsequence $\{a_n\}$ and a function $A(x) \in \mathrm{BV}\ (0 \leq x \leq 1)$, to which $A_{a_n}(x)$ converges substantially, such that

$$\lim_{n\to\infty} \int_b^c F^{(b,c)}(x, t; y, a_n)\, dA_{a_n}(y) = \int_b^c F^{(b,c)}(x, t; y, 0)\, dA(y).$$

Then (1.2) becomes

$$(2.2)\ u(x, t) = \int_b^c F^{(b,c)}(x, t; y, 0)\, dA(y) - \int_0^t F^{(b,c)}(x, t; b, s)\, u_x(b, s)\, ds$$
$$+ \int_0^t F^{(b,c)}(x, t; c, s)\, u_x(c, s)\, ds,$$

where the existence of the two latter integrals is guaranteed by condition (2).

Furthermore,

$$B_b(t) = \int_0^t u_x(b, s)\, ds \text{ and } C_c(t) = \int_0^t u_x(c, s)\, ds \in \mathrm{BV}[0 \leq t \leq t_0]$$

for every $t_0 < T$ uniformly for $0 < b \leq x_0$ and $x_1 \leq c < 1$. Hence the uniformity holds for any sequence of values of b tending to zero and of c tending to one. Hence there exist subsequences $\{b_n\}$ and $\{c_n\}$ and functions

$$B(t), C(t) \in \mathrm{BV}\ (0 \leq t \leq t_0)$$

such that

$$\lim_{n\to\infty} \int_0^t F^{(b_n,c_n)}(x, t; b_n, s)\, dB_{b_n}(s) = \int_0^t F(x, t; 0, s)\, dB(s)$$

and

$$\lim_{n\to\infty}\int_0^t F^{(b_n,c_n)}(x,\ t;\ c_n,\ s)\,dC_{c_n}(s) = \int_0^t F(x,\ t;\ 1,\ s)\,dC(s).$$

Hence $u(x,\ t)$ has the representation asserted.

We will later show that A, B, C are independent of the particular sequences of a, b, c used here (see Theorem 3).

To prove the necessity of condition (1) we must differentiate under the integral sign. The only difficulty encountered in this is the disposition of the terms which arise from the variable upper limit. If, however, one forms a difference quotient it is easy to see that the contribution arising from the variability of the upper limit must always vanish, due to the strong convergence to zero of the kernel as $s \longrightarrow t-0$.

To establish (2) we write

$$\begin{aligned} u_x(x,\ t) &= \int_0^1 F_x(x,\ t;\ y,\ 0)\,dA(y) - \int_0^t F_x(x,\ t;\ 0,\ s)\,dB(s) \\ &\qquad + \int_0^t F_x(x,\ t;\ 1,\ s)\,dC(s) \\ &= \int_0^1 F_x(x,\ t;\ y,\ 0)\,dA(y) + \int_0^t G_y(x,\ t;\ 0,\ s)\,dB(s) \\ &\qquad + \int_0^t G_y(1-x,\ t;\ 0,\ s)\,dC(s) \\ &= U_1(x,\ t) + U_2(x,\ t) + U_3(x,\ t). \end{aligned}$$

Now

$$|U_2(x,\ t)| \le \int_0^t G_y(x,\ t;\ 0,\ s)|dB(s)| = v_2(x,\ t)$$

and

$$|U_3(x,\ t)| \le \int_0^t G_y(1-x,\ t;\ 0,\ s)\,|dC(s)| = v_3(x,\ t),$$

where $v_2(x,\ t)$ and $v_3(x,\ t)$ are nonnegative solutions of (1.1). Then, by [3, p. 22-23] and [7, p. 373], $v_2(x,\ t)$ and $v_3(x,\ t)$ must satisfy condition (2). Hence so must $U_2(x,\ t)$ and $U_3(x,\ t)$.

To examine $U_1(x,\ t)$ we need to note that, by (1.3), for $0 < x,\ y < 1$,

$$F_x(x,\, t;\, y,\, 0) = -\frac{x-y}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x-y)^2}{4t}\right] - \frac{(x+y)^2}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x+y)^2}{4t}\right]$$
$$-\frac{(x+y-2)}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x+y-2)^2}{4t}\right] + \bar{u}_1(x,\, y,\, t),$$

where $\bar{u}_1$ is bounded, say $|\bar{u}_1| \leq B_1$. Then

$$\int_0^t |U_1(x,\, s)|\, ds \leq \int_0^t\int_0^1 \sum_{n=1}^{3} \frac{|a_n|}{4\pi^{1/2}s^{3/2}}\exp\left[-\frac{a_n^2}{4s}\right] |dA(y)|\, ds$$
$$+\, tB_1\, V_A(1),$$

where $a_1 = x - y$, $a_2 = x + y$, $a_3 = x + y - 2$, and $V_A(1)$ is the variation of A. Then

$$\int_0^t |U_1(x,\, s)|\, ds \leq \frac{1}{4\pi^{1/2}}\int_0^1\int_0^t \sum_{n=1}^{3} \frac{|a_n|}{s^{3/2}}\exp\left[-\frac{a_n^2}{4s}\right] ds\, |dA(y)|$$
$$+\, tB_1\, V_A(1),$$

$$= \frac{1}{2\pi^{1/2}}\int_0^1 \sum_{n=1}^{3} \int_{a_n^2/4t}^{\infty} e^{-s}\, s^{-1/2}\, ds\, |dA(y)| + tB_1 V_A(1),$$

$$\leq \frac{3}{2\pi^{1/2}}\int_0^1\int_0^{\infty} e^{-s}\, s^{-1/2}\, ds\, |dA(y)| + tB_1\, V_A(1),$$

$$= (3/2 + tB_1)\, V_A(1),$$

the change of order of integration being permissible by Fubini's theorem. Since $U_1(x,\, t)$, $U_2(x,\, t)$, and $U_3(x,\, t)$ separately satisfy condition (2), so must their sum, $u_x(x,\, t)$.

To verify condition (3) we write

$$u(x,\, t) = \int_0^1 - \int_0^t + \int_0^t = u_1(x,\, t) + u_2(x,\, t) + u_3(x,\, t),$$

and first consider

$$u_2(x,\, t) = -\int_0^t F(x,\, t;\, 0,\, s)\, dB(s).$$

But, by (1.3),

$$F(x, t; 0, s) = \pi^{-1/2}(t-s)^{-1/2} \exp\left[-\frac{x^2}{4(t-s)}\right] + \bar{u}_2(x, t, s),$$

where $\bar{u}_2$ is bounded, say by B_2. Then

$$\int_0^1 |u_2(x, t)|\, dx \le \int_0^t \pi^{-1/2}(t-s)^{-1/2} \int_0^1 \exp\left[-\frac{x^2}{4(t-s)}\right] dx\, |dB(s)| + B_2 V_B(t),$$

$$\le \int_0^t \pi^{-1/2}(t-s)^{-1/2} \int_0^\infty \exp\left[-\frac{x^2}{4(t-s)}\right] dx\, |dB(s)| + B_2 V_B(t),$$

$$= (1 + B_2) V_B(t).$$

Similarly,

$$\int_0^1 |u_3(x, t)|\, dx \le (1 + B_3) V_C(t).$$

We turn now to $u_1(x, t)$:

$$\int_0^1 |u_1(x, t)|\, dx \le \int_0^1 \int_0^1 F(x, t; y, 0)\, dx\, |dA(y)|.$$

But, again by (1.3),

$$F(x, t; y, 0) = \frac{1}{2}(\pi t)^{-1/2} \left\{ \exp\left[-\frac{(x-y)^2}{4t}\right] + \exp\left[-\frac{(x+y)^2}{4t}\right] + \exp\left[-\frac{(x+y-2)^2}{4t}\right]\right\} + \bar{u}_4(x, y, t),$$

where $\bar{u}_4$ is bounded by, say, B_4. Then

$$\int_0^1 |u_1(x, t)|\, dx \le \frac{1}{2}(\pi t)^{-1/2} \int_0^1 \int_0^1 \left\{\exp\left[-\frac{(x-y)^2}{4t}\right] + \exp\left[-\frac{(x+y)^2}{4t}\right] + \exp\left[-\frac{(x+y-2)^2}{4t}\right]\right\} dx\, |dA(y)| + B_4 V_A(1),$$

$$\le \frac{1}{2}(\pi t)^{-1/2} \int_0^1 \left\{\int_{-1}^1 \exp\left[-\frac{(x-y)^2}{4t}\right] dx + \int_0^1 \exp\left[-\frac{(x+y-2)^2}{4t}\right] dx\right\} |dA(y)| + B_4 V_A(1),$$

$$\leq \frac{1}{2}(\pi t)^{-1/2}\int_0^1 2\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4t}\right] dx\,|dA(y)| + B_4 V_A(1),$$

$$\leq V_A(1)(2+B_4)$$

Hence, for $0 < t \leq t_0$,

$$\int_0^1 |u(x, t)|\,dx \leq V_A(1)\,(2+B_4) + V_B(t_0)(1+B_2) + V_C(t_0)(1+B_3) = M$$

This completes the proof.

3. The behavior at the boundary. We are now prepared to examine in detail the behavior near the boundary of solutions of our generalized boundary value problem considered in section 2. The main result of the section is:

THEOREM 2. *If $u(x, t)$ is representable in R by (2.1), then*

(1) $$\lim_{t \to 0+} u(x, t) = A'(x)$$

and

(2) $$\lim_{x \to 0+} u_x(x, t) = B'(t-0); \quad \lim_{x \to 1-0} u_x(x, t) = C'(t-0)$$

$$\left(\textit{where } B'(t-0) = \lim_{h \to 0+} \frac{B(t-0) - B(t-h)}{h}, \textit{ and similarly for } C'(t-0)\right),$$

wherever the derivatives in question exist.

Proof. If $u(x, t)$ is representable by (2.1), let

$$u(x, t) = \int_0^1 - \int_0^t + \int_0^t = u_1(x, t) + u_2(x, t) + u_3(x, t)$$

as before. Let I be any open interval whose closure is contained in $\{0 < x < 1\}$. Then for $x \in I$, $F(x, t; 0, s)$ and $F(x, t; 1, s)$ both converge uniformly to zero as $t \longrightarrow 0+$, as can be seen from (1.3). Then clearly $u_2(x, t)$, $u_3(x, t) \longrightarrow 0$ as $t \longrightarrow 0+$, for $x \in I$.

Also, for $x \in I$, by (1.3),

$$F(x, t; y, 0) = (4\pi t)^{-1/2} \exp\left[-\frac{(x-y)^2}{4t}\right] + o(1)$$

uniformly as $t \longrightarrow 0+$. Hence

$$u_1(x, t) = \int_0^1 (4\pi t)^{-1/2} \exp\left[-\frac{(x-y)^2}{4t}\right] dA(y) + o(1).$$

Then (see [3, p. 25-26 and 65-66] and [7, p. 393-394])

$$\lim_{t \to 0+} u_1(x, t) = A'(x)$$

wherever this derivative exists. Since any $x \in \{0 < x < 1\}$ can be caught in such an I, this establishes (1).

To verify conclusion (2) we write, as before,

$$u_x(x, t) = \int_0^1 F_x(x, t; y, 0)\, dA(y) + \int_0^t G_y(x, t; 0, s)\, dB(s)$$
$$+ \int_0^t G_y(1-x, t; 0, s)\, dC(s),$$
$$= U_1(x, t) + U_2(x, t) + U_3(x, t).$$

As $x \longrightarrow 0+$, $U_1(x, t)$ and $U_3(x, t)$ vanish since the kernels converge uniformly to zero, and as $x \longrightarrow 1-0$, $U_1(x, t)$ and $U_2(x, t)$ vanish for the same reason. Then by [5], $u_x(x, t)$ tends to $B'(t-0)$ or $C'(t-0)$ according as x tends to zero or one, whenever the derivatives exist.

We can now give criteria for the existence of boundary values of the function $u(x, t)$ itself on the sides $x = 0$, and $x = 1$.

COROLLARY 1. *If $u(x, t)$ is representable in R by (2.1), then $u(0+, t)$ exists if $B'(t-0)$ does.*

Proof. Let $0 < x_0 < 1$; then

$$u(x, t) = \int_{x_0}^x u_x(y, t)\, dy + u(x_0, t) \qquad (0 < x < 1),$$

and the integrand is bounded in $0 < x \le x_0$. Hence the integral exists for $x = 0$ and defines $u(0+, t)$.

We might also note in passing that for such t, the x difference quotient at the boundary also tends to $B'(t-0)$; for, by the mean value theorem,

$$\frac{u(h, t) - u(0+, t)}{h} = u_x(\bar{h}, t) \longrightarrow B'(t-0)$$

as $h \longrightarrow 0$.

From Theorem 1 we have:

COROLLARY 2. *For $u(0+, t)$ to exist it is sufficient that*

$$\int_0^{t-0} (t-s)^{-1/2} \, |dB(s)|$$

converge.

Proof. Define

$$f(t) = \int_0^1 F(0, t; y, 0)\, dA(y) + \int_0^t F(0, t, 1, s)\, dC(s)$$
$$+ 2\int_0^t \pi^{-1/2} (t-s)^{-1/2} \sum_{n=1}^{\infty} \exp\left[-\frac{n^2}{(t-s)}\right] dB(s)$$
$$- \pi^{-1/2} \int_0^{t-0} (t-s)^{-1/2}\, dB(s),$$

and consider

$$\limsup_{x \to 0+} |u(x, t) - f(t)|$$
$$\leq \pi^{-1/2} \limsup_{x \to 0+} \int_0^{t-0} (t-s)^{-1/2} \left\{1 - \exp\left[-\frac{x^2}{4(t-s)}\right]\right\} |dB(s)|.$$

Now given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\pi^{-1/2} \int_{t-\delta}^{t-0} (t-s)^{-1/2} \exp\left[-\frac{x^2}{4(t-s)}\right] |dB(s)|$$
$$\leq \pi^{-1/2} \int_{t-\delta}^{t-0} (t-s)^{-1/2} \, |dB(s)| \leq \epsilon,$$

so that

$$\limsup_{x \to 0+} |u(x, t) - f(t)|$$
$$\leq \pi^{-1/2} \limsup_{x \to 0+} \int_0^{t-\delta} (t-s)^{-1/2} \left\{1 - \exp\left[-\frac{x^2}{4(t-s)}\right]\right\} |dB(s)| + 2\epsilon = 2\epsilon.$$

Let $\epsilon \longrightarrow 0$ to get

$$\lim_{x \to 0+} u(x, t) = u(0+, t) = f(t),$$

which completes the proof.

We may also note that if $B(s)$ were monotone, then, since $\exp[-x^2/4(t-s)]$ converges to unity in a monotone way, we could invoke, the monotone convergence theorem to obtain the convergence of the integral as a necessary and sufficient condition for the existence of $u(0+, t)$.

Also with Theorem 2 at our disposal we can now prove:

THEOREM 3. *Let $u(x, t)$ be representable by* (2.1); *then the functions $A(x)$, $B(t)$, $C(t)$ are uniquely determined by $u(x, t)$, so that, at every point of continuity,*

$$A(x) = \lim_{a \to 0+} \int_0^x u(y, a)\, dy$$

and

$$B(t) = \lim_{b \to 0+} \int_0^t u_x(b, s)\, ds; \quad C(t) = \lim_{c \to 1-0} \int_0^t u_x(c, s)\, ds.$$

Proof. For suppose $B_1(t)$ and $B_2(t)$ arise from two distinct sequences. Then clearly if $B_3(t) = B_1(t) - B_2(t)$, we have

$$\int_0^t F(x, t; 0, s)\, dB_3(s) \equiv 0 \text{ in } R.$$

Hence, differentiating, we get

$$\int_0^t G_y(x, t; 0, s)\, dB_3(s) \equiv 0$$

We first show $B_3(s)$ is continuous: suppose it has a jump σ at t_0; then, for $t > t_0$,

$$0 = \int_0^t G_y(x, t; 0, s)\, dB_4(s) + \sigma\, G_y(x, t; 0, t_0),$$

where $B_4(s)$ is the boundary function remaining after the jump σ at t_0 is removed. Then

$$0 = \int_0^t \frac{x}{2\pi^{1/2}(t-s)^{3/2}} \exp\left[-\frac{x^2}{4(t-s)}\right] dB_4(s) + \frac{x\sigma}{2\pi^{1/2}(t-t_0)^3} \exp\left[-\frac{x^2}{4(t-t_0)}\right] + o(1).$$

Choose δ so small that

$$V_{B_4}(t_0+\delta) - V_{B_4}(t_0-\delta) < \frac{|\sigma|}{2},$$

set $t - t_0 = x^2/4$, and take x so small that $x^2/4 < \delta$. Then

$$0 = \int_{t_0-\delta}^{t_0+x^2/4} \frac{x}{2\pi^{1/2}(t_0-s+x^2/4)^{3/2}} \exp\left[-\frac{x^2}{4(t_0-s+x^2/4)}\right] dB_4(s) + \frac{4\sigma}{e\pi^{1/2}x^2} + o(1),$$

$$0 \geq \frac{4|\sigma|}{e\pi^{1/2}x^2} - \int_{t_0-\delta}^{t_0+x^2/4} \frac{x}{2\pi^{1/2}(t_0-s+x^2/4)^{3/2}} \exp\left[-\frac{x^2}{4(t_0-s+x^2/4)}\right] |dB_4(s)| + o(1).$$

The maximum of the integrand is at $s = t_0$, so that

$$0 \geq \frac{4|\sigma|}{e\pi^{1/2}x^2} - \frac{2|\sigma|}{e\pi^{1/2}x^2} + o(1) = \frac{2|\sigma|}{e\pi^{1/2}x^2} + o(1),$$

and we have a contradiction as $x \longrightarrow 0+$.

Similarly the jumps of $C(t)$ are determined.

Suppose $A_1(x)$ and $A_2(x)$ arise from two distinct sequences, and $A_3(x) = A_1(x) - A_2(x)$; then, as before,

$$0 \equiv \int_0^1 F(x, t; y, 0)\, dA_3(y) \text{ in } R.$$

And suppose it has a jump of σ at x_0; then, as before,

$$0 = \sigma(4\pi t)^{-1/2} + (4\pi t)^{-1/2} \int_0^1 \exp\left[-\frac{(y-x_0)^2}{4t}\right] dA_4(y) + o(1).$$

If δ is so small that

$$V_{A_4}(x_0+\delta) - V_{A_4}(x_0-\delta) \leq |\sigma|/2,$$

then

$$0 = \sigma(4\pi t)^{-1/2} + (4\pi t)^{-1/2} \int_{x_0-\delta}^{x_0+\delta} \exp\left[-\frac{(y-x_0)^2}{4t}\right] dA_4(y) + o(1),$$

$$0 \geq |\sigma|(4\pi t)^{-1/2} - |\sigma|(4\pi t)^{-1/2}/2 + o(1) = |\sigma|(4\pi t)^{-1/2}/2 + o(1),$$

and as $t \longrightarrow 0+$ we get a contradiction.

Then A_3, B_3, C_3 are continuous functions of bounded variation, and by Theorem 2 their derivatives are zero almost everywhere. Each of them must then have an infinite derivative on a nondenumerable set. (See e.g. [8, p. 128].) This then implies that $\lim u(x, t)$ and $\lim u_x(x, t)$ must become infinite on a nondenumerable set, which is a contradiction, and the functions A_3, B_3, C_3 are constants. Hence, since every sequence of a's, b's, or c's contains a subsequence for which $A_a(x)$, etc., converges to a common limit, the limit must also be attained for continuous approach. Thus the last statement of the theorem is established.

4. The Lebesgue integral representation. We are now in a position to establish:

THEOREM 4. *For $u(x, t)$ to be representable in R by*

$$u(x,t) = \int_0^1 F(x, t; y, 0)\, a(y)\, dy - \int_0^t F(x, t; 0, s)\, b(s)\, ds + \int_0^t F(x, t; 1, s)\, c(s)\, ds, \tag{4.1}$$

where $a(y) \in L\ (0 \leq y \leq 1)$ and $b(s), c(s) \in L\ (0 \leq s \leq s_0 < T \leq \infty)$ for every s_0, $(0 < s_0 < T)$, it is necessary and sufficient that

(1) $$u(x, t) \in H \text{ in } R,$$

(2) $$\lim_{y, y' \to 0+} \int_0^t |u_x(y, s) - u_x(y', s)|\, ds = 0$$

and

$$\lim_{y, y' \to 1-0} \int_0^t |u_x(y, s) - u_x(y', s)|\, ds = 0$$

for every t $(0 < t < T)$, and

$$\lim_{s,s'\to 0} \int_0^1 |u(y,s) - u(y,s')|\, dy = 0. \tag{3}$$

Proof. For the sufficiency, let the closed finite interval $I \subset \{0 \le s < T\}$ be prescribed, and let e be any measurable set in I. Given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, I)$ such that

$$\int_e |u_x(y,s) - u_x(y',s)|\, ds \le \epsilon/2 \quad \text{for } y, y' < \delta.$$

Then

$$\int_e |u_x(y,s)|\, ds \le \int_e |u_x(y',s)|\, ds + \int_e |u_x(y,s) - u_x(y',s)|\, ds$$
$$\le \int_e |u_x(y', s|\, ds + \epsilon/2.$$

Now keep y' fixed and take $m(e)$ so small that

$$\int_e |u_x(y',s)|\, ds \le \epsilon/2,$$

so that, for $0 < y < \delta$,

$$\int_e |u_x(y,s)|\, ds \le \epsilon$$

if $m(e)$ is sufficiently small. Hence $B_b(s)$ are uniformly absolutely continuous; consequently, so is $B(s)$, and $dB(s)$ can be replaced by $b(s)\, ds$, where $B'(s) = b(s)$ almost everywhere. Similarly $dC(s) = c(s)ds$ and $dA(y) = a(y)\, dy$.

The necessity of (1) follows by Theorem 1. To prove that of (2) we write

$$b(s) = b_1(s) - b_2(s),$$

where $b_1(s)$ and $b_2(s)$ are both nonnegative, say, for example,

$$b_1(s) = |b(s)| \quad\text{and}\quad b_2(s) = |b(s)| - b(s).$$

Let

$$u^{(i)}(x,t) = -\int_0^t F(x,t;0,s)\, b_i(s)\, ds \qquad (i = 1, 2).$$

Then

$$u_x^{(i)}(x,t) = \int_0^t G_y(x,t;0,s)\, b_i(s)\, ds \qquad (i = 1, 2).$$

We know from Theorem 2 that

$$\lim_{x \to 0+} u_x^{(i)}(x, t) = b_i(t) \qquad (i = 1, 2),$$

almost everywhere, and, by Theorem 3,

$$\lim_{x \to 0+} \int_0^t u_x^{(i)}(x, s)\, ds = \int_0^t b_i(s)\, ds \qquad (i = 1, 2).$$

Since the $u_x^{(i)}(x, t)$ are nonnegative (see [4, Remark 1, p. 975]), we can say (see [4, p. 977])

$$\lim_{x \to 0+} \int_0^t |u^i(x, s) - b_i(s)|\, ds = 0 \qquad (i = 1, 2). \tag{4.2}$$

Now consider

$$\begin{aligned} \int_0^t |u_x(x, s) - b(s)|\, ds \leq & \int_0^t |u_x^{(1)}(x, s) - b_1(s)|\, ds \\ & + \int_0^t |u_x^{(2)}(x, s) - b_2(s)|\, ds + \int_0^t \left| \int_0^1 F_x(x, s; y, 0)\, a(y) \right| dy\, ds \\ & + \int_0^t \left| \int_0^s G_y(1 - x, s; 0, \tau)\, c(\tau)\, d\tau \right| ds. \end{aligned} \tag{4.3}$$

As $x \longrightarrow 0+$, the first and second integrals on the right vanish by (4.2), and the fourth since $G_y(1 - x, s; 0, \tau)$ tends to zero uniformly in s and τ as $x \longrightarrow 0+$. To estimate the third we note

$$\begin{aligned} F_x(x, s; y, 0) = - \frac{1}{4\pi^{1/2} s^{3/2}} & \left\{ (x - y) \exp\left[-\frac{(x - y)^2}{4s} \right] \right. \\ & \left. + (x + y) \exp\left[-\frac{(x + y)^2}{4s} \right] \right\} + \bar{u}(x, y, s), \end{aligned}$$

where $\bar{u} = o(1)$ uniformly in y and s as $x \longrightarrow 0+$. Then

$$\begin{aligned} |F_x(x, s; y, 0)| \leq \frac{1}{4\pi^{1/2} s^{3/2}} & \left\{ |x - y| \exp\left[-\frac{(x - y)^2}{4t} \right] \right. \\ & \left. + (x + y) \exp\left[-\frac{(x + y)^2}{4t} \right] \right\} + |\bar{u}|. \end{aligned}$$

But

$$\int_0^t s^{-3/2} \exp\left[-\frac{a^2}{4s}\right] ds = \frac{2}{|a|} \int_{a^2/4t}^{\infty} e^{-v} v^{-1/2} dv \leq \frac{2\pi^{1/2}}{|a|}.$$

Hence

$$\int_0^t |F(x, s; y, 0)| ds \leq 1 + o(1) \leq 2$$

for x sufficiently small. Thus the third integral on the right side of (4.3) is dominated by

$$\int_0^1 |a(y)| \int_0^t |F_x(x, s; y, 0)| ds\, dy \leq 2 \int_0^1 |a(y)| dy.$$

Then by the dominated convergence theorem we can pass to the limit under the integral sign, by which we get zero as a limit, since $F_x(x, s; y, 0)$ tends to zero. This proves

$$\lim_{x \to 0+} \int_0^t |u_x(x, s) - b(s)| ds,$$

from which condition (2) follows immediately.

Condition (3) follows similarly, but more easily.

5. **Uniqueness.** We now turn to the question of the extent to which the boundary data uniquely determine the solution of the boundary-value problem. We get one result as an immediately corollary of our Theorem 4.

COROLLARY 3. *If $u(x, t)$ is representable by (4.1) in R, and has zero boundary values almost everywhere for approach along the normal, then $u(x, t) \equiv 0$ in R.*

Proof. By Theorem 2, $a(y)$, $b(s)$, and $c(s)$ vanish almost everywhere.

The situation in the case of the Stieltjes representation is not so simple (see [6]): We can have a function representable in R by (2.1) which has boundary values identically zero for approach along the normal, yet which is itself not identically zero; for example, for $0 \leq t_0$, let

$$u(x, t) = \begin{cases} 0 & (0 \leq t \leq t_0), \\ -F(x, t; 0, t_0) & (t_0 < t). \end{cases}$$

This is a nontrivial solution of the heat equation, representable by (2.1), for which

$$u(x, 0+) \equiv 0, \quad u_x(0+, t) \equiv 0, \quad u_x(1-0, t) \equiv 0.$$

However we can assert:

THEOREM 3. *Suppose $u(x, t)$ is representable in R by (2.1), that*

$$u(x, 0+) \equiv 0 \qquad (0 < x < 1),$$

and that $B(s)$ and $C(s)$ are monotone for $0 \leq s < T \leq \infty$. Let

$$\lim u_x(x, t) = 0 \quad as \quad (x, t) \longrightarrow (0, s)$$

along a parabolic arc of the form $t - s = ax^2$, $(a > 0)$ and

$$\lim u_x(x, t) = 0 \quad as \quad (x, t) \longrightarrow (1, s)$$

along a parabolic arc of the form $t - s = b(x-1)^2$ $(b > 0)$ for every s $(0 \leq s < T)$. Then $u(x, t) \equiv 0$ in R

Proof. Let x be fixed, $0 < x < 1$. Then, by (2.1) and (1.3),

$$u(x, t) = \int_0^1 F(x, t; y, 0)\, dA(y) + o(1) \quad \text{as} \quad t \longrightarrow 0+.$$

Choose $0 < \delta < (1/2) \min(x, 1-x)$, so that

$$\begin{aligned}
u(x, t) &= \int_{x-\delta}^{x+\delta} (4\pi t)^{-1/2} \exp\left[-\frac{(y-x)^2}{4t}\right] dA(y) + o(1), \\
&= \int_{x-\delta}^{x+\delta} (4\pi t)^{-1/2} \exp\left[-\frac{(y-x)^2}{4t}\right] d[A(y) - A(x)] + o(1), \\
&= \int_{x-\delta}^{x+\delta} \frac{y-x}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{(y-x)^2}{4t}\right] [A(y) - A(x)]\, dy + o(1), \\
&= \int_{-\delta}^{\delta} \frac{z^2}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{z^2}{4t}\right] \frac{A(x+z) - A(x)}{z}\, dz + o(1).
\end{aligned}$$

Then

$$\begin{aligned}
u(x, t) &\geq \operatorname*{Inf}_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z} \int_{-\delta}^{\delta} \frac{z^2}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{z^2}{4t}\right] dz + o(1), \\
&= \operatorname*{Inf}_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z} \int_{-\delta/2t^{1/2}}^{\delta/2t^{1/2}} \frac{2}{\pi^{1/2}} \zeta^2 e^{-\zeta^2}\, d\zeta + o(1).
\end{aligned}$$

Let $t \longrightarrow 0+$:

$$u(x, 0+) = 0 \geq \operatorname*{Inf}_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z}$$

Let $\delta \longrightarrow 0$:

$$0 \geq \underline{D} A(x).$$

Similarly,

$$0 \leq \overline{D} A(x)$$

for every x $(0 < x < 1)$. Now $A(x)$ is continuous, for if it had a jump it would violate one or the other of these conditions. Then by [**1**, p. 580], it must be both nonincreasing and nondecreasing, and hence constant.

Furthermore,

$$u_x(x, t) = \int_0^t G_y(x, t\colon 0, s)\, dB(s) + o(1) \quad \text{as} \quad (x, t) \longrightarrow (0, s).$$

Then, by [6], $B(s)$ is constant. Similarly one sees $C(s)$ is constant. This completes the proof.

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THE UNIVERSITY OF MINNESOTA

ORTHOGONAL HARMONIC POLYNOMIALS

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1. Introduction. In this paper we develop sets of harmonic polynomials in x, y, z which are orthogonal over prolate and oblate spheroids. The orthogonality is taken in several different norms, each of which leads to the discussion of a partial differential equation by means of the kernel of the orthogonal system corresponding to that norm. The principal point of interest is that the orthogonality of the harmonic polynomials in question does not depend on the shape of the spheroids, but only on their size. More precisely, the polynomials depend only on the location of the foci of the ellipse generating the spheroid, and not on its eccentricity.

The importance of constructing these polynomials stems from the role which they play in the calculation of the kernel functions and Green's functions of the Laplace and biharmonic equations in a spheroid. One can compute from the kernels, in turn, the solution of the basic boundary-value problems for these equations. As a particular case, one arrives at formulas for the solution of the partial differential equation

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial z^2} = 0$$

which arises in discussion of axially symmetric flow.

Results of the type presented here have occurred previously in the work of Zaremba [**10**], and are related to recent developments of Friedrichs [**3**, **4**] and the author [**5**]. The polynomials investigated in this earlier work are in two independent real variables and yield formulas for solving the Laplace and biharmonic equations in two dimensions. Thus it is natural to suggest that the basic results generalize to n-dimensional space. In this connection, it is

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easily verified that a part of the theory carries over to arbitrary ellipsoids in three-dimensional space.

2. Notation and definitions. We shall make use of rectangular coordinates x, y, z, cylindrical coordinates ρ, ϕ, z, and spherical coordinates r, θ, ϕ. Thus

$$x = \rho \cos \phi = r \sin \theta \cos \phi,$$
$$y = \rho \sin \phi = r \sin \theta \sin \phi,$$
$$z = r \cos \theta.$$

The Laplace integral formula

$$P_n^h(\cos \theta) = \frac{(n+h)!}{\pi i^h\, n!} \int_0^{\pi} (\cos \theta + i \sin \theta \cos t)^n \cos ht\, dt$$

for the Legendre polynomials $P_n(\cos \theta) = P_n^0(\cos \theta)$ and the associated Legendre functions $P_n^h(\cos \theta)$ is basic for our work. In terms of Laplace's integral we obtain the solid spherical harmonics in the form

$$r^n P_n^h(\cos \theta) \cos h\phi = \frac{(n+h)!}{\pi\, i^h\, n!} \int_0^{\pi} (z + i\rho \cos t)^n \cos h\phi \cos ht\, dt,$$

$$r^n P_n^h(\cos \theta) \sin h\phi = \frac{(n+h)!}{\pi\, i^h\, n!} \int_0^{\pi} (z + i\rho \cos t)^n \sin h\phi \cos ht\, dt.$$

They are homogeneous harmonic polynomials of degree n in x, y, z.

We shall be interested in obtaining complete orthogonal systems of harmonic polynomials in the interior of the prolate spheroid

$$(1) \qquad \frac{z^2}{\operatorname{ch}^2 \alpha} + \frac{\rho^2}{\operatorname{sh}^2 \alpha} = 1,$$

and in the interior of the oblate spheroid

$$(2) \qquad \frac{z^2}{\operatorname{sh}^2 \alpha} + \frac{\rho^2}{\operatorname{ch}^2 \alpha} = 1.$$

Thus it is convenient to introduce coordinates u, v defined by the relations

$$z + i\rho = \cos(u - iv) = \cos u \operatorname{ch} v + i \sin u \operatorname{sh} v$$

for the prolate case, and defined by

$$\rho + iz = \sin(u + iv) = \sin u \operatorname{ch} v + i \cos u \operatorname{sh} v$$

for the oblate case. In both cases, the boundaries of the above spheroids have the equation $v = \alpha$.

We define

$$U_{n,h}(\rho, z) = \left[\frac{(n+h)!}{(n-h)!}\right]^{1/2} \frac{1}{\pi i^h} \int_0^\pi P_n(z + i\rho \cos t) \cos ht \, dt,$$

$$V_{n,h}(\rho, z) = \left[\frac{(n+h)!}{(n-h)!}\right]^{1/2} \frac{i^{n-h}}{\pi} \int_0^\pi P_n(iz - \rho \cos t) \cos ht \, dt.$$

By the addition theorem for the Legendre polynomials we obtain the well-known expressions

$$U_{n,h}(\rho, z) = \left[\frac{(n-h)!}{(n+h)!}\right]^{1/2} P_n^h(\cos u) P_n^h(\operatorname{ch} v),$$

$$V_{n,h}(\rho, z) = \left[\frac{(n-h)!}{(n+h)!}\right]^{1/2} i^{n-h} P_n^h(\cos u) P_n^h(i \operatorname{sh} v),$$

where in the first case u, v are coordinates in the prolate spheroid (1) and in the second case u, v are coordinates in the oblate spheroid (2).

Here

$$P_n^h(\operatorname{ch} v) = \operatorname{sh}^h v \, P_n^{(h)}(\operatorname{ch} v),$$

$$P_n^h(i \operatorname{sh} v) = \operatorname{ch}^h v \, P_n^{(h)}(i \operatorname{sh} v).$$

The expressions

$$U_{n,h}(\rho, z) \cos h\phi, \qquad U_{n,h}(\rho, z) \sin h\phi,$$

$$V_{n,h}(\rho, z) \cos h\phi, \qquad V_{n,h}(\rho, z) \sin h\phi$$

are harmonic polynomials in x, y, z of degree n.

We shall be concerned here with the new polynomials

$$X_{n,h} = \frac{\partial}{\partial z} U_{n+1,h}$$

$$= \left[\frac{(n+1+h)!}{(n+1-h)!}\right]^{1/2} \frac{1}{\pi i^h} \int_0^\pi P'_{n+1}(z + i\rho\cos t)\cos ht\, dt$$

and

$$Y_{n,h} = \frac{\partial}{\partial z} V_{n+1,h}$$

$$= -\left[\frac{(n+1+h)!}{(n+1-h)!}\right]^{1/2} \frac{i^{n-h}}{\pi} \int_0^\pi P'_{n+1}(iz - \rho\cos t)\cos ht\, dt.$$

The functions

$$X_{n,h}(\rho, z)\cos h\phi, \qquad X_{n,h}(\rho, z)\sin h\phi,$$

$$Y_{n,h}(\rho, z)\cos h\phi, \qquad Y_{n,h}(\rho, z)\sin h\phi$$

are linear combinations of the classical spherical harmonics. The functions $X_{n,0}$ and $Y_{n,0}$ involve only zonal harmonics and satisfy the partial differential equation

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial z^2} = 0$$

of axially symmetric flow.

Let us denote by D either the prolate or the oblate spheroid described above, and let us denote the Dirichlet integral over D by

$$(f, g) = \iiint_D \left\{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right\} dx\, dy\, dz$$

$$= \iint_S f \frac{\partial g}{\partial \nu}\, d\sigma, \qquad (\Delta g = 0),$$

where S is the surface of D, and where ν and $d\sigma$ denote outer normal and area elements on S. Since $z + i\rho = \cos(u - i\ v)$ and $\rho + iz = \sin(u + i\ v)$ are isogonal mappings, we obtain, on the spheroid S,

$$d\sigma \frac{\partial}{\partial \nu} = \rho\, d\phi\, du \frac{\partial}{\partial v}.$$

Hence

$$(3) \qquad (f, g) = \iint_S f \frac{\partial g}{\partial v} \rho\, d\phi\, du = \int_0^\pi \int_0^{2\pi} f \frac{\partial g}{\partial v} \rho\, d\phi\, du.$$

3. **Orthogonality.** If $h \neq k$, we have by the orthogonality of ordinary Fourier series

$$(U_{n,h} \cos h\phi, \quad U_{m,k} \cos k\phi) = 0,$$

$$(U_{n,h} \sin h\phi, \quad U_{m,k} \sin k\phi) = 0,$$

$$(U_{n,h} \cos h\phi, \quad U_{m,k} \sin k\phi) = 0,$$

$$(U_{n,h} \cos h\phi, \quad U_{m,h} \sin h\phi) = 0,$$

and similarly for $V_{n,h}$. For $h = k$ we obtain in the prolate spheroid

$$(U_{n,h} \cos h\phi, U_{m,h} \cos h\phi) = \int_0^\pi \int_0^{2\pi} U_{n,h} \frac{\partial U_{m,h}}{\partial v} (\cos^2 h\phi) \rho\, d\phi\, du$$

$$= \pi(1 + \delta_{0h}) \frac{(n-h)!}{(n+h)!} P_n^h(\operatorname{ch} \alpha) \left[\operatorname{sh} \alpha\, P_m^{h+1}(\operatorname{ch} \alpha) + h \operatorname{ch} \alpha\, P_m^h(\operatorname{ch} \alpha)\right]$$

$$\cdot \int_0^\pi P_n^h(\cos u)\, P_m^h(\cos u) \sin u\, du$$

$$= \frac{2\pi(1 + \delta_{0h})}{2n+1} P_n^h(\operatorname{ch} \alpha) \left[\operatorname{sh} \alpha P_n^{h+1}(\operatorname{ch} \alpha) + h \operatorname{ch} \alpha\, P_n^h(\operatorname{ch} \alpha)\right] \delta_{nm},$$

where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$.

Similarly

$$(U_{n,h} \sin h\phi, \quad U_{m,h} \sin h\phi)$$

$$= \frac{2\pi}{2n+1} P_n^h(\text{ch}\,\alpha) \left[\text{sh}\,\alpha P_n^{h+1}(\text{ch}\,\alpha) + h \,\text{ch}\,\alpha\, P_n^h(\text{ch}\,\alpha)\right] \delta_{nm}.$$

For the oblate spheroid we have in like manner

$$(V_{n,h} \cos h\phi, \quad V_{m,h} \cos h\phi)$$

$$= \pi(1+\delta_{0h}) \frac{(n-h)!}{(n+h)!} i^{n+m-2h} P_n^h(i\,\text{sh}\,\alpha) \left[i\,\text{ch}\,\alpha\, P_m^{h+1}(i\,\text{sh}\,\alpha)\right.$$

$$\left. + h\,\text{sh}\,\alpha\, P_m^h(i\,\text{sh}\,\alpha)\right] \int_0^\pi P_n^h(\cos u) P_m^h(\cos u) \sin u \, du$$

$$= \frac{2\pi(1+\delta_{0h})}{2n+1} (-1)^{n-h} P_n^h(i\,\text{sh}\,\alpha) \left[i\,\text{ch}\,\alpha\, P_n^{h+1}(i\,\text{sh}\,\alpha)\right.$$

$$\left. + h\,\text{sh}\,\alpha\, P_n^h(i\,\text{sh}\,\alpha)\right] \delta_{nm}.$$

Also

$$(V_{n,h} \sin h\phi, \quad V_{m,h} \sin h\phi)$$

$$= \frac{2\pi}{2n+1} (-1)^{n-h} P_n^h(i\,\text{sh}\,\alpha) \left[i\,\text{ch}\,\alpha\, P_n^{h+1}(i\,\text{sh}\,\alpha) + h\,\text{sh}\,\alpha\, P_n^h(i\,\text{sh}\,\alpha)\right] \delta_{nm}.$$

We have therefore proved:

THEOREM 1. *The harmonic polynomials $U_{n,h} \cos h\phi$, $U_{n,h} \sin h\phi$ form a complete orthogonal system for the interior of the prolate spheroid* (1) *in the sense of the Dirichlet integral. The harmonic polynomials $V_{n,h} \cos h\phi$, $V_{n,h} \sin h\phi$ form a similar system inside the oblate spheroid* (2). *The polynomials $U_{n,0}$ and $V_{n,0}$ alone form, respectively, complete orthogonal systems for the equation of axially symmetric flow inside the spheroids* (1) *and* (2).

We turn next to a less obvious result for the polynomials $X_{n,h}$ and $Y_{n,h}$. Let

$$[f, g] = \iiint_D f g \, dx\, dy\, dz.$$

Then clearly, if $h \neq k$,

$$[X_{n,h} \cos h\phi,\ X_{m,k} \cos k\phi] = 0,$$

$$[X_{n,h} \sin h\phi,\ X_{m,k} \sin k\phi] = 0,$$

$$[X_{n,h} \cos h\phi,\ X_{m,k} \sin k\phi] = 0,$$

$$[X_{n,h} \cos h\phi,\ X_{m,h} \sin h\phi] = 0,$$

and similarly for $Y_{n,h}$. Now

$$\frac{\partial}{\partial z} = \frac{\partial u}{\partial z}\frac{\partial}{\partial u} + \frac{\partial v}{\partial z}\frac{\partial}{\partial v}$$

when $z + i\rho = \cos(u - iv)$. Also

$$\frac{\partial u}{\partial z} - i\frac{\partial v}{\partial z} = \frac{d(u-iv)}{d(z+i\rho)} = \frac{d(z-i\rho)}{d(u+iv)}\,\frac{d(u+iv)}{d(z-i\rho)}\,\frac{d(u-iv)}{d(z+i\rho)}$$

$$= -\frac{\partial(u, v)}{\partial(z, \rho)} \sin(u + iv).$$

Therefore

$$[X_{n,h} \cos h\phi, f]$$

$$= -\iiint_D f \cos h\phi \left\{ \frac{\partial U_{n+1,h}}{\partial u} \sin u \operatorname{ch} v - \frac{\partial U_{n+1,h}}{\partial v} \cos u \operatorname{sh} v \right\} \cdot \frac{\partial(u, v)}{\partial(z, \rho)}\, \rho\, d\phi\, d\rho\, dz$$

$$= \left[\frac{(n+1-h)!}{(n+1+h)!}\right]^{1/2} \int_0^{\alpha}\int_0^{\pi}\int_0^{2\pi} f \cos h\phi \sin u \operatorname{sh} v \cdot \Big[P_{n+1}^{h}(\operatorname{ch} v)\, P_{n+1}^{h+1}(\cos u) \sin u \operatorname{ch} v + P_{n+1}^{h}(\cos u)\, P_{n+1}^{h+1}(\operatorname{ch} v) \cos u \operatorname{sh} v \Big]\, d\phi\, du\, dv.$$

The last integral vanishes when f is a harmonic polynomial of the form

$$P_m^h(\cos u)\, P_m^h(\operatorname{ch} v) \cos h\phi$$

with $m < n$, since

$$\int_0^\pi P_{n+1}^{h+1}(\cos u) \sin u\, P_m^h(\cos u) \sin u\, du = 0,$$

$$\int_0^\pi P_{n+1}^h(\cos u) \cos u\, P_m^h(\cos u) \sin u\, du = 0.$$

Hence for $n \neq m$

$$[X_{n,h} \cos h\phi,\ X_{m,h} \cos h\phi] = 0,$$

and similarly

$$[X_{n,h} \sin h\phi,\ X_{m,h} \sin h\phi] = 0.$$

For $m = n$, we have

$$f = X_{n,h} \cos h\phi = \left[\frac{n+1+h}{n+1-h}\right]^{1/2} (2n+1)\, U_{n,h} \cos h\phi + \cdots,$$

where the dots indicate harmonic polynomials of lower degree, which are orthogonal to $X_{n,h} \cos h\phi$. Thus

$$[X_{n,h} \cos h\phi,\ X_{n,h} \cos h\phi]$$

$$= (2n+1)\, \frac{(n-h)!}{(n+h)!} \int_0^\alpha \int_0^\pi \int_0^{2\pi} \cos^2 h\phi \sin u \operatorname{sh} v\, P_n^h(\cos u)\, P_n^h(\operatorname{ch} v)$$

$$\cdot\, [P_{n+1}^{h+1}(\cos u)\, P_{n+1}^h(\operatorname{ch} v) \sin u \operatorname{ch} v$$

$$+\, P_{n+1}^h(\cos u)\, P_{n+1}^{h+1}(\operatorname{ch} v) \cos u \operatorname{sh} v]\, d\phi\, du\, dv$$

$$= \pi(1+\delta_{0h})\, \frac{(n-h)!}{(n+h)!} \int_0^\alpha \int_0^\pi P_{n+1}^{h+1}(\cos u)^2\, P_n^h(\operatorname{ch} v)$$

$$P_{n+1}^h(\operatorname{ch} v) \operatorname{sh} v \operatorname{ch} v \sin u\, du\, dv$$

$$+ \pi(1+\delta_{0h}) \frac{(n-h+1)!}{(n+h)!} \int_0^\alpha \int_0^\pi P_{n+1}^h(\cos u)^2 P_n^h(\operatorname{ch} v)$$

$$P_{n+1}^{h+1}(\operatorname{ch} v) \operatorname{sh}^2 v \sin u \, du \, dv$$

$$= \frac{2\pi(1+\delta_{0h})}{2n+3} (n+1+h) \int_0^\alpha P_n^h(\operatorname{ch} v) \operatorname{sh} v$$

$$\cdot [(n+2+h) P_{n+1}^h(\operatorname{ch} v) \operatorname{ch} v + P_{n+1}^{h+1}(\operatorname{ch} v) \operatorname{sh} v] \, dv.$$

The same value is obtained if we replace $\cos h\phi$ by $\sin h\phi$ throughout, $h > 0$.

For the oblate spheroids, we have, on the other hand,

$$\frac{\partial}{\partial z} = \frac{\partial u}{\partial z}\frac{\partial}{\partial u} + \frac{\partial v}{\partial z}\frac{\partial}{\partial v},$$

with $\rho + iz = \sin(u + iv)$. Hence

$$\frac{\partial u}{\partial z} - i\frac{\partial v}{\partial z} = \frac{d(u-iv)}{d(z+i\rho)} = \frac{d(\rho+iz)}{d(u+iv)}\frac{d(u+iv)}{d(\rho+iz)}\frac{d(u-iv)}{d(z+i\rho)}$$

$$= -i\frac{\partial(u,v)}{\partial(\rho,z)} \cos(u+iv).$$

Therefore

$$[Y_{n,h} \cos h\phi, f]$$

$$= -\iiint_D f \cos h\phi \left\{ \frac{\partial V_{n+1,h}}{\partial u} \sin u \operatorname{sh} v - \frac{\partial V_{n+1,h}}{\partial v} \cos u \operatorname{ch} v \right\}$$

$$\cdot \frac{\partial(u,v)}{\partial(\rho,z)} \rho \, d\phi \, d\rho \, dz$$

$$= i^{n+1-h} \left[\frac{(n+1-h)!}{(n+1+h)!}\right]^{1/2} \int_0^\alpha \int_0^\pi \int_0^{2\pi} f \cos h\phi \sin u \operatorname{ch} v$$

$$\cdot [P_{n+1}^h(i \operatorname{sh} v) P_{n+1}^{h+1}(\cos u) \sin u \operatorname{sh} v$$

$$+ i P_{n+1}^{h}(\cos u) P_{n+1}^{h+1}(i \operatorname{sh} v) \cos u \operatorname{ch} v] \, d\phi \, du \, du.$$

This integral vanishes when f is a harmonic polynomial

$$P_m^h(\cos u) P_m^h(i \operatorname{sh} v) \cos h\phi$$

of degree $m < n$, since

$$\int_0^{\pi} P_{n+1}^{h+1}(\cos u) \sin u \, P_m^h(\cos u) \sin u \, du = 0,$$

$$\int_0^{\pi} P_{n+1}^{h}(\cos u) \cos u \, P_m^h(\cos u) \sin u \, du = 0.$$

Hence, for $n \neq m$,

$$[Y_{n,h} \cos h\phi, \; Y_{m,h} \cos h\phi] = 0,$$

and also

$$[Y_{n,h} \sin h\phi, \; Y_{m,h} \sin h\phi] = 0.$$

For $m = n$, we note that

$$f = Y_{n,h} \cos h\phi = - \left[\frac{n+1+h}{n+1-h} \right]^{1/2} (2n+1) V_{n,h} \cos h\phi + \cdots,$$

where the dots represent harmonic polynomials of lower degree, which are orthogonal to $Y_{n,h} \cos h\phi$. Therefore

$$[Y_{n,h} \cos h\phi, \; Y_{n,h} \cos h\phi]$$

$$= - (2n+1) \, i^{2n-2h+1} \frac{(n-h)!}{(n+h)!} \int_0^{\alpha} \int_0^{\pi} \int_0^{2\pi} \cos^2 h\phi \sin u \operatorname{ch} v$$

$$P_n^h(\cos u) P_n^h(i \operatorname{sh} v) \cdot [P_{n+1}^{h}(i \operatorname{sh} v) P_{n+1}^{h+1}(\cos u) \sin u \operatorname{sh} v$$

$$+ i P_{n+1}^{h}(\cos u) P_{n+1}^{h+1}(i \operatorname{sh} v) \cos u \operatorname{ch} v] \, d\phi \, du \, dv$$

$$= \frac{2\pi(1+\delta_{0h})}{2n+3}\,(-1)^{n-h+1}\, i\,(n+1+h) \int_0^{\alpha} P_n^h(i \operatorname{sh} v) \operatorname{ch} v$$

$$\cdot \left[(n+2+h)\, P_{n+1}^h(i \operatorname{sh} v) \operatorname{sh} v + i\, P_{n+1}^{h+1}(i \operatorname{sh} v) \operatorname{ch} v \right] dv.$$

We obtain the same value if $\cos h\phi$ is replaced by $\sin h\phi$.

This completes the proof of:

THEOREM 2. *The harmonic polynomials* $X_{n,h} \cos h\phi$, $X_{n,h} \sin h\phi$ *form a complete orthogonal system for the interior of the prolate spheroid* (1) *in the sense of the scalar product*

$$[f, g] = \iiint_D f\, g\, dx\, dy\, dz.$$

The corresponding system in the oblate spheroid (2) *is*

$$Y_{n,h} \cos h\phi, \quad Y_{n,h} \sin h\phi.$$

The zonal polynomials $X_{n,0}$ *and* $Y_{n,0}$ *are complete and orthogonal for the equation of axially symmetric flow in their respective domains* (1) *and* (2).

Friedrichs [4] has investigated the eigenvalue problem

$$\frac{[f_z, f_z]}{(f, f)} = \frac{\iiint_D (\partial f/\partial z)^2\, dx\, dy\, dz}{\iiint_D \{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + (\partial f/\partial z)^2\}\, dx\, dy\, dz} = \text{maximum}$$

for harmonic functions f in quite general regions D of space. It is clear from Theorem 1 and Theorem 2 that we have:

THEOREM 3. *The eigenfunctions for the problem*

$$\frac{[f_z, f_z]}{(f, f)} = \text{maximum}, \quad \Delta f = 0,$$

in the prolate spheroid (1) *are*

$$U_{n,h} \cos h\phi, \quad U_{n,h} \sin h\phi,$$

and in the oblate spheroid (2) *they are*

$$V_{n,h} \cos h\phi, \quad V_{n,h} \sin h\phi.$$

The corresponding eigenvalues are

$$(n+1+h)\frac{\int_0^\alpha P_n^h(\mathrm{ch}\, v)\,\mathrm{sh}\, v\,[(n+2+h)P_{n+1}^h(\mathrm{ch}\, v)\,\mathrm{ch}\, v + P_{n+1}^{h+1}(\mathrm{ch}\, v)\,\mathrm{sh}\, v]\,dv}{P_{n+1}^h(\mathrm{ch}\,\alpha)\,[\mathrm{sh}\,\alpha\, P_{n+1}^{h+1}(\mathrm{ch}\,\alpha) + h\,\mathrm{ch}\,\alpha\, P_{n+1}^h(\mathrm{ch}\,\alpha)]}$$

for the prolate spheroids and $(n+1+h)Q$, *where* Q *is the expression*

$$\frac{i\int_0^\alpha P_n^h(i\,\mathrm{sh}\, v)\,\mathrm{ch}\, v[(n+2+h)P_{n+1}^h(i\,\mathrm{sh}\, v)\,\mathrm{sh}\, v + i\,P_{n+1}^{h+1}(i\,\mathrm{sh}\, v)\,\mathrm{ch}\, v]\,dv}{P_{n+1}^h(i\,\mathrm{sh}\,\alpha)\,[i\,\mathrm{ch}\,\alpha\, P_{n+1}^{h+1}(i\,\mathrm{sh}\,\alpha) + h\,\mathrm{sh}\,\alpha\, P_{n+1}^h(i\,\mathrm{sh}\,\alpha)]},$$

for the oblate spheroids.

Friedrichs was led to this extremal problem through his investigation of Korn's inequality and existence theorems for the partial differential equations of elasticity. We shall show in the following how the eigenfunctions can be used to solve the biharmonic equation.

One sees easily from Theorem 3 that

$$U_{n,h} \cos h\phi, \quad U_{n,h} \sin h\phi$$

and

$$V_{n,h} \cos h\phi, \quad V_{n,h} \sin h\phi$$

are also orthogonal in the norm

$$\iiint\limits_D \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2\right\} dx\, dy\, dz = (f, f) - [f_z, f_z].$$

However, we do not go into details since this norm leads to no apparent application.

One can obtain quite interesting results, on the other hand, by using the orthogonality of the $X_{n,h}$ and the $Y_{n,h}$ over the interior of the ellipses (1) and

(2) for all values of α to obtain a corresponding orthogonality of the same polynomials over the surface of the spheroids with respect to a suitable weight function. Indeed, we have

$$\frac{d}{d\alpha}[X_{n,h}\cos h\phi,\ X_{m,k}\cos k\phi]$$

$$=\frac{2\pi(1+\delta_{0h})(n+1+h)\,\delta_{hk}\,\delta_{nm}}{2n+3}\ \frac{d}{d\alpha}\int_0^\alpha P_n^h(\operatorname{ch} v)\operatorname{sh} v$$

$$\cdot\,[(n+2+h)\,P_{n+1}^h(\operatorname{ch} v)\operatorname{ch} v+P_{n+1}^{h+1}(\operatorname{ch} v)\operatorname{sh} v]\,dv,$$

whence

$$\iint_S \{X_{n,h}\cos h\phi\, X_{m,k}\cos k\phi\}\,|1-(z+i\rho)^2|^{1/2}\,d\sigma$$

$$=\frac{2\pi(1+\delta_{0h})(n+h+1)}{2n+3}\ P_n^h(\operatorname{ch}\alpha)\operatorname{sh}\alpha$$

$$\cdot\,[(n+2+h)\,P_{n+1}^h(\operatorname{ch}\alpha)\operatorname{ch}\alpha+P_{n+1}^{h+1}(\operatorname{ch}\alpha)\operatorname{sh}\alpha]\,\delta_{hk}\,\delta_{nm}.$$

Likewise, by the same reasoning,

$$\iint_S \{Y_{n,h}\cos h\phi\, Y_{m,k}\cos k\phi\}\,|1-(\rho+iz)^2|^{1/2}\,d\sigma$$

$$=\frac{2\pi(1+\delta_{0h})(n+h+1)}{2n+3}\ i(-1)^{n-h+1}\,P_n^h(i\operatorname{sh}\alpha)\operatorname{ch}\alpha$$

$$\cdot\,[(n+2+h)\,P_{n+1}^h(i\operatorname{sh}\alpha)\operatorname{sh}\alpha+i\,P_{n+1}^{n+1}(i\operatorname{sh}\alpha)\operatorname{ch}\alpha]\,\delta_{hk}\,\delta_{nm},$$

with exactly the same formulas in both cases if $\cos h\phi$ is replaced by $\sin h\phi$.

This calculation yields:

THEOREM 4. *The polynomials* $X_{n,h}\cos h\phi$, $X_{n,h}\sin h\phi$ *are complete and orthogonal over the surface of the spheroid* (1) *in the sense of the scalar product*

$$\{f, g\} = \iint_S f g \, |1 - (z + i\rho)^2|^{1/2} \, d\sigma$$

with weight function $|1 - (z + i\rho)|^{1/2}$ *equal to the square root of the product of the distances from* (ρ, ϕ, z) *to the points* $(0, 0, 1)$ *and* $(0, 0, -1)$. *The harmonic polynomials* $Y_{n,h} \cos h\phi$, $Y_{n,h} \sin h\phi$ *are complete and orthogonal over the surface of the oblate spheroid* (2) *in the sense of the scalar product*

$$\{f, g\} = \iint_S f g \, |1 - (\rho + iz)^2|^{1/2} \, d\sigma.$$

There exist quite clearly further orthogonality properties of the polynomials $U_{n,h}$ and $V_{n,h}$ which do not depend on the shape of the spheroids (1) and (2). However, we make no pretense here at tabulating all possible orthogonal harmonic polynomials of this type (cf. [8]), but proceed rather to apply the results already obtained to the Laplace and biharmonic equations.

4. The kernels. The Green's function $G(P, Q)$ for the Laplace equation in a region D is a harmonic function of the coordinates x, y, z of the point P in D, except at Q, where

$$G(P, Q) = \frac{1}{r(P, Q)} + \text{harmonic terms},$$

and it vanishes for P on the surface S of D. Here $r(P, Q)$ denotes the distance from P to Q. The Neumann's function $N(P, Q)$ has a similar fundamental singularity,

$$N(P, Q) = \frac{1}{r(P, Q)} + \text{harmonic terms},$$

while its normal derivative is constant on S and

$$\iint_S N(P, Q) \, d\sigma(P) = 0.$$

The harmonic kernel function $K(P, Q)$ is defined by the formula [2]

$$K(P, Q) = \frac{1}{4\pi} \{N(P, Q) - G(P, Q)\}.$$

If $f_n(P)$ is a complete orthonormal system of harmonic functions in D in the sense

$$(f_n, f_m) = \delta_{nm},$$

with

$$\iint_S f\, d\sigma = 0,$$

then one has the Bergman expansion

$$K(P, Q) = \sum_{n=1}^{\infty} f_n(P) f_n(Q).$$

On the other hand, if $g_n(P)$ is a complete orthonormal system of harmonic functions in D in the sense of the scalar product

$$\{f, g\} = \iint_S f g\, \omega\, d\sigma$$

corresponding to an arbitrary positive weight function ω on S, then the kernel

$$H(P, Q) = \sum_{n=1}^{\infty} g_n(P) g_n(Q)$$

is given by [7]

$$H(P, Q) = \frac{1}{(4\pi)^2} \iint_S \frac{1}{\omega(T)} \frac{\partial G(T, P)}{\partial \nu(T)} \frac{\partial G(T, Q)}{\partial \nu(T)}\, d\sigma(T).$$

For P on S we have

$$\omega(P) H(P, Q) = -\frac{1}{4\pi} \frac{\partial G(P, Q)}{\partial \nu(P)}.$$

The Green's function $\Gamma(P, Q)$ of the biharmonic equation

$$\Delta \Delta F = 0$$

is a biharmonic function of the coordinates of P, except at Q, where

$$\Gamma(P, Q) = - r(P, Q) + \text{biharmonic terms},$$

and for P on S it satisfies

$$\Gamma(P, Q) = \frac{\partial \Gamma(P, Q)}{\partial \nu(P)} = 0.$$

If $h_n(P)$ is a complete orthonormal system of harmonic functions in the sense

$$[h_n, h_m] = \delta_{nm},$$

then the kernel function

$$k(P, Q) = \sum_{n=1}^{\infty} h_n(P) h_n(Q)$$

is given by the identity [5, 10]

$$k(P, Q) = -\frac{1}{8\pi} \Delta(P) \Delta(Q) \Gamma(P, Q).$$

The relation here between the harmonic functions h_n and the biharmonic kernel function k is a consequence of the nature of the energy integral

$$\iiint_D (\Delta F)^2 \, dx \, dy \, dz$$

for the biharmonic equation.

We discuss here the expansion of the kernels K, H, and k in terms of the orthogonal polynomials of § 3 for the case where D is a prolate or oblate spheroid. One obtains easily from Theorems 1, 2, and 4, together with the computation of the related normalization constants, the following results:

THEOREM 5. *In the prolate spheroid* (1) *we have*

$K(\rho, z, \phi; \rho', z', \phi')$

$$= \sum_{n=1}^{\infty} \sum_{h=0}^{n} \frac{(2n+1) U_{n,h}(\rho, z) U_{n,h}(\rho', z') \cos h(\phi - \phi')}{2\pi(1+\delta_{0h}) P_n^h(\operatorname{ch}\alpha)[\operatorname{sh}\alpha\, P_n^{h+1}(\operatorname{ch}\alpha) + h \operatorname{ch}\alpha\, P_n^h(\operatorname{ch}\alpha)]} + C,$$

where C is a constant chosen to agree with the normalization of Neumann's function. In the oblate spheroid (2),

$$K(\rho, z, \phi; \rho', z', \phi') = \sum_{n=1}^{\infty} \sum_{h=0}^{n} \frac{(-1)^{n-h}(2n+1)}{2\pi(1+\delta_{0h})} \frac{V_{n,h}(\rho, z)\, V_{n,h}(\rho', z') \cos h(\phi - \phi')}{P_n^h(i \operatorname{sh} \alpha)[i \operatorname{ch} \alpha\, P_n^{h+1}(i \operatorname{sh} \alpha) + h \operatorname{sh} \alpha\, P_n^h(i \operatorname{sh} \alpha)]} + C,$$

where again C is a suitable constant.

THEOREM 6. *In the prolate spheroid* (1),

$$k(\rho, z, \phi; \rho', z', \phi') = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(2n+3)}{2\pi(1+\delta_{0h})(n+1+h)} \frac{X_{n,h}(\rho, z) X_{n,h}(\rho', z') \cos h(\phi - \phi')}{\int_0^\alpha P_n^h(\operatorname{ch} v) \operatorname{sh} v[(n+2+h) P_{n+1}^h(\operatorname{ch} v) \operatorname{ch} v + P_{n+1}^{h+1}(\operatorname{ch} v) \operatorname{sh} v]\, dv}.$$

In the oblate spheroid (2),

$$k(\rho, z, \phi; \rho', z', \phi') = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(-1)^{n-h}\, i\,(2n+3)}{2\pi(1+\delta_{0h})(n+1+h)} \frac{Y_{n,h}(\rho, z)\, Y_{n,h}(\rho', z') \cos h(\phi - \phi')}{\int_0^\alpha P_n^h(i \operatorname{sh} v) \operatorname{ch} v[(n+2+h) P_{n+1}^h(i \operatorname{sh} v) \operatorname{sh} v + i P_{n+1}^{h+1}(i \operatorname{sh} v) \operatorname{ch} v]\, dv}.$$

THEOREM 7. *In the prolate spheroid* (1),

$$H(\rho, z, \phi; \rho', z', \phi') = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(2n+3)}{2\pi(1+\delta_{0h})(n+1+h)} \frac{X_{n,h}(\rho, z)\, X_{n,h}(\rho', z') \cos h(\phi - \phi')}{P_n^h(\operatorname{ch} \alpha) \operatorname{sh} \alpha\, [(n+2+h) P_{n+1}^h(\operatorname{ch} \alpha) \operatorname{ch} \alpha + P_{n+1}^{h+1}(\operatorname{ch} \alpha) \operatorname{sh} \alpha]},$$

when $\omega = |1 - (z + i\rho)^2|^{1/2}$. *If* $\omega = |1 - (\rho + iz)^2|^{1/2}$, *we have, for the oblate spheroid* (2),

$$H(\rho, z, \phi; \rho', z', \phi') = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(2n+3)i(-1)^{n-h}}{2\pi(1+\delta_{0h})(n+1+h)}$$

$$\frac{Y_{n,h}(\rho, z)\, Y_{n,h}(\rho', z') \cos h(\phi - \phi')}{P_n^h(i \operatorname{sh}\alpha)\operatorname{ch}\alpha[(n+2+h)P_{n+1}^h(i\operatorname{sh}\alpha)\operatorname{sh}\alpha + iP_{n+1}^{h+1}(i\operatorname{sh}\alpha)\operatorname{ch}\alpha]}.$$

Theorem 7 is of interest because it yields, say for (1), the relation

$$-\frac{1}{4\pi}\frac{\partial G(\rho, z, \phi; \rho', z', \phi')}{\partial \nu} = |1-(z+i\rho)^2|^{1/2} \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(2n+3)}{2\pi(1+\delta_{0h})(n+1+h)}$$

$$\frac{X_{n,h}(\rho, z)\, X_{n,h}(\rho', z') \cos h(\phi - \phi')}{P_n^h(\operatorname{ch}\alpha)\operatorname{sh}\alpha[(n+2+h)P_{n+1}^h(\operatorname{ch}\alpha)\operatorname{ch}\alpha + P_{n+1}^{h+1}(\operatorname{ch}\alpha)\operatorname{sh}\alpha]}$$

when the point ρ, z, ϕ lies on S. This formula can be compared with the corresponding, more classical, formula which follows from Theorem 5.

Theorem 6 permits one to calculate the biharmonic Green's function for prolate or oblate spheroids, and thus in turn to solve the biharmonic boundary-value problem in this case. Indeed, we have (cf. [5])

$$\Gamma(P, Q) = \frac{1}{2\pi}\iiint_D \frac{d\sigma(T)}{r(T, P)r(T, Q)} - \frac{1}{2\pi}\iiint_D \iiint_D \frac{k(T, R)d\sigma(T)d\sigma(R)}{r(T, P)r(R, Q)}.$$

It is significant to note in this connection that all our results can be extended to the case of the region outside a spheroid. One has merely to replace for this purpose the Legendre functions P_n^h by the Legendre functions Q_n^h of second kind [**6**]. Thus $U_{n,h}$ should be replaced, for example, by

$$\int_0^{\pi} Q_n(z + i\rho \cos t)\cos ht\, dt,$$

and $V_{n,h}$ should be replaced by

$$\int_0^{\pi} Q_n(iz - \rho\cos t)\cos ht\, dt.$$

Finally, by combining both kinds of functions, one can obtain orthonormal systems in the region between two confocal spheroids. Thus one might develop elaborate formulas for the solution of the biharmonic equation in such shell regions using the basic method of this paper.

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DISTRIBUTION OF ROUND-OFF ERRORS FOR RUNNING AVERAGES

R. E. Greenwood and A. M. Gleason

1. Statement of the problem. Let $G_1, G_2, \cdots$ be scores (positive integers) obtained in a sequence of plays in a certain game. For purposes of handicapping matches it is desired to use running averages, and on the hypothesis that the score of the last play is more significant than any prior score, the following formula is used for computing the running averages $\{S_n\}$:

$$S_{n+1} = \frac{(k-1)\,S_n + G_{n+1}}{k} \tag{1.1}$$

where k is a positive integer. Certain modifications in (1.1) may be necessary when $n < k$.

The running averages defined by (1.1) are not necessarily integers. It is therefore convenient to define a rounded running average (which will be integral) by the relation

$$T_{n+1} = \frac{(k-1)\,T_n + G_{n+1} + D}{k}. \tag{1.2}$$

It is convenient to use three set of values for D in the foregoing relation.

Case A. For k odd, $D = A \in \left\{\frac{-k+1}{2}, \frac{-k+3}{2}, \cdots, \frac{k-1}{2}\right\}$.

Case B. For k even, $D = B \in \left\{\frac{-k}{2} + 1, \frac{-k}{2} + 2, \cdots, \frac{k}{2}\right\}$.

Case C. For k even, $D = C \in \left\{\frac{-k}{2}, \frac{-k}{2} + 1, \cdots, \frac{k}{2}\right\}$.

For each $n \geq k$ define the error E_n by the relation

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$$E_n = T_n - S_n. \tag{1.3}$$

(For $n < k$, the error would depend on the modifications made in relation (1.1).) For $n \geq k$, then,

$$E_{n+1} = T_{n+1} - S_{n+1} = \frac{(k-1)\,E_n + D}{k}. \tag{1.4}$$

For Case A, if at some stage $|E_n| \leq (k-1)/2$, then

$$|E_{n+1}| \leq \frac{(k-1)\,(k-1)/2 + (k-1)/2}{k} = \frac{k-1}{2}. \tag{1.5}$$

For cases B and C, if at some stage $|E_n| \leq k/2$, then by a similar procedure one obtains

$$|E_{n+1}| \leq k/2. \tag{1.6}$$

Thus the errors introduced by the rounding off process are bounded if $|E_k| \leq (k-1)/2$ or $k/2$ for the odd and even values of k respectively.

It is assumed that the scores $\{G_i\}$ are such that equal probability values are realistic. In case C, where there will sometimes be a choice for round-off, one might choose to round-off to the even integer. Thus, one would sometimes add $k/2$ and sometimes subtract $k/2$, corresponding to the two end-values with probabilities $1/(2k)$, while the intermediate values would have probabilities $1/k$. It is desired to find a limiting distribution for the error E_n; in this paper such limiting distributions are found for a few special cases.

Allowing one's intuition free rein, one sees that limiting distributions for the error E_n exist in all three cases. If such distributions exist, then relation (1.4) may be used to determine means and variances, if any. Thus

$$k\mu(E_{n+1}) = (k-1)\,\mu(E_n) + \mu(D), \tag{1.7}$$

$$k^2 \operatorname{Var}(E_{n+1}) = (k-1)^2 \operatorname{Var}(E_n) + \operatorname{Var}(D). \tag{1.8}$$

It is easy to verify that

$$\begin{aligned} \mu(A) &= 0, & \operatorname{Var}(A) &= (k^2-1)/12, \\ \mu(B) &= 1/2, & \operatorname{Var}(B) &= (k^2-1)/12, \\ \mu(C) &= 0, & \operatorname{Var}(C) &= (k^2+2)/12. \end{aligned}$$

Then for the limiting distributions E_A, E_B, E_C for the three cases one gets

$$\begin{aligned} &\mu(E_A) = 0, && \mathrm{Var}\,(E_A) = (k^2-1)/12(2k-1), \\ &\mu(E_B) = 1/2, && \mathrm{Var}\,(E_B) = (k^2-1)/12(2k-1), \\ &\mu(E_C) = 0, && \mathrm{Var}\,(E_C) = (k^2+2)/12(2k-1), \end{aligned} \tag{1.10}$$

2. Distribution of the round-off error for $k = 2$, Case B. For the special value $k = 2$ and for Case B, one may take $E_1 \equiv 0$. Let $F_n(x)$ be the cumulative distribution for E_n, and let $\{f_{i,n}\}$ be the jumps in $F_n(x)$ at the points of discontinuity. One readily obtains the functions

$$F_2(x) = \begin{cases} 0, & x < 0, \\ 1/2, & 0 \le x < 1/2 \\ 1, & 1/2 \le x. \end{cases} \tag{2.1}$$

$$\{f_{i,2}\} = \{1/2 \text{ at } 0,\ 1/2 \text{ at } 1/2\}.$$

$$F_3(x) = \begin{cases} 0, & x < 0, \\ j/4, & (j-1)/4 \le x < j/4 \qquad (j = 1, 2, 3), \\ 1, & 3/4 \le x. \end{cases} \tag{2.2}$$

$$\{f_{i,3}\} = \{1/4 \text{ at } 0,\ 1/4 \text{ at } 1/4,\ 1/4 \text{ at } 1/2,\ 1/4 \text{ at } 3/4\}.$$

By induction one gets

$$F_{n+1}(x) = \begin{cases} 0, & x < 0 \\ j/2^n, & (j-1)/2^n \le x < j/2^n \qquad (j = 1, \cdots, 2^n - 1), \\ 1, & (2^n - 1)/2^n \le x. \end{cases} \tag{2.3}$$

$$\{f_{i,n+1}\} = \{\text{jumps of } 1/2^n \text{ at points } j/2^n,\ j = 0, 1, \cdots, 2^n - 1\}.$$

In this simple example, heuristic considerations suggest that there is a limiting cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x < 1, \\ 1, & 1 \le x, \end{cases} \tag{2.4}$$

and its associated distribution function

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0 & \text{elsewhere}. \end{cases} \tag{2.5}$$

In order to deal with continuous functions insofar as possible, it is convenient to take Fourier transforms of the jumps $\{f_{j,n}\}$. The finite Fourier transform may be defined by relations

$$\phi_n(u) = \int_{-\infty}^{\infty} e^{iut}\, d\, F_n(t) \tag{2.6}$$

$$= \sum_{\text{all } j} f_{j,n} \exp(iuj).$$

Thus we get

$$\phi_2(u) = 1/2 + (1/2)\exp(iu/2) \tag{2.7}$$

$$= \exp(iu/4)\cos(u/4) = \frac{1}{2}\exp(iu/4)\,\frac{\sin(u/2)}{\sin(u/4)},$$

$$\phi_{n+1}(u) = \frac{1}{2^n}\sum_{j=0}^{j=2^n-1}\exp\left(\frac{iju}{2^n}\right) \tag{2.8}$$

$$= \frac{1}{2^n}\,\frac{1-\exp(iu)}{1-\exp(iu/2^n)}$$

$$= \frac{1}{2^n}\,\frac{\sin(u/2)}{\sin(u/2^{n+1})}\exp\left(iu\frac{2^n-1}{2^{n+1}}\right).$$

The sequence of transforms $\{\phi_n\}$ has a limit $\phi(u)$,

$$\phi(u) = \frac{\sin u/2}{u/2}\exp(iu/2). \tag{2.9}$$

In order to transform back, it is convenient to use another definition of the Fourier transform,

$$\phi(u) = \int_{-\infty}^{\infty} e^{iut} f(t)\, dt. \tag{2.10}$$

Then, whenever $f(x)$ is of class $L_2(-\infty, \infty)$ and of bounded variation in the neighborhood of t [1, p. 83, Theorem 58],

$$\frac{1}{2}[f(t+0)+f(t-0)] = \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{-\lambda}^{\lambda} e^{-iut}\phi(u)\, du. \tag{2.11}$$

Direct computation of the inverse transform (using 2.11) of $\phi(u)$ as defined by (2.9) might be troublesome. However, the Fourier transform (2.10) of the supposed limiting distribution function of (2.5)

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0 & \text{elsewhere} \end{cases}$$

is just the limiting function $\phi(u)$ as given by (2.9). Since $f(x)$ is of class $L_2(-\infty, \infty)$ and is of bounded variation, the theorem quoted above enables one to identify (2.5) as the limiting distribution function of the error for Case B, except for the values $f(0)$ and $f(1)$ where f should be chosen as 1/2.

The use of the Fourier transform $\phi_n(u)$, (as defined by (2.6)), is equivalent to the use of the characteristic functions of the jump distributions $\{f_{j,n}\}$. With this interpretation, it is possible to use Lévy's theorem [2, p. 101-102] to the effect that convergence of $\phi_n(u)$ to $\phi(u)$ implies the convergence of $F_n(x)$ to the limiting form $F(x)$ given by (2.4) and that $\phi(u)$ is the characteristic function of the cumulative distribution function $F(x)$.

The mean and variance of $f(x)$ as given by (2.5) (with or without modifications at 0 and 1) are 1/2 and 1/12 respectively, and thus agree with the values called for by relations (1.10).

3. Distribution of round-off errors for $k = 2$, Case C. Case C has symmetry noticeably lacking in Case B. For convenience, take $E_1 \equiv 0$ as before. Let $G_n(x)$ and $\{g_{j,n}\}$ be the cumulative and point-wise distribution functions. For this case

$$G_2(x) = \begin{cases} 0, & x < -1/2, \\ 1/4, & -1/2 \leq x < 0, \\ 3/4, & 0 \leq x < 1/2, \\ 1, & 1/2 \leq x, \end{cases} \tag{3.1}$$

$$\{g_{j,2}\} = \{1/4 \text{ at } -1/2,\ 1/2 \text{ at } 0,\ 1/4 \text{ at } 1/2\}.$$

Designate the finite Fourier transform (2.6) by $\psi_2(u)$. Then

$$\begin{aligned} \psi_2(u) &= (1/4)\exp(-iu/2) + 1/2 + (1/4)\exp(iu/2) \\ &= (1/4)[\exp(iu/4) + \exp(-iu/4)]^2 = \cos^2(u/4). \end{aligned} \tag{3.2}$$

This may be written in the form

$$\psi_2(u) = (1/4)[x + 1/x]^2 \quad \text{where} \quad x = \exp(iu/4). \tag{3.3}$$

Notice that to get $\{g_{j,3}\}$ from $\{g_{j,2}\}$ and the set $\{C\}$, $\{C\} = \{-1, 0, 1\}$ with probabilities $\{1/4, 1/2, 1/4\}$ respectively, one merely takes 1/4 of the set $\{g_{j,2}\}$ on a smaller range at one end of the new range, 1/2 of the set $\{g_{j,2}\}$ on a smaller range at the middle, and 1/4 of the set $\{g_{j,2}\}$ on a smaller range at the other end of the new range. In effect, one goes from $\psi_2(u)$ to $\psi_3(u)$ by replacing x by x^2, multiplying by

$$[(1/4)x^2 + 1/2 + 1/(4x^2)] = (1/4)[x + 1/x]^2.$$

and then identifying $x = \exp(iu/8)$.

By this rule, one gets

$$\psi_3(u) = (1/4)^2 (x + 1/x)^2 (x^2 + 1/x^2)^2 = \cos^2(u/4)\cos^2(u/8). \tag{3.4}$$

Proceeding by induction, one gets

$$\psi_{n+1}(u) = \cos^2(u/4)\cos^2(u/8)\cdots\cos^2(u/2^{n+1}). \tag{3.5}$$

The sequence of transforms $\{\psi_n(u)\}$ has a limit,

$$\psi(u) = \lim_{n\to\infty} \psi_n(u) = \frac{\sin^2(u/2)}{(u/2)^2}, \tag{3.6}$$

by use of a well-known infinite product.

Direct computation of the inverse transform of (3.6) may be troublesome. However, it may be verified quite readily that if

$$g(x) = \begin{cases} 1 + x, & -1 < x < 0, \\ 1 - x, & 0 \le x < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.7}$$

the Fourier transform of $g(x)$ is just $\psi(u)$ of (3.6). Then, by use of (2.11), it follows that $g(x)$ as defined above may be taken as the pointwise distribution function for the limiting distribution E_C.

Direct computations show that

$$\mu(E_C) = 0, \quad \mathrm{Var}(E_C) = 1/6,$$

which values are in agreement with relations (1.10).

4. Conclusion. For higher values of k, the limits of the Fourier transforms may be difficult to obtain.

A somewhat more general problem would be to take

$$S_{n+1} = \frac{(k-m)\, S_n + mG_{n+1}}{k} \tag{4.1}$$

instead of (1.1), where k and m are both positive integers. In effect, however, this merely allows the k in (1.1) to be a positive rational number instead of a positive integer.

An equivalent statement of the problem would be to consider the distribution of $M(d)$, where

$$M(d) = \frac{1}{k} \sum_{i=0}^{\infty} d_i \left(\frac{k-1}{k}\right)^i, \tag{4.2}$$

and where $\{d_i\}$ is selected from the set D according to the value of k and the end-point choice. For the expansion of $M(d)$ is

$$M(d) = (1/k)\{d_0 + (k-1)/k\{d_1 + (k-1)/k\{d_2 + \cdots\}\}\}, \tag{4.3}$$

and this is just the scoring used in (1.4) but with reversed numerical ordering. Thus for $k = 2$ and Case B, M is uniformly distributed on $(0, 1)$, while for Case C, M has a house-top distribution on $(-1, 1)$.

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THE SPACE H^p, $0 < p < 1$, IS NOT NORMABLE

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1. Introduction. For $p > 0$, the space H^p is defined to be the class of functions $x(z)$ of the complex variable z, which are analytic in the interior of the unit circle, and satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta < \infty.$$

Set

$$A_p(r; x) = \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta\right)^{1/p}$$

and

$$||x|| = \sup_{0 \leq r < 1} A_p(r; x).$$

S. S. Walters has shown [2] that H^p, $0 < p < 1$, is a linear topological space under the topology: $U \subset H^p$ is open if $x_0 \in U$ implies the existence of a "sphere" S: $||x - x_0|| < r$ such that $S \subset U$. He conjectured in [3] that H^p, $0 < p < 1$, does not have an equivalent normed topology, and it is shown here that this conjecture is correct. Since the conjugate space $(H^p)^*$ has sufficiently many members to distinguish elements of H^p, the space H^p, $0 < p < 1$, affords an interesting nontrivial example of a locally bounded linear topological space which is not locally convex.

2. *Proof.* For $x \in H^p$, $p > 0$, it is known [4, 160] that $A_p(r; x)$ is a nondecreasing function of r. Consequently, if $P(z)$ is a polynomial, then $P \in H^p$ and $||P|| = A_p(1; P)$. This observation will be used below.

According to a theorem of Kolmogoroff [1], a linear topological space has an equivalent normed topology if and only if the space contains a bounded open convex set. It will be shown here that the "sphere" K_1: $||x|| < 1$ of H^p,

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$0 < p < 1$, contains no convex neighborhood of the origin; this is clearly sufficient to show that H^p, $0 < p < 1$, contains no bounded open convex set, and hence is not normable.

To accomplish this, the contrary is assumed. Thus, it is assumed that K_1 contains a convex neighborhood V of the origin. Since V is open, V contains a "sphere" K_ϵ: $\|x\| < \epsilon$. There will be exhibited $x_1, \cdots, x_N \in K_\epsilon$, and $a_1 > 0, \cdots, a_N > 0$, with $\sum a_k = 1$, such that $\sum a_k x_k \notin K_1$ and, *a fortiori*, $\sum a_k x_k \notin V$, in contradiction to the assumed convexity of V.

If $x(\theta)$ is a complex function of the real variable $\theta \in I$: $0 \leq \theta \leq 2\pi$, define

$$A(x) = \left(\frac{1}{2\pi}\int_0^{2\pi} |x(\theta)|^p \, d\theta\right)^{1/p}.$$

Once and for all, k is any integer in the range $1, \cdots, N$. Let I_k denote the interval

$$\frac{2\pi(k-1)}{N} < \theta < \frac{2k\pi}{N},$$

and let i_k denote the degenerate interval consisting of the point $(2\pi/N)(k-1/2)$. Define the continuous function $c_k(\theta)$ to be zero on $I - I_k$, to be equal to $\epsilon N^{1/p}$ on i_k, and to be linear on each of the two intervals in $I_k - i_k$. Let

$$a_k = B_N k^{-1/p}, \; B_N = \left(\sum_1^N k^{-1/p}\right)^{-1},$$

so that $a_k > 0$ and $\sum a_k = 1$. It is easily verified that

$$A(c_k) = \epsilon(p+1)^{-1/p} < \epsilon$$

and

$$A(\sum a_k c_k) = \epsilon B_N (p+1)^{-1/p} \left(\sum_1^N k^{-1}\right)^{1/p}.$$

Since B_N is bounded away from zero below, N can be chosen such that

$$A(\sum a_k c_k) > 1.$$

Each $c_k(\theta)$ is absolutely continuous on I. Given $\alpha > 0$, it follows that there is a trigonometrical polynomial

$$T_k(\theta) = \sum_{n=-m_k}^{m_k} a_{nk}\, e^{in\theta}$$

such that

$$|T_k(\theta) - c_k(\theta)| < \alpha$$

uniformly in θ. Setting

$$p_k(\theta) = e^{im_k\theta}\, T_k(\theta)$$

gives

$$|p_k(\theta) - e^{im_k\theta}\, c_k(\theta)| < \alpha$$

uniformly in θ. Set

$$C_k(\theta) = e^{im_k\theta}\, c_k(\theta).$$

It is clear that $A(C_k) = A(c_k)$ and $A(\sum a_k\, C_k) = A(\sum a_k\, c_k)$. Since $A(x)$ is a continuous function of x, it follows, if α is small enough, that $A(p_k) < \epsilon$ and $A(\sum a_k\, p_k) > 1$.

Let

$$P_k(z) = \sum_{n=-m_k}^{m_k} a_{nk}\, z^{n+m_k},$$

so that $P_k(e^{i\theta}) = p_k(\theta)$. As previously remarked,

$$||P_k|| = A_p(1; P_k) = A(p_k)$$

and

$$||\sum a_k\, P_k|| = A_p(1; \sum a_k\, P_k) = A(\sum a_k\, p_k).$$

Since $P_1, \cdots, P_N \in K_\epsilon \subset V \subset K_1 \subset H^p$, $a_1 > 0, \cdots, a_N > 0$, $\sum a_k = 1$, and $\sum a_k\, P_k \notin K_1$, we have obtained the required contradiction.

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ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET OF POLYNOMIALS

M. N. MIKHAIL

1. Introduction. For the general terminology used in this paper the reader is referred to J. M. Whittaker [2], [3]. Let

$$p_n(z) = \sum_i p_{ni}\, z^i$$

be a basic set, and let

$$z^n = \sum_{i=0}^{D_n} \pi_{ni}\, p_i(z).$$

The order ω and type γ of $\{p_n(z)\}$ are defined as follows. Let $M_i(R)$ be the maximum modulus of $p_i(z)$ in $|z| \le R$. Let

$$\omega_n(R) = \sum_i |\pi_{ni}|\, M_i(R), \tag{1}$$

$$\omega(R) = \limsup_{n \to \infty} \frac{\log \omega_n(R)}{n \log n}, \tag{2}$$

$$\omega = \lim_{R \to \infty} \omega(R); \tag{3}$$

and, for $0 < \omega < \infty$, let

$$\gamma(R) = \limsup_{n \to \infty} \{\omega_n(R)\}^{1/(n\omega)}\, e/(n\omega), \tag{4}$$

$$\gamma = \lim_{R \to \infty} \gamma(R). \tag{5}$$

If

$$P_n(z) = \sum_i \pi_{ni}\, z^i,$$

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then $\{P_n(z)\}$ is called the reciprocal set of $\{p_n(z)\}$. We shall establish for certain basic sets new formulas expressing upper bounds of the order of the reciprocal set in terms of the data of the original set.

2. **Theorem.** The following theorem holds only if an infinity of $\pi_{nn} \neq 0$; then the whole proof should be carried out for those values of n for which $\pi_{nn} \neq 0$. This is a genuine restriction since there are basic sets such that $\pi_{nn} = 0$ for all n; for example, for $h = 0, 1, 2, \cdots$, let

$$p_{3h}(z) = -\frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+1}(z) = \frac{1}{2} z^{3h} - \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+2}(z) = \frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} - \frac{1}{2} z^{3h+2}.$$

NOTATION. For a fixed n, let $p_{nh'}$ be the set of all nonzero elements p_{nh}, and let

$$\min_{h'} p_{nh'} = p_{n'}.$$

THEOREM 1. *Let $\{p_n(z)\}$ be a basic set of polynomials, such that*

$$\limsup_{n\to\infty} \frac{D_n}{n} = a \qquad (a \geq 1),$$

and of increase less than order ω and type γ, and suppose that

$$\kappa = \liminf_{n\to\infty} \frac{\log|\pi_{nn}|}{n \log n}$$

and

$$k = \liminf_{n\to\infty} \frac{\log|p_{n'}|}{n \log n}.$$

Then its reciprocal set is of order Ω, where

i) *if $k > \omega$, then $\Omega \leq \omega - \kappa$;*

ii) *if* $k \leq \omega$, *then* $\Omega \leq 2\omega - \kappa - k$.

Proof. Let $\gamma_1 > \gamma$; then in view of (4) we have

$$\omega_n(R) \leq \left(\frac{n\omega\gamma_1}{e}\right)^{n\omega} \tag{6}$$

for values of $n > n_0$ and for sufficiently large values of $R > R_0 > 1$. From (1), we have

$$|\pi_{nn}|\, M_n(R) \leq \omega_n(R).$$

Then

$$|\pi_{nn}|\, |p_{ni}|\, R^i \leq \omega_n(R);$$

that is

$$|p_{ni}| \leq \frac{\omega_n(R)}{|\pi_{nn}|}. \tag{7}$$

Also

$$|\pi_{ij}|\, M_j(R) \leq \omega_i(R);$$

then

$$|\pi_{ij}| \leq \frac{\omega_i(R)}{M_j(R)} \leq \frac{\omega_i(R)}{\min\limits_{h'} |p_{ih'}|} = \frac{\omega_i(R)}{|p_{i'}|}. \tag{8}$$

From the definition of a reciprocal set, and in view of (1), we get

$$\Omega_n(R) \leq \sum_{i=0}^{D_n} |p_{ni}| \sum_j |\pi_{ij}|\, R^j \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} \sum_j |\pi_{ij}|$$

by (7); that is, by (8),

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} N_i \frac{\omega_i(R)}{|p_{i'}|}.$$

Then

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{\omega_i(R)}{|p_i'|} \right\}$$

$$\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{(i\omega\gamma_1)^{i\omega}}{|p_i'|} \right\} \quad \text{by (6),}$$

where $F(R)$ is a function independent of n.

Then for sufficiently large values of $n > n_0$ and $R > R_0$, we get

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + D_n \left(\frac{n\omega\gamma_1}{n^{k_1/\omega}} \right)^{n\omega} \right\} \quad (\text{where } k_1 \geq k).$$

Hence:

i) If $k > \omega$ (this implies $k_1 > \omega$), then $(n\omega\gamma_1/n^{k_1/\omega})^{n\omega}$, for values of $n > n_0$, will be a small quantity compared to $F(R)$. Therefore,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \Omega_n(R)}{n \log n}$$

$$\leq \lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \frac{\log \omega_n(R)}{n\log n} + \frac{D_n \log R}{n \log n} - \frac{\log|\pi_{nn}|}{n\log n} + \frac{\log D_n}{n\log n} + \frac{\log F(R)}{n\log n} \right\},$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa.$$

ii) If $k \leq \omega$, then as k_1 approaches k we find that $F(R)$ will be very small compared to $\{n\omega\gamma_1/n^{k_1/\omega}\}^{n\omega}$ for $n > n_0$. Therefore,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \omega_n(R)}{n\log n} \leq \lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \frac{\log \omega_n(R)}{n\log n} - \frac{\log|\pi_{nn}|}{n\log n} \right.$$

$$\left. + \frac{D_n \log R + 2\log D_n}{n\log n} + \frac{n\omega\left(1 - \frac{k_1}{\omega}\right)\log n}{n\log n} + \frac{n\omega \log \omega\gamma_1}{n\log n} \right\}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa + \omega - k = 2\omega - \kappa - k.$$

N. B. *In the case of simple sets, the restriction mentioned above for π_{nn} is satisfied. In this case we have*

$$-\kappa = \limsup_{n\to\infty} \frac{\log |p_{nn}|}{n \log n}.$$

COROLLARY. *If $\{p_n(z)\}$ is a simple set of polynomials,*

$$\left.\begin{array}{ll} \text{i)} & \text{if } k > \omega, \text{ then } \Omega \leq \omega - \kappa \\ \text{ii)} & \text{if } k \leq \omega, \text{ then } \Omega \leq 2\omega - \kappa - k \end{array}\right\} \quad \text{where } \kappa = -\limsup_{n\to\infty} \frac{\log |p_{nn}|}{n \log n}.$$

3. Examples. We shall look at four examples.

i) Let $p_n(z) = n^{3n} z^n - n^{2n} z^{n-1} - n^{3n} z^{n+1}$ (n odd),

$p_n(z) = n^{2n} z^n - n^{3n}$ (n even),

$p_0(z) = 1.$

then

$$z^n = n^{-3n} p_n(z) + n^{-n} (n-1)^{-2(n-1)} p_{n-1}(z) + (n+1)^{-2(n+1)} p_{n+1}(z)$$
$$+ \{(n-1)^{(n-1)} n^{-n} + (n+1)^{(n+1)}\} p_0(z) \quad (n \text{ odd}),$$

$$z^n = n^{-2n} p_n(z) + n^n p_0(z) \quad (n \text{ even}).$$

By Theorem (1) of [1], we get $\omega = 1$. Since $\kappa = -3$, $k = 2$, we get, according to case i) of the theorem, $\Omega \leq 1 + 3 = 4$. This is true because $\Omega = 4$ by Corollary (1.1) of [1].

N. B. *This example and the following examples show that the values given in the conclusion of the above theorem are "best possible."*

ii) Let $p_n(z) = n^{2n} z^n - n^{3n/2} z^{2n} - n^{2n}$ (n odd),

$p_n(z) = \left(\frac{n}{2}\right)^{3n/2} z^n - \left(\frac{n}{2}\right)^{2n}$, with $p_0(z) = 1$ (n even), Then

$$z^n = n^{-2n}\, p_n(z) + n^{-7n/2}\, p_{2n}(z) + (1 + n^{n/2})\, p_0(z) \qquad (n \text{ odd}),$$

$$z^n = \left(\frac{n}{2}\right)^{-3n/2} p_n(z) + \left(\frac{n}{2}\right)^{n/2} p_0(z) \qquad (n \text{ even}),$$

Applying theorem (1) of [1], we get $\omega = 1/2$. Now $\kappa = -2$, $k = 3/2$. Then according to case i), of the theorem, we get

$$\Omega \leq \frac{1}{2} + 2.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

iii) Let $p_n(z) = n^n z^n - n^{n/2} z^{n-1} - n^{3n/2}$ $\qquad (n \text{ odd}),$

$$p_n(z) = (n+1)^{(n+1)} z^n - (n+1)^{2(n+1)} z^{(n+1)} - (n+1)^{5(n+1)/2} \qquad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = \frac{1}{1 - n^{n/2}} \left\{ n^{-n}\, p_n(z) + n^{-3n/2}\, p_{n-1}(z) + (n^{n/2} + n^n)\, p_0(z) \right\} \qquad (n \text{ odd}),$$

$$z^n = \frac{1}{1 - (n+1)^{(n+1)/2}} \left\{ (n+1)^{-(n+1)}\, p_n(z) + p_{n+1}(z) + 2(n+1)^{3(n+1)/2}\, p_0(z) \right\} \qquad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = -1$, $k = 1/2$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 + 1 - \frac{1}{2} = \frac{5}{2}.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

iv) Let $$p_n(z) = \frac{2^{(n-1)}}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}\, z^n + \frac{2^{(n-1)}\, n^n}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}$$

$$+ \frac{2^{2(n-1)}\,(n-1)^{(n-1)}}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}\, z^{n-1} \qquad (n \text{ odd}),$$

$$p_n(z) = \frac{2^{2n}\,(n+1)^{2(n+1)}}{2^n\, n^n\,(n+1)^{2(n+1)} + n^{4n}}\, z^n - \frac{n^n}{2^n\,(n+1)^{2(n+1)} + n^{3n}}\, z^{n+1}$$

$$- \frac{n^n (n+1)^{(n+1)}}{2^n (n+1)^{2(n+1)} + n^{3n}}\, z^{2n+2} \qquad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = n^{2n}\, p_n(z) - n^{3n}\, p_{2n}(z)$$

$$- (n-1)^{2(n-1)}\, p_{n-1}(z) - n^{5n}\, p_{2n+1}(z) \qquad (n \text{ odd}),$$

$$z^n = \left(\frac{1}{2}\, n\right)^n p_n(z) + \left(\frac{1}{2}\, n\right)^{2n} p_{n+1}(z) \qquad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = 2$, $k = -3$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 - 2 + 3 = 3.$$

This is true because $\Omega = 3$ by Corollary (1.1) of [1].

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ON THE LINEAR INDEPENDENCE OF ALGEBRAIC NUMBERS

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1. Introduction. Besicovitch [1] has proved by elementary methods involving only the concept of the irreducibility of equations the following:

THEOREM. *Let*

$$a_1 = b_1 p_1,\ a_2 = b_2 p_2, \cdots, a_s = b_s p_s,$$

where $p_1, p_2, \cdots, p_s$ *are different primes, and* $b_1, b_2, \cdots, b_s$ *are positive integers not divisible by any of these primes. If* $x_1, x_2, \cdots, x_s$ *are positive real roots of the equations*

$$x^{n_1} - a_1 = 0,\ x^{n_2} - a_2 = 0, \cdots, x^{n_s} - a_s = 0,$$

and $P(x_1, x_2. \cdots, x_s)$ *is a polynomial with rational coefficients of degree less than or equal to* $n_1 - 1$ *with respect to* x_1*, less than or equal to* $n_2 - 1$ *with respect to* x_2*, and so on, then* $P(x_1 x_2, \cdots, x_s)$ *can vanish only if all its coefficients vanish.*

It is rather surprising that this has not been proved before, since results of this kind occur as particular cases of a general investigation in the theory of algebraic numbers, and some have been known for many years. We have the well-known general problem:

PROBLEM. Let K be an algebraic number field, and let $x_1, x_2, \cdots, x_s$ be algebraic numbers of degrees $n_1, n_2, \cdots, n_s$ over K. When does the field $K(x_1, x_2, \cdots, x_s)$ have degree $n_1 n_2 \cdots n_s$ over K?

This holds if either the degrees or the discriminants over K of the fields $K(x_1), K(x_2), \cdots, K(x_s)$ are relatively prime in pairs. The first part is a simple consequence of the usual theory of reducibility when $s = 2$, and the extension is obvious. The second part for $s = 2$ is given as Theorem 87 in Hilbert's report on algebraic number fields, and its proof depends on algebraic number

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theory. The result for general s follows easily.

We discuss here the special case when $x_1, x_2, \cdots, x_s$ are specified roots of the respective equations

$$(1) \qquad x^{n_1} = a_1,\ x^{n_2} = a_2, \cdots, x^{n_s} = a_s,$$

where $a_1, a_2, \cdots, a_s$ are numbers in K. In the particular case when $n_1 a_1$, $n_2 a_2, \cdots, n_s a_s$ are relatively prime in pairs, the discriminants of the fields $K(x_1)$, $K(x_2), \cdots, K(x_s)$ are certainly relatively prime in pairs, and the foregoing conclusion holds. We consider two types of more general fields K. For the first, K and $x_1, x_2, \cdots, x_s$ are all real. For the second, K includes all the n_1th, n_2th, $\cdots$, n_sth roots of unity, and then the fields

$$K(x_1), K(x_2), \cdots, K(x_s)$$

are the so-called Kummer fields and have been known for many years. The elementary ideas used in their discussion are similar to those employed by Besicovitch. We have now the result really asked for in the problem, but stated as follows:

THEOREM. *A polynomial $P(x_1, x_2, \cdots, x_s)$ with coefficients in K and of degrees in $x_1, x_2, \cdots, x_s$, less than $n_1, n_2, \cdots, n_s$, respectively, can vanish only if all its coefficients vanish provided that the algebraic number field K is such that there exists no relation of the form*

$$(2) \qquad x_1^{\nu_1} x_2^{\nu_2} \cdots x_s^{\nu_s} = a,$$

where a is a number in K, unless

$$\nu_1 \equiv 0 \pmod{n_1},\ \nu_2 \equiv 0 \pmod{n_2}, \cdots, \nu_s \equiv 0 \pmod{n_s}.$$

If K is of the first type, then a particular case, which includes the result of Besicovitch and is equivalent to it, arises when K is the rational number field, the x's are all real, the a's are integers, a_r $(r = 1, 2, \cdots, s)$ is exactly divisible by a prime power $p_r^{\alpha_r}$ (that is, by no higher power of p) with $(\alpha_r, n_r) = 1$, the p_r are all different, and p_r is prime to a_t when $r \neq t$. The condition implied in (2) is satisfied, as follows easily from the lemma below.

When K is of the second type, the theorem is given by Hasse [2], in the equivalent form that K includes all the nth roots of unity, where n is the least common multiple of $n_1, n_2, \cdots, n_s$. Hasse, however, is also investigating the relation of the Galois group of the field $K(x_1, x_2, \cdots, x_s)$ to those of $K(x_1)$, $K(x_2)$, and so on, and so his proof is not particularly elementary. In view of all this, an elementary proof of the theorem may be worth while.

2. **Lemma.** We prove first, for completeness, a well-known result:

LEMMA. *Let K be an algebraic number-field such that either K is real and the equation $x^n - a = 0$, where a is in K, has a real root, or K contains all the nth roots of unity. Then the equation $x^n - a = 0$ is reducible in K if and only if a is the Nth power of a number in K for some $N > 1$ dividing n. When K is of the first type, a real root is the root of an irreducible binomial equation in K. When K is of the second type, $x^n - a$ factorizes completely into binomial factors $x^m - b$ in K and irreducible in K.*

Proof. Let us suppose that $x^n - a = 0$ is reducible in K. Write it as

$$x^n - a = f(x)\, g(x),$$

where

$$f(x) = x^m + b_1 x^{m-1} + \cdots + b_m,$$

the b's are in K, and $f(x)$ is irreducible in K. When K is of the first type, we may suppose x', a specified real root of $x^n - a = 0$, is a root of $f(x) = 0$. All the roots of $f(x) = 0$ are roots of $x^n - a = 0$, and so they have the form $\epsilon' x'$, where ϵ' is an nth root of unity and x' is any specified root of $x^n - a = 0$, but the specified real root when K is of the first type. From the product of the roots of $f(x) = 0$,

$$x'^m = \pm \epsilon b_m,$$

where ϵ is an nth root of unity. Hence x' is also the root of an equation

$$x^m = b,$$

where b is in K since, for the first type, $\epsilon = \pm 1$. Hence the irreducible equation $f(x) = 0$ of degree m must be the same as the binomial equation $x^m - b = 0$.

Further, the equations $x^n - a = 0$, $x^m - b = 0$ have a common root. Write $d = (m, n)$, $n = dN$, $m = dM$, where $(N, M) = 1$ and $a^M = b^N$. There exist rational integers u, v such that $uM + vN = 1$. Then

$$a = a^{uM+vN} = (b^u a^v)^N,$$

where $N \mid n$. Conversely if $a = A^N$, where A is in K and $N \mid n$, the equation $x^n - a = 0$ is obviously reducible in K.

This proves the lemma.

3. **Proof of theorem.** The ideas involved are not essentially different from those of Besicovitch. The given conditions imply that the theorem holds for $s = 1$. It will be proved by induction on s, and so it may be assumed that no such relation as $P = 0$ holds for s or fewer roots of equations satisfying the given conditions. We then prove it for $s + 1$ roots. Suppose a relation such as

$$P(x_1, x_2, \cdots, x_{s+1}) = 0 \tag{3}$$

holds, so that x_1 is a root of the equation, supposed irreducible in K,

$$P_0 x^r + P_1 x^{r-1} + \cdots + P_r = 0, \tag{4}$$

where $P_0, P_1, \cdots, P_r$ are polynomials with coefficients in K, and of degrees in $x_2, x_3, \cdots, x_{s+1}$ respectively less than $n_2, n_3, \cdots, n_{s+1}$. Since $1/P_0$ can be expressed as a polynomial in $x_2, x_3, \cdots, x_{s+1}$ with coefficients in K, we may take $P_0 = 1$. We write

$$P_1 = P_1(x_2) = P_1(x_2, x_3, \cdots, x_{s+1}),$$

and so on, according to the variable occuring in P_1 which we wish to emphasize.

Each root of the equation (4) in x can be written as

$$x = \epsilon' x_1, \qquad \text{where } \epsilon'^{n_1} = 1.$$

Hence from the product of the roots of (4), x_1 is also a root of an equation

$$\epsilon x^r = \pm P_r, \qquad \text{where } \epsilon^{n_1} = 1,$$

Also $\epsilon = 1$ when the field K is of the first type. Write

$$X_1 = \epsilon x_1^r, \quad \text{and so} \quad X_1^{n_1} = a_1^r.$$

Then by the lemma, X_1 is a root of an equation irreducible in K,

$$X_1^{N_1} = A_1 ,$$

where A_1 is in K. Also,

$$(5) \qquad X_1 = \pm P_r = Q = Q(x_2) = Q(x_2, x_3, \cdots, x_{s+1}),$$

say. Hence the relation (3) is replaced, when the new variable X_1 is introduced, by the relation (5) which is in general simpler. The equation

$$(6) \qquad (Q(x))^{N_1} - A_1 = 0$$

has a root $x = x_2$; and since $x^{n_2} - a_2$ is irreducible in the field $K(x_3, x_4, \cdots, x_{s+1})$ by the hypothesis for s variables, each root of $x^{n_2} - a_2 = 0$, for example, the conjugate x_2' of x_2, must be a root of (6); so

$$Q(x_2') = X_1' ,$$

where X_1' is one of the conjugates of X_1 since $X^{N_1} - A_1$ is irreducible in K.

Now $X_1 = Q(x_2)$ is the root of the equation in $K(x_3, x_4, \cdots, x_{s+1})$,

$$F = (X - Q(x_2))\ (X - Q(x_2'))\cdots = 0,$$

where the product is extended to all the conjugates of x_2. Since all the roots of the equation $F = 0$ in X are conjugates of X_1, and since, by the hypothesis for s variables, $X^{N_1} - A_1$ is irreducible in $K(x_3, x_4, \cdots, x_{s+1})$, we must have

$$F = (X^{N_1} - A_1)^{M_1}$$

for some integer $M_1 > 0$, and so $n_2 = M_1 N_1$. Since $N_1 > 1$, on comparing coefficients of $X^{n_2 - 1}$, we obtain

$$(7) \qquad \sum Q(x_2') = 0, \ \sum X_1' = 0,$$

where the sum is extended over all the conjugates of x_2 and X_1, respectively. There are of course exactly M_1 conjugates of x_2 which give the same X_1.

Write now

$$X = Q(x) = B_0 x^{n_2 - 1} + B_1 x^{n_2 - 2} + , \cdots , + B_{n_2 - 1},$$

where $B_0 = B_0(x_3, x_4, \cdots, x_{s+1})$, and consider all the relations obtained by changing x into x_2 and all its conjugates. By addition, on noting (7), we get $B_{n_2-1} = 0$. Write now

$$X_1/x_2 = X_1'.$$

Then by our condition and by our lemma, X_1' must be the root of an irreducible equation in K,

$$X'^{N'} = A',$$

and the conditions involved in (2) still hold. Proceeding as before, we get $B_{n_2-2} = 0$, and so on until $B_1 = 0$. By the theorem for s variables, a relation such as

$$X_1/x_2^{n_2-1} = B_0$$

is impossible since $X_1/x_2^{n_2-1}$ is the root of an irreducible binomial equation. This finishes the proof.

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ALTERNATING METHOD ON ARBITRARY RIEMANN SURFACES

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1. Introduction. Schwarz gave the first rigorous construction of harmonic functions with given singularities on closed Riemann surfaces, by means of his alternating method for domains with annular intersection [**16**]. The method also is directly applicable to open Riemann surfaces of finite genus, since these can always be continued so as to form closed surfaces [**7**; **8**]. For surfaces of infinite genus, this continuation is no longer possible. But if the surface is of parabolic type, Schwarz's method can still be used, a "null boundary" having no effect on the behaviour of the alternating functions [**5**; **11**]. In the general case, there are two obstacles which prevent using Schwarz's method as such. First, if the surface has a large (ideal) boundary, the alternating functions are not determined by their values on the relative boundaries. Second, Schwarz's convergence proof fails, since the Poisson integral is inapplicable on arbitrary Riemann domains. We are going to show that, by certain changes of Schwarz's original method, these difficulties can be overcome.

This paper is a detailed exposition of a reasoning outlined in preliminary notes [**9**-**11**]. The manuscript of the paper was communicated (in French) to the Helsinki University in December, 1949. In the meanwhile, the author published a linear operator method [**13**], which also can be used to establish the results of these notes. A presentation of the classical alternating method for arbitrary Riemann surfaces seems, however, to have independent interest from a methodological viewpoint; such a presentation is the purpose of this paper.

The alternating method on Riemann surfaces, as sketched in [**9**-**11**], was referred to also in the recent papers of Kuramochi [**1**], Kuroda [**2**], Mori [**3**], and Ohtsuka [**6**]. A historical note on the method was given in [**15**].

2. Functions with vanishing conjugate a_0-periods. We start with two lemmas, which are basic for the alternating procedure.

Let R be an arbitrary Riemann surface, and G a subdomain, compact or not. The relative boundary a_0 of G, that is, the set of boundary points of G, interior

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to R, is assumed to consist of a finite number of closed analytic Jordan curves. On a_0, let f be a real single-valued function, harmonic in an open set containing a_0.

LEMMA 1. *There exists always a harmonic single-valued function u in G with the following properties*:

1°. *u takes on the values f on a_0.*

2°. *u is bounded in G and satisfies*

$$\min_{a_0} f \leq u \leq \max_{a_0} f. \tag{1}$$

3°. *u has a finite Dirichlet integral over G,*

$$D(u) = \iint_G |\operatorname{grad} u|^2 \, dx\, dy < \infty. \tag{2}$$

Here $z = x + iy$ is a local uniformizer of R.

4°. *The period along a_0 of the harmonic function v, conjugate to u, vanishes,*

$$\int_{a_0} dv = 0. \tag{3}$$

Proof. If G is compact, the lemma is evident. Suppose now G is noncompact. We form an exhaustion $G_1 \subset G_2 \subset G_3 \subset \cdots$ of G, such that the boundary of G_n consists of a_0 and a set a_n of closed analytic Jordan curves tending, for $n \longrightarrow \infty$, to the ideal boundary of G. Let u_n be a harmonic function in G_n which coincides with f on a_0 and assumes on a_n a constant value c_n. By Schwarz's reflexion principle, it is easy to see that u_n is harmonic still on $a_0 + a_n$.

We fix the constant c_n as follows. We observe that u_n depends continuously on c_n. The same is also true for the normal derivative $\partial u_n/\partial n$ on a_0 and, consequently, for

$$\int_{a_0} dv_n = \int_{a_0} \frac{\partial v_n}{\partial s}\, ds = \int_{a_0} \frac{\partial u_n}{\partial n}\, ds, \tag{4}$$

where the meaning of v_n and ds is evident. If we choose

$$c_n = \min_{a_0} f,$$

then obviously $\partial u_n/\partial n \geq 0$, if n denotes the interior normal of a_n with respect to

G_n. Hence, the period (4) is nonpositive in this case. If we had chosen

$$c_n = \max_{a_0} f,$$

we would have found by the same reasoning that the integral (4) is nonnegative. Thus, there must exist a value c_n such that

$$\min f \leq c_n \leq \max f,$$

and such that the integral (4) vanishes. In the sequel we suppose that the constant c_n has been selected according to this condition. We have then in G_n the uniform estimate

$$\min f \leq u_n \leq \max f. \tag{5}$$

In the sequence of the uniformly bounded functions u_n, there is a subsequence which converges uniformly in every closed subdomain of G to a single-valued function u, harmonic on $G + a_0$.

In order to see that u is uniquely determined, we shall prove that the sequence $\{u_n\}$ itself, not only a subsequence, converges. Let x_n be the harmonic function in G_n with $x_n = 0$ on a_0, $x_n = 1$ on a_n. The sequence $\{x_n\}$ decreases monotonically, converging to a harmonic function x on G with $x = 0$ on a_0. If $x \equiv 0$, u is necessarily unique, since the difference $u' - u''$ of two functions u would assume, by

$$\int d(v' - v'') = 0,$$

both positive and negative values, and would be dominated by a multiple of x. Hence we can confine our attention to the case $x \not\equiv 0$.

By Green's formula

$$\int_{-a_0+a_n} x_n \, d\overline{v}_n - u_n \, d\overline{y}_n = 0,$$

where y_n is the harmonic conjugate of x_n, we have

$$c_n = -\frac{\int_{a_0} f \, d\overline{y}_n}{\int_{a_0} d\overline{y}_n}. \tag{6}$$

Hence c_n converges to a unique constant c. Now let z be an arbitrary (fixed) point of G. For sufficiently large n, z is an interior point of G_n. Let g_n be the Green's function of G_n with the logarithmic pole at z. Draw a small circle C about z. The Green's formula

$$\int_{-a_0+a_n+C} u_n\, d\bar{h}_n - g_n\, d\bar{v}_n = 0,$$

where h_n is the harmonic conjugate of g_n, yields, if we let C shrink to the point z,

$$u_n(z) = \frac{1}{2\pi}\left[\int_{a_0} f\, dh_n + c_n\left(2\pi - \int_{a_0} d\bar{h}_n\right)\right]. \tag{7}$$

This shows the convergence of $u_n(z)$ and thus uniqueness of u.

In order to prove that the function u satisfies the conditions $1° - 3°$, we note that the u_n converge uniformly even on the closure of G_n. In fact, for $\epsilon > 0$ and for m, q sufficiently large, we have $(u_m - u_q) < \epsilon$ on a_n; and on a_0 this difference vanishes. By the maximum principle, the Cauchy criterion is fulfilled on the closure of G_n. In view of the harmonic boundary values and Schwarz's reflexion principle, the convergence is uniform even in a domain slightly extended beyond a_0 and a_n. From this we conclude that all derivatives of u_m converge in the closure of G_n.

From the uniform convergence it follows that u takes on the value f on a_0. The condition $2°$ is guaranteed by (5). In order to study the condition $3°$ we observe that, for $p > 0$,

$$D(u) = \lim_{n\to\infty} D_n(u) = \lim_{n\to\infty}\lim_{p\to\infty} D_n(u_{n+p}), \tag{8}$$

where D refers to G and D_n to G_n. We have

$$D_n(u_{n+p}) \le D_{n+p}(u_{n+p}) = \int_{-a_0+a_{n+p}} u_{n+p}\, dv_{n+p}. \tag{9}$$

In this expression, we have

$$\int_{a_{n+p}} u_{n+p}\, dv_{n+p} = c_{n+p}\int_{a_{n+p}} dv_{n+p} = c_{n+p}\int_{a_0} dv_{n+p} = 0. \tag{10}$$

Since the integral on the right in (9) extended over a_0 converges because of the uniform convergence of the u_n and grad v_n, this integral is uniformly bounded. Hence, the condition $3°$ is fulfilled.

The condition $4°$ follows again by the uniform convergence of grad v_n on a_0.

This completes the proof of Lemma 1.

3. Functions with nonvanishing conjugate a_0-periods. Suppose now that the region G is not compact.

LEMMA 2. *There exists always a single-valued harmonic function u on G which coincides with f on a_0, and whose conjugate function v has the period*

$$\int_{a_0} dv = 1. \tag{11}$$

Proof. It suffices to consider the case $f \equiv 0$; in order to pass to the general case we have only to add to the constructed function a function furnished by Lemma 1.

Let now u_n be a harmonic function in G_n which vanishes on a_0 and assumes a constant value d_n on a_n, such that the period of the conjugate function v_n of u_n is

$$\int_{a_0} dv_n = 1. \tag{12}$$

This choice is always possible, since the value of the foregoing integral is proportional to d_n. Obviously u_n is a multiple of the harmonic measure of a_n.

By Green's formula

$$\int_{-a_0 + a_n} (u_{n+p}\, dv_n - u_n\, dv_{n+p}) = 0,$$

we have

$$\int_{a_n} u_{n+p}\, dv_n = d_n. \tag{13}$$

On the other hand, for the functions u_{n+p}, positive in G_{n+1}, we can use Harnack's principle, which can be expressed, in the present case, as follows. For all the functions u_{n+p}, there is a constant $M < \infty$ such that, on a_n, interior to G_{n+1},

$$\max_{a_n} u_{n+p} < M \min_{a_n} u_{n+p}. \tag{14}$$

Hence, by (13) and

$$\int_{a_n} dv_n = 1,$$

we have

$$\max_{a_n} u_{n+p} < M \int_{a_n} u_{n+p} \, dv_n = M \, d_n. \tag{15}$$

Thus, by the maximum principle, the functions u_{n+p} are uniformly bounded in G_n and form a compact family.

4. Oscillation of functions. In order to prove the convergence of the alternating functions, we still need a lemma concerning oscillations of functions.

Let R be an arbitrary Riemann surface and R_0 a compact closed point-set on R. Consider all single-valued harmonic functions u on R.

LEMMA 3. *There exists a positive constant $q < 1$, independent of u, such that for every u the oscillations of u on R and R_0,*

$$\begin{aligned} S(u, R) &= \sup_R u - \inf_R u \\ S(u, R_0) &= \max_{R_0} u - \min_{R_0} u, \end{aligned} \tag{16}$$

satisfy the inequality

$$S(u, R_0) \le q \, S(u, R). \tag{17}$$

Proof. For the two cases $S(u, R) = 0$ and $S(u, R) = \infty$, the proposition (17) is evident; thus, it suffices to consider bounded nonconstant functions u. We normalize these functions, without loss of generality, by adding a constant and multiplying by a constant such that

$$\sup_R u = 1, \quad \inf_R u = 0. \tag{18}$$

This being done, we have to prove the existence of a constant $q < 1$ such that $S(u, R_0) < q$. If such a constant did not exist, there would be a sequence of functions $u_1, u_2, u_3, \cdots$ such that

$$\lim_{n \to \infty} S(u_n, R_0) = 1, \tag{19}$$

and, consequently,

$$\max_{R_0} u_n \longrightarrow 1, \quad \min_{R_0} u_n \longrightarrow 0. \tag{20}$$

Among the functions u_n, uniformly bounded on R, one can select a subsequence, say again $\{u_n\}$, which tends uniformly to a function u^*, harmonic and single-valued on R. The points P_n and Q_n where u_n assumes maximum and minimum values, respectively, on the closed set R_0, accumulate at some points P^* and Q^* of R_0,

$$(21) \qquad P_n \longrightarrow P^*, \qquad Q_n \longrightarrow Q^*.$$

It is easily seen that

$$(22) \qquad u^*(P^*) = 1 \text{ and } u^*(Q^*) = 0.$$

In fact, if $u^*(P^*)$ were < 1, let ϵ be a positive constant, $\epsilon < 1/2(1 - u^*(P^*))$. By the continuity of u^*, there would be a neighborhood K of P^* such that, at each point P of K,

$$u^*(P) < u^*(P^*) + \epsilon.$$

On the other hand, by the definition of P_n, for sufficiently large n,

$$u_n(P_n) > 1 - \epsilon,$$

and the points P_n lie on K. Thus, at these points P_n, one would have

$$u_n(P_n) - u^*(P_n) > 1 - u^*(P^*) - 2\epsilon = \text{const.} > 0,$$

in contradiction to the uniform convergence of the u_n to u^*. This proves the first equality (22). The second one is proved in the same manner.

Consequently, the function u^* would be harmonic, single-valued, and nonconstant on R, and would assume its maximum and minimum values at interior points of R. This violation of the maximum principle disproves our antithesis. The lemma follows.

5. The existence theorem. After these preparations we are able to establish existence of the harmonic functions in question on the whole surface. Let R_0 be a subdomain of R whose relative boundary, that is, the set of boundary points interior to R, consists of a finite set of closed analytic Jordan curves. The complement $G = R - R_0$ then consists of a finite number m of disjoint domains G_i $(i = 1, 2, \cdots, m)$, compact or not. Let now a_i be the common part of the boundaries of R_0 and G_i.

In each G_i, let u_i be a given function, vanishing on a_i, harmonic, single-valued and nonconstant in a neighborhood of a_i, having otherwise arbitrary singularities and, in case G_i is noncompact, an arbitrary behaviour at the common (ideal) part of the boundaries of R and G_i. Denote by ds an arc element of a_i, and by $\partial u_i/\partial n$ the normal derivative of u_i in the interior direction of G_i.

THEOREM. *If R_0 is compact, then the condition*

$$\sum_{i=1} \int_{a_i} \frac{\partial u_i}{\partial n}\, ds = 0 \tag{23}$$

guarantees the existence of a function f on the whole surface R, satisfying the following conditions:

1°. *The function is harmonic, single-valued and nonconstant outside the possible singularities of the u_i.*

2°. *The difference $f - u_i$ is harmonic, single-valued, and bounded in the whole region G_i, and has a finite Dirichlet integral over G_i.*

In case R_0 is noncompact, the existence of f satisfying 1° *and* 2° *is always assured, independently of the condition* (23). *If this is satisfied, f is bounded in R_0 and has there a finite Dirichlet integral.*

Proof. Consider first the case where R_0 is compact. Let R' be another compact region ($\subset R$), containing the closure of R_0 in its interior, and bounded by a finite number of closed analytic Jordan curves. The intersection $H_i = R' \cap G_i$ is supposed to consist of one single region, bounded by a_i and the intersection b_i of G_i and the boundary of R'. Denote, for the time being, u_i by u_{i0}. In R', let f_0 be the harmonic function coinciding with u_{i0} on b_i. In G_i, form, by the procedure of Lemma 1, a function h_{i1}, harmonic and single-valued in G_i, coinciding with f_0 on a_i, bounded by the inequalities

$$\min_{a_i} f_0 \le h_{i1} \le \max_{a_i} f_0,$$

possessing a finite Dirichlet integral over G_i,

$$D_i(h_{i1}) < \infty,$$

and satisfying the condition

$$\int_{a_i} dk_{i1} = 0,$$

where k_{i1} is the harmonic conjugate of h_{i1}. Write, in G_i,

$$u_{i1} = u_{i0} + h_{i1}. \tag{24}$$

Let f_1 be the harmonic function in R' coinciding with u_{i1} on b_i. We then form again by the procedure of Lemma 1 a harmonic function h_{i2} in G_i which assumes the values f_1 on a_i and has the corresponding boundedness properties. We thus obtain successively a sequence of functions h_{in} and u_{in} in G_i, and f_n in R', determined by the conditions

$$\text{(25)} \qquad \begin{cases} f_n = u_{in} \text{ on } b_i, \\ h_{i(n+1)} = f_n \text{ on } a_i \\ u_{i(n+1)} = u_{i0} + h_{i(n+1)} \text{ in } G_i, \end{cases} \qquad (n = 0, 1, 2, \cdots),$$

and having the properties

$$\text{(26)} \qquad \min_{a_i} f_n \le h_{i(n+1)} \le \max_{a_i} f_n,$$

$$\text{(27)} \qquad D_i(h_{in}) < \infty,$$

$$\text{(28)} \qquad \int_{a_i} dk_{in} = 0,$$

where k_{in} is the conjugate function of h_{in}.

One has to prove the convergence of the functions f_n and u_{in} toward a desired common function f. We shall show first the convergence of the functions f_n on the closure $\overline{R}'$ of R'.

By Cauchy's criterion, this convergence is assured as soon as the difference $f_{n+p} - f_n$ tends, for $n, p \longrightarrow \infty$, toward zero on the boundary b of R'. In order to use Lemma 3, we shall reduce estimation of this difference to that of its oscillation on b,

$$\text{(29)} \qquad |f_{n+p} - f_n| \le S(f_{n+p} - f_n;\ b),$$

this inequality being valid as soon as

$$\text{(30)} \qquad \min_{R'} |f_{n+p} - f_n| = 0.$$

We shall now prove the latter relation.

Let x_i be, in $H_i = R' \cap G_i$, the harmonic function vanishing on a_i and assuming the constant value 1 on b_i. Let y_i be the conjugate function of x_i. The condition (30) is satisfied if

$$\sum_i \int_{b_i} (f_{n+p} - f_n)\, dy_i = 0. \tag{31}$$

In order to establish this equation, we make use of Green's formula

$$\int_{-a_i+b_i} (f_n\, dy_i - x_i\, dg_n) = 0,$$

where g_n is the harmonic conjugate of f_n. It follows, by

$$\sum \int_{b_i} dg_n = 0,$$

that

$$\sum \int_{a_i} f_n\, dy_i = \sum \int_{b_i} f_n\, dy_i. \tag{32}$$

On the other hand, the formula

$$\int_{-a_i+b_i} (u_{i(n+1)}\, dy_i - x_i\, dv_{i(n+1)}) = 0$$

gives, in view of (23), (28), and, accordingly, of

$$\sum \int_{b_i} dv_{i(n+1)} = \sum \int_{b_i} dv_{i0} = 0,$$

the relation

$$\sum \int_{a_i} u_{i(n+1)}\, dy_i = \sum \int_{b_i} u_{i(n+1)}\, dy_i. \tag{33}$$

By (25), (32), and (33), we have

$$\sum \int_{b_i} f_n\, dy_i = \sum \int_{b_i} f_{n+1}\, dy_i. \tag{34}$$

This yields the desired equality (31).

The problem of convergence of f_n has herewith been reduced to the estimation of the oscillation $S(f_{n+p} - f_n;\, b)$. We have first

$$(35)\qquad S(f_{n+p}-f_n;\ b)\le\sum_{m=1}^{p} S(f_{n+m}-f_{n+m-1};\ b).$$

To estimate $S(f_{n+1}-f_n;\ b)$, note first that

$$(36)\qquad f_{n+1}-f_n=h_{i(n+1)}-h_{in}\ \text{ on }\ b_i,$$

$$(37)\qquad f_n-f_{n-1}=h_{i(n+1)}-h_{in}\ \text{ on }\ a_i.$$

Since the functions $h_{i(n+1)}$ and h_{in} were constructed by the procedure of Lemma 1 as limits of certain harmonic functions coinciding with f_n and f_{n-1}, respectively, on a_i, and satisfying the condition (26), the difference $h_{i(n+1)}-h_{in}$ can be considered as defined by the procedure of Lemma 1, with the boundary values f_n-f_{n-1} on a_i. Thus, this difference satisfies the corresponding condition in the whole G_i:

$$(38)\qquad \min_{a_i}(f_n-f_{n-1})\le h_{i(n+1)}-h_{in}\le\max_{a_i}(f_n-f_{n-1}).$$

The relations (36) - (38) yield

$$(39)\qquad S(f_{n+1}-f_n;\ b)\le S(f_n-f_{n-1};\ a),$$

where a is the boundary of R_0.

On the other hand, by Lemma 3, applied to the difference f_n-f_{n-1}, the domain R', and the boundary of R_0, we have

$$(40)\qquad S(f_n-f_{n-1};\ a)\le q\cdot S(f_n-f_{n-1};\ b),$$

q being a positive constant <1. Thus,

$$(41)\qquad S(f_{n+1}-f_n;\ b)\le q\ S(f_n-f_{n-1};\ b).$$

By repetition of the same reasoning starting from f_n-f_{n-1}, and so on, we obtain the desired estimate

$$(42)\qquad S(f_{n+1}-f_n;\ b)\le q^n\,S_0,$$

where S_0 signifies the constant $S(f_1-f_0;\ b)$.

Applied to (35), this yields

$$(43)\qquad S(f_{n+p}-f_n;\ b)<q^n\,\frac{S_0}{1-q}.$$

The right side tends to zero, independently of p. By (29), Cauchy's criterion is satisfied and the uniform convergence of the functions f_n to a single-valued harmonic function f in R' has been proved.

The convergence of the functions h_{in} follows immediately. In fact, the relation (38), applied to the difference $h_{i(n+p)} - h_{in}$, gives

$$\max_{G_i} |h_{i(n+p)} - h_{in}| \leq \max_{a} |f_{n+p-1} - f_{n-1}|. \tag{44}$$

This implies, by the convergence of f_n, that of h_{in}. The limit function h_i is harmonic and single-valued in G_i. The corresponding limit of the functions u_{in} is $u_i + h_i$, where we use again u_i instead of u_{i0}.

The functions f and $u_i + h_i$ are identical in $H_i = R' \cap G_i$. In fact, the difference $f_n - u_{in}$ vanishes on b_i and coincides with $f_n - f_{n-1}$ on a_i, thus tending to zero on $a_i + b_i$ and hence on H_i.

Denote in the sequel by f the function thus obtained on the whole surface R. It remains to show that it satisfies the conditions $1° - 2°$ of the theorem.

Since the difference $f - u_i = h_i$ is harmonic and single-valued in G_i, the same is true of the function f except at the singularities of u_i. We shall show that f does not reduce to a constant.

Let h_i' be the harmonic function in G_i, constructed by the procedure of Lemma 1 to coincide with $h_i = f - u_i$ on a_i. Then $h_i' - h_{in}$ is the function in G_i corresponding, by this procedure, to the values $h_i' - h_{in}$ on a_i. By the relations

$$\min_{a_i} (h_i - h_{in}) \leq h_i' - h_{in} \leq \max_{a_i} (h_i - h_{in}),$$

valid in G_i, and by the convergence $h_{in} \longrightarrow h_i$ on a_i, the functions h_{in} converge uniformly to h_i' in G_i; that is, $h_i' \equiv h_i$. By Lemma 1, this implies that

$$\min_{a_i} f \leq h_i \leq \max_{a_i} f. \tag{45}$$

If now f were constant, the same would be the case with h_i, hence also with $u_i = f - h_i$, contrary to our assumptions. This proves the property $1°$.

Property $2°$ follows from the fact just mentioned that $h_i = f - u_i$ is a harmonic function in G_i constructed by the procedure of Lemma 1. This completes the proof of the theorem for the case where R_0 is compact.

The case where R_0 is noncompact reduces simply to the preceding case. We only have to isolate the common part of the boundaries of R_0 and R from $R - R_0$

by a finite set α_0 of simple analytic Jordan curves which divide R_0 into a compact domain R_0^* and a noncompact domain G_0. By Lemmas 1 and 2, there exists in G_0 a function u_0, harmonic and single-valued, vanishing on α_0 and having a prescribed period for the conjugate function v_0. We select this period in accordance with the condition

$$\int_{\alpha_0} dv_0 = -\sum_1^m \int_{\alpha_i} dv_i. \tag{46}$$

All the assumptions of the first part of our theorem are thus satisfied. Hence, there exists a function f on R which fulfills the conditions stated in the latter part of our theorem.

6. **Applications.** The theorem thus proved has applications to the classification of Riemann surfaces and to the theory of Abelian integrals as announced in [10; 11]. Here we confine our attention to some typical corollaries.

COROLLARY 1. *There are Green's functions on a Riemann surface R if and only if the boundary has a positive harmonic measure.*

Proof. Suppose there is a Green's function g on R. Let P: $z = 0$ be its logarithmic pole, in a parameter disc K: $|z| \leq 1$. In $G_1 = R - \overline{K}$, let u be the harmonic function constructed by the procedure of Lemma 1 for values $u = g$ on α_1: $|z| = 1$. Then $g - u$ is bounded in G_1,

$$|g - u| \leq M < \infty,$$

and has a nonvanishing conjugate period. This clearly implies the existence of a nonvanishing harmonic measure ω in G_1.

Conversely, suppose $\omega \not\equiv 0$ in G_1. Multiply ω by such a constant that the conjugate of the function u_1 thus obtained has the period 2π along α_1. Take as the domain R_0 of our theorem the annulus $1/2 < |z| < 1$. In G_2: $|z| < 1/2$ write $u_2 = \log 1/|z|$. By our theorem, there is a function g' on R with the pole $\log 1/|z|$ at $z = 0$ and such that $g' - u_1$ is bounded in G. The existence of a Green's function follows.

This result [10], proved later also by Virtanen [18] and Kuroda [2], shows that the classification of Riemann surfaces in those with "null-boundary" and "positive boundary" coincides with Riemann's classification on the basis of existence or nonexistence of Green's functions.

Another application of our theorem is a criterion for the existence of single-valued nonconstant harmonic functions which are bounded (HB) or have a finite Dirichlet integral (HD). It was stated by Nevanlinna [4] and Virtanen [17] that there are functions HB or HD on R if and only if R has a null-boundary. This assertion has been disproved by Ahlfors and Royden. A correct criterion follows:

COROLLARY 2. *There are functions* HB *or* HD *on a given Riemann surface R if and only if some function u of class* HB *or* HD *respectively in G satisfies the conditions $u = 0$ on a, $\int_a dv = 0$.*

The condition of a positive harmonic measure is equal to the first condition given above [8]. Thus, the inadequacy of Nevanlinna's statement is due to the lack of the second condition.

A further application deals with Abelian integrals. The following problem was stated by Myrberg in 1948 (October 13, at Helsinki University): Does there exist a nonconstant harmonic function with a finite Dirichlet integral on an arbitrary open Riemann surface R. The above theorem gives [11]:

COROLLARY 3. *On an arbitrary Riemann surface of positive genus there exist Abelian integrals of the first, second, and third kind which possess a finite Dirichlet integral outside a neighborhood of the singularities.*

The Abelian integrals, the existence of which was thus proved, have later been investigated by Virtanen and Nevanlinna. The existence proof can also be performed by adapting the classical reasoning of Weyl.

Another immediate consequence of the foregoing theorem is the following result, proved first by Nevanlinna [5] using integral equations. Let R be an open Riemann surface of parabolic type, and let A and B be two noncompact subdomains such that $A \cap B$ is a doubly connected region, bounded by two analytic Jordan curves. Let a and b be two single-valued harmonic functions in $A \cap B$.

COROLLARY 4. *If the difference of the conjugate functions of a and b is single-valued, then there exists a harmonic function f on R such that $f - a$ in A, $f - b$ in B, are harmonically continuable, single-valued, and bounded.*

To prove this, we have only to select as the domain R_0 of our theorem a region interior to $A \cap B$, separating the two boundary curves of the latter, and the existence of f is assured.

In several related problems, an extremal method [14] seems to be more powerful than the alternating methods. A comparative survey on these methods was given in [15].

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ITERATES OF ARITHMETIC FUNCTIONS AND A PROPERTY OF THE SEQUENCE OF PRIMES

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1. Introduction. In a previous paper [2], the author has investigated certain properties of the iterates of arithmetic functions which are of the following form. For $n = \prod p_i^{\alpha_1}$,

$$f(n) = \prod f(p_i)\,[A(p_i)]^{\alpha_i - 1}, \tag{1.1}$$

where $f(p_i)$ is an integer, $1 < f(p_i) < p_i$, and $A(p_i)$ is an integer $\leq p_i$, for odd primes p_i; whereas $f(2) = 1$, $A(2) = 2$. We shall denote the set of these arithmetic functions by K. These conditions ensure that for $n > 2$, $f(n) < n$, and hence if $f^k(n)$ denotes the k-th iterate of f there is a unique integer k such that

$$f^k(n) = 2. \tag{1.2}$$

For this k we write $k = C_f(n)$. We define

$$C_f(1) = C_f(2) = 0.$$

In this paper we propose to consider the problem of determining a $g \in K$ such that for all odd primes p, and all $f \in K$,

$$C_g(p) \geq C_f(p). \tag{1.3}$$

The solution to this problem produces an interesting property of the sequence of primes in that we shall show that (1.3) is equivalent to having g skip down through the primes. More precisely, if $p_1 = 2$, $p_2 = 3, \cdots$, and in general p_i denotes the i-th prime, (1.3) is equivalent to having $g(3) = 2$, $g(5) = 4$ or 3, and

$$g(p_i) = p_{i-1} \quad \text{for } i > 3. \tag{1.4}$$

2. A theorem concerning functions of K. In carrying out the proof of the result

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stated in the introduction, we shall require a certain property of the iterates of the functions of K, which we now derive.

For $n = \prod p_i^{\alpha_i}$, we define the arithmetic function $A(n)$ as

$$A(n) = \prod [A(p_i)]^{\alpha_i},$$

where the $A(p_i)$ are as given in (1.1). It then follows that, for all integers m and n,

$$A(mn) = A(m)\,A(n) \quad \text{and} \quad A(n) \leq n.$$

LEMMA 2.1. *For any divisor d of n, we have for $f \in K$,*

$$\frac{A(d)\,f(n)}{f(d)} \leq n, \tag{2.1}$$

where $A(d)\,f(n)/f(d)$ is an integer.

Proof. We can write

$$\begin{aligned} f(n) &= A(n) \prod_{p \mid n} \frac{f(p)}{A(p)} \\ &= A(n) \prod_{p \mid d} \frac{f(p)}{A(p)} \prod_{\substack{p \mid n \\ p \nmid d}} \frac{f(p)}{A(p)} \\ &= A(n)\, \frac{f(d)}{A(d)} \cdot \frac{f(d')}{A(d')}, \end{aligned}$$

where

$$d' = \prod_{\substack{p \mid n \\ p \nmid d}} p,$$

so that d' divides n. Since $A(n)$ is completely multiplicative, we have then

$$A(n) = A\left(\frac{n}{d'}\right) A(d'), \quad \text{or} \quad \frac{A(n)}{A(d')} = A\left(\frac{n}{d'}\right).$$

Hence

$$\frac{A(d)\,f(n)}{f(d)} = A\left(\frac{n}{d'}\right) f(d') \leq n,$$

where clearly $A(d)\,f(n)/f(d)$ is an integer.

LEMMA 2.2. *For $f \in K$, if $e(n) = 0$ or 1 according as n is odd or even,*

$$C_f(2n) \leq C_f(n) + e(n). \tag{2.2}$$

Proof. Since $f \in K$, we have $f(2) = 1$, $A(2) = 2$, and hence

$$f(2n) = 2f(n) \quad \text{or} \quad f(n),$$

where if n is odd $f(2n) = f(n)$ and $C_f(2n) = C_f(n)$. Otherwise, continuing, we have

$$f^2\,(2n) = 2f^2(n) \text{ or } f^2(n)$$
$$\cdots\cdots$$

and in general

$$f^k(2n) = 2f^k(n) \quad \text{or} \quad f^k(n).$$

Then taking $k = C_f(n)$ we get

$$f^k(2n) = 4 \text{ or } 2,$$

so that

$$C_f(2n) \leq k + 1 = C_f(n) + 1.$$

THEOREM 2.1. *If x is such that for all $z < x$, $C_f(z) < C_f(x)$, where $f \in K$, then for all y,*

$$C_f(xy) \leq C_f(x) + C_f(y) + e(x). \tag{2.3}$$

Proof. We have

$$f(xy) = \frac{f(x)\,f(y)\,A(d)}{f(d)},$$

where $d = (x, y)$. Letting

$$\beta_1 = \frac{f(x)\,A(d)}{f(d)},$$

we know from Lemma 2.1 that β_1 is an integer less than or equal to x; and

$$f(xy) = \beta_1\,f(y).$$

Then similarly

$$f^2(xy) = \beta_2 f^2(y),$$

where

$$\beta_2 = \frac{f(\beta_1)A(y)}{f(y)} \leq \beta_1 \leq x,$$

$$y = (\beta_1, f(y)).$$

Thus in general we have:

$$f^k(xy) = \beta_k f^k(y), \qquad \beta_k \leq \beta_{k-1} \leq \cdots \leq \beta_1 \leq x,$$

so that, letting $k = C_f(y)$, we get

$$f^k(xy) = 2\beta_k, \qquad \beta_k \leq x.$$

Then if $\beta_k < x$ we have via Lemma 2.2, and our hypothesis,

$$\begin{aligned} C_f(xy) &= C_f(y) + C_f(2\beta_k) \\ &\leq C_f(y) + C_f(\beta_k) + 1 \\ &\leq C_f(y) + C_f(x). \end{aligned}$$

On the other hand, if $\beta_k = x$ we have

$$\begin{aligned} C_f(xy) &= C_f(y) + C_f(2x) \\ &\leq C_f(y) + C_f(x) + e(x), \end{aligned}$$

and the theorem is proved.

3. Derivation of the main result. In carrying out the proof of the equivalence of (1.3) and (1.4) we shall need certain estimates from elementary prime number theory. These results are given in the following lemma. As is conventional, we shall write $p_1 = 2$, $p_2 = 3, \cdots$, and let p_i denote the i-th prime.

LEMMA 3.1. *Letting $\pi(x)$ = the number of primes $\leq x$, we have*

(a) $2p_{i-2} > p_i$ *for* $i > 5$,

(b) *for all positive integers* $x > 2$,

$$\pi(x) - \pi\left(\frac{x}{7}\right) > \sqrt{x}. \tag{3.1}$$

Proof. Both of the above are deducible from a result of Ramanujan [1] which asserts that for $x > 300$,

$$\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{x/6 - 3\sqrt{x}}{\log x} \tag{3.2}$$

Ramanujan gives explicitly the result that for $x \geq 11$,

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 2,$$

which implies (a). As for (b), we note that since, for $x \geq 10{,}590$,

$$\frac{1}{\log x}\left(\frac{1}{6}x - 3\sqrt{x}\right) > \sqrt{x},$$

we have (3.1) for all $x > 10{,}590$. We can check (3.1) for all $x < 10{,}590$ very quickly. We check up to $x = 17$. Then let

$$\begin{array}{lll} a_0 = 10{,}590, & a_1 = 2{,}309, & a_2 = 653, \\ a_3 = 229, & a_4 = 103, & a_5 = 59, \\ a_6 = 37, & a_7 = 23, & a_8 = 17; \end{array}$$

inspecting tables of primes, we see that these numbers have the property that

$$\pi(a_{i+1}) - \pi\left(\frac{a_i}{7}\right) > \sqrt{a_i},$$

which completes the proof of (b).

We now give our main result as:

THEOREM 3.1. *If $g(x)$, $g \in K$, is such that $C_g(x)$ is maximal for all primes p, that is $C_g(p) \geq C_f(p)$ for all $f \in K$ and all p, then $g(3) = 2$, $g(5) = 4$ or 3, and, for $i > 3$, $g(p_i) = p_{i-1}$.*

Proof. Since $g \in K$, we clearly have $g(2) = g(1) = 1$; and $g(3) = 2$. Now in choosing $g(5) < 5$, we consider all possible values and choose the one which makes $C_g(5)$ a maximum. Symbolically, we may write

$$g(5) = C^{-1}\{\max[C(j),\ 0 < j < 5]\} = 4 \text{ or } 3.$$

Thus $g(5)$ has two possible values 4 or 3. Similarly proceeding to $p_4 = 7$ and $p_5 = 11$ we have

$$g(7) = C^{-1}\{\max[C(j),\ 0 < j < 7]\} = 5$$

and

$$g(11) = C^{-1}\{\max[C(j),\ 0 < j < 11]\} = 7.$$

In general, for the i-th prime we must have

$$g(p_i) = C^{-1}\{\max[C(j),\ 0 < j < p_i]\}. \tag{3.3}$$

Now it would seem that the determination of this value $g(p_i)$, since it depends upon the $C(j)$, which in turn may require the values of $g(n)$ for composite n, would remain undetermined so long as nothing is said about the function $A(n)$. However, as we shall see, the *maximum* of these $C(j)$, required in (3.3), will turn out to be completely independent of $A(n)$.

We have noted that the theorem is true for $i = 4, 5$. Proceeding by induction, assume it true for all i', $4 \le i' < n$, and consider $n > 5$. From (3.3) we see that in order to complete the proof we need only show that for any x such that

$$p_n > x > p_{n-1} \tag{3.4}$$

we must have

$$C(x) < C(p_{n-1}) = n - 2. \tag{3.5}$$

Assume that for some x satisfying (3.4), (3.5) is false, and let x be the smallest one satisfying (3.4) for which

$$C(x) \ge n - 2. \tag{3.6}$$

Then we have also

$$C(g(x)) \ge n - 3. \tag{3.7}$$

We shall now show that $g(x) \ne p_{n-1}$. For suppose that $g(x) = p_{n-1}$. Then x must have a prime divisor q such that $g(q) = p_{n-1}$. But from (3.4) we see that $q \le p_{n-1}$, which is impossible.

If $g(x) < p_{n-1}$, by our inductive hypothesis we would have

$$C(g(x)) \le C(p_{n-2}) = n - 3.$$

Now if $C(g(x)) = n - 3$, it would follow that $g(x) = p_{n-1}$. This in turn implies that p_{n-1} divides x. Since $x \ne p_{n-1}$, this yields

$$x \geq 2p_{n-1} > p_n,$$

which is a contradiction. The only alternative left is that $C(g(x)) < n - 3$, which contradicts (3.7). Thus we conclude that $g(x) > p_{n-1}$ so that we must have

$$p_n > x > g(x) > p_{n-1}.$$

Since x is the smallest integer satisfying (3.4) and not (3.5), we must have $C(g(x)) < n - 2$ or $C(x) \leq n - 2$; hence

$$C(x) = n - 2. \tag{3.10}$$

Now x is not even, for if it were we would have

$$g(x) \leq \frac{x}{2} < \frac{p_n}{2} < p_{n-1},$$

which is a contradiction. Also x is not divisible by 3 for $n > 5$; for if it were, $g(x)$ would be even and we would get, using Lemma 3.1 (a),

$$g^2(x) \leq \frac{g(x)}{2} < \frac{p_n}{2} < p_{n-2}.$$

But then

$$C(g^2(x)) \leq C(p_{n-3}) = n - 4.$$

If the inequality sign holds, this implies $C(x) < n - 2$ in contradiction to (3.10). On the other hand, if the equality sign holds then $g^2(x) = p_{n-3}$. This in turn implies that p_{n-2} divides $g(x)$. If $g(x) \neq p_{n-2}$, then

$$g(x) \geq 2p_{n-2} > p_n,$$

which is impossible. Finally, $g(x) = p_{n-2}$ implies that x is divisible by p_{n-1}, which is impossible.

Also, if x is not divisible by 5, the argument is the same as for 3. On the other hand if $g(5) = 3$, and x is divisible by 5, it results that $g^2(5)$ is even, and hence

$$g^3(x) \leq \frac{1}{2}\, g^2(x) \leq \frac{1}{2} \cdot \frac{2}{3}\, g(x) \leq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5}\, x \leq \frac{p_n}{5} < p_{n-4}.$$

But this again implies that $C(x) < n - 2$, which is impossible.

Suppose then that $p \geq 7$ is the smallest prime which divides x. Since x is composite, $1 < x/p < x$ and $p \leq \sqrt{x}$. It is clear from (3.3) and our inductive hypothesis that for $z < p$, $C(z) < C(p)$. Hence via Theorem 2.1 we have

$$n - 2 = C(x) \leq C\left(\frac{x}{p}\right) + C(p).$$

Via our inductive hypothesis we see that, since $p \leq \sqrt{x}$,

$$C(p) \leq \pi(\sqrt{x}),$$

so that

$$C\left(\frac{x}{p}\right) + \pi(\sqrt{x}) \geq n - 2. \tag{3.11}$$

Since

$$x > p_{n-1} > \frac{x}{7} \geq \frac{x}{p},$$

and

$$C(x) = C(p_{n-1}) = n - 2,$$

we have

$$C(x) - C\left(\frac{x}{p}\right) \geq \pi(x) - \pi\left(\frac{x}{7}\right) > \sqrt{x},$$

by Lemma 3.1 (b); and

$$C\left(\frac{x}{p}\right) < n - 2 - \sqrt{x}. \tag{3.12}$$

Combining (3.11) and (3.12) yields $\pi(\sqrt{x}) > \sqrt{x}$, an obvious contradiction: thus the proof of the theorem is completed.

4. Some remarks and generalizations. From the above we note that imposing the condition that the function $C_f(n)$ be maximal at the primes determines uniquely the values of $f(n)$ at the primes without restricting $A(n)$ in any way. This is natural from a certain point of view, since the function $A(n)$ plays a role only in evaluating $f(n)$ for powers of a prime. This might lead one to suspect that requiring that $C_f(n)$ be maximal at the p_i^2 in addition to the p_i would also determine the values of $A(n)$. This is in fact the case, and one may prove (we omit the proof since it is long and very similar to that of § 3):

THEOREM 4.1. *If* $C_g(x)$ *is maximal at the primes and squares of primes, then* $A_g(3) = 2$ *or* $3, A_g(5) = 5$ *or* 4, *and for* $p_i > 5$, $A_g(p_i) = p_i$ *or* p_{i-1}. *Furthermore this same maximal* $C_g(x)$ *is realized for any admissible choice of the* $A_g(p_i)$ *(that is, as either* p_i *or* p_{i-1}*).*

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CONVEXITY PROPERTIES OF INTEGRAL MEANS OF ANALYTIC FUNCTIONS

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1. Introduction. Let $f = f(z)$ denote an analytic function of the complex variable z in the open circle $|z| < R$. For each positive number t, the mean of order t of the modulus of $f(z)$ is defined as follows:

$$\mathfrak{M}_t(r; f) = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^t \, d\theta\right]^{1/t}, \qquad (0 \leq r < R).$$

The reader might consult [5, p. 143-144; 3; and 4, p. 134-146] for some of the properties of this mean value function $\mathfrak{M}_t(r; f)$.

We consider the question: does the analyticity in $|z| < R$ of the function f imply the convexity of the mean $\mathfrak{M}_t(r; f)$ as a function of r in the interval $0 \leq r < R$? It is known [1] that:

(A) Unless the function f is suitably restricted, the set of positive values t for which the question may be answered affirmatively has a finite upper bound.

(B) If the number t is of the form $2/k$, with k a positive integer, then, for every analytic function f, the mean of order t is convex.

(C) If the function f vanishes at the origin, then the mean $\mathfrak{M}_t(r; f)$ is convex for every fixed positive number t.

(D) If the function f has no zero in the circle, then its mean of order t is convex, provided that the positive number satisfies $t \leq 2$.

(E) If the function f has at most k zeros, $k \geqq 1$, in the circle, then the mean of order t is convex provided that the positive number t satisfies $t \leqq 2/k$.

The main purpose of this paper is to prove that, for every analytic function f, the mean of order four is convex. Moreover, we show by example that if the number t is greater than 5.66, then there is an analytic function whose mean of order t is not convex.

2. Means of nonvanishing functions. Assume that $g(z)$ is analytic in $|z| < R$,

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and that the expansion for $g(z)$ in the given circle is

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then the integral

$$h(r; g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta$$

has the expansion

$$h(r; g) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

valid in $r < R$. Let

$$Q(r; g, c) = hh'' - c(h')^2,$$

where primes denote differentiation with respect to r, h is the function $h(r; g)$, and c is a constant independent of the variable r and of the function g. If C is a class of functions $\{g(z)\}$, and if, for all functions g in this class C, for all $r < R$, and for a particular positive value c_0, the inequality

$$Q(r; g, c_0) \geq 0$$

holds, then the inequality

$$Q(r; g, c) \geq 0$$

holds for all $c < c_0$, all $r < R$, and all functions g in the class C. We now specify the class C to be the class of all functions $g(z)$ which are analytic and do not vanish in $|z| < R$. If $f(z)$ is in class C, then any single-valued branch of $[f(z)]^{\alpha}$ where α is an arbitrary real number, is also in class C. Given a function $f_0(z)$ in class C, and a fixed positive number t, let $g_0(z)$ be a single-valued branch of $[f_0(z)]^{t/2}$; and let

$$h_0(r) = \frac{1}{2\pi} \int_0^{2\pi} |g_0(z)|^2 \, d\theta.$$

Then

$$\mathfrak{M}_t(r; f_0) = [h_0]^{1/t};$$

and since h_0 is a nonvanishing function of r, we have

$$\frac{d^2\mathfrak{M}_t(r; f_0)}{dr^2} = P \cdot Q[r; g_0, (1 - 1/t)],$$

where

$$P = \frac{\mathfrak{M}_t(r; f_0)}{th_0^2} > 0.$$

Every function $g(z)$ in class C is a single-valued branch of $[f(z)]^{t/2}$, where $f(z)$ is some appropriate function in class C. Therefore, for positive values t, the mean $\mathfrak{M}_t(r; f)$ is a convex function of r for all functions f in class C if and only if

$$Q[r; g, (1 - 1/t)] \geq 0$$

for all functions g in class C. Since the inequality $1 - 1/t < 1 - 1/t_0$ holds for all t and t_0 satisfying $0 < t < t_0$, we conclude from the preceding remarks that, if the positive value t_0 is such that the mean $\mathfrak{M}_{t_0}(r; f)$ is convex for all nonvanishing $f(z)$, then the mean $\mathfrak{M}_t(r; f)$ is convex for all nonvanishing $f(z)$, provided that t is any positive value not exceeding t_0.

For a simple example of a function $\mathfrak{M}_t(r; f)$ which is not convex, consider the mean of order eight of a single-valued branch of

$$f(z) = \sqrt{1 + z} \qquad \text{in } |z| < 1.$$

In this case, we have

$$h(r) = 1 + 4r^2 + r^4;$$

and $[h(r)]^{1/8}$ is not convex in $0 \leq r < 1$.

Since, for every analytic function f, the mean of order two is convex, it now follows that there exists a greatest positive value t_0, in the range $2 \leq t_0 < 8$, such that $\mathfrak{M}_{t_0}(r; t)$ is convex for all nonvanishing analytic functions. It will be a corollary of our result that this greatest value t_0 satisfies the inequalities $4 \leq t_0 < 5.66$.

3. **Preliminary lemmata.** The proof of our main theorem will be based on the following lemmata.

LEMMA 1. *Let a_i $(i = 1, 2, \cdots)$ be a sequence of positive numbers such* *that the sum*

$$\sum_{i=1}^{\infty} 1/a_i$$

converges to the finite value M. If the sequence of real variables x_i $(i = 1, 2, \cdots)$ *is restricted to satisfy the inequality*

$$\sum_{i=1}^{\infty} a_i x_i^2 \leq B,$$

then the maximum value of the function

$$f = \sum_{i=1}^{\infty} x_i$$

is $(BM)^{1/2}$.

Proof. We consider first maximizing

$$f_n = \sum_{i=1}^{n} x_i,$$

with the variables subject to the condition

$$\sum_{i=1}^{n} a_i x_i^2 = B.$$

Let

$$M_n = \sum_{i=1}^{n} 1/a_i.$$

The critical points of the function f_n are at the solutions of the simultaneous equations

$$a_i x_i = a_j x_j \qquad (i, j = 1, \cdots, n),$$

which are given by

$$x_i^2 = B(M_n a_i^2), \qquad (i = 1, \cdots, n).$$

Therefore, the maximum f_n is $M_n (B/M_n)^{1/2}$ or $(BM_n)^{1/2}$. Since $M_n < M$, and all the values a_i are positive, it follows that for all n the partial sums f_n are bounded by $(BM)^{1/2}$ and the conclusion of the lemma follows.

LEMMA 2. *Let S be the sum*

$$S = \sum_{n=3}^{\infty} 1/(6n^2 - 9n + 2).$$

Then this sum S is less than 0.09504.

Proof. The function $f(n) = 1/(6n^2 - 9n + 2)$ has the following expansion in powers of $1/(n-1)$:

$$f(n) = \sum_{k=2}^{\infty} a_k/(n-1)^k,$$

with $a_2 = 1/6$, $a_3 = -1/12$, $a_4 = 5/72$. For determining subsequent values of a_k, it is convenient to use the recursion formula:

$$a_{k+2} = (a_k - 3a_{k+1})/6.$$

The coefficients a_2 and a_3 are positive and negative respectively. Therefore it follows directly from the recursion formula that the general coefficients a_k alternate in sign. By another use of the recursion formula, we see that the sum $a_k + a_{k+1}$ is equal to $(a_{k-2} - a_{k-1})/12$, and therefore that the sign of the sum $a_k + a_{k+1}$ is the same as that of the coefficient a_{k-2}, or of the coefficient a_k. Since the inequalities $|a_2| > |a_3| > |a_4|$ hold, it now follows that the numerical values of the coefficients all decrease with increasing k. Let $\zeta(k)$ be the Riemann zeta-function, and let $s(k) = \zeta(k) - 1$. Since the foregoing expansion for $f(n)$ is an absolutely convergent series, the sum S may be expanded in an alternating series of the form

$$S = \sum_{k=2}^{\infty} a_k\, s(k),$$

whose terms decrease in numerical value with increasing k. Using (see [2]) the approximations $s(2) = 0.644935$, $s(4) = 0.082324$, $s(6) = 0.017344$, $s(8) = 0.004078$, $s(10) = 0.000995$, which are too large, and the approximations $s(3) =$

0.202056, $s(5) = 0.036927$, $s(7) = 0.008349$, $s(9) = 0.002008$, which are too small, we obtain the value 0.09504 stated in the lemma by summing this last series up to and including the term for $k = 10$.

LEMMA 3. *Let*

$$y = \sqrt{x} + \sqrt{0.04752}\,\sqrt{9x^2 - 10x + 1}\,,$$

where x lies in the range $0 \leq x \leq 1/9$. Then the maximum value of y is less than $(\sqrt{2} - 1)$.

Proof. Setting the derivative of y equal to zero, we find that the value of x maximizing y is the solution of the equation

$$0.04752x(10 - 18x)^2 - (9x^2 - 10x + 1) = 0.$$

This critical value of x lies between 0.07 and 0.08. Therefore

$$\max y < \sqrt{0.08} + \sqrt{0.04752\,[9(0.07)^2 - 10(0.07) + 1]}$$

$$< 0.283 + 0.129 = 0.412.$$

Since $(\sqrt{2} - 1)$ is greater than 0.414, the conclusion of the lemma follows.

4. The mean of order four. Let

$$g(z) = [f(z)]^2$$

have the expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

valid in $|z| < R$. Following the ideas developed in § 2, we see that

$$\mathfrak{M}_4(r; f) = [h(r)]^{1/4},$$

with

$$h(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

and that $\mathfrak{M}_4(r; f)$ is convex in $r < R$ if and only if

$$Q(r) \equiv hh'' - \frac{3}{4}(h')^2 = \sum_{i,j=0}^{\infty} Q_{ij}\, p_i\, p_j\, r^{2(i+j)-2},$$

with

$$Q_{ij} = i(2i-1) + j(2j-1) - 3ij \text{ and } p_i = |a_i|^2,$$

is nonnegative in the interval $0 \leqq r < R$. The only coefficient Q_{ij} which is negative is $Q_{11} = -1$. That the mean of order four is convex may be concluded from the following theorem.

THEOREM. *If a function $g(z)$ is analytic in the circle $|z| < R$, and the function*

$$\left[\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2\, d\theta\right]^{1/4}$$

is not convex as a function of r in the interval $r < R$, then $g(z)$ is not the square of an analytic function in $|z| < R$.

Proof. It is pointed out in the introduction that if $f(0) = 0$, then the mean $\mathfrak{M}_t(r; f)$ is convex for all t. Therefore we may assume that

$$[f(0)]^2 = g(0) = p_0$$

is not zero. The hypothesis of the theorem implies that

$$Q(r) = \sum_{i,j=0}^{\infty} Q_{ij}\, p_i\, p_j\, r^{2(i+j)-2}$$

takes on negative values; since Q_{11} is the only negative coefficient, this is possible only if the value $p_1 = |a_1|^2$ is not zero. Therefore, we may make the normalizations

$$a_0 = 1,\ a_1 = \sqrt{2},\ p_0 = 1, \text{ and } p_1 = 2.$$

Let

$$Q_1(r) = 2p_0 p_1 + (12p_0 p_2 - p_1{}^2)r^2 + 2p_1\, p_2\, r^4$$

$$+ 2\sum_{n=3}^{\infty} (Q_{0n}\, p_0\, p_n\, r^{2n-2} + Q_{1n}\, p_1\, p_n\, r^{2n}),$$

with $Q_{0n} = n(2n-1)$ and $Q_{1n} = 2n^2 - 4n + 1$. Since $Q(r) \geq Q_1(r)$, and $Q_1(r)$ can be negative only for values of r satisfying

$$2p_0\, p_1 - p_1^2 r^2 < 0,$$

we have in the normalized case the result that $Q_1(r)$ is negative for some $r > 1$; and the expression

$$Q_2(r) = 4 + (12p_2 - 4)r^2 + \left[4p_2 + \sum_{n=3}^{\infty} (12n^2 - 18n + 4)p_n\right] r^4$$

also takes on negative values. The discriminant of $Q_2(r)$ as a quadratic form in r^2 must be positive. Therefore we have the inequality

$$\sum_{n=3}^{\infty} (6n^2 - 9n + 2)\, p_n < (9p_2^2 - 10p_2 + 1)/2,$$

and the result that p_2 is less than 1/9. Applying Lemma 1, we see that

$$\sum_{n=3}^{\infty} |a_n| < \sqrt{S(9p_2^2 - 10p_2 + 1)/2},$$

with

$$S = \sum_{n=3}^{\infty} 1/(6n^2 - 9n + 2).$$

By use of Lemma 2, we have

$$\sum_{n=2}^{\infty} |a_n| < \sqrt{p_2} + \sqrt{0.04752}\sqrt{9p_2^2 - 10p_2 + 1};$$

and, by use of Lemma 3, we have

$$\sum_{n=2}^{\infty} |a_n| < \sqrt{2} - 1.$$

Applying Rouché's Theorem to the function

$$g(z) = 1 + \sqrt{2}\, z + \sum_{n=2}^{\infty} a_n z^n$$

we see that, if the function $g(z)$ is analytic in the circle $|z| \leq 1$, then $g(z)$ has exactly one zero within this circle, and therefore that $g(z)$ is not the square of an analytic function in this circle. Since the convexity of the mean must break down only for values of r greater than one, we have established the theorem.

5. Examples of nonconvex means. Let $f(z)$ be a single-valued branch of the function $[(1-z)^2/(1-\epsilon z)]^{2/t}$, with $\epsilon = 0.19$. We shall show that if $t \geq 5.66$, then the mean $\mathfrak{M}_t(r; f)$ is not convex in $r < 1$. Since

$$[f(z)]^{t/2} = 1 + (-2 + \epsilon)\, z + [(1-\epsilon)^2 z^2/(1-\epsilon z)],$$

it follows that

$$\mathfrak{M}_t(r; f) = [h(r)]^{1/t},$$

with

$$h(r) = 1 + (4 - 4\epsilon + \epsilon^2)\, r^2 + [(1-\epsilon)^4 r^4/(1-\epsilon^2 r^2)].$$

By straight-forward calculation, we have

$$(1+\epsilon)\, h(1) = 6 - 2\epsilon = 5.62;\ (1+\epsilon)^2\, h'(1) = 12 - 4\epsilon^2 = 11.8556;$$

$$(1+\epsilon)^3\, h''(1) = 20 + 4\epsilon - 4\epsilon^2 - 4\epsilon^3 = 20.588164;$$

and

$$\begin{aligned}(1+\epsilon)^4\, Q(r) &= (1+\epsilon)^4\, [hh'' - (1 - 1/t)\, (h')^2] \\ &\leq (1+\epsilon)^4\, [115.71 - (1 - 1/t)\, (140.55)] \\ &< 0, \text{ if } t > 140.55/24.84, \text{ and therefore if } t \geq 5.66.\end{aligned}$$

Thus we have examples of nonconvex means $\mathfrak{M}_t(r; f)$ for $t \geq 5.66$ even under the restriction that $f(z)$ does not vanish in its circle of analyticity.

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PLANE GEOMETRIES FROM CONVEX PLATES

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1. Introduction. It is shown below that to each member of a general class of two-dimensional convex bodies there corresponds an affine geometry in the sense of Artin [1] and an S. L. space in the sense of Busemann [4].

A two-dimensional convex body is called a *convex plate.* For the few elementary properties of such plates assumed here, see [3].

Let K be a convex plate, and let K^0 denote its boundary curve. All constructions are to be made in the plane E of K. Consider an arbitrary direction ϕ in E and the two lines of support to K in this direction. Let t_0 be the line of support whose associated half-plane in the direction $\phi + \pi/2$ contains K. Let t_1 be the other line of support. For $0 < i < 1$, let t_i be the line parallel to t_0 which divides line segments extending from t_0 to t_1 in the ratio of i to $1 - i$. Let t_i cut K^0 at points R_i and T_i so that the directed segment $R_i T_i$ has direction ϕ.

For $0 < i < 1$ and $0 < j < 1$, let S_{ij} be the point which divides $R_i T_i$ in the ratio of j to $1 - j$. The set $s_j = \bigcup_i S_{ij}$ is an open Jordan arc whose endpoints are points of contact of t_0 and t_1 with K. A set s_j is called a *strut.* Other struts may be obtained by varying ϕ. When the direction needs emphasis, the above notations are modified by affixing the angle in parentheses, for example, $R_i(\phi)$ or $s_j(\phi)$. Two struts with no common points or all points in common are called *parallel.* Clearly $s_j(\phi)$ and $s_k(\phi)$ are parallel.

Under the name *Durchlinien,* Zindler [6] studied struts of the form $s_{1/2}(\phi)$. It is easy to see that $s_{1/2}(\phi)$ halves the area of K, and that the centroid of K is contained in the convex hull of this strut.

2. A preliminary theorem. This section is devoted to a proof of the following theorem. An edge of K is defined as a *(maximal) line segment in* K^0.

THEOREM. *If for distinct directions ϕ and ψ, struts $s_i(\phi)$ and $s_j(\psi)$ meet at distinct points P and Q, they meet at all points of the segment PQ. Such segments of intersection occur if and only if K has at least two edges.*

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Proof. Let $i = 1/(1+a)$ and $j = 1/(1+b)$. From the affine invariant nature of the problem, we may assume ϕ and ψ are respectively the positive x- and positive y-directions in E, where P has been chosen as the origin. We may assume the chords passing through P along the axes are P_3P_1 and P_4P_2, where P_1, P_2, P_3, and P_4 have respectively the coordinates $(a, 0)$, $(0, b)$, $(-1, 0)$, and $(0, -1)$. If P_4P_1 is parallel to P_3P_2, let n be the line parallel to these lines which passes through P. Otherwise let n be the line on P and the point of intersection of these lines. Finally, we may assume that Q lies in the first quadrant on or above the line n. Let Q have coordinates (r, s).

Let the chords through Q parallel to the axes be Q_3Q_1 and Q_4Q_2. Coordinates of Q_1, Q_2, Q_3, and Q_4 have respectively the form $(r+ap, s)$, $(r, s+bq)$, $(r-p, s)$, and $(r, s-q)$. We note that P_4P_1, n, and P_3P_2 have respectively the equations

$$ay = x - a, \quad a(b+1)y = b(a+1)x, \quad \text{and} \quad y = bx + b.$$

Since Q is on or above n,

$$b(a+1)r \leqq a(b+1)s. \tag{1}$$

Because K is convex, Q_2 cannot be above P_2P_3; that is,

$$s + bq \leqq b(r+1). \tag{2}$$

Multiply (2) by a and add to (1). This gives

$$r - a \leqq a(s-q); \tag{3}$$

that is, Q_4 is on or above P_1P_4. Moreover, equality in (3) implies equality in (1) and (2).

Consider first the case $r < a$. Here, since Q_4 cannot be above P_1P_4, it lies on P_1P_4. Thus equality holds in (2), and Q_2 lies on P_2P_3. Since P_4, Q_4, and P_1 are distinct and collinear, they are on an edge of K. Similarly, Q_2, P_2, and P_3 lie on an edge.

In the case $r \geqq a$,

$$\text{slope } P_4P_1 \leqq \text{slope } Q_4Q_1, \quad 1/a \leqq q/ap, \quad \text{and} \quad p \leqq q.$$

If $s < b$, Q_3 cannot be below P_2P_3; that is,

$$b(r+1) \leqq s + bp. \tag{4}$$

Together with (2) this yields $q \leqq p$. Hence $p = q$, and equality holds in both (2) and (4). This shows that Q_2, P_2, Q_3, P_3 are collinear, and hence on an edge of K. Furthermore, slope P_4P_1 = slope Q_4Q_1, and P_4, P_1, Q_4, Q_1 are on an edge. If $s \geqq b$, Q_3 cannot be above P_2P_3, slope $Q_3Q_2 \leqq$ slope P_3P_2, $bq/p \leqq b$, and $q \leqq p$. Again $p = q$, slope P_4P_1 = slope Q_4Q_1, and slope P_3P_2 = slope Q_3Q_2. An edge of K contains P_4, P_1, Q_4, and Q_1, and another edge contains Q_2, Q_3, P_2, and P_3.

3. Affine geometries. Consider a convex plate K with the properties:

(i) K has at most one edge;

(ii) K has no corners.

Let I be the set of inner points of K. Consider distinct points P and Q in I. Assume, for a given ϕ, that P is on $s_j(\phi)$ and Q is on $s_k(\phi)$, $j < k$. Clearly P is on $s_{1-j}(\phi + \pi)$, and Q is on $s_{1-k}(\phi + \pi)$. From considerations of continuity, there exists a direction ψ such that P and Q are on a strut $s_i(\psi)$. From this and from the previous section we have the following result.

PROPERTY I. Two distinct points in I lie on one and only one strut.

Consider now a strut $s_i(\phi)$ and a point P in I. The strut $s_j(\phi)$ which passes through P is parallel to $s_i(\phi)$. On the other hand, let $s_k(\psi)$, $\psi \neq \phi$, pass through P. Since $s_i(\phi)$ and $s_k(\psi)$ have endpoints which separate one another on K^0. these struts have some point of I in common, and the following holds.

PROPERTY II. Given a strut s and a point P in I, there is one and only one strut through P and parallel to s.

PROPERTY III. There are three points of I not on a strut.

These three properties are Axioms I, II, and III of Artin [1]. Listed in *Lattice theory* [2, p. 110] as APG1, APG2, and APG3, they classify I as a plane affine geometry.

It would be of interest to know what sets I satisfy Artin's Axiom IV (see Appendix), or even what sets have nontrivial dilatations. An ellipse K yields an I with all the desired properties. To show this it is sufficient to consider the case where K is the circle

$$x^2 + y^2 \leqq a^2.$$

Consider the sphere

$$S: x^2 + y^2 + (z - a)^2 = a^2,$$

resting on the origin of the xy-plane E. The line

$$x \cos \phi + y \sin \phi = R$$

in E projects from the center of S into a great half-circle on S. This half-circle projects perpendicularly on E into a half-ellipse, the strut $s_i(\phi)$, where

$$2i = 1 + R/\sqrt{R^2 + a^2}.$$

Thus the mapping which takes (r, θ) in I into the point (R, θ) of E, where

$$R\sqrt{a^2 - r^2} = ar,$$

places the struts in one-to-one correspondence with straight lines. In this example, we have a finite model for Euclidean geometry.

4. Other geometries. In general we may obtain a plane projective geometry from a plane affine geometry by adjoining an ideal line (see [**2**, p. 110]). In this case K^0 serves as the ideal line. The affine and projective geometries associated with K are examples of matroid lattices.

In § 3 we mapped an elliptical I onto the Euclidean plane E. A similar mapping may be defined for any I so that struts map on curves in E which satisfy the hypotheses of [**4**, p. 89, Th. 1]. It follows that a metric may be introduced (in E and hence) in I which makes of I an S. L. Space of Busemann: I will be finitely compact, convex in the sense of Menger, externally convex in the sense of Busemann, and the struts will be geodesics under this metric. This S. L. space also satisfies the Euclidean Parallel Axiom. In fact, all Hilbert's (plane) Axioms [**5**] are satisfied except the congruence axioms. The determination of the conditions under which the latter hold is an open problem.

5. Appendix. Artin's Axiom IV, not readily available to all readers, is given below after necessary introductory material. Using Axioms I-IV, we may assign coordinates (α, β) to points so that the equation of a "strut" is linear.

The set of points considered is called a *plane*. A mapping σ associates with every point P a point $P' = \sigma(P)$. A mapping is called a *dilatation* if to each pair of points P, Q correspond parallel struts s and s' such that P and Q lie on s, and P' and Q' lie on s'. The identity mapping of the plane is denoted by 1. A *translation* is a dilatation which is either 1 or else has no fixed points. A *trace* of a dilatation σ is a strut which contains a point P and its image P'. (If $P \neq P'$, there is a unique trace on P.) A *homomorphism* is a correspondence from translation τ to translation τ^α such that each trace of τ is a trace of τ^α and such that

$$(\tau_1 \tau_2)^\alpha = \tau_1^\alpha \cdot \tau_2^\alpha.$$

AXIOM IVa. Given P and Q, there exists a translation carrying P into Q.

AXIOM IVb. Given translations τ_1 and τ_2 (neither equal to 1) with the same traces, there exists a homomorphism τ^α such that $\tau_1^\alpha = \tau_2$.

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